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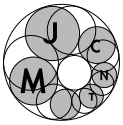

Voronoi's Conjecture for extensions of Voronoi parallelohedra

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Abstract: In 1908 Voronoi conjectured that every parallelohedron is a a Voronoi parallelohedron for some Euclidean metric in $\mathbb{E}^d$. Although the conjecture is still neither proved, nor disproved, there are several positive results for some special classes of parallelohedra. In this paper we extend the list of such classes by one new case.

Let I be a segment in the d-dimensional Euclidean space $\mathbb{E}^d$. Let P and $P+I$ be parallelohedra in $\mathbb{E}^d$, where the plus sign denotes the Minkowski sum. We prove that, if Voronoi's Conjecture holds for P, then Voronoi's Conjecture holds for $P+I$ as well.

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1. Introduction

1.1. Voronoi's Conjecture and previous partial solutions

A parallelohedron is, by definition, a convex polytope that admits a face-to-face tiling of an affine space by its translates (parallel copies). The notion of parallelohedra was first introduced by E. S. Fedorov [15] in 1885.

In 1908 G. F. Voronoi [31] posed his famous conjecture on parallelohedra. Voronoi's Conjecture has several equivalent statements (one may either use the language of lattices, or the language of quadratic forms, or combine both). First we give the most common, "lattice", version (see, for example, [14]).

CONJECTURE (VORONOI'S CONJECTURE). Every d-dimensional parallelohedron P is affinely equivalent to a Dirichlet–Voronoi domain of some d-dimensional lattice $\Lambda \subset \mathbb{E}^d$.

However, later in the paper we will use the "mixed" version (see Section 2).

We will say that *Voronoi's Conjecture is true for a certain class of parallelohedra*, if every representative of this class is affinely equivalent to a Dirichlet–Voronoi cell of some lattice.

Any parallelohedron, which is affinely equivalent to a Dirichlet–Voronoi cell of some lattice will be referred to as *Voronoi parallelohedron*. Perhaps, a more accurate term would be *affinely Voronoi parallelohedron*, but we shorten it for the sake of brevity, and because we do not want to care much about the actual affine maps. Further justification of choosing the shorter term is in Section 2.

Voronoi's Conjecture remains unresolved so far, despite a number of significant advances. The conjecture is known to be true in dimensions $d \leq 4$ (see Subsection 1.3 for discussion). Other major partial results include those by Voronoi [31], Zhitomirskii [32], and Ordine [25]. The definitions of the respective classes of parallelohedra fall outside of the scope of this paper. For details, the reader may refer to [17, Section 1].

Another major partial result is the one by R. Erdahl [14]. We will state it explicitly, as it is closely related to this paper.

First we recall that the Minkowski sum of two point sets $X, Y \in \mathbb{R}^d$ is the point set

$$X + Y = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in X, \mathbf{y} \in Y\}.$$

Taking the sum of two points is assumed to be coordinatewise, or, equivalently, the radius vector of the point $\mathbf{x} + \mathbf{y}$ is the sum of radius vectors of the points $\mathbf{x}$ and $\mathbf{y}$.

A convex polytope is called a *zonotope*, if it is a Minkowski sum of a finite number of segments. If a zonotope is a parallelohedron, then it will be called a *space-filling zonotope*.

THEOREM (ERDAHL, 1999). *Voronoi's Conjecture is true for all space-filling zonotopes.*

1.2. Extensions of parallelohedra and Voronoi's Conjecture

In [19, 20] V. Grishukhin considered a special class of parallelohedra. A parallelohedron P' belongs to this class if $P' = P + I$, where P is a parallelohedra of

the same dimension as P', I is a segment. If such an identity holds, then, following Á. Horváth [23], we will call P' an *extension* of P.

In [20] Grishukhin posed as an open problem to prove or disprove Voronoi's Conjecture for all possible extensions of Voronoi parallelohedra. The main result of this paper is that we give an affirmative answer to that problem.

In fact, Grishukhin found a necessary and sufficient condition for an extension of a Voronoi parallelohedron to satisfy Voronoi's Conjecture (see Theorems 11 and 12 of this paper). However, he could not decide if this condition holds for all possible extensions of Voronoi parallelohedra.

We will also mention that A. Végh [28] also claimed a partial result in Grishukhin's problem, see Corollary 13 and the subsequent remark.

It is not hard to see that every space-filling zonotope can be obtained by a sequence of extensions from a parallelepiped. Therefore, once Grishukhin's problem is solved, then Erdahl's theorem follows immediately.

It is also an interesting observation that space-filling zonotopes were the only class of parallelohedra such that Voronoi's Conjecture was proved for them, but without using the technique of canonical scaling. The term and the technique are due to Voronoi [31], but we also recommend the reader the paper [6] to get introduced into the technique. We claim that the results of this paper provide an effective way to compute a canonical scaling for a zonotope. Indeed, the proofs of Theorems 11 and 12 in [20] are by finding an explicit canonical scaling for the extension $P + I$, given the canonical scaling for P. Hence we can find a canonical scaling of any space-filling zonotope inductively, starting from a parallelepiped.

1.3. Examples and further open problems

Parallelohedra of dimensions 1, 2 and 3. 1-dimensional parallelohedra are segments (there are no other 1-dimensional polytopes). Every 2-dimensional parallelohedron is either a parallelogram or a centrally symmetric hexagon. Every 3-dimensional parallelohedron belongs to one of the following 5 types: parallelepiped, centrally symmetric hexagonal prism, rhombic dodecahedron, elongated dodecahedron, and truncated octahedron (see [9, 15]). Therefore for $d \leq 3$ all parallelohedra are zonotopes. By Erdahl's theorem, Voronoi's Conjecture is true for all parallelohedra of dimension $d \leq 3$.

4-dimensional parallelohedra. The classification is due to B. Delaunay [4], who found 51 combinatorial types of 4-dimensional parallelohedra, and M. Stogrin [27], who

added the missing 52-nd type and thus finished the list. Among these 52 types only 17 correspond to space-filling zonotopes. Parallelohedra of the other 35 types have the form $C_{24} + Z$, where C_{24} is an affine image of the standard 24-cell (the Dirichlet–Voronoi cell for the lattice A_4, see [2]), Z is a zonotope with $0 \leq \dim Z \leq 4$.

This fact makes the 24-cell extremely important, because it is the simplest example of a parallelohedron which is not a zonotope.

As for Voronoi's Conjecture, it is still true for 4-dimensional parallelohedra. All known proofs deduce this fact from the combinatorial classification (see [4, 16]).

Minkowski sum with a segment. The less specialized reader may benefit from the following, mostly informal description of a Minkowski sum $P + I$, where P is a d-polytope in $\mathbb{R}^d$ (not necessarily a parallelohedron), $I \subset \mathbb{R}^d$ is a segment. We also may assume that $I = [-\mathbf{x}, \mathbf{x}]$, and it is a vertical segment with $\mathbf{x}$ above $-\mathbf{x}$.

Suppose that the light falls vertically from above. Then there is a completely lightened part of ∂P (the *upper cap* $\mathrm{Cap}_{\mathbf{x}}(P)$), the opposite, completely dark, part of ∂P (the *lower cap* $\mathrm{Cap}_{-\mathbf{x}}(P)$), and several vertical facets in between, which will be called *semi-shaded facets* in our paper.

To get the Minkowski sum $P + I$ one has to move the upper cap upwards by $\mathbf{x}$, to move the lower cap downwards by $-\mathbf{x}$, and to take the convex hull of the displaced caps. That is,

$$P + I = \mathrm{conv}\big((\mathrm{Cap}_{\mathbf{x}}(P) + \mathbf{x}) \cup (\mathrm{Cap}_{-\mathbf{x}}(P) - \mathbf{x})\big).$$

Totally contracted parallelohedra. The 24-cell is also an example of a parallelohedron which is not an extension of any other parallelohedron. P. Engel [13] calls such parallelohedra *totally contracted*.

Non-extendable parallelohedra. Let E_6^* be the 6-dimensional lattice dual to the lattice E_6 (see [2]). Then its Dirichlet–Voronoi cell $P_V(E_6^*)$ does not allow any extension (see [21]), i.e, is *non-extendable*. An open problem is whether this is the minimal example (by dimension). If Voronoi's Conjecture is true for all 5-dimensional parallelohedra, then $P_V(E_6^*)$ is indeed a minimal example, since all Voronoi parallelohedra of dimension $d \leq 5$ allow an extension. (The last result is a consequence of the recent classification of 5-dimensional Voronoi parallelohedra, see [11].)

The following question is, certainly of interest (even despite its statement is not explicit): can one choose a non-extendable parallelohedron so that none of the

known theorems would imply Voronoi's conjecture for it? This is, of course, not the case for $P_V(E_6^*)$, because it satisfies Zhitomirskii's condition (see [32]).

Extensions and projections. If a parallelohedron $P' = P + I$ is an extension of a parallelohedron P, it is known that the projection of P along I, $\mathrm{proj}_I(P)$, is a parallelohedron. Can one prove that Voronoi's Conjecture is true for both P and $P + I$, if $\mathrm{proj}_I(P)$ is a Voronoi parallelohedron? If the answer is affirmative, then every minimal (by dimension) counterexample to Voronoi's Conjecture (if such a counterexample exists) is non-extendable.

Minkowski sums of two high-dimensional parallelohedra. While in this paper we consider Minkowski sums of a parallelohedron and a segment, we can also treat the second summand, the segment, as a particular case of a parallelohedron. Therefore a natural problem would be to investigate Minkowski sums of two parallelohedra with respect to the questions, whether the sum is a parallelohedron, or whether the sum satisfies Voronoi's Conjecture.

If the sum and the summands are supposed to be Voronoi parallelohedra, then it is a well-understood case (see [26]). However, without that additional assumptions almost nothing is known.

1.4. Outline of the proof

Among the statements of this paper there are four that are most important, namely Theorem 3 (the main result), Theorems 14, 4 and 5. Denote by $\mathrm{M}(d)$ ($\mathrm{A}(d)$, $\mathrm{B}(d)$, $\mathrm{C}(d)$) the following proposition: Theorem 3 (14, 4, 5 respectively) is true for parallelohedra of all dimensions up to d. The whole proof will consist of 5 main parts.

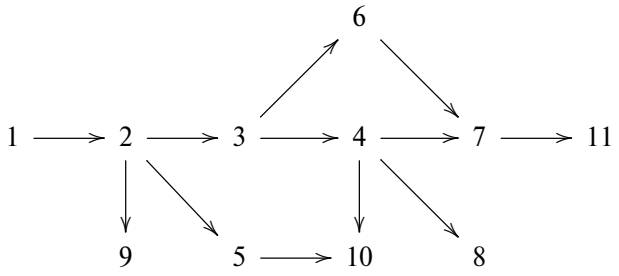
1) $\mathrm{M}(d) \Leftrightarrow \mathrm{A}(d)$.
2) $\mathrm{A}(d)$ is true for $d \leq 4$; $\mathrm{B}(d)$ and $\mathrm{C}(d)$ are true for $d \leq 2$.
3) $\mathrm{B}(d-3) \Longrightarrow \mathrm{C}(d-2)$;
4) $\big(\mathrm{A}(d-2), \mathrm{C}(d-2)\big) \Longrightarrow \mathrm{B}(d-2)$;
5) $\big(\mathrm{B}(d-2), \mathrm{C}(d-2)\big) \Longrightarrow \mathrm{A}(d)$.

Implications 3–5 make together an inductive step, while statement 2 is the base of the induction. The induction implies that the statements $\mathrm{A}(d)$, $\mathrm{B}(d)$, and $\mathrm{C}(d)$ are true for any positive integer d. In turn, it gives that $\mathrm{M}(d)$ is true for any d due to the equivalence 1, hence Theorem 3 follows.

The proof of equivalence 1 is given in Section 4, right after the introducing all the necessary notions in Sections 2 and 3.

Statements 2–5 are proved in Sections 8–11, with one section per each statement. In Sections 5, 6, and 7 we prepare all the auxiliary facts we need to perform the inductive step.

For convenience of the reader, we visualize the dependence of sections in this paper.



2. Preliminary facts

2.1. Combinatorial and affine properties of parallelohedra

Throughout the paper we assume that the origin $\mathbf{0}$ is given. Every vector $\mathbf{v}$ is considered as a radius-vector and identified with its endpoint.

We will use the notation $\operatorname{lin} L$ for the linear space associated with the affine space L. If $\mathcal{S}$ is a set of points, then $\operatorname{aff} \mathcal{S}$ denotes the affine hull of $\mathcal{S}$, i.e. the minimal affine space containing all its points.

The particularly important usage of the notation is $\operatorname{lin}\operatorname{aff} \mathcal{S}$ when $\mathcal{S}$ is a set of vectors. One can check that $\operatorname{lin}\operatorname{aff} \mathcal{S}$ is the space of all linear combinations of vectors of $\mathcal{S}$ with sum of coefficients equal to 0.

We will also need the notation for linear projections. In this paper proj_p denotes the projection *along* the linear subspace p onto some complementary affine subspace p'. If needed, p' is specified separately, otherwise it is chosen arbitrarily.

For the sake of brevity, we will also write proj_M instead of $\operatorname{proj}_{\operatorname{lin}\operatorname{aff} M}$ when M is a polytope (in this paper the common cases are that M is a segment or a face of a parallelohedron).

We recall some general properties of parallelohedra.

1) If $T(P)$ is a face-to-face tiling of $\mathbb{E}^d$ by translates of P, then

$$\Lambda(P) = \{\mathbf{t} : P + \mathbf{t} \in T(P)\}$$

is a d-dimensional lattice.

2) P has a center of symmetry.

3) Each facet of P (i.e. a $(d-1)$-dimensional face of P) has a center of symmetry.

DEFINITION 1. *Consider an arbitrary face $F \subset P$ of dimension $d-2$. The set of all facets of P parallel to F is called the belt of P determined by F and denoted by* $\mathrm{Belt}(F)$. *Each facet of* $\mathrm{Belt}(F)$ *contains two $(d-2)$-faces parallel and congruent to F, and each $(d-2)$-dimensional face of P parallel to F is shared by two facets of* $\mathrm{Belt}(F)$.

4) For every $(d-2)$-dimensional face $F \subset P$ the belt $\mathrm{Belt}(F)$ consists of 4 or 6 facets. It means that $\mathrm{proj}_F(P)$ is a parallelogram or a centrally symmetric hexagon.

Properties 1–4 were established in [24]. B. Venkov [29] proved that every convex polytope satisfying conditions 2, 3 and 4 (*Minkowski–Venkov conditions*) is a parallelohedron.

For the rest of the subsection let P be a cell of $T(P)$. Without loss of generality, we can also assume that the origin $\mathbf{0}$ is chosen at the center of P.

DEFINITION 2. *A standard face of P is a face that can be represented as $P \cap P'$, where $P' \in T(P)$.*

DEFINITION 3. *Let F be a standard face of P. Choose a cell $P' \in T(P)$ such that $F = P \cap P'$. Then choose a vector $\mathbf{s}(F)$ satisfying the identity $P + \mathbf{s}(F) = P'$. The vector $\mathbf{s}(F)$ is called the standard vector of the standard face F.*

Every standard face of a parallelohedron P has a unique standard vector. This is a consequence of the following result.

THEOREM (HORVÁTH [22]; DOLBILIN [8] – LATER AND INDEPENDENTLY) *Let F be a standard face of a parallelohedron P. Then F is centrally symmetric. More precisely, let $P' \in T(P)$ be such that $F = P \cap P'$. If $P + \mathbf{s}(F) = P'$, then the center of symmetry of the face F is the point $\mathbf{s}(F)/2$.*

The theorem above means that there is an equivalent definition of the standard vector.

DEFINITION 4. *Let F be a standard face of P. Let $\mathbf{s}(F)$ be such a vector that the point $\mathbf{s}(F)/2$ is the center of symmetry of F. Then the vector $\mathbf{s}(F)$ is called the standard vector of the standard face F.*

We mainly need the notation $\mathbf{s}(F)$ for the cases described in the two examples below.

EXAMPLE 1. If F is a facet of P, then F is necessarily standard. Then the notion of a standard vector $\mathbf{s}(F)$ coincides with the notion of a *facet vector* of F [24].

EXAMPLE 2. If F is a $(d-2)$-dimensional face of P, then F is standard iff $\mathrm{Belt}(F)$ consists of 4 facets.

2.2. Voronoi's Conjecture and Voronoi parallelohedra

In Section 1 we defined a Voronoi parallelohedron to be a polytope that is affinely equivalent to a Dirichlet–Voronoi domain for some lattice. This definition requires the existence of some predefined Euclidean metric in $\mathbb{R}^d$, since the notion of a Dirichlet–Voronoi domain indeed requires some metric.

We can give an equivalent definition, without using any predefined metric.

DEFINITION 5. *Given a lattice $\Lambda \subset \mathbb{R}^d$ with $\mathbf{0} \in \Lambda$, and a positive quadratic form Ω in $\mathbb{R}^d$, let $P(\Lambda, \Omega)$ be the Dirichlet–Voronoi domain of the point $\mathbf{0}$ with respect to the lattice Λ and the Euclidean metric given by Ω. We will call a parallelohedron P Voronoi, if it is a translate of $P(\Lambda, \Omega)$ for some Λ and Ω.*

Then we can restate Voronoi's Conjecture as follows.

CONJECTURE (VORONOI'S CONJECTURE, EQUIVALENT STATEMENT). Every d-dimensional parallelohedron is a Voronoi parallelohedron.

The language of flexible quadratic forms is the crucial point of Section 8. In addition, we benefit from using this language in Lemma 16, which is important for Section 6. For these reasons we choose this language as primary. I.e., given a parallelohedron, we look for a Euclidean metric (equivalently, a positive quadratic form) that makes the parallelohedron be a Dirichlet–Voronoi cell.

We also need to recall several notions concerning lattice Delaunay tilings.

Let Λ be a d-dimensional lattice in the space $\mathbb{R}^d$, and let a Euclidean distance $\|\cdot\|$ be given. We call a sphere

$$S(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^d : \|\mathbf{y} - x\| = r\}$$

empty, if $||\mathbf{z} - \mathbf{x}|| \geq r$ for every $\mathbf{z} \in \Lambda$.

If $S(\mathbf{x}, r)$ is an empty sphere and

$$\dim \operatorname{aff}(S(\mathbf{x}, r) \cap \Lambda) = d,$$

then the set

$$\operatorname{conv}(S(\mathbf{x}, r) \cap \Lambda)$$

is called a *lattice Delaunay d-cell*. It is known (see, for example, [5]) that all lattice Delaunay d-cells for a given lattice Λ form a face-to-face tiling D_Λ of $\mathbb{R}^d$.

Each k-face of a Delaunay d-cell is affinely equivalent to some Delaunay k-cell (see [7, § 13.2] for details). Thus, for simplicity, we can call all faces of D_Λ just Delanay cells.

There is a duality between the Delaunay tiling D_Λ and the Voronoi tiling V_Λ. Namely, for every face F of V_Λ there is a cell $D(F)$ of D_Λ such that

1) If P is a d-parallelohedron, $D(P)$ is the center of P.
2) $D(F) \subset D(F')$ iff $F' \subset F$.

Let F be a face of $T(P)$ and let $\dim F = d - k$. Consider a k-dimensional plane p that intersects F transversally. In a small neighborhood of F the section of $T(P)$ by p coincides with a complete k-dimensional polyhedral fan, which is called the *fan of a face* F and denoted by $\operatorname{Fan}(F)$. By duality, the combinatorics of $\operatorname{Fan}(F)$ is uniquely determined by the combinatorics of $D(F)$ and vice versa.

We will particularly need the classification of Delaunay k-cells for $k = 2, 3$ (or, equivalently, all possible structures of fans $\operatorname{Fan}(F)$ of dimension 2 or 3). There are two possible combinatorial types of two-dimensional fans and five possible combinatorial types of three-dimensional fans. They are shown in Figure 1. These types are listed, for example, by B. Delaunay [4, §8], who solved a more complicated problem — to find all possible combinatorial types of 3-dimensional fans $\operatorname{Fan}(F)$ without assumption that P is Voronoi.

An explicit classification of all affine types of Delaunay k-cells exists for $k \leq 6$ [10].

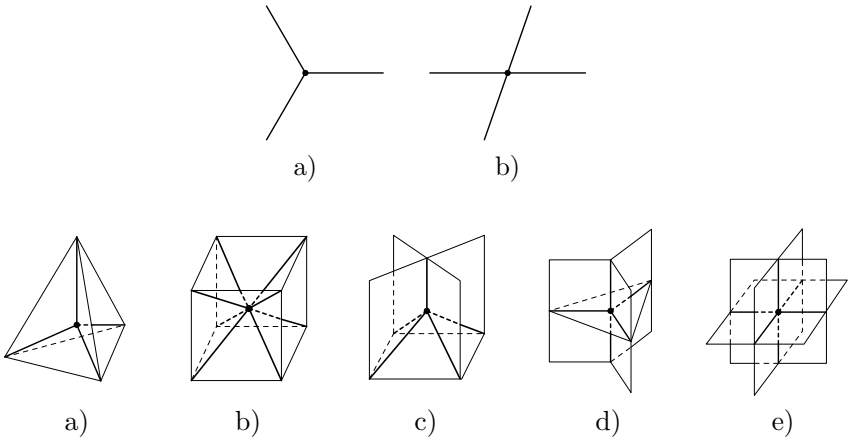


Fig. 1. Fans of $(d-2)$- and $(d-3)$-faces

Notice that a $(d-2)$-face F of a parallelohedron P is standard iff it determines a four-belt, or (assuming that P is a Voronoi parallelohedron) iff the dual Delaunay 2-cell $D(F)$ is a rectangle.

2.3. Reducibility of parallelohedra

DEFINITION 6. *A parallelohedron P is called reducible, if $P = P_1 \oplus P_2$, where P_1 and P_2 are convex polytopes of smaller dimension.*

From [18, Lemma 3 and Proposition 4] it follows that P_1 and P_2 are parallelohedra and if P is Voronoi, then P_1 and P_2 are Voronoi as well.

A. Ordine proved the following criterion of reducibility for parallelohedra.

THEOREM 1 (A. ORDINE, [25]). *Let P be a parallelohedron. Suppose that each facets of P is colored either with red or with blue so that*

1) *Opposite facets of P (with respect to the central symmetry of P) are of the same color.*
2) *If two facets of P belong to a common six-belt, then they are of the same color.*
3) *Not all facets of P are colored with the same color.*

Then one can represent P as $P_1 \oplus P_2$ such that blue facets form $P_1 \oplus \partial P_2$ and red facets form $\partial P_1 \oplus P_2$.

The reversed statement also holds. Namely, if $P = P_1 \oplus P_2$, assume that the facets of $P_1 \oplus \partial P_2$ form the blue part of ∂P and $\partial P_1 \oplus P_2$ form the red part. Then the resulting coloring satisfies conditions 1–3.

We need to mention another Ordine's result (Theorem 2 right below), which was a key ingredient in the proof of Theorem 1. It will be also one of the ingredients of our proof, used a the very end.

THEOREM 2 (A. ORDINE, [25]). *Let P be a parallelohedron in $\mathbb{E}^d$. Suppose that q_1 and q_2 are linear spaces of dimension at least* 1 *such that $q_1 \oplus q_2 = \mathbb{E}^d$. Assume that for every facet $F \subset P$ with a facet vector $\mathbf{s}(F)$ holds*

$$\mathbf{s}(F) \in q_1 \quad \textit{or} \quad \mathbf{s}(F) \in q_2.$$

Then $P = P_1 \oplus P_2$ and $\operatorname{lin}\operatorname{aff} P_i = q_i$ *for $i = 1, 2$.*

2.4. Main results

In this paper we prove the following three theorems simultaneously.

THEOREM 3. *Let I be a segment. Suppose that P and $P + I$ are parallelohedra and P is Voronoi in the standard Euclidean metric of $\mathbb{E}^d$. Then $P + I$ is Voronoi in some other Euclidean metric.*

THEOREM 4. *Let P be a Voronoi parallelohedron in $\mathbb{E}^d$ and let Π_1, Π_2 be hyperplanes. Assume that for every facet $F \subset P$ holds*

$$\mathbf{s}(F) \in \Pi_1 \quad \textit{or} \quad \mathbf{s}(F) \in \Pi_2.$$

Then P is reducible.

THEOREM 5. *Let P be a Voronoi parallelohedron in $\mathbb{E}^d$ and let Π_1, Π_2 be hyperplanes. Assume that the following conditions hold.*

1) *$P = P_1 \oplus P_2 \oplus \ldots \oplus P_k$, where $k > 1$ and all P_i are irreducible.*
2) *$\mathbf{s}(F) \in \Pi_1$ or $\mathbf{s}(F) \in \Pi_2$ for every facet $F \subset P$.*

Then for each $i = 1, 2, \ldots, k$ one has $\operatorname{aff} P_i \parallel \Pi_1$ *or* $\operatorname{aff} P_i \parallel \Pi_2$

Theorems 4 and 5 require that P has a special property. Since this property is extremely important for us, we give a definition.

DEFINITION 7. *Let P be a parallelohedron in $\mathbb{E}^d$. We say that a pair of hyperplanes (Π_1, Π_2) is a cross for P if for every facet $F \subset P$ holds*

$$\mathbf{s}(F) \in \Pi_1 \quad \textit{or} \quad \mathbf{s}(F) \in \Pi_2.$$

As we have seen in Section 1, Theorem 3 has an immediate corollary that Voronoi's Conjecture is true for e very space-filling zonotope (in other words, it implies Erdahl's theorem). Theorems 4 and 5 are also related to known results; namely, they generalize Theorem 2, if we are restricted to the class of Voronoi parallelohedra.

3. Free segments and free spaces of parallelohedra

DEFINITION 8. *Let P be a d-dimensional parallelohedron. Let I be a segment such that $P + I$ is a d-dimensional parallelohedron as well. Then I is called a free segment for P.*

DEFINITION 9. *Let P be a d-dimensional parallelohedron. A linear space p is called a free space for P if every segment $I \parallel p$ is free for P.*

We will extensively use the following criterion of free segments.

THEOREM 6 (V. GRISHUKHIN, [19]). *Let P be a parallelohedron and I be a segment. Then I is free for P if and only if every six-belt of P contains a facet parallel to I.*

We mention that the proof of Theorem 6 in [19] was incomplete. M. Dutour noticed that not all belts of $P + I$ were checked to have 4 or 6 facets. Namely, the belts spanned by $(d-2)$-faces of form $E \oplus I$, where E is a $(d-3)$-face of P, were not considered. The same remark refers to Theorem 11 as well. However, the missing case is considered in [12], where the complete proof of Theorem 6 is given, and the same case analysis gives the proof of Theorem 11. See also Lemma 25.

Theorem 6 has an immediate corollary which motivates introducing the notion of free space.

COROLLARY 7. *Let P be a parallelohedron and let $F_1, F_2, \ldots, F_k$ be facets of P with the property that each six-belt of P contains at least one F_i. Then*

$$\text{lin aff } F_1 \cap \text{lin aff } F_2 \cap \ldots \cap \text{lin aff } F_k \tag{3.1}$$

is a free space for P.

DEFINITION 10. *Let P be a d-dimensional parallelohedron. A free space for P of form* (3.1) *is called perfect.*

The notions and statements above concerning free segments and free spaces do not require that P is Voronoi. If, however, P is Voronoi, then

1) I is free for P if and only if each triple of facet vectors corresponding to a six-belt contains a vector $\mathbf{s}(F) \perp I$.
2) If $\mathbf{s}(F_1), \mathbf{s}(F_2), \ldots, \mathbf{s}(F_k)$ are facet vectors of P and each triple of facet vectors corresponding to a six-belt contains some $\mathbf{s}(F_i)$ or some $-\mathbf{s}(F_i)$, then the orthogonal complement

$$\langle \mathbf{s}(F_1), \mathbf{s}(F_2), \ldots, \mathbf{s}(F_k) \rangle^{\perp} \tag{3.2}$$

is a perfect free space for P.

Now return from the Voronoi case to the case of general parallelohedra.

DEFINITION 11. *Let P be a parallelohedron of dimension d and let I be a free segment for P. We call a $(d-2)$-dimensional face F of P semi-shaded by I if $F \oplus I$ is a facet of $P + I$.*

The following statement also immediately follows from 6.

COROLLARY 8. *Let P be a d-dimensional parallelohedron and let I be a free segment for P. Then every $(d-2)$-dimensional face of P semi-shaded by I is standard.*

Consequently, the standard vector $\mathbf{s}(F)$ is defined for every $(d-2)$-dimensional face F semi-shaded by I.

Introduce the notation

$$\mathcal{A}_I(P) = \{\mathbf{s}(F) : \dim \operatorname{aff} F = d-2 \text{ and } F \text{ is semi-shaded by } I\},$$

$$\mathcal{B}_I(P) = \{\mathbf{s}(F) : \dim \operatorname{aff} F = d-1 \text{ and } F \parallel I\}.$$

Working with the sets $\mathcal{A}_I(P)$ and $\mathcal{B}_I(P)$ we will need a theorem by B. Venkov and a corollary emphasized by Á. Horváth. We state both results below.

DEFINITION 12. *Let P be a d-dimensional parallelohedron and p be a linear space of dimension d', where $0 < d' < d$. Assume that for every point $\mathbf{x} \in \mathbb{E}^d$ the set $P \cap (\mathbf{x}+p)$ is either a d'-dimensional polytope or empty. Then we say that P has positive width along p.*

THEOREM 9 (B. VENKOV, [30]). *Assume that P is a d-dimensional parallelohedron with positive width along a d'-dimensional linear space p. Let $F_1, F_2, \dots, F_k$ be all facets of P parallel to p and let $\mathbf{s}_i = \mathbf{s}(F_i)$. Finally, let proj_p denote the projection along p onto the complementary space q. Then*

1) *proj_p is a bijection of $\langle \mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_k \rangle$ and q. In particular,*

$$\dim \langle \mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_k \rangle = d - d'.$$

2) *The set*

$$\{\mathrm{proj}_p(P + \mathbf{t}) : \mathbf{t} \in \mathbb{Z}(\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_k)\}$$

is a face-to-face tiling of q by parallelohedra. All tiles are translates of $\mathrm{proj}_p(P)$.

If I is free for P, then $P + I$ has positive width along I. Therefore Theorem 9 has the following corollary.

COROLLARY 10 (Á. HORVÁTH, [23]). *Suppose P is a d-dimensional parallelohedron and a segment I is free for P. Then*

$$\dim \langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle = d - 1.$$

In addition, if proj_I is a projection along I onto a complementary $(d-1)$-space, then $Q = \mathrm{proj}_I(P)$ is a parallelohedron and

$$\Lambda(Q) = \mathrm{proj}_I\big(\mathbb{Z}(\mathcal{A}_I(P) \cup \mathcal{B}_I(P))\big).$$

4. Free segments and Voronoi's Conjecture

The following two theorems were stated by V. Grishukhin. Placed together, they answer the question if an extension of a Voronoi parallelohedron satisfies Voronoi's Conjecture.

THEOREM 11 (V. GRISHUKHIN, [20]). *Let P and $P + I$ be parallelohedra. Suppose that P is an irreducible Voronoi parallelohedron, and Ω is the metric that makes P a Dirichlet–Voronoi cell. If we write the sign $\perp$ for orthogonality in the metric Ω, then the following statements are equivalent.*

1) *Voronoi's Conjecture holds for $P + I$.*

2) *For every standard* $(d-2)$*-face* F *such that* $\mathbf{s}(F) \in \mathcal{A}_I(P)$ *one has* $I \perp \mathbf{s}(F)$.

THEOREM 12 (V. GRISHUKHIN, [20]). *Let* P *and* $P+I$ *be parallelohedra. Suppose that* P *is Voronoi and reducible so that* $P = P_1 \oplus P_2$. *Define*

$$I_1 = \mathrm{proj}_{\mathrm{lin\,aff}\, P_2}(I), \qquad I_2 = \mathrm{proj}_{\mathrm{lin\,aff}\, P_1}(I),$$

assuming that $\mathrm{proj}_{\mathrm{lin\,aff}\, P_2}$ *is a projection along* $\mathrm{lin\,aff}\, P_2$ *onto* $\mathrm{aff}\, P_1$ *and similarly for* $\mathrm{proj}_{\mathrm{lin\,aff}\, P_1}$. *Then*

1) $P_1 + I_1$ *and* $P_2 + I_2$ *are parallelohedra.*
2) *Voronoi's Conjecture holds for* $P+I$ *iff it holds for both* $P_1 + I_1$ *and* $P_2 + I_2$.

Theorem 11 has an important consequence, Corollary 13.

COROLLARY 13. *Suppose that* P *and* $P+I$ *are parallelohedra and* P *is Voronoi. Then Voronoi's Conjecture for* $P+I$ *holds if*

$$\dim \langle \mathcal{B}_I(P) \rangle = d-1. \tag{4.1}$$

Remark. An almost identical statement to Corollary 13 is given without a proof in [28, Theorem 3.18].

PROOF. From (4.1) and Corollary 10 follows that

$$\mathcal{A}_I(P) \subset \langle \mathcal{B}_I(P) \rangle .$$

For the rest of the proof the orthogonality is related to the Euclidean metric that makes P Voronoi.

I is orthogonal to $\langle \mathcal{B}_I(P) \rangle$. Thus, in particular, I is orthogonal to each vector of $\mathcal{A}_I(P)$. Now Corollary 13 immediately follows from Theorem 11. □

By combining Corollary 13 and Theorem 12, we can give an equivalent restatement of Theorem 3 (Theorem 14 below).

We will not give the proof of Theorem 14 immediately. Restricted to dimensions $\leq d$, the statement of Theorem 14 will be denoted as $\mathrm{A}(d)$. Along with two other theorems, the proof of Theorem 14 will result from an inductive argument in Sections 8–11.

However, we will prove (see Lemma 15) the equivalence $\mathrm{A}(d) \Leftrightarrow \mathrm{M}(d)$, where $\mathrm{M}(d)$ is the statement of Theorem 3 for parallelohedra of dimension $\leq d$. Hence we will complete the first step of the proof as sketched in Section 1.

THEOREM 14. *If a Voronoi parallelohedron P has a 2-dimensional free space, then P is reducible.*

LEMMA 15. *Theorems* 3 *and* 14 *are equivalent.*

PROOF. Indeed, let Theorem 3 be true. Suppose that there exists an irreducible Voronoi parallelohedron P with a 2-dimensional free space. Then P has a perfect free space q of dimension at least 2. Further, there are finitely many possibilities for $\langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle$. Therefore there is only a finite number of directions for $I \parallel q$ such that

$$I \perp \langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle .$$

But from Theorems 3 and 11 it follows that the orthogonality should hold for every $I \parallel q$. The contradiction shows that Theorem 14 follows from Theorem 3.

Let Theorem 14 be true. If Theorem 3 is false, consider a counterexample $P + I$ with the least possible dimension of P. If P is reducible, then by Theorem 12 either $P_1 + I_1$ or $P_2 + I_2$ is a smaller counterexample, which is a contradiction to the minimality. If P is irreducible, then, by Theorem 14, P has no free spaces of dimension greater than 1. Therefore I is parallel to a perfect free line and the identity (4.1) holds. Hence $P + I$ is Voronoi, so it is not a counterexample to Theorem 3. As a result, Theorem 3 follows from Theorem 14. □

LEMMA 16. *Let I be a segment and let P and $P + I$ be Voronoi parallelohedra (possibly, for different Euclidean metrics). Then* $\mathrm{proj}_I(P)$ *is a Voronoi parallelohedron for every possible choice of the image space of the projection.*

PROOF. Indeed, it is enough to prove Lemma 16 for any image space of proj_I, since changing the image space results in the affine transformation of the projection.

Let $\Pi = \langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle$ be the image space of proj_I. There exists a Euclidean norm $\| \cdot \|_\Omega$, obtained from a positive definite quadratic form Ω as

$$\|\mathbf{x}\|_\Omega^2 = \mathbf{x}^T \Omega \mathbf{x},$$

such that, with respect to $\|\cdot\|_\Omega$, P is a Dirichlet-Voronoi cell, and, in addition, I is orthogonal to Π. For the irreducible case it is a consequence of Theorem 11, and for reducible parallelohedra see [20, §9]. Then $\mathrm{proj}_I(P)$ is Voronoi with respect to the restriction of $\|\cdot\|_\Omega$ to Π (see the details in [20, Proposition 5]).

Next we are starting a block of three sections in which we will prepare the tools for the main induction. For convenience, the reader may not reading this block right at the moment. Instead he or she may move to the induction and return back to check any particular statement referenced in the induction. The order of these three sections corresponds to the order in which the auxiliary statements from them appear in the proof of main results. □

5. Dilatation of Voronoi parallelohedra

Assume that Λ is a d-dimensional lattice and Ω is a positive definite quadratic form. By $P(\Lambda, \Omega)$ we will denote a parallelohedron, which is a Dirichlet–Voronoi cell for the lattice Λ with respect to the Euclidean Metric $\|\cdot\|_\Omega$.

Let $\mathbf{n}$ be a vector. Consider a quadratic form

$$\Omega_{\mathbf{n}} = \Omega + \Omega^T \mathbf{n}\mathbf{n}^T \Omega. \tag{5.1}$$

For every nonzero vector $\mathbf{x}$ one has

$$\mathbf{x}^T \Omega_{\mathbf{n}} \mathbf{x} = \mathbf{x}^T \Omega \mathbf{x} + (\mathbf{n}^T \Omega \mathbf{x})^2 > 0,$$

thus $\Omega_{\mathbf{n}}$ is a positive definite quadratic form. If not otherwise stated, everywhere below we assume that $\mathbf{n} \neq \mathbf{0}$.

Definition 13. *All parallelohedra of form $P(\Lambda, \Omega_{\mathbf{n}})$ will be called dilatations of $P(\Lambda, \Omega)$.*

Let $\mathcal{F}(\Lambda, \Omega)$ be the set of all facet vectors of $P(\Lambda, \Omega)$. For what follows, we will need an another description of facet vectors. Namely, the points $\mathbf{x}, \mathbf{x}' \in \Lambda$ are adjoint by a facet vector of $P(\Lambda, \Omega)$ iff the ball

$$B_\Omega(\mathbf{x}, \mathbf{x}') = \left\{ \mathbf{y} : \left\| \mathbf{y} - \frac{\mathbf{x} + \mathbf{x}'}{2} \right\|_\Omega \leq \frac{1}{2} \|\mathbf{x} - \mathbf{x}'\|_\Omega \right\}$$

contains no points of Λ other than $\mathbf{x}$ and $\mathbf{x}'$. This is because $[\mathbf{x}, \mathbf{x}']$ with $\mathbf{x}, \mathbf{x}' \in \Lambda$ is a Delaunay 1-cell iff $\mathbf{x}' - \mathbf{x}$ is a facet vector and, moreover, the empty sphere for the segment $[\mathbf{x}, \mathbf{x}']$ is centered at its midpoint (see [7, Lemma 13.2.7]).

Define

$$\mathcal{F}_{\mathbf{n}}(\Lambda, \Omega) = \{\mathbf{s} : \mathbf{s} \in \mathcal{F}(\Lambda, \Omega) \text{ and } \mathbf{n}^T \Omega \mathbf{s} \neq 0\}.$$

The following lemma is expressed by a single formula, however, its meaning is explained in Corollary 18.

LEMMA 17. $\langle \mathcal{F}_{\mathbf{n}}(\Lambda, \Omega_{\mathbf{n}}) \rangle \subseteq \langle \mathcal{F}_{\mathbf{n}}(\Lambda, \Omega) \rangle$.

PROOF. Before starting the proof we emphasize an important property. For every vector $\mathbf{x}$ and every real λ the conditions

$$\mathbf{n}^T \Omega \mathbf{x} = 0 \quad \text{and} \quad \mathbf{n}^T \Omega_{\lambda \mathbf{n}} \mathbf{x} = 0$$

are equivalent. This is an immediate consequence of the formula (5.1).

Consider the Delaunay tiling with vertex set Λ in the Euclidean metric given by a quadratic form $\Omega_{\lambda \mathbf{n}}$. We will observe the change of the set $\mathcal{F}_{\mathbf{n}}(\Lambda, \Omega_{\lambda \mathbf{n}})$ as λ grows from 0 to 1.

Suppose that at some $\lambda_0 \in (0, 1)$ a new vector of $\mathcal{F}_{\mathbf{n}}(\Lambda, \Omega_{\lambda \mathbf{n}})$ emerges. Thus there is a pair of points $\mathbf{x}, \mathbf{x}' \in \Lambda$ with the following properties.

1) For $\lambda \searrow \lambda_0$ the ball $B_{\Omega_{\lambda \mathbf{n}}}(\mathbf{x}, \mathbf{x}')$ contains no points of Λ other than $\mathbf{x}$ and $\mathbf{x}'$.
2) For $\lambda \nearrow \lambda_0$ the ball $B_{\Omega_{\lambda \mathbf{n}}}(\mathbf{x}, \mathbf{x}')$ contains some other points of Λ.
3) $\mathbf{n}^T \Omega (\mathbf{x}' - \mathbf{x}) \neq 0$.

If we prove that for a sufficiently small $\varepsilon > 0$ the inclusion

$$\mathbf{x}' - \mathbf{x} \in \mathcal{F}_{\mathbf{n}}(\Lambda, \Omega_{(\lambda_0 - \varepsilon)\mathbf{n}})$$

holds, then we are done. Indeed, the inclusion means that $\mathcal{F}_{\mathbf{n}}(\Lambda, \Omega_{\lambda \mathbf{n}})$ never expands as λ grows from 0 to 1.

Consider the ball $B_{\Omega_{\lambda_0 \mathbf{n}}}(\mathbf{x}, \mathbf{x}')$. By continuity, it contains some points of Λ distinct from $\mathbf{x}$ and $\mathbf{x}'$, but only on the boundary. Thus

$$D = \operatorname{conv}(B_{\Omega_{\lambda_0 \mathbf{n}}}(\mathbf{x}, \mathbf{x}') \cap \Lambda)$$

is a centrally symmetric Delaunay cell for the metric $\|\cdot\|_{\Omega_{\lambda_0\mathbf{n}}}$ of dimension at least 2. It is not hard to see that all edges of D are also Delaunay edges for every metric $\|\cdot\|_{\Omega_{(\lambda_0-\varepsilon)\mathbf{n}}}$ if ε is positive and small enough.

We say that a point $\mathbf{y} \in \Lambda$ is above (below, on the same level with) a point $\mathbf{y}' \in \Lambda$ if $\mathbf{n}\Omega(\mathbf{y}-\mathbf{y}')$ is positive (negative, zero respectively). As $\mathbf{x}$ and $\mathbf{x}'$ are not on the same level, we will assume that $\mathbf{x}'$ is above $\mathbf{x}$.

We aim to prove that $\mathbf{x}$ and $\mathbf{x}'$ can be adjoint by a sequence of edges of D in such a way that every edge of a sequence goes between two vertices of different levels. This will imply that $\mathbf{x}'-\mathbf{x}$ is a combination of facet vectors of $P(\Lambda, \Omega_{(\lambda_0-\varepsilon)\mathbf{n}})$, and Lemma 17 will proved.

Observe that a vertex of D is inside $B_{\Omega_{(\lambda_0-\varepsilon)\mathbf{n}}}$ if it is above $\mathbf{x}'$ or below $\mathbf{x}$. Since D has a center of symmetry at $\frac{\mathbf{x}+\mathbf{x}'}{2}$, D has vertices both above $\mathbf{x}'$ and below $\mathbf{x}$.

Further, D has no point $\mathbf{z} \neq \mathbf{x}$ on the same level with $\mathbf{x}$. Indeed, assume the converse. Then the points $\mathbf{x}$, $\mathbf{x}'$, $\mathbf{z}$ and $\mathbf{z}' = \mathbf{x}+\mathbf{x}'-\mathbf{z}$ lie on the sphere $B_{\Omega_{\lambda_0\mathbf{n}}}$. Thus

$$\left\|\mathbf{x}-\frac{\mathbf{x}+\mathbf{x}'}{2}\right\|_{\Omega_{\lambda_0\mathbf{n}}} = \left\|\mathbf{x}'-\frac{\mathbf{x}+\mathbf{x}'}{2}\right\|_{\Omega_{\lambda_0\mathbf{n}}} = \left\|\mathbf{z}-\frac{\mathbf{x}+\mathbf{x}'}{2}\right\|_{\Omega_{\lambda_0\mathbf{n}}} = \\ = \left\|\mathbf{z}'-\frac{\mathbf{x}+\mathbf{x}'}{2}\right\|_{\Omega_{\lambda_0\mathbf{n}}}. \tag{5.2}$$

Since $\mathbf{z}$ is on the same level with $\mathbf{x}$ and $\mathbf{z}'$ is on the same level with $\mathbf{x}'$, it is clear that

$$\mathbf{n}^T\Omega\left(\mathbf{x}-\frac{\mathbf{x}+\mathbf{x}'}{2}\right) = \mathbf{n}^T\Omega\left(\mathbf{z}-\frac{\mathbf{x}+\mathbf{x}'}{2}\right) = -\mathbf{n}^T\Omega\left(\mathbf{x}'-\frac{\mathbf{x}+\mathbf{x}'}{2}\right) = \\ = -\mathbf{n}^T\Omega\left(\mathbf{z}'-\frac{\mathbf{x}+\mathbf{x}'}{2}\right).$$

Therefore (5.2) holds after substituting all instances of $\lambda_0\mathbf{n}$ with $\lambda\mathbf{n}$ for every real λ. As a result, $[\mathbf{x}, \mathbf{x}']$ is never a Delaunay edge, because the empty sphere centered at its midpoint necessarily contains at least two more points.

A well-known fact from linear programming [33, § 3.2] tells that $\mathbf{x}'$ can be connected with at least one of the highest vertices (call this vertex $\mathbf{y}$) of D by a sequence of edges going strictly upwards. We have proved that $\mathbf{x}'$ cannot be the highest point of D, so $\mathbf{y} \neq \mathbf{x}'$.

Consider the segment $[\mathbf{x}, \mathbf{y}]$. If it is an edge of D, the proof is over. Otherwise (see [7, Lemma 13.2.7] again) $[\mathbf{x}, \mathbf{y}]$ is a diagonal of a centrally symmetric face $D' \subset D$.

Suppose that $\mathbf{y}$ is not the only highest point of D'. Then there is a vertex $\mathbf{y}' \in D'$ on the same level with $\mathbf{y}$. But due to the central symmetry, the point $\mathbf{z} = \mathbf{x} + \mathbf{y} - \mathbf{y}'$ is a vertex of D' and is on the same level with $\mathbf{x}$. This is impossible, so $\mathbf{y}$ is the only highest point of D'.

Thus one can connect $\mathbf{x}$ and $\mathbf{y}$ by a sequence of edges going strictly upwards. As a result, we have connected $\mathbf{x}$ and $\mathbf{x}'$ by a sequence of edges going first strictly upwards and then strictly downwards. Thereby we have completed the remaining part of the proof. □

COROLLARY 18. *Assume that the parallelohedron $P(\Lambda, \Omega)$ has a cross of hyperplanes Π, Π' (by Definition 7, it means that every facet vector of $P(\Lambda, \Omega)$ is parallel to Π or to Π'). Let $\mathbf{n}$ be a normal vector to Π in the metric $\|\cdot\|_\Omega$. Then $P(\Lambda, \Omega_\mathbf{n})$ has the same cross (Π, Π').*

PROOF. The property of $P(\Lambda, \Omega)$ to have a cross Π, Π' means that

$$\langle \mathcal{F}_\mathbf{n}(\Lambda, \Omega) \rangle \subseteq \Pi'.$$

By Lemma 17,

$$\langle \mathcal{F}_\mathbf{n}(\Lambda, \Omega_\mathbf{n}) \rangle \subseteq \langle \mathcal{F}_\mathbf{n}(\Lambda, \Omega) \rangle \subseteq \Pi'.$$

But the set $\mathcal{F}(\Lambda, \Omega_\mathbf{n}) \setminus \mathcal{F}_\mathbf{n}(\Lambda, \Omega_\mathbf{n})$ lies in the orthogonal complement to $\mathbf{n}$ (which is the same in both $\|\cdot\|_\Omega$ and $\|\cdot\|_{\Omega_\mathbf{n}}$), i.e. in Π. Thus

$$\mathcal{F}(\Lambda, \Omega_\mathbf{n}) \subset \Pi \cup \Pi',$$

which means that $P(\Lambda, \Omega_\mathbf{n})$ has the cross (Π, Π'). □

6. Layering of parallelohedra with free segments

DEFINITION 14. *Let P be a d-dimensional parallelohedron and I be a free segment for P. Fix a vector $\mathbf{e}_I \parallel I$. Define the cap of P visible by I, or, simply, the I-cap of P as a homogeneous $(d-1)$-dimensional complex $\mathrm{Cap}_I(P)$ consisting of all facets F of P*

satisfying the condition

$$\mathbf{e}_I \cdot \mathbf{n}(F) < 0$$

and all subfaces of those facets. (*Obviously, each I defines two caps centrally symmetric to each other.*)

For a parallelohedron P and its free segment I define

$$\mathcal{C}_I(P) = \{\mathbf{s}(F) : F \text{ is a facet of } \mathrm{Cap}_I(P)\}.$$

LEMMA 19. *Let P be a parallelohedron and I be its free segment. Then*

$$\text{lin aff}\, \mathcal{C}_I(P) \subseteq \langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle .$$

PROOF. Let $I = [-\mathbf{x}, \mathbf{x}]$ and $\mathbf{e}_I = 2\mathbf{x}$.

The proof is by contradiction. Suppose that F_1 and F_2 are facets of $\mathrm{Cap}_I(P)$ and

$$\mathbf{s}(F_1) - \mathbf{s}(F_2) \notin \langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle .$$

One can easily see that for every $\lambda > 0$ the segment λI is free for P. Moreover, $F_1 + \lambda\mathbf{x}$ and $F_2 + \lambda\mathbf{x}$ are facets of $P + \lambda I$ with facet vectors

$$\mathbf{s}(F_1) + \lambda\mathbf{e}_I \quad \text{and} \quad \mathbf{s}(F_1) + \lambda\mathbf{e}_I$$

respectively. Hence

$$\mathcal{A}_I(P) \cup \mathcal{B}_I(P) \cup \{\mathbf{s}(F_1) - \mathbf{s}(F_2)\} \subset \Lambda(P + \lambda I).$$

By assumption, the lattice generated by $\mathcal{A}_I(P)$, $\mathcal{B}_I(P)$ and $\mathbf{s}(F_1) - \mathbf{s}(F_2)$ is d-dimensional and does not depend on λ. Let V be the fundamental volume of this lattice. Then the volume of $P + \lambda I$, which is the fundamental volume of $\Lambda(P + \lambda I)$, is at most V. But as $\lambda \to \infty$, the volume of $P + \lambda I$ becomes arbitrtarily large, a contradiction. □

LEMMA 20. *Let P be a d-dimensional parallelohedron centered at $\mathbf{0}$ and let I be its free segment. Then*

$$\Lambda(P)/\mathbb{Z}(\mathcal{A}_I(P) \cup \mathcal{B}_I(P)) = \mathbf{Z}.$$

PROOF. Consider the sublattice $\Lambda_0 = \Lambda(P) \cap \langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle$. It is enough to prove that

$$\Lambda_0 = \mathbb{Z}(\mathcal{A}_I(P) \cup \mathcal{B}_I(P)). \tag{6.1}$$

Assume that (6.1) does not hold.

Let $\mathbf{t} = \mathbf{s}(F)$, where F is a facet of $\mathrm{Cap}_I(P)$. By Lemma 19,

$$\mathrm{proj}_{\mathrm{lin}\,\Lambda_0}(\mathbf{s}(F')) \in \{\mathbf{0}, \pm\mathbf{t}\}$$

for all F' being facets of P. Here $\mathrm{proj}_{\mathrm{lin}\,\Lambda_0}$ is a projection along $\mathrm{lin}\,\Lambda_0$ onto $\mathbb{R} \cdot \mathbf{t}$.

Since the set of all facet vectors of P generates $\Lambda(P)$,

$$\mathrm{proj}_{\mathrm{lin}\,\Lambda_0}\,\Lambda(P) = \mathbb{Z} \cdot \mathbf{t}.$$

Set $I = [-\mathbf{x}, \mathbf{x}]$, $\mathbf{e}_I = 2\mathbf{x}$. Consider the tiling $T(P + \lambda I)$ for an arbitrary $\lambda > 0$. We will show that

$$\Lambda(P + \lambda I) = \Lambda_0 \oplus \mathbb{Z} \cdot (\mathbf{t} + \lambda \mathbf{e}_I).$$

To prove this, it is enough to check that all facet vectors of $P + \lambda I$ belong to

$$\Lambda_0 \oplus \mathbb{Z} \cdot (\mathbf{t} + \lambda \mathbf{e}_I).$$

Indeed, each facet vector of $P + \lambda I$ is either from $\mathcal{A}_I(P)$, or from $\mathcal{B}_I(P)$, or it has the form

$$\pm(\mathbf{s}(F) + \lambda \mathbf{e}_I),$$

where F is a facet of P and the sign is chosen to be plus, if $F \in \mathrm{Cap}_I(P)$ and minus if $-F \in \mathrm{Cap}_I(P)$. In the first two cases the facet vectors belong to Λ_0, and in the third case the facet vector belongs to $\pm(\Lambda_0 + \mathbf{t} + \mathbf{e}_I)$.

From Corollary 10 follows that for sufficiently large λ the hyperplane $\mathrm{aff}\,\Lambda_0$ is covered, except for a lower-dimensional subset, by interior parts of parallelohedra

$$\{P + \lambda I + \mathbf{u} : \mathbf{u} \in \mathbb{Z}(\mathcal{A}_I(P) \cup \mathcal{B}_I(P))\}.$$

Let $\mathbf{v} \in \Lambda_0 \setminus \mathbb{Z}(\mathcal{A}_I(P) \cup \mathcal{B}_I(P)$. Then the same holds for

$$\{P + \lambda I + \mathbf{v} + \mathbf{u} : \mathbf{u} \in \mathbb{Z}(\mathcal{A}_I(P) \cup \mathcal{B}_I(P))\}.$$

This is impossible since $T(P + \lambda I)$ is a tiling. Hence $\mathbf{v}$ does not exist and (6.1) holds. □

LEMMA 21. *Let P be a parallelohedron with a free segment I. Let F be a facet of P such that*

$$\mathbf{s}(F) \in \langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle .$$

Then F is parallel to I.

PROOF. Assume the converse. Then $\mathbf{s}(F) \in \mathcal{C}_I(P)$. Thus

$$\operatorname{aff} \mathcal{C}_I(P) \cap \operatorname{aff}(\mathcal{A}_I(P) \cup \mathcal{B}_I(P)) \neq \varnothing,$$

since the intersection contains $\mathbf{s}(F)$. Application of Lemma 19 gives

$$\langle \mathcal{C}_I(P) \rangle \subset \langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle .$$

This immediately gives

$$\dim \langle \mathcal{B}_I(P) \cup \mathcal{C}_I(P) \rangle \leq d - 1.$$

But $\mathcal{B}_I(P)$ together with $\mathcal{C}_I(P)$ generate a d-lattice $\Lambda(P)$, a contradiction. □

LEMMA 22. *Let P be a d-dimensional parallelohedron centered at $\mathbf{0}$ and let I be its free segment. Choose a vector $\mathbf{t}$ so that*

$$\Lambda(P) = \mathbb{Z}(\mathcal{A}_I(P) \cup \mathcal{B}_I(P)) \oplus \mathbb{Z} \cdot \mathbf{t}.$$

Let $\mathbf{v} \in \mathbb{Z}(\mathcal{A}_I(P) \cup \mathcal{B}_I(P))$. Then

$$\operatorname{proj}_I(P \cap (P + \mathbf{v} + \mathbf{t})) = \operatorname{proj}_I(P) \cap \operatorname{proj}_I(P + \mathbf{v} + \mathbf{t}).$$

PROOF. From Lemma 20 immediately follows that

$$\mathcal{C}_I(P) \subset \pm\mathbf{t} + \mathbb{Z}(\mathcal{A}_I(P) \cup \mathcal{B}_I(P)).$$

Without loss of generality assume that the sign is "+".

Consider a homogeneous $(d-1)$-dimensional complex $\mathcal{K}$, all faces of which are faces of $T(P)$, and satisfying

$$|\mathcal{K}| = \bigcup_{\mathbf{v}} (\mathrm{Cap}_I(P) + \mathbf{v}),$$

where $\mathbf{v}$ runs through the lattice $\mathbb{Z}(\mathcal{A}_I(P) \cup \mathcal{B}_I(P))$ and $|\mathcal{K}|$ denotes the support of $\mathcal{K}$. Informally, $\mathcal{K}$ splits two layers

$$\mathcal{L}_0 = \{P + \mathbf{v} : \mathbf{v} \in \mathbb{Z}(\mathcal{A}_I(P) \cup \mathcal{B}_I(P))\} \quad \text{and}$$
$$\mathcal{L}_1 = \{P + \mathbf{t} + \mathbf{v} : \mathbf{v} \in \mathbb{Z}(\mathcal{A}_I(P) \cup \mathcal{B}_I(P))\}. \tag{6.2}$$

$\mathcal{K}$ has the following properties.

1) The projection proj_I onto a hyperplane Π transversal to I is a homeomorphism between $|\mathcal{K}|$ and Π.
2) $|\mathcal{K}| = \left(\bigcup_{P' \in \mathcal{L}_0} P' \right) \bigcap \left(\bigcup_{P' \in \mathcal{L}_1} P' \right)$.
3) $\mathrm{proj}_I(P' \cap |\mathcal{K}|) = \mathrm{proj}_I(P')$ for every $P' \in \mathcal{L}_0 \cup \mathcal{L}_1$.

Statement 1 follows from A. D. Alexandrov's tiling theorem [1]. We apply it to the complex spanned by polytopes

$$\{F + \mathbf{v} : F \in \mathrm{Cap}_I(P),\ \mathbf{v} \in \mathbb{Z}(\mathcal{A}_I(P) \cup \mathcal{B}_I(P))\}.$$

One can easily check that the set of $(d-1)$-polytopes above locally forms a local tiling around each face of dimension $(d-3)$. Hence this set is a tiling of an affine $(d-1)$-space.

Therefore each line parallel to I is split by $|\mathcal{K}|$ into two rays, say, the lower and the upper one with respect to some fixed orientation of I. We will call the union of all lower closed rays *the lower part of* $\mathbb{R}^d$ and the union of all upper closed rays *the upper part of* $\mathbb{R}^d$ with respect to $|\mathcal{K}|$.

To prove statement 2 notice that all parallelohedra of $\mathcal{L}_0$ lie in one (say, lower) part of $\mathbb{R}^d$, respectively, all parallelohedra of $\mathcal{L}_1$ lie in the upper part. Thus the intersection is contained in the intersection of lower and upper parts, i.e. in $|\mathcal{K}|$. On the other hand, every point of $|\mathcal{K}|$ is an intersection of some parallelohedron from $\mathcal{L}_0$ and some parallelohedron from $\mathcal{L}_1$.

Statement 3 is an immediate corollary of definitions of a cap and $\mathcal{K}$.

In the notation of Lemma 22, let $P' = P + \mathbf{v} + \mathbf{t}$. Thus

$$P \in \mathcal{L}_0 \quad \text{and} \quad P' \in \mathcal{L}_1.$$

From statement 2 follows that

$$P \cap P' = (P \cap |\mathcal{K}|) \cap (P' \cap |\mathcal{K}|).$$

Since proj_I is a homeomorphism of $|\mathcal{K}|$, one has

$$\begin{aligned}\mathrm{proj}_I(P \cap P') &= \mathrm{proj}_I\big((P \cap |\mathcal{K}|) \cap (P' \cap |\mathcal{K}|)\big) = \\ &= \mathrm{proj}_I(P \cap |\mathcal{K}|) \cap \mathrm{proj}_I(P' \cap |\mathcal{K}|) = \mathrm{proj}_I(P) \cap \mathrm{proj}_I(P').\end{aligned}$$

The last identity is due to statement 3. □

7. Two-dimensional perfect free spaces

In this section we study the following construction. Let P be a Voronoi parallelohedron and let p be a two-dimensional perfect free space of P. This case is extremely important in our argument, so we aim to establish several consequences.

We need some more notation. Define

$$\mathcal{B}_p(P) = \{\mathbf{s}(F) : F \text{ is a facet of } P \text{ and } \mathbf{s}(F) \perp p\},$$

$$\mathcal{A}_p(P) = \{\mathbf{s}(F) : F \text{ is a standard } (d-2)\text{-face of } P \text{ and } \mathbf{s}(F) \perp p\}.$$

Here the orthogonality is related to the Euclidean metric that makes P Voronoi.

Since $\mathbf{s}(F) \perp F$, then from the definition of a perfect space follows that

$$\dim \left\langle \mathcal{B}_p(P) \right\rangle = d - 2.$$

Therefore $\left\langle \mathcal{B}_p(P) \right\rangle$ is the orthogonal complement to p and hence

$$\mathcal{A}_p(P) = \mathcal{A}_I(P) \cap \left\langle \mathcal{B}_p(P) \right\rangle$$

for every $I \parallel P$.

LEMMA 23. *Let P be a Voronoi parallelohedron and let p be a two-dimensional perfect free space of P. Let I be a segment rotating in p. Then I_0 is parallel to a perfect line iff the hyperplane*

$$\langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle$$

as a function of I is discontinuous at $I = I_0$.

PROOF. Notice that for every $I \parallel p$ one has $\mathcal{B}_p(P) \subseteq \mathcal{B}_I(P)$.

Suppose I_0 is not parallel to a perfect line. Then $\dim \left\langle \mathcal{B}_{I_0}(P) \right\rangle < d - 1$, so

$$\mathcal{B}_{I_0}(P) = \mathcal{B}_p(P).$$

The same holds for all I close enough to I_0. In addition, for all I close enough to I_0 holds

$$\mathcal{A}_{I_0}(P) = \mathcal{A}_I(P).$$

Thus the hyperplane $\langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle$ is the same for all I close enough to I_0.

Suppose that I_0 is parallel to a perfect line. Then $\dim \left\langle \mathcal{B}_{I_0}(P) \right\rangle = d - 1$.

The hyperplane function $\langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle$ takes only finitely many values, as P has finitely many standard vectors. Therefore to prove the discontinuity of this function it is enough to prove

$$\langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle \neq \left\langle \mathcal{A}_{I_0}(P) \cup \mathcal{B}_{I_0}(P) \right\rangle, \tag{7.1}$$

if I is close enough to I_0, but $I \nparallel I_0$.

Indeed, P has a facet F such that

$$\mathbf{s}(F) \in \mathcal{B}_{I_0}(P) \setminus \mathcal{B}_p(P). \tag{7.2}$$

If I satisfies the conditions above, then $I \nparallel F$. But then, by Lemma 21,

$$\mathbf{s}(F) \notin \langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle . \tag{7.3}$$

To finish the proof we observe that (7.1) follows from comparing (7.2) and (7.3). □

LEMMA 24. *Let P be a Voronoi parallelohedron and let p be a two-dimensional perfect free space of P. Then*

1) *p contains exactly two perfect lines — ℓ_1 and ℓ_2.*
2) *Every facet F of P is parallel either to ℓ_1 or to ℓ_2, or to both. (The last case means $F \parallel p$.)*

PROOF.

Start with the proof of statement 1. Choose a segment $I_0 \parallel p$ such that $\dim \left\langle \mathcal{B}_{I_0}(P) \right\rangle = d - 2$. It is possible, moreover, I_0 can be chosen arbitrarily, except for a finite number of directions.

Let G be a standard $(d-2)$-face of P such that

$$\mathbf{s}(G) \in \mathcal{A}_{I_0}(P) \setminus \mathcal{A}_p(P).$$

G adjoins two facets F and F'. These facets belong to antipodal caps of P with respect to I_0, so

$$\mathbf{s}(F), \mathbf{s}(F') \notin \left\langle \mathcal{A}_{I_0}(P) \cup \mathcal{B}_{I_0}(P) \right\rangle .$$

Let G' be the $(d-2)$-face of P defined by

$$\mathrm{Belt}(G') = \mathrm{Belt}(G) \quad \text{and} \quad \operatorname{rel\,int} G' \subset \operatorname{rel\,int} \mathrm{Cap}_{I_0}(P).$$

Rotating the segment $I \parallel p$, one can observe that

$$\langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle = \left\langle \mathcal{B}_p(P) \cup \{\mathbf{v}\} \right\rangle ,$$

where $\mathbf{v} \in \{\mathbf{s}(F), \mathbf{s}(F'), \mathbf{s}(G), \mathbf{s}(G')\}$ and the cases $\{\mathbf{s}(F), \mathbf{s}(F')\}$ happen only if

$I \parallel F$ and $I \parallel F'$ respectively. Thus $I \parallel F$ and $I \parallel F'$ are the only cases of discontinuity of

$$\langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle .$$

Therefore p contains exatly two perfect lines, namely those parallel to F and F'.

To prove statement 2 suppose that F is a facet of P and $F \nparallel p$. Let I be a segment satisfying $I \parallel F$ and $I \parallel p$. We have

$$\dim \langle \mathcal{B}_I(P) \rangle \geq \dim \left\langle \mathcal{B}_p(P) \cup \mathbf{s}(F) \right\rangle = d - 1.$$

Hence I is parallel to a perfect line. By assumption, $F \parallel I$, so F is parallel to a perfect line. □

In the following lemma we will reproduce from [12] the analysis of all possible arrangements of a free segment and a $(d-3)$-face of a parallelohedron. Additionally, we will emphasize the arrangements that appear if a segment is parallel to a perfect two-dimensional free plane transversal to a $(d-3)$-face or to a perfect line of such plane.

LEMMA 25. *Let E be a $(d-3)$-face of a parallelohedron P and let I be a free segment for P. Let the image space of* proj_E *be the* 3*-space where* $\mathrm{Fan}(E)$ *lies. Then*

1) *If* $\mathrm{proj}_E(I)$ *does not degenerate into a point, then* $\mathrm{proj}_E(I)$ *and* $\mathrm{Fan}(E)$ *are arranged together in one of the ways shown in Figures* 2–3.
2) *If p is a perfect two-dimensional free plane of P and p is transversal to E, then* $\mathrm{proj}_E(p)$ *is arranged with* $\mathrm{Fan}(E)$ *in one of the ways shown in Figure* 4.
3) *Finally, if I is parallel to a perfect free line in p, then* $\mathrm{proj}_E(I)$ *is arranged as one of the highlighted segments in Figure* 4.

PROOF. Item 1 is verified by inspection. One should check if the condition of Theorem 6 holds for all six-belts associated with E. For the proof of item 2, one should enumerate all the 2-planes in the image space of proj_E such that all segments parallel to such a plane are enlisted in Figures 2–3. Finally, in order to select segments parallel to perfect free lines, one should apply Lemma 24.

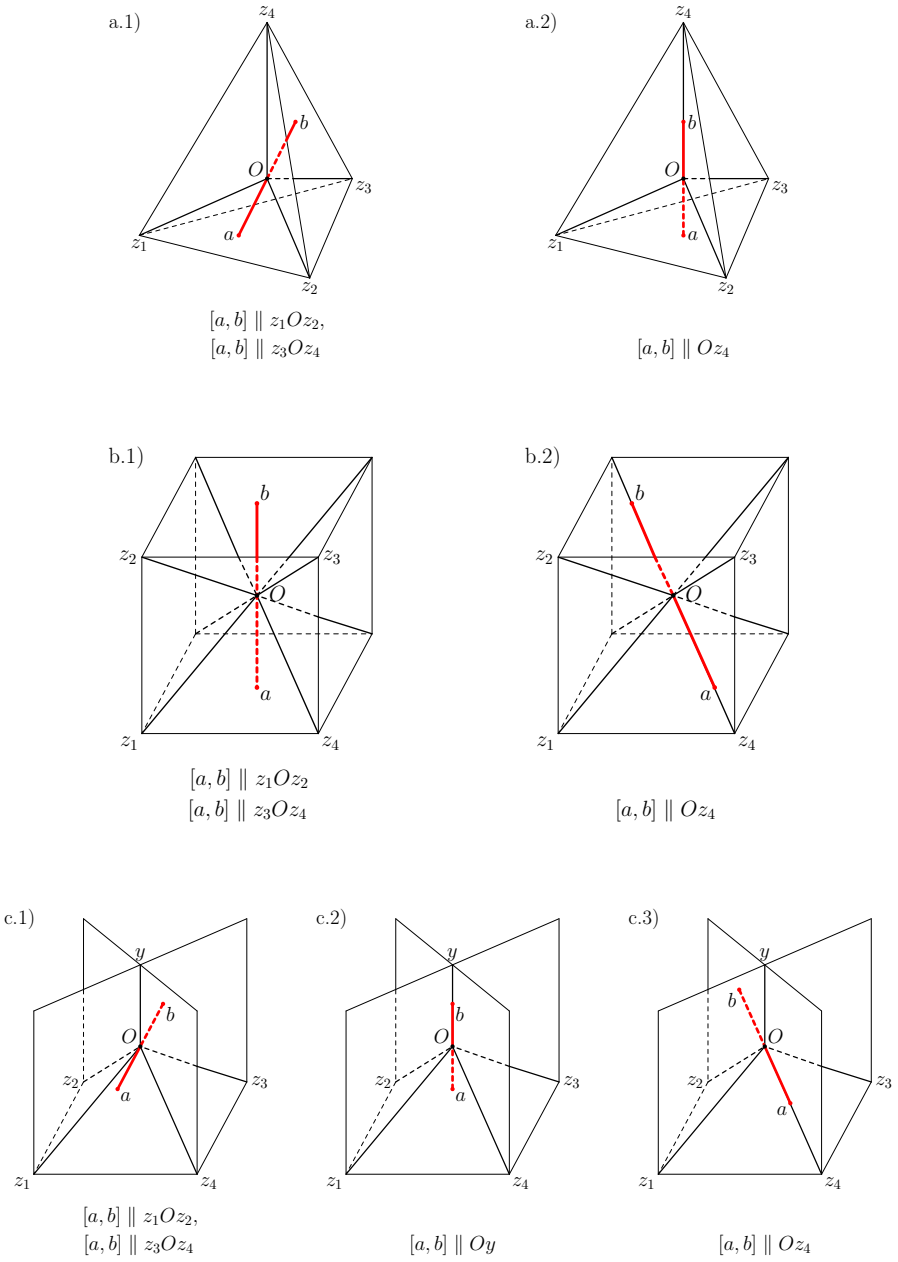


Fig. 2. Possible arrangements of free segments and $(d-3)$-faces

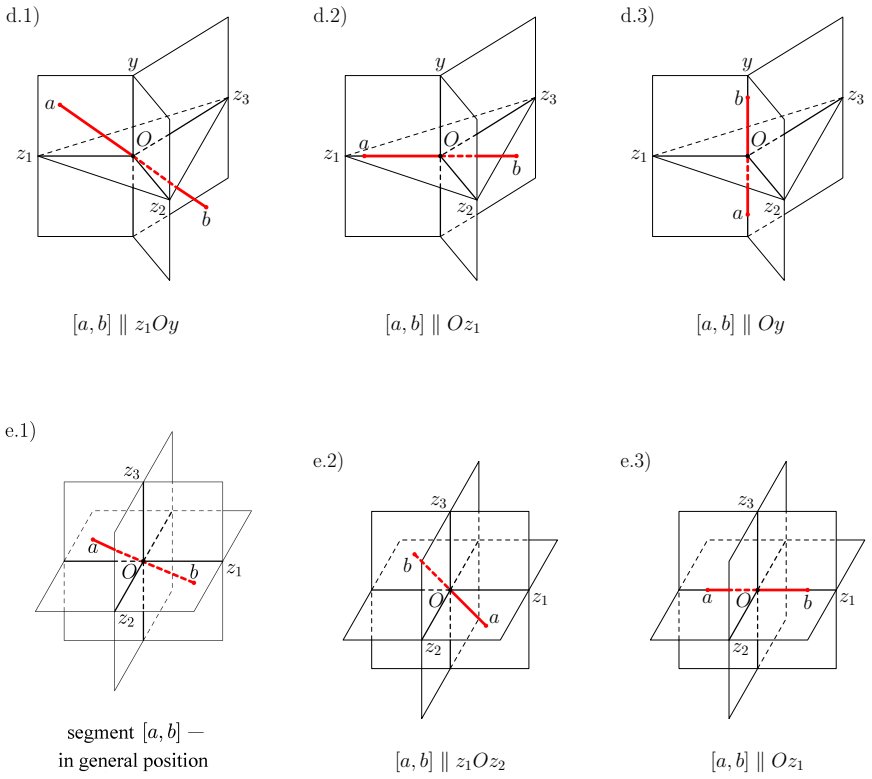


Fig. 3. Possible arrangements of free segments and $(d-3)$-faces, continued

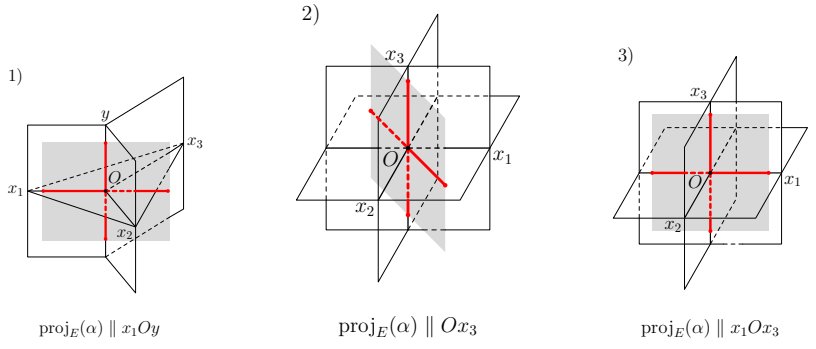


Fig. 4. Possible arrangements of $(d-3)$-faces and transversal free planes

□

LEMMA 26. *Let P be a Voronoi parallelohedron and let p be a two-dimensional perfect free space of P. In the notation of Lemma* 24 *let I, Y_1, Y_2 be segments such that $I \parallel p$, $Y_1 \parallel \ell_1$, $Y_2 \parallel \ell_2$. Then*

1) *If G is a standard $(d-2)$-face of P and $\mathbf{s}(G) \in \mathcal{A}_{Y_1}(P)$, then $\mathbf{s}(G) \in \mathcal{A}_p(P)$.*
2) *$P + Y_1$ is Voronoi in some Euclidean metric.*
3) *p is a perfect free plane and ℓ_1 and ℓ_2 are perfect free lines of $P + Y_1$.*
4) *$P + Y_1 + Y_2 + I$ is a parallelohedron.*

PROOF. If, on the contrary, $\mathbf{s}(G) \notin \mathcal{A}_p(P)$, then, by the argument of Lemma 24, $I = Y_1$ is the continuity point of

$$\langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle$$

as a function of I. But Y_1 is parallel to a perfect line, so by Lemma 23, $I = Y_1$ is not a continuity point. A contradiction gives statement 1.

Statement 2 is an immediate consequence of the definition of a perfect line and Corollary 13.

For the proof of statement 3, we first need to prove that every six-belt of $P + Y_1$ contains a facet parallel to p. Consider several cases.

Let a six-belt of $P + Y_1$ be inherited from a six-belt of P. Since p is a perfect space and $Y_1 \parallel p$, then indeed such a six-belt contains a facet parallel to p.

Let a six-belt of $P + Y_1$ be inherited from a four-belt of P. Such a six-belt contains a facet $G \oplus Y_1$, where G is a standard $(d-2)$-face of P. G spans a four-belt of P with no facet of this belt parallel to Y_1. According to Lemma 24, statement 2, all facets of the four-belt of P spanned by G are parallel to Y_2. Thus $G \parallel Y_2$. As a result we have

$$G \oplus Y_1 \parallel Y_1 \quad \text{and} \quad G \oplus Y_1 \parallel Y_2,$$

and hence $G \oplus Y_1 \parallel p$.

The last possible case occurs if a six-belt of $P + Y_1$ is spanned by a $(d-2)$-face of form $E \oplus Y_1$, where E is a $(d-3)$-face of P. In this case E and Y_1 can be arranged in the following ways reflected in Figures 2–3: c.1), d.1), e.1) (here we refer to Lemma 25).

Consider two subcases. First let p not be transversal to E. Then $\ell = p \cap \operatorname{lin\,aff} E$ is a line. Consider an arbitrary facet of the six-belt of $P + Y_1$ spanned by $E \oplus Y_1$. It has one of the forms $F + Y_1$ or $G \oplus Y_1$, where F is a facet of P, respectively,

G is a standard $(d-2)$-face of P. This facet is parallel to E and therefore to ℓ. Besides, it is parallel to Y_1. So it is parallel to p. Consequently, $\mathbf{s}(F)$ or $\mathbf{s}(G)$ is orthogonal to p.

In the second subcase p is transversal to E. $P+Y_1$ has a $(d-2)$-face $E\oplus Y_1$ only if the arrangement of p corresponds to the case 2) in Figure 4 and the arrangement of Y_1 corresponds to the case e.2) in Figure 3. But then $E \oplus Y_1$ spans a four-belt of P. Hence no six-belt is possible in this subcase.

In addition notice that each of the lines ℓ_1 and ℓ_2 is parallel to more facets of $P + Y_1$ than a generic line in p. Hence ℓ_1 and ℓ_2 are perfect free lines for $P + Y_1$.

By the same argument applied to $P + Y_1$, the parallelohedron $P + Y_1 + Y_2$ is Voronoi and has p as a perfect free space. Thus $P+Y_1+Y_2+I$ is a parallelohedron, which is exactly statement 4. □

LEMMA 27. *Let P be a Voronoi parallelohedron and let p be its perfect two-dimensional free space. Then* $\mathrm{proj}_p(P)$ *is a* $(d-2)$*-dimensional Voronoi parallelohedron.*

PROOF. In fact, we want to check that P, $P+Y_1$, $\mathrm{proj}_{Y_1}(P)$ and $\mathrm{proj}_{Y_1}(P+Y_2)$ are Voronoi parallelohedra.

P is Voronoi by assumption. $P + Y_1$ is Voronoi by Lemma 26, statement 2. Application of Lemma 16 gives that $\mathrm{proj}_{Y_1}(P)$ is Voronoi. Further, by the argument of Lemma 26, $P + Y_2$ is Voronoi and ℓ_1 is its perfect free line. Thus statement 2 of Lemma 26 gives that $P + Y_1 + Y_2$ is Voronoi. Lemma 16 applied to $P + Y_2$ and $P + Y_1 + Y_2$ implies that $\mathrm{proj}_{Y_1}(P + Y_2)$ is Voronoi. It remains to apply Lemma 16 for the third time — to $\mathrm{proj}_{Y_1}(P)$ and

$$\mathrm{proj}_{Y_1}(P) + \mathrm{proj}_{Y_1}(Y_2) = \mathrm{proj}_{Y_1}(P + Y_2). \qquad \square$$

Let I be a segment parallel to p, but not parallel to Y_1 and Y_2. For $j = 1, 2$ let

$$\mathcal{C}^j_I(P) = \{\mathbf{s}(F) : F \parallel Y_j \text{ and } \mathbf{s}(F) \in \mathcal{C}_I(P)\}.$$

In other words,

$$\mathcal{C}^j_I(P) = \mathcal{C}_I(P) \cap \mathcal{B}_{Y_j}(P).$$

The last formula immediately implies the following.

LEMMA 28. $\operatorname{lin\,aff} \mathcal{C}_I^j(P) \parallel \langle \mathcal{B}_p(P) \rangle$ *for* $j = 1, 2$.

PROOF. Indeed,

$$\operatorname{lin\,aff} \mathcal{C}_I^j(P) \subseteq \langle \mathcal{A}_I(P) \cup \mathcal{B}_I(P) \rangle \cap \langle \mathcal{A}_{Y_j}(P) \cup \mathcal{B}_{Y_j}(P) \rangle .$$

The right part is an intersection of two different hyperplanes, each parallel to $\langle \mathcal{B}_p(P) \rangle$. Thus the intersection is exactly $\langle \mathcal{B}_p(P) \rangle$. □

LEMMA 29. *Let P be a Voronoi parallelohedron and let $\mathfrak{p}$ be its perfect two-dimensional free space. In addition, let P be centered at $\mathbf{0}$. Assume that I is a segment parallel to P, but not parallel to Y_1 and Y_2. With $\mathcal{C}_I^j(P)$ defined as above, choose*

$$\mathbf{w}_j \in \mathcal{C}_I^j(P) \quad \textit{for} \quad j = 1, 2 \quad \textit{and}$$

$$\mathbf{t}_j \in \Lambda(P) \cap \left(\mathbf{w}_j + \langle \mathcal{A}_p(P) \cup \mathcal{B}_p(P) \rangle \right) .$$

Then, if $P_j = P + \mathbf{t}_j$, one has

$$\operatorname{proj}_p(P \cap P_1 \cap P_2) = \operatorname{proj}_p(P) \cap \operatorname{proj}_p(P_1) \cap \operatorname{proj}_p(P_2).$$

PROOF. Consider the complex $\mathcal{K}$ defined in Section 6. Recall that $\mathcal{K}$ splits two layers $\mathcal{L}_0, \mathcal{L}_1 \subset T(P)$ given by formulae (6.2). Since $P \in \mathcal{L}_0$ and $P_1, P_2 \in \mathcal{L}_1$,

$$P \cap P_1 \cap P_2 \subset |\mathcal{K}|.$$

Set $Q = \operatorname{proj}_I(P)$. Q is a Voronoi parallelohedron with a free segment $\operatorname{proj}_I(Y_1)$. (Or $\operatorname{proj}_I(Y_2)$, which has the same direction.) One can easily see that the sets

$$\mathcal{M}_j = \{Q + \operatorname{proj}_I \mathbf{t} : \mathbf{t} \in \operatorname{aff} \mathcal{C}_I^j(P) \cap \Lambda(P)\} \quad \text{for} \quad j = 1, 2$$

compose two layers of the same tiling of $\mathbb{R}^{d-1}$ by translates of Q. The notion of layers is the same as described in Section 6. Call them $\mathcal{M}_1$- *and* $\mathcal{M}_2$-*layers*, respectively.

These layers are neighboring. Indeed, choose an arbitrary standard $(d-2)$-face G of P with

$$\mathbf{s}(G) \in \mathcal{A}_I(P) \setminus \mathcal{A}_p(P).$$

Then take a face G' spanning the same belt as G such that rel int $G' \subset$ rel int $\mathrm{Cap}_I(P)$. It is not hard to see that $\mathrm{proj}_I(G')$ belongs to the common boundary of the $\mathcal{M}_1$- and $\mathcal{M}_2$-layers. Consequently,

$$\begin{aligned}\mathrm{proj}_p(P_1 \cap P_2 \cap \mathcal{K}) &= \mathrm{proj}_p((P_1 \cap \mathcal{K}) \cap (P_2 \cap \mathcal{K})) = \\ &= \mathrm{proj}_{\mathrm{proj}_I(Y_1)}((Q + \mathrm{proj}_I(\mathbf{t}_1)) \cap (Q + \mathrm{proj}_I(\mathbf{t}_2))) = \\ &= \mathrm{proj}_{\mathrm{proj}_I(Y_1)}(Q + \mathrm{proj}_I(\mathbf{t}_1)) \cap \mathrm{proj}_{\mathrm{proj}_I(Y_1)}(Q + \mathrm{proj}_I(\mathbf{t}_2)) = \\ &= \mathrm{proj}_p(P_1) \cap \mathrm{proj}_p(P_2).\end{aligned}$$

It remains to prove that

$$P_1 \cap P_2 \cap \mathcal{K} \cap \mathrm{proj}_I^{-1}(P) \subset P.$$

So, it suffices to show that the the common boundary of the $\mathcal{M}_1$- and $\mathcal{M}_2$-layers separates the $\mathrm{proj}_I(Y_1)$-caps of Q from each other. It follows from the fact that each of these two caps is covered by its layer — one by the $\mathcal{M}_1$-layer and the other — by the $\mathcal{M}_2$-layer.

We recall that $\dim Q = d - 1$, so the caps of Q are homogeneous $(d-2)$-dimensional complexes. Each cap is connected, so we need to prove that every two facets of a cap (of dimension $d-2$) sharing a $(d-3)$-face belong to the same layer. This $(d-3)$-face is, obviously of form $\mathrm{proj}_I(E)$, where E is a $(d-3)$-face of P.

Of course, p is transversal to E. By Lemma 25, E as a face of $T(P)$ can have only cubic or prismatic type of coincidence and, if E is cubic, P has a facet F or a standard $(d-2)$-face G related to the dual cell $D(E)$ such that $\mathbf{s}(F) \in \mathcal{B}_p(P)$ (respectively, $\mathbf{s}(F) \in \mathcal{A}_p(P)$).

In each case $\mathrm{proj}_I(E)$ adjoins two facets of Q covered by the same layer (either $\mathcal{M}_1$- or $\mathcal{M}_2$-). Further, if a facet of a $\mathrm{proj}_I(Y_1)$-cap of Q is covered by the $\mathcal{M}_1$-layer, its antipodal is covered by the $\mathcal{M}_2$-layer and vice versa. Thus the caps are covered by different layers. This finishes the proof. □

8. Sketch of the further argument

Before we start with the induction, recall the notation for the statements that we prove and the outline of the induction.

- The statement $M(d)$ says that Voronoi's Conjecture is true for all extensions of Voronoi parallelohedra of all dimensions up to d. (If $M(d)$ holds for all d, then Theorem 3 is true.)
- The statement $A(d)$ says that every Voronoi parallelohedron of dimension not greater than d is reducible, once it has a 2-dimensional free plane. (If $A(d)$ holds for all d, then Theorem 14 is true.)
- The statement $B(d)$ says that every Voronoi parallelohedron of dimension not greater than d is reducible, once it has a cross. (If $B(d)$ holds for all d, then Theorem 4 is true.)
- The statement $C(d)$ says that if a reducible Voronoi parallelohedron of dimension not greater than d, has a cross, then every its irreducible direct summand is parallel to at least one plane of the cross. (If $C(d)$ holds for all d, then Theorem 5 is true.)

We have already proved that $M(d) \Leftrightarrow A(d)$. Here we will show that $A(d)$ is true for $d \leq 4$, and also $B(d)$ and $C(d)$ are both true for $d \leq 2$.

Then we will follow with the three implications

- $B(d-3) \Longrightarrow C(d-2)$;
- $\big(A(d-2), C(d-2)\big) \Longrightarrow B(d-2)$;
- $\big(B(d-2), C(d-2)\big) \Longrightarrow A(d)$.

Once we are finished with that, then, by induction, all main results are proved. Now we check the induction base.

LEMMA 30. *$A(d)$ is true for $d \leq 4$; $B(d)$ and $C(d)$ are true for $d \leq 2$.*

PROOF. Voronoi's Conjecture is true for all 1-, 2-, 3-, and 4-dimensional parallelohedra, including all extensions. Therefore $M(d)$ holds for $d \leq 4$. But $M(d) \Leftrightarrow A(d)$, so $A(d)$ is also true for $d \leq 4$.

Among the 1- and 2-dimensional parallelohedra the only ones with a cross are parallelograms. They are reducible, and the summands are segments which are parallel to the planes (1-dimensional lines) of the cross. Hence $B(d)$ and $C(d)$ are true for $d \leq 2$. □

9. Proof of implication B($d-3$) $\Rightarrow$ C($d-2$)

The proof of this implication is easy and short, however, we devote a separate section for it, as well as for proofs of two other implications. Here and in the next section we will write n for $d-2$.

LEMMA 31. *If Theorem* 4 *is true for parallelohedra of all dimensions up to* $n-1$ *then Theorem* 5 *is true for* n*-dimensional parallelohedra.*

PROOF. Let $\dim P = n$ and $P = P_1 \oplus P_2 \oplus \ldots \oplus P_k$, where $k > 1$ and all P_i are irreducible and let (Π_1, Π_2) be a cross for P. We have to prove that aff $P_i \parallel \Pi_1$ or aff $P_i \parallel \Pi_2$ for each $i = 1, 2, \ldots, k$.

Assume the converse, say, aff $P_1 \nparallel \Pi_1$ and aff $P_i \nparallel \Pi_2$. Then

$$(\text{lin aff } P_1 \cap \Pi_1,\ \text{lin aff } P_1 \cap \Pi_2)$$

is a pair of hyperplanes in lin aff P_1 being a cross for P_1. But $\dim P_1 < n$, therefore by Theorem 4 the parallelohedron P_1 is reducible, a contradiction. □

10. Proof of implication $\big($A($d-2$), C($d-2$)$\big) \Rightarrow$ B($d-2$)

The required implication is the outcome of Lemma 33. For that lemma, we need to establish a connection between Voronoi parallelohedra with a cross and Voronoi parallelohedra with a 2-dimensional free plane, which is done in Lemma 32.

LEMMA 32. *Assume that a Voronoi* n*-parallelohedron* $P(\Lambda, \Omega)$ *has a cross* (Π_1, Π_2) *and the lattices*

$$\Lambda \cap \Pi_1 \quad \textit{and} \quad \Lambda \cap \Pi_2$$

are $(n-1)$*-dimensional. Then there are vectors* $\mathbf{n}_1$ *and* $\mathbf{n}_2$ *such that*

1) $\mathbf{n}_1$ *is orthogonal to* Π_1 *in* $\|\cdot\|_\Omega$.
2) $\mathbf{n}_2$ *is orthogonal to* Π_2 *in* $\|\cdot\|_{\Omega_{\mathbf{n}_1}}$.
3) *The twofold dilatation* $P(\Lambda, (\Omega_{\mathbf{n}_1})_{\mathbf{n}_2})$ *has a free space* $\langle \mathbf{n}_1, \mathbf{n}_2 \rangle$.

PROOF. The lattice $\Lambda \cap \Pi_1 \cap \Pi_2$ is $(n-2)$-dimensional. Indeed, Π_1 and Π_2 have bases consisting of integer vectors, so they can be restricted to hyperplanes in $\mathbb{Q}^n$. Therefore $\Pi_1 \cap \Pi_2$ restricted to $\mathbb{Q}^n$ is a $(d-2)$-dimensional linear space. Hence it has a rational basis and, under a proper scaling, an integer basis.

For every possible choice of $(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}$ its restriction to $\Pi_1 \cap \Pi_2$ coincides with the restriction of Ω to the same space. Denote this rectriction by Ω'. Let ρ be the radius of the largest empty sphere for the lattice $\Lambda \cap \Pi_1 \cap \Pi_2$ with respect to the metric $\|\cdot\|_{\Omega'}$.

Let $\mathbf{n}_1'$ be an arbitrary normal to Π_1 in $\|\cdot\|_\Omega$. Then the whole lattice Λ can be covered by a bundle $\mathcal{X}_1$ of hyperplanes

$$(\mathbf{n}_1')^T \Omega \mathbf{x} = m\alpha, \quad m \in \mathbb{Z}.$$

Set $\mathbf{n}_1 = \left|\frac{\rho}{\alpha}\right| \mathbf{n}_1'$. Then the distance between the hyperplanes

$$(\mathbf{n}_1')^T \Omega \mathbf{x} = m\alpha \quad \text{and} \quad (\mathbf{n}_1')^T \Omega \mathbf{x} = (m+1)\alpha$$

in $\|\cdot\|_{\Omega_{\mathbf{n}_1}}$ equals $|\alpha|\sqrt{1+\frac{\rho^2}{\alpha^2}} > \rho$.

Similarly, let $\mathbf{n}_2'$ be an arbitrary normal to Π_2 in $\|\cdot\|_{\Omega_{\mathbf{n}_1}}$. Then the whole lattice Λ can be covered by a bundle $\mathcal{X}_2$ of hyperplanes

$$(\mathbf{n}_2')^T \Omega_{\mathbf{n}_1} \mathbf{x} = m\beta, \quad m \in \mathbb{Z}.$$

Set $\mathbf{n}_2 = \left|\frac{\rho}{\beta}\right| \mathbf{n}_2'$. Then the distance between the hyperplanes

$$(\mathbf{n}_2')^T \Omega_{\mathbf{n}_1} \mathbf{x} = m\beta \quad \text{and} \quad (\mathbf{n}_2')^T \Omega_{\mathbf{n}_1} \mathbf{x} = (m+1)\beta$$

in $\|\cdot\|_{(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}}$ equals $|\beta|\sqrt{1+\frac{\rho^2}{\beta^2}} > \rho$.

Changing the metric from $\|\cdot\|_{\Omega_{\mathbf{n}_1}}$ to $\|\cdot\|_{(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}}$ does not decrease the distances, so the $(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}$-distance between two consecutive planes of $\mathcal{X}_1$ is still greater than ρ.

Consider the Delaunay tiling $\mathcal{D}$ for lattice Λ and metric $\|\cdot\|_{(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}}$. We prove that every triangle $\Delta \in \mathcal{D}$ has an edge parallel to $\Pi_1 \cap \Pi_2$.

By Corollary 18, every edge of $\mathcal{D}$ is parallel to Π_1 or Π_2. By Pigeonhole principle, Δ has two edges parallel to the same hyperplane, say, Π_2. Then $\operatorname{aff}\Delta \parallel \Pi_2$.

Assume that no edge of Δ is parallel to $\Pi_1 \cap \Pi_2$. Then no edge of Δ is parallel to Π_1. Then the vertices of Δ belong to pairwise different planes of $\mathcal{X}_1$. Denote the vertices of Δ by $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$. Without loss of generality assume that the plane of $\mathcal{X}_1$ passing through $\mathbf{x}_2$ lies between the planes of $\mathcal{X}_1$ passing through $\mathbf{x}_1$ and $\mathbf{x}_3$.

Consider a subbundle $\mathcal{X}_1' \subset \mathcal{X}_1$ consisting of those hyperplanes of $\mathcal{X}_1$ that have at least one integer point in common with $\operatorname{aff}\Delta$. Of course, the hyperplanes of $\mathcal{X}_1'$

are equally spaced, and the intersection of each with Π_2 contains a $(d-2)$-lattice. Finally, the hyperplanes of $\mathcal{X}_1$ passing through $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are in $\mathcal{X}_1'$.

Let the interval $(\mathbf{x}_1, \mathbf{x}_3)$ be intersected by exactly m hyperplanes of $\mathcal{X}_1'$ (obviously, $m \geq 1$). Set

$$\mathbf{x}_4 = \frac{m-1}{m+1}\mathbf{x}_1 + \frac{2}{m+1}\mathbf{x}_3.$$

Obviously, $\mathbf{x}_4 \in [\mathbf{x}_1, \mathbf{x}_3]$.

Since $[\mathbf{x}_1, \mathbf{x}_3]$ is an edge of $\mathcal{D}$, $\partial B_{(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}}(\mathbf{x}_1, \mathbf{x}_3)$ is an empty sphere. Perform a homothety with center $\mathbf{x}_1$ and coefficient $\frac{2}{m+1}$. The ball $B_{(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}}(\mathbf{x}_1, \mathbf{x}_3)$ goes to the ball $B_{(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}}(\mathbf{x}_1, \mathbf{x}_4)$. Since $\frac{2}{m+1} \leq 1$,

$$B_{(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}}(\mathbf{x}_1, \mathbf{x}_4) \subset B_{(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}}(\mathbf{x}_1, \mathbf{x}_3)$$

and therefore $\partial B_{(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}}(\mathbf{x}_1, \mathbf{x}_4)$ is an empty sphere.

By choice of $\mathbf{x}_4$, the point $\frac{\mathbf{x}_1+\mathbf{x}_4}{2}$ lies in a plane of $\mathcal{X}_1'$. Thus the $(n-2)$-plane

$$\frac{\mathbf{x}_1 + \mathbf{x}_4}{2} + (\Pi_1 \cap \Pi_2)$$

contains an $(n-2)$-lattice with all empty spheres not greater than ρ in radius. But the sphere

$$\partial B_{(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}}(\mathbf{x}_1, \mathbf{x}_4) \cap \left(\frac{\mathbf{x}_1 + \mathbf{x}_4}{2} + (\Pi_1 \cap \Pi_2) \right)$$

is empty and has radius

$$\frac{1}{2}\|\mathbf{x}_4 - \mathbf{x}_1\|_{(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}} > \rho,$$

because $\mathbf{x}_1$ and $\mathbf{x}_4$ belong to two non-consecutive planes of $\mathcal{X}_1$. A contradiction, thus every triangle of $\mathcal{D}$ has an edge parallel to $\Pi_1 \cap \Pi_2$.

Hence, by Theorem 6, the orthogonal complement to $\Pi_1 \cap \Pi_2$ in $\|\cdot\|_{(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}}$ is a free space for $P(\Lambda, (\Omega_{\mathbf{n}_1})_{\mathbf{n}_2})$. It is not hard to check that $\mathbf{n}_1$ and $\mathbf{n}_2$ are independent and both orthogonal to $\Pi_1 \cap \Pi_2$. □

LEMMA 33. *Assume that Theorems* 14 *and* 5 *are true for dimension* n. *Then all* n*-dimensional Voronoi parallelohedra with crosses are reducible.*

PROOF. Let $P(\Lambda, \Omega)$ have a cross. Then $\mathcal{F}(\Lambda, \Omega)$ can be partitioned into two subsets $\mathcal{F}_1, \mathcal{F}_2$ of dimension less than n each. If necessary, append $\mathcal{F}_1$ and $\mathcal{F}_2$ by several

vectors of Λ to obtain generating sets of two hyperplanes Π_1 and Π_2 respectively. By construction, (Π_1, Π_2) is a cross for $P(\Lambda, \Omega)$ satisfying the conditions of Lemma 32.

Consider the parallelohedron $P(\Lambda, (\Omega_{\mathbf{n}_1})_{\mathbf{n}_2})$ introduced in Lemma 32. It has a two-dimensional free space $\langle \mathbf{n}_1, \mathbf{n}_2 \rangle$. In addition, by Corollary 18, (Π_1, Π_2) is a cross for $P(\Lambda, (\Omega_{\mathbf{n}_1})_{\mathbf{n}_2})$ as well.

By Theorem 14 for dimension n,

$$P(\Lambda, (\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}) = P_1 \oplus P_2 \oplus \ldots \oplus P_k.$$

In turn, Theorem 5 says that aff $P_j \parallel \Pi_1$ or aff $P_j \parallel \Pi_2$.

Let R_1 be the sum of all summands that are parallel to Π_1 and R_2 be the sum of the remaining summands. Then

$$P(\Lambda, (\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}) = R_1 \oplus R_2,$$

where aff $R_j \parallel \Pi_j$ $(j = 1, 2)$. Obviously, aff R_1 and aff R_2 are orthogonal with respect to $(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2}$.

Thus $(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2} = \Omega_1 + \Omega_2$, where Ω_1 and Ω_2 are positive semidefinite quadratic forms with kernels lin aff R_2 and lin aff R_1 respectively.

The kernel of $(\Omega_{\mathbf{n}_1})_{\mathbf{n}_2} - \Omega_{\mathbf{n}_1}$ contains lin aff R_2. Thus the kernel of

$$\Omega_1' = \Omega_1 - (\Omega_{\mathbf{n}_1})_{\mathbf{n}_2} + \Omega_{\mathbf{n}_1}$$

contains lin aff R_2. But the form Ω_1' is positive definite on lin aff R_1, otherwise the form $\Omega_{\mathbf{n}_1} = \Omega_2 + \Omega_1'$ is not positive definite.

Ω_1' and Ω_2 have complementary kernels lin aff R_2 and lin aff R_1 respectively, therefore $P(\Lambda, \Omega_{\mathbf{n}_1}) = R_1' \oplus R_2$, where aff $R_1' \parallel \Pi_1$. Thus, in addition, (Π_1, Π_2) is a cross for $P(\Lambda, \Omega_{\mathbf{n}_1})$.

Repeating the same argument for $P(\Lambda, \Omega_{\mathbf{n}_1})$ we obtain that $P(\Lambda, \Omega)$ is reducible and has the cross (Π_1, Π_2). □

11. Proof of implication $\big(\mathbf{B}(d-2), \mathbf{C}(d-2)\big) \Rightarrow \mathbf{A}(d)$

We will use the results of Section 7 extensively. In order to do this, we prove the following.

LEMMA 34. *If a parallelohedron has a free two-dimensional plane, then it has a free perfect two-dimensional plane.*

PROOF. Assume that P be a parallelohedron and p is a free plane for P. Let $\pm F_1, \pm F_2, \ldots, \pm F_k$ be all the facets of P parallel to p. Obviously, each six-belt of P contains at least one pair $\pm F_j$, otherwise p is not free. Further,

$$\dim (\text{lin aff } F_1 \cap \text{lin aff } F_2 \cap \ldots \cap \text{lin aff } F_k) \geq 2,$$

as the intersection contains p.

If necessary, add facets $\pm F_{k+1}, \pm F_{k+2}, \ldots \pm F_m$ so that

$$\dim p' = 2, \quad \text{where} \quad p' = (\text{lin aff } F_1 \cap \text{lin aff } F_2 \cap \ldots \cap \text{lin aff } F_m)\,.$$

Then the conditions of Theorem 6 hold for every segment $I \parallel p'$. Thus p' is a free plane for P. □

The remaining part of the proof is presented as a series of lemmas.

LEMMA 35. *Let $R = P(\Lambda, \Omega)$ be a Voronoi parallelohedron. Assume in addition that R is centered at the origin and $\mathbf{0} \in \Lambda$.*

Let $\mathbf{v}$ be a vector. Call a facet $F \subset R$ good, if the point $\mathbf{v} + \frac{1}{2}\mathbf{s}(F)$, which is the center of the facet $F + \mathbf{v} \subset R + \mathbf{v}$, is disjoint from all facets of $T(R)$ parallel to F. Otherwise call F bad.

Finally, let

$$\mathbf{v}' \in (\Lambda + \mathbf{v}) \cap R.$$

Then the vector $\mathbf{v}'$ is parallel to all bad facets of R.

PROOF. Let $F \subset R$ be a bad facet. Then, by definition of a bad facet, the point $\mathbf{v} + \frac{1}{2}\mathbf{s}(F)$ belongs to some facet $F + \mathbf{t}$, where $\mathbf{t} \in \Lambda$. It means that the polytopes $F + \mathbf{t}$ and $F + \mathbf{v}$ have a common point $\mathbf{v} + \frac{1}{2}\mathbf{s}(F)$.

Therefore the polytopes F and $F + \mathbf{v} - \mathbf{t}$ share a common point

$$\mathbf{v} - \mathbf{t} + \frac{1}{2}\mathbf{s}(F).$$

Hence (see [3] for details),

$$\mathbf{v} - \mathbf{t} \in \frac{1}{2}F + \frac{1}{2}(-F). \tag{11.1}$$

The inclusion (11.1) has two immediate consequences. First of all, $\mathbf{v} - \mathbf{t} \parallel F$. Secondly, since $-F$ is also a face of R, $\mathbf{v} - \mathbf{t} \in R$. Thus we have found a particular vector from $(\Lambda + \mathbf{v}) \cap R$ which is parallel to F. Now we have to prove the same parallelity for all other vectors of $(\Lambda + \mathbf{v}) \cap R$.

R is a fundamental domain for the translation group Λ. Consequently, if $\mathbf{v} - \mathbf{t} \in \operatorname{rel\,int} R$, then $(\Lambda + \mathbf{v}) \cap R$ consists of the only vector $\mathbf{v} - \mathbf{t}$, which is parallel to F, as proved above.

Now suppose that $\mathbf{v} - \mathbf{t} \in \partial R$. Let E be the minimal face of R containing the point $\mathbf{v} - \mathbf{t}$. All the elements of $(\Lambda + \mathbf{v}) \cap R$ are representable as $\mathbf{v} - \mathbf{t} + \mathbf{t}'$, where $\mathbf{t}' \in \Lambda$ and $E + \mathbf{t}' \subset R$. For a Voronoi parallelohedron R it is well-known that $E \subset R$ and $E + \mathbf{t}' \subset R$ together give $\mathbf{t}' \perp E$ with orthogonality related to $\|\cdot\|_\Omega$.

On the other hand, $\frac{1}{2}F + \frac{1}{2}(-F)$ is a mid-section of the prism $\operatorname{conv}(F \cup (-F))$. Therefore (11.1) guarantees that if $\mathbf{v} - \mathbf{t} \in E$, then necessarily

$$\left[\mathbf{v} - \mathbf{t} - \frac{1}{2}\mathbf{s}(F), \mathbf{v} - \mathbf{t} + \frac{1}{2}\mathbf{s}(F)\right] \subset E.$$

Hence $\mathbf{s}(F) \in \operatorname{lin\,aff} E$ and, consequently, $\mathbf{t}' \perp \mathbf{s}(F)$. As a result, $\mathbf{t}' \parallel F$, and finally, $\mathbf{v} - \mathbf{t} + \mathbf{t}' \parallel F$. □

LEMMA 36. *Let a Voronoi parallelohedron P have a free perfect two-dimensional plane p. Then P is a prism, or the parallelohedron $R = \operatorname{proj}_p(P)$ has a cross.*

PROOF. Recall that p contains two perfect free lines ℓ_1 and ℓ_2 and let the segment I be parallel to p, but non-parallel to both ℓ_j. Again, let the segments Y_j to be parallel to ℓ_j. In Section 7 we have defined the sets $\mathcal{C}_I^j(P)$ for $j = 1, 2$. As in Lemma 29, let

$$\mathbf{w}_j \in \mathcal{C}_I^j(P), \quad \text{and}$$
$$\Lambda_j = \Lambda(P) \cap \left(\mathbf{w}_j + \left\langle \mathcal{A}_p(P) \cup \mathcal{B}_p(P) \right\rangle\right).$$

By Lemma 26, $P + Y_1 + Y_2$ is a parallelohedron. Since it has a nonzero width in the direction p, the sets

$$T_j = \{\operatorname{proj}_p(P + \mathbf{t}) : \mathbf{t} \in \Lambda_j\} \quad (j = 1, 2)$$

both are tilings of $\mathbb{R}^{d-2}$ by translates of a parallelohedron $R = \mathrm{proj}_p(P)$. Choose $\mathbf{v}_1$ and $\mathbf{v}_2$ so that

$$R - \mathbf{v}_j \in T_j.$$

Let $\mathbf{v}_1 \in \Lambda(R)$. Then R is a tile of $\mathbf{T}_1$, and it is the only tile of T_1 to have a $(d-2)$-dimensional intersection with R.

From Lemma 29 it immediately follows that

$$|\mathcal{C}_I^j(P)| = 1.$$

Thus all but two facets of P are parallel to ℓ_2. As an immediate consequence we get that P is a prism. Similarly, P is a prism if $\mathbf{v}_2 \in \Lambda(R)$.

By Lemma 27, $R = P(\Lambda(R), \Omega)$ for some positive quadratic form Ω of $(d-2)$ variables. Now in terms of Lemma 35, assume that every facet of R is good with respect at least to one vector $\mathbf{v}_1$ or $\mathbf{v}_2$. Choose

$$\mathbf{v}_j' \in R \cap (\mathbf{v}_j + \Lambda(R)).$$

The cases $\mathbf{v}_j' = \mathbf{0}$ have been considered before, so assume that $\mathbf{v}_j' \neq \mathbf{0}$ for $j = 1, 2$.

Then, according to Lemma 35, each facet of R is parallel to $\mathbf{v}_1'$ or $\mathbf{v}_2'$. Equivalently, each facet vector of R is orthogonal to $\mathbf{v}_1'$ or $\mathbf{v}_2'$ in the metric $\|\cdot\|_\Omega$. Thus orthogonal complements to $\mathbf{v}_1'$ and $\mathbf{v}_2'$ form a cross for R.

We will prove that nothing else is possible. Namely, no facet of R can be bad with respect both to $\mathbf{v}_1$ and $\mathbf{v}_2$.

Assume that E' is a facet of R that is bad with respect to $\mathbf{v}_1$ and $\mathbf{v}_2$. Then, obviously there exist $R_1 \in T_1$ and $R_2 \in T_2$ satisfying

$$E' \cap \mathrm{rel\,int}\, R_1 \cap \mathrm{rel\,int}\, R_2 \neq \varnothing. \tag{11.2}$$

Indeed, in the sense of $(d-3)$-dimensional Lebesgue measure, almost every point of E' close enough to its center is covered by exactly one tile of T_1 and exactly one tile of T_2.

Let $R_j = \mathrm{proj}_p(P_j)$, where $P_j = P + \mathbf{t}_j$, $\mathbf{t}_j \in \Lambda_j$ and $j = 1, 2$. Then, by Lemma 29, the face $P \cap P_1 \cap P_2$ is $(d-2)$-dimensional and has a $(d-3)$-subface E such that

$$\mathrm{proj}_p(E) = E' \cap R_1 \cap R_2.$$

Since

$$\dim \operatorname{aff} E = \dim \operatorname{aff} \operatorname{proj}_p(E) = d - 3,$$

the plane p is transversal to E. This corresponds to one of the cases of Lemma 25, item 2. But none of these cases matches with (11.2), a contradiction. Hence R cannot have facets which are bad with respect both to $\mathbf{v}_1$ and $\mathbf{v}_2$. □

LEMMA 37. *Let a Voronoi d-parallelohedron P have a free two-dimensional plane p. Assume that Theorems* 4 *and* 5 *hold for dimension $n = d - 2$. Then P is reducible.*

PROOF. Lemma 34 asserts that P has a perfect free plane. Therefore let p be perfect for the rest of the proof.

We will use the notation of Lemma 36. We also assume that the image space of proj_p is $\langle \mathcal{A}_p(P) \cup \mathcal{B}_p(P) \rangle$.

If P is a prism, then, obviously, the assertion of Lemma 37 is true. By Lemma 36, if P is not a prism, then every facet vector of the parallelohedron $R = \operatorname{proj}_p(P)$ is orthogonal to at least one of the two vectors $\mathbf{v}_1'$ and $\mathbf{v}_2'$. Thus R is reducible, and Theorem 5 gives that

$$R = S_1 \oplus S_2, \quad \text{where} \quad \operatorname{aff} S_j \perp \mathbf{v}_j', \quad j = 1, 2.$$

Hence $\mathbf{v}_1' \in \operatorname{lin} \operatorname{aff} S_2$ and $\mathbf{v}_2' \in \operatorname{lin} \operatorname{aff} S_1$. Consequently, if $\mathbf{t} \in \Lambda_1$ and $\dim \operatorname{aff}((R+\mathbf{t}) \cap R) = d-2$ (respectively, $\mathbf{t} \in \Lambda_2$ and $\dim \operatorname{aff}((R+\mathbf{t}) \cap R) = d-2$), then

$$\operatorname{proj}_p(\mathbf{t}) \in \operatorname{lin} \operatorname{aff} S_2 \quad (\text{respectively, } \operatorname{proj}_p(\mathbf{t}) \in \operatorname{lin} \operatorname{aff} S_1).$$

But if F is a facet of P and $\mathbf{s}(F) \in \mathcal{C}_I(P)$, then

$$\operatorname{proj}_p\big(P \cap (P + \mathbf{s}(F))\big) = R \cap (R + \mathbf{t}),$$

where $\mathbf{t}$ denotes $\mathbf{s}(F)$. In particular, this gives

$$\dim \operatorname{aff}\big(R \cap (R + \mathbf{t})\big) = d - 2.$$

As a result,

$$\mathcal{C}_I^1(P) \subset \mathbf{w}_1 + \operatorname{lin} \operatorname{aff} S_2, \qquad \mathcal{C}_I^2(P) \subset \mathbf{w}_2 + \operatorname{lin} \operatorname{aff} S_1. \tag{11.3}$$

Further, every vector of $\mathcal{B}_p(P)$ corresponds to a facet vector of R, so

$$\mathcal{B}_p(P) \in \text{lin aff } S_1 \cup \text{lin aff } S_2. \tag{11.4}$$

Combining (11.3) and (11.4), we obtain that every facet vector of P belongs to one of the two complementary spaces

$$\langle \mathbf{w}_1 \rangle \oplus \text{lin aff } S_2 \quad \text{and} \quad \langle \mathbf{w}_2 \rangle \oplus \text{lin aff } S_1.$$

By Theorem 2, P is reducible. □

At this final point we have proved all the implications involved in our induction. Hence the proof of all main results (Theorems 3, 4, 5 and 14) is complete.

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