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The Voronoi Conjecture for Parallelohedra with Simply Connected δ-Surfaces

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Abstract We show that the Voronoi conjecture is true for parallelohedra with simply connected δ-surfaces. That is, we show that if the boundary of parallelohedron P remains simply connected after removing closed nonprimitive faces of codimension 2, then P is affinely equivalent to a Dirichlet–Voronoi domain of some lattice. Also, we construct the π-surface associated with a parallelohedron and give another condition in terms of a homology group of the constructed surface. Every parallelohedron with a simply connected δ-surface also satisfies the condition on the homology group of the π-surface.

Keywords Parallelohedron · Voronoi conjecture · Fundamental group · Homology group · Canonical scaling

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1 The Voronoi Conjecture

Definition 1.1 A d-dimensional polytope P is called a *parallelohedron* if $\mathbb{R}^d$ can be tiled by nonoverlapping translates of P.

A tiling by parallelohedra is *face-to-face* if the intersection of any two copies of P is a face of both, and it is *non-face-to-face* otherwise. Venkov [18] and, later, independently, McMullen [13] proved that if there is a non-face-to-face tiling by P, then there is also a face-to-face tiling. It is clear that the face-to-face tiling by P is unique up to translation. We will denote this tiling by $\mathcal{T}(P)$ or simply $\mathcal{T}$ when the generating polytope of the tiling is obvious.

In 1897 Minkowski [15] proved that every parallelohedron P is centrally symmetric, and all facets of P are centrally symmetric. Later, Venkov [18] added the third necessary condition: he proved that the projection of P along any face of codimension 2 is a two-dimensional parallelohedron, i.e. a parallelogram or a centrally symmetric hexagon. Also, Venkov proved that these three conditions are sufficient for a convex polytope to be a parallelohedron. In 1980, McMullen [13] gave an independent proof that these three conditions are necessary and sufficient; see also [14] for acknowledgment of priority.

The centers of all tiles of $\mathcal{T}(P)$ form a d-dimensional lattice $\Lambda(P)$. If the fundamental domain of $\Lambda(P)$ has volume 1, then the homothetic polytope $2P$ is centrally symmetric with respect to the origin, has volume 2^d, and contains no lattice points in the interior. This, by definition, means that $2P$ is an *extremal body*. Additional information about extremal bodies can be found in Gruber and Lekkerkerker [9, Chap. 2, §12].

On the other hand, given a d-dimensional lattice Λ, the Dirichlet–Voronoi polytope $P(\Lambda)$ is a parallelohedron; the Dirichlet–Voronoi polytope is the set of points that are closer to a given lattice point O than to any other point of Λ.

Conjecture 1 (Voronoi [19]) *Any d-dimensional parallelohedron P is affinely equivalent to a Dirichlet–Voronoi polytope $P(\Lambda')$ for some d-dimensional lattice Λ'.*

Voronoi's conjecture has been proved for several families of parallelohedra with special local combinatorial properties.

Denote the set of all k-faces of a tiling $\mathcal{T}$ by $\mathcal{T}^k$, and denote the set of all k-faces of a polytope P by P^k.

Definition 1.2 A face $F \in \mathcal{T}^{d-k}$ is called *primitive* if F belongs to exactly $k+1$ tiles in $\mathcal{T}$.

For example, any facet of $\mathcal{T}$ is a primitive because it belongs to exactly two tiles. For a cubic tiling of $\mathbb{R}^d$, the facets are the only primitive faces.

A face F of a parallelohedron P, with codimension 2, is primitive if a projection of P along F is a hexagon; otherwise it is nonprimitive; in the nonprimitive case, F belongs to four tiles in $\mathcal{T}(P)$. The set of facets of P that are parallel to a given face F with codimension 2 is called a *belt* of P. A belt can have four or six facets and accordingly is called a 4- or 6-*belt*.

Definition 1.3 A parallelohedron P is called k-*primitive* if all the k-faces of $\mathcal{T}(P)$ are primitive. A 0-primitive parallelohedron is usually called a *primitive* parallelohedron.

Clearly, any k-primitive parallelohedron is also $(k+1)$-primitive.

In 1909, Voronoi [19] (see also [9, Chap. 2, §12]) proved Conjecture 1 for primitive parallelohedra. In 1929, Zhitomirskii [20] proved Voronoi's conjecture for $(d-2)$-primitive parallelohedra. In this case, all the projections along the $(d-2)$-faces are hexagons.

Definition 1.4 Assume that $k > 1$, and that $G \in \mathcal{T}^{d-k}(P)$. Let $\mathcal{N}_G$ be the set of normal vectors for all facets containing G. Then P is *k-irreducible* if for each $G \in \mathcal{T}^{d-k}(P)$ the set $\mathcal{N}_G$ cannot be represented as $\mathcal{N}_G = \mathcal{N}_1 \cup \mathcal{N}_2$, where $\mathcal{N}_1$, $\mathcal{N}_2$ are nonempty, and $\operatorname{lin}\mathcal{N}_1 \cap \operatorname{lin}\mathcal{N}_2 = \{\mathbf{0}\}$.

For example, the result of Zhitomirskii [20] establishes Voronoi's conjecture for 2-irreducible parallelohedra.

In 2005, Ordine [16] proved Voronoi's conjecture for 3-irreducible parallelohedra. To date, no improvements to this result are known.

We also mention a result by Erdahl [5], who proved Voronoi's conjecture for space-filling zonotopes. A *zonotope* is a Minkowski sum of finitely many segments. Erdahl's proof is based on a technique of unimodular vector representations, and he constructed the appropriate affine transformation for a zonotope using normals to its facets. This approach significantly differs from the original approach introduced by Voronoi and developed later by Zhitomirskii and Ordine, where the main focus is on constructing an auxiliary polyhedral surface called a generatrissa and building a quadratic form associated to the surface.

Definition 1.5 The surface ∂P of a d-dimensional parallelohedron P is homeomorphic to the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$. After deletion of all closed nonprimitive faces of codimension 2 of P, we obtain new $(d-1)$-dimensional manifold without boundary (in the topological sense of the notion of a "manifold without boundary"). We call this manifold the *δ-surface* of P and denote it by P_δ.

The manifold P_δ is centrally symmetric because ∂P is centrally symmetric. If we glue together every pair of opposite points of the P_δ, then we obtain another $(d-1)$-dimensional manifold that is a subset of real projective space $\mathbb{RP}^{d-1}$. We call this manifold *π-surface* of P and denote it by P_π.

The δ-surface is related to notions of Venkov graphs and Venkov subgraphs of parallelohedra. Consider a graph with vertices corresponding to pairs of the antipodal (opposite) facets of parallelohedron P. Edges connect any two different vertices corresponding to four facets of P belonging to a single belt. The edges are colored in blue and red. If the facets belong to a 4-belt, then the edge is blue; otherwise they belong to a 6-belt and the edge is red. The graph is called a *Venkov graph* of parallelohedron P. A subgraph with the same set of vertices and just red edges remaining is called a *red Venkov subgraph* of P (for details see [8]).

Ordine proved a theorem [16, Theorem 2] that a parallelohedron P is irreducible if and only if its red Venkov subgraph is connected. The "if" part is easy. For the "only if," Ordine used the technique of gain functions and an (additive) version of the quality translation theorem by Ryshkov and Rybnikov [17]. Firstly, Ordine proved that the lattice of centers can be decomposed into a direct sum of sublattices so that each facet vector of P falls into one of these sublattices, and two facet vectors fall into the same sublattice if and only if their facets correspond to the same connected component of the red Venkov subgraph. Second, he showed that a union of parallelohedra centered in points of such a sublattice is convex and invariant under translations of the sublattice affine hull. Essentially that completes the proof. The preprint [11] contains a rewritten proof of Ordine's result, but it is in Russian.

The theorem is equivalent to a condition whereby P is irreducible iff its boundary remains connected after deletion of all nonprimitive faces of codimension 2. Thus, the δ-surface of P is connected iff P cannot be represented as a direct sum of two parallelohedra of lower dimensions. The same holds for π-surface P_π. Ordine also showed that the general case follows if Voronoi's conjecture is proved for the irreducible case where a parallelohedron cannot be represented as a direct sum. Therefore, we restrict our attention to parallelohedra with connected δ-surfaces. If a parallelohedron can be represented as the direct sum of two parallelohedra of lower dimensions, which is what we called reducible, then it is reducible at every vertex in the sense of Definition 1.4.

In this paper we will prove (Theorem 4.3) Voronoi's conjecture for parallelohedra with simply connected δ-surfaces. Also, we prove (Theorem 4.6) Voronoi's conjecture for a family of parallelohedra with a first $\mathbb{Q}$-homology group of π-surfaces generated by "trivial" cycles. These conditions are global, whereas all preceding conditions using the canonical scaling method of Voronoi (Sect. 2) were local. Our results generalize the theorems of Voronoi and Zhitomirskii, but for now it is unclear whether or not our theorems are applicable to Ordine's case.

However, a comparison of Ordine's conditions and the conditions of Theorem 4.6 for irreducible parallelohedra in small dimensions can be made. For reducible parallelhedra $P = P_1 \oplus P_2$, Voronoi's conjecture is inherited from the Voronoi conjecture for Minkowski summands P_1 and P_2. For dimension 3, this is done in Sect. 5, and both conditions are true for all three possible irreducible three-dimensional parallelohedra. The four-dimensional case is done by Garber in [6], and it is shown there that all four-dimensional parallelohedra satisfy conditions of Theorem 4.6. Using the methods described by Garber in [6, Sect. 5] for space-filling zonotopes (i.e., a zonotope and a parallelohedron at the same time), one can show that a five-dimensional zonotope with zone vectors (i.e., vectors representing the corresponding segments in the Minkowski sum; see also [12] for the definition) represented by columns of the matrix below does not satisfy Ordine's conditions. It is a Π-zonotope (see the same paper [6, Sect. 5]) with a graph on six vertices, with seven edges forming a 6-cycle with a long diagonal:

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the three- and four-dimensional cases (Sect. 5 and [6]) Theorem 4.6 is equivalent to Voronoi's conjecture, but we do not know of any proof for such equivalence for arbitrary dimensions, or at least for five-dimensional parallelohedra. Also, we do not know of any example of a parallelohedron that satisfies the Voronoi conjecture and does not satisfy Theorem 4.6.

It seems that ideas of a topological extension of Voronoi's method were developed earlier, but we found no published articles or available preprints. As a reference, we can give the link [4], where in the last three paragraphs the result of oral communication of Dienst and Venkov's son on the connection between Voronoi's conjecture and a certain cohomology group is revealed. In particular, Venkov said that his father "showed that for a given parallelohedron, the conjecture follows from the vanishing of a certain cohomology group assigned to this polytope. By this, he was able to prove the conjecture for all polytopes with no more than one belt of length 4" (this is a precise quote from the Web site [4]).

2 Canonical Scaling and Surface of a Parallelohedron

This section is devoted to the notion of a canonical scaling and its connection with Voronoi's conjecture.

Definition 2.1 A function $\mathfrak{s} : \mathcal{T}^{d-1} \longrightarrow \mathbb{R}_+$ is called a *canonical scaling* for the tiling $\mathcal{T}$ (for a parallelohedron P) if for each $(d-2)$-dimensional face $G \in \mathcal{T}^{d-2}$ the direction of the unit normal $\mathbf{n}_i$ for the facets F_i that contain G can be chosen such that

$$\sum_{i=1}^{3 \text{ or } 4} \mathfrak{s}(F_i)\mathbf{n}_i = \mathbf{0}; \tag{1}$$

the sum ranges to 3 if G is primitive and to 4 if G is not primitive.

Lemma 2.2 *Voronoi's conjecture is true for a parallelohedron P iff the corresponding tiling $\mathcal{T}(P)$ admits a canonical scaling.*

Remark This property was used by Voronoi [19], Zhitomirskii [20], and Ordine [16] to prove Voronoi's conjecture 1 for particular classes of parallelohedra. For a proof of Lemma 2.2, see [3,7,17,19].

A primitive $(d-2)$-face G belongs to exactly three facets – F_1, F_2, F_3. Unit normals $\mathbf{n}_i$ to these facets span a two-dimensional plane since they are orthogonal to G and no two of them are collinear, so there is exactly one (up to a nonzero factor) linear dependence:

$$\alpha_1\mathbf{n}_1 + \alpha_2\mathbf{n}_2 + \alpha_3\mathbf{n}_3 = \mathbf{0}.$$

Therefore, a canonical scaling $\mathfrak{s}$ for $\mathcal{T}$ (if it exists) must satisfy the local rule:

$$\frac{\mathfrak{s}(F_i)}{\mathfrak{s}(F_j)} = \frac{|\alpha_i\mathbf{n}_i|}{|\alpha_j\mathbf{n}_j|} = \left|\frac{\alpha_i}{\alpha_j}\right|. \tag{2}$$

Definition 2.3 The fraction $\left|\alpha_j/\alpha_i\right|$ is the value of the *gain function* $\mathfrak{g}(F_i, F_j)$, defined on pairs of facets that share a primitive face of codimension two.

As we have just seen, the gain function is uniquely defined for pairs of facets linked by a primitive $(d-2)$-face. The gain function shows how a canonical scaling changes along a path that travels facet to facet across such primitive faces. The gain function has the property that

$$\mathfrak{g}(F_i, F_j) = \frac{1}{\mathfrak{g}(F_j, F_i)}.$$

In the remainder of this section we use the gain function to obtain necessary and sufficient conditions for the existence of a canonical scaling.

Lemma 2.4 *If there exists a positive-valued function $\mathfrak{s}'$ on the set of all facets of P that satisfies condition* (2) *for every two facets with a common primitive* $(d-2)$*-face, then there exists a canonical scaling $\mathfrak{s} : \mathcal{T}^{d-1} \longrightarrow \mathbb{R}_+$.*

Remark The inverse statement is trivial since we can take as $\mathfrak{s}'$ the restriction of $\mathfrak{s}$ on the surface of one copy of P.

Proof Consider an arbitrary facet F and its opposite facet F' of P.

If F belongs to some 6-belt, then the application of rule (2) to this belt immediately gives us $\mathfrak{s}'(F) = \mathfrak{s}'(F')$. If F does not belong to any 6-belt, and $\mathfrak{s}'(F') \neq \mathfrak{s}'(F)$, then set $\mathfrak{s}'(F') := \mathfrak{s}'(F)$. This modification of $\mathfrak{s}'$ preserves the hypothesis that condition (2) holds at all primitive $(d-2)$-faces.

Now the function $\mathfrak{s}'$ is invariant with respect to the central symmetry of P. We translate this function on all copies of P in the tiling $\mathcal{T}$. This translation is correctly defined since different tiles of $\mathcal{T}$ are glued together by opposite facets of P and values of $\mathfrak{s}'$ are equal on these facets. The constructed function $\mathfrak{s} : \mathcal{T}^{d-1} \longrightarrow \mathbb{R}_+$ satisfies condition (1) for every primitive $(d-2)$-face because $\mathfrak{s}'$ satisfies rule (2). For a nonprimitive $(d-2)$-face, condition (1) holds as well because every such face lies in two pairs of opposite facets with equal values of $\mathfrak{s}'$ in each pair. Consequently, we can choose opposite normal directions for the facets in each pair and obtain a zero sum. Thus the function $\mathfrak{s}$ is a canonical scaling. □

Definition 2.5 A sequence of facets $\gamma = [F_0, \ldots, F_k]$ is a *primitive combinatorial path* on P if consecutive facets F_i and F_{i+1} are linked by a common primitive face of codimension 2. We call γ a *primitive cycle* if $F_0 = F_k$.

Define the gain function $\mathfrak{g}$ for every primitive path of P by the formula

$$\mathfrak{g}(\gamma) = \prod_{i=1}^{k} \mathfrak{g}(F_{i-1}, F_i).$$

We call a curve γ on the δ-surface of P (Definition 1.5) *generic* if the endpoints of γ are interior to the facets of P, γ does not intersect any face of dimension less than $d-2$, and the intersection of γ with a $(d-2)$-dimensional face of P is transversal.

For every generic curve γ on the δ-surface of P, we define a *supporting primitive combinatorial path* γ' consisting of facets that support γ. This allows us to define a gain function on generic curves: $\mathfrak{g}(\gamma) := \mathfrak{g}(\gamma')$. Obviously, for the union of two curves $\gamma = \gamma_1 \cup \gamma_2$, we have $\mathfrak{g}(\gamma) = \mathfrak{g}(\gamma_1) \cdot \mathfrak{g}(\gamma_2)$.

Lemma 2.6 *Voronoi's conjecture 1 is true for a parallelohedron P iff $\mathfrak{g}(\gamma) = 1$ holds for every generic cycle γ on the δ-surface of P.*

Proof Assume that $\mathfrak{g}(\gamma) = 1$ holds for every generic cycle γ. We will construct a function $\mathfrak{s}'$ from Lemma 2.4 in the following way. Consider an arbitrary facet F and set $\mathfrak{s}'(F) = 1$. Now for every facet G consider an arbitrary generic curve γ that starts in the center of F and ends in the center of G, and set $\mathfrak{s}'(G) = \mathfrak{g}(\gamma)$. We show that $\mathfrak{s}'$ is defined correctly and satisfies the conditions of Lemma 2.4.

Assume that two different curves γ_1 and γ_2 produce different values of $\mathfrak{s}'$ on a facet G. Then $\mathfrak{g}(\gamma_1) \neq \mathfrak{g}(\gamma_2)$. For a cycle $\gamma = \gamma_1 \cup \gamma_2^{-1}$ (here γ_2^{-1} denotes the reversed curve γ_2), we have $\mathfrak{g}(\gamma) = \mathfrak{g}(\gamma_1)/\mathfrak{g}(\gamma_2) \neq 1$, a contradiction. Thus, the function $\mathfrak{s}'$ is defined correctly on the set of all facets of P.

To show that $\mathfrak{s}'$ satisfies the conditions of Lemma 2.4 consider two arbitrary facets F_1 and F_2 with a common nonprimitive $(d-2)$-face. Assume that values on facets F_i were obtained with paths γ_i, and consider a path γ_3 that connects the centers of F_1 and F_2 through their common $(d-2)$-face. We have the cycle $\gamma = \gamma_3 \cup \gamma_2^{-1} \cup \gamma_1$, so

$$\mathfrak{g}(\gamma_3) = \frac{\mathfrak{g}(\gamma_2)}{\mathfrak{g}(\gamma_1)} = \frac{\mathfrak{s}'(F_2)}{\mathfrak{s}'(F_1)}.$$

By definition, $\mathfrak{g}(\gamma_3) = \mathfrak{g}(F_1, F_2)$; therefore, $\mathfrak{s}'$ satisfies the conditions of Lemma 2.4, and there exists a canonical scaling of the tiling, so by Lemma 2.2, Voronoi's conjecture is true for P.

On the other hand, if Voronoi's conjecture is true for P, then there is a canonical scaling $\mathfrak{s} : \mathcal{T}^{d-1} \longrightarrow \mathbb{R}_+$, from which we can determine the corresponding gain function. It easily follows using (2) that $\mathfrak{g}(\gamma) = 1$ for every generic cycle γ on the δ-surface of P. □

3 Dual 3-Cells and Local Consistency of Canonical Scaling

Definition 3.1 Consider an arbitrary face G of codimension k of the tiling $\mathcal{T}(P)$. Then the corresponding *dual k-cell* $\mathcal{D}_G$ is the convex hull of the centers of all tiles that share G.

For example, the dual cell corresponding to a d-dimensional polytope $P' \in \mathcal{T}(P)$ is a single point – its center.

Definition 3.2 We say that a dual cell $\mathcal{D}_G$ *is a face* of a dual cell $\mathcal{D}_H$ iff H is a face of G. Using that we can introduce the notion of *incidence* for dual cells. A cell $\mathcal{D}_G$ is *incident* to a dual cell $\mathcal{D}_H$ iff H is incident to G.

It is clear that a face $\mathcal{D}_G$ of a dual cell $\mathcal{D}_H$ belongs to it as a point set, but it is not proved and there is no known counterexample showing that $\mathcal{D}_G$ is a polyhedral face of $\mathcal{D}_H$. Such defined incidences on dual cells induce an inverse face lattice structure compared with the face lattice of $\mathcal{T}(P)$.

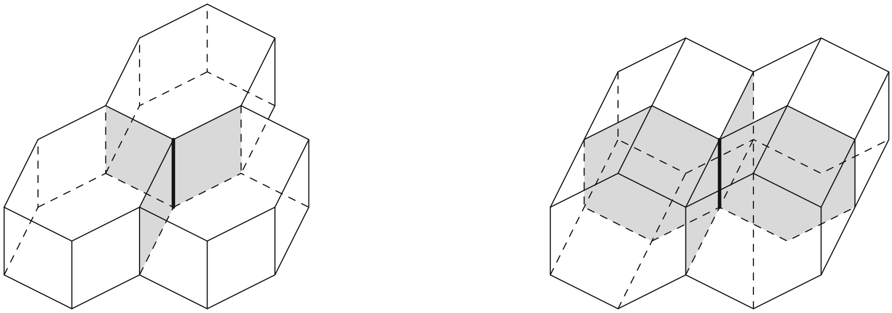


Fig. 1 Local structure of primitive and nonprimitive $(d-2)$-faces

Definition 3.3 $\mathcal{D}_F$ is said to be *combinatorially equivalent* to a polytope in case the polytope has an identical face lattice.

For example, a dual 2-cell $\mathcal{D}_F$ is combinatorially equivalent to either a triangle for a primitive F or to a parallelogram for a nonprimitive F (Fig. 1).

We will use the following classification theorem on $(d-3)$-faces of parallelohedra or, equivalently, dual 3-cells. The first part of the theorem was originally stated by Delone, and the second is a reformulation in terms of dual cells.

Theorem 3.4 (Delone [1]; see also [16]) *There are five possible combinatorial types of coincidence of parallelohedra at* $(d-3)$*-faces, and each of the five types* (*Fig.* 2) *has a representative when* $d = 3$.

In other words, every dual 3-cell is combinatorially equivalent to one of the five three-dimensional polytopes: cube, triangular prism, tetrahedron, octahedron, and quadrangular pyramid.

Every $(d-3)$-face G of a d-polytope P is a face of codimension 2 in the boundary complex ∂P, which is homeomorphic to a $(d-1)$-sphere. We consider a union U_G of all facets F_i of P containing G as a neighborhood of G in ∂P. U_G is naturally homeomorphic to a product $G \times D^2$ where D^2 is a two-dimensional disk. Each facet F_i containing G could be coherently represented as $G \times S_i \subset G \times D^2$, where S_i is a two-dimensional sector of D^2. The union of sectors is the entire D^2. Thus, all facets F_i containing G form a closed combinatorial path where each F_i appears just once along it. Clearly, the closed path is unique up to two choices of direction for a bypass along it.

Definition 3.5 The closed combinatorial path generated by a $(d-3)$-face G of a d-polytope P on the surface of P, as described previously, we call a *cycle around* G.

Lemma 3.6 *Let* G *be a* $(d-3)$*-face of a parallelohedron* P*. If the* $(d-2)$*-faces containing* G *are all primitive, then the cycle around* F *is primitive, and the gain function along this path has a value of 1.*

Remark The cycle around G is a primitive cycle in sense of Definition 2.5.

In the proof of Lemma 3.6 we will use the following lemma.

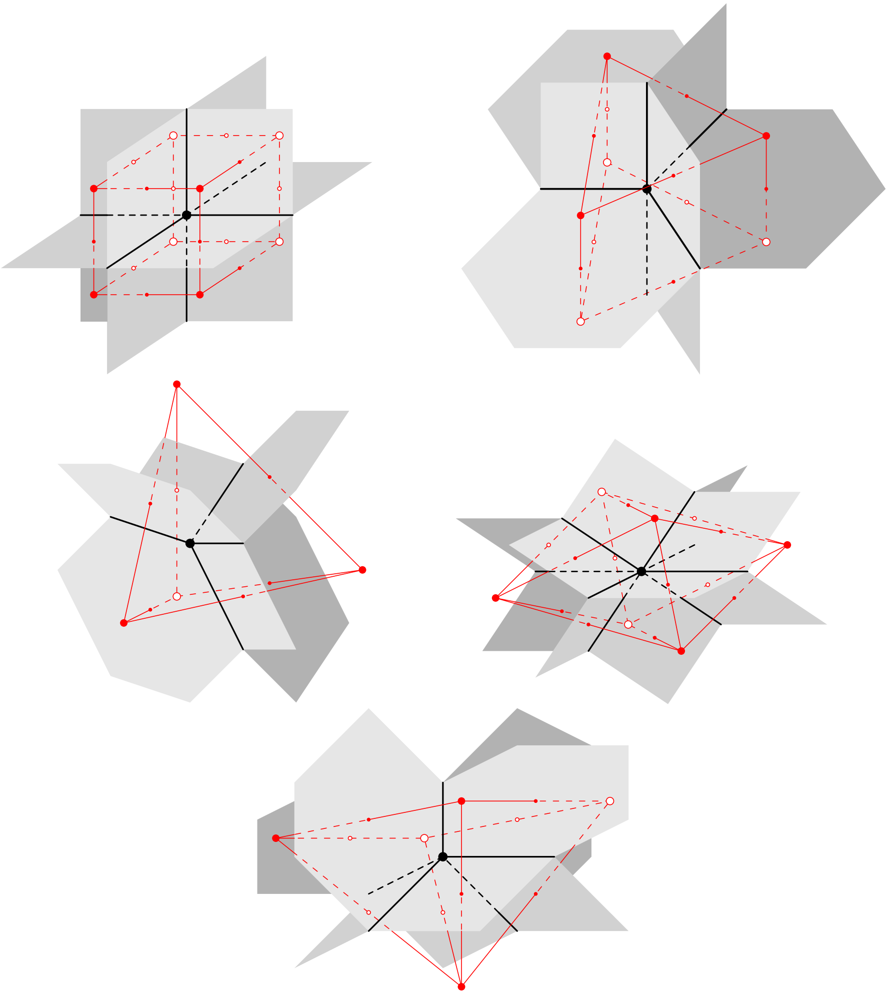


Fig. 2 Dual 3-cells

Lemma 3.7 *Let G be a* $(d-3)$*-face of parallelohedron P. Suppose that all* $(d-2)$*-faces of P containing G are primitive. Let* $P = P_0,\ P_1,\ P_2, \ldots$ *be all the parallelohedra of* $\mathcal{T}$ *that share G. Then there exist affine functions* $U_0, U_1, U_2, \ldots$ *with the following property: given any pair of parallelohedra* (P_i, P_j) *with a common facet, the functions* U_i *and* U_j *coincide on the affine hull of the facet* $P_i \cap P_j$ *and nowhere else.*

Remark This lemma is an extension of a statement by Zhitomirskii [20, Sects. 1 and 4], who needed all $(d-2)$-faces of the entire tiling containing G to be primitive. We relax the condition and demand that just some of these $(d-2)$-faces be primitive, namely, those belonging to P. As a result, we admit an extra case (3) for a dual cell of G.

Proof Theorem 3.4 implies that there are three possible cases.

(1) The dual cell of G is combinatorially equivalent to a tetrahedron.
(2) The dual cell of G is combinatorially equivalent to an octahedron.
(3) The dual cell of G is combinatorially equivalent to a quadrangular pyramid, and P corresponds to its apex.

We will construct U_i to be constant on every $(d-3)$-dimensional affine plane parallel to G. Therefore, we will restrict ourselves to the three-dimensional images of parallelohedra P_i under the projection π_G along face G.

The projections $\{\pi_G(P_i) : i = 0, 1, 2, \ldots\}$ split the neighborhood of the point $\pi_G(G)$ in the same way as a three-dimensional polyhedral fan $\mathcal{C}$ does. The combinatorics of $\mathcal{C}$ is completely prescribed by the type of the dual cell of G.

Our goal is, in fact, to construct a piecewise affine continuous function U on $\mathcal{C}$ such that the restriction of U to each three-dimensional cone of $\mathcal{C}$ is an affine function, and the affine functions for different cones are different. For Cases 1 and 2, this was done by Zhitomirskii; however, to make our proof more transparent, we will consider these cases again along with Case 3. For simplicity, suppose that $\pi_G(G)$ is the origin. We will also require that $U(\pi_G(G)) = 0$; therefore, we can speak of linear functions rather than affine ones.

Case 1. $\mathcal{C}$ has four one-dimensional faces (rays). Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ be the unit vectors along these rays. For every $\lambda \geq 0$ and $j = 1, 2, 3, 4$, set $U(\lambda \mathbf{x}_j) = \lambda$. It defines U because every three-dimensional cone C of the fan $\mathcal{C}$ is three-sided, and three linear functions on extreme rays of C can be extended in a unique way to a linear function on C.

Case 2. Let C be one of the three-dimensional cones of the fan $\mathcal{C}$. C is four-sided. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ be the vectors along the extreme rays of C in the order in which they are met by a path around $\pi_G(G)$ on ∂C. After a proper scaling of these vectors, we can assume that $\mathbf{x}_1 + \mathbf{x}_3 = \mathbf{x}_2 + \mathbf{x}_4$. Now, for every $\lambda \in \mathbb{R}$ and $j = 1, 2, 3, 4$, set

$$U(\lambda \mathbf{x}_j) = |\lambda|. \tag{3}$$

Let C' be a four-sided three-dimensional cone with the vertex at the origin. Let $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4$ be the vectors along its extreme rays such that $\mathbf{y}_1 + \mathbf{y}_3 = \mathbf{y}_2 + \mathbf{y}_4$. Suppose that u_i, $i = 1, 2, 3, 4$, are linear functions on the extreme rays of C' corresponding to $\mathbf{y}_i$. One can check that u_1, u_2, u_3, and u_4 can be extended to the same linear function U' if and only if

$$u_1(\mathbf{y}_1) + u_3(\mathbf{y}_3) = u_2(\mathbf{y}_2) + u_4(\mathbf{y}_4). \tag{4}$$

The preceding condition can be applied to any of the six cones of the fan $\mathcal{C}$. Therefore, the function U can be recovered from its values on all one-dimensional faces of the fan $\mathcal{C}$ if those values are as in Eq. (3).

Case 3. Let C be the only four-sided cone of the fan $\mathcal{C}$. Once again, we can assume that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ are the vectors along the extreme rays of C satisfying $\mathbf{x}_1 + \mathbf{x}_3 = \mathbf{x}_2 + \mathbf{x}_4$. Let $\mathbf{y}$ be the vector along the remaining one-dimensional face of $\mathcal{C}$. Because of the structure of $\mathcal{C}$, the 2-face of $\mathcal{C}$ between $\mathbf{x}_1$ and $\mathbf{y}$, and the 2-face between $\mathbf{x}_3$ and $\mathbf{y}$

lie in the same plane. Therefore, the vectors $\mathbf{x}_1$, $\mathbf{x}_3$, and $\mathbf{y}$ are coplanar. Similarly, the vectors $\mathbf{x}_2$, $\mathbf{x}_4$, and $\mathbf{y}$ are coplanar as well. Consequently, $\mathbf{y}$ is a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_3$ and a linear combination of $\mathbf{x}_2$ and $\mathbf{x}_4$ as well. This is possible only if $\mathbf{y}$ is collinear to the vector $-\mathbf{x}_1 - \mathbf{x}_3 = -\mathbf{x}_2 - \mathbf{x}_4$.

Since $\mathcal{C}$ consists of convex cones, $\mathbf{y}$ has the same direction as $-\mathbf{x}_1 - \mathbf{x}_3$, but not as $\mathbf{x}_1 + \mathbf{x}_3$. Hence, the directions of rays of $\mathcal{C}$ are the directions of vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$, and $-\mathbf{x}_1 - \mathbf{x}_3$.

For every $\lambda \geq 0$ and $j = 1, 2, 3, 4$, set $U(\lambda\mathbf{x}_j) = \lambda$ and $U(-\lambda\mathbf{x}_1 - \lambda\mathbf{x}_3) = \lambda$. Because of condition (4), we can recover U on cone C. Further, as in Case 1, we can recover U on all other cones of $\mathcal{C}$ because these cones are three-sided.

Thus, in every case we have constructed the function U. Returning from the projections to the parallelohedra P_i, we get the required set of functions U_j. □

Proof of Lemma 3.6 Lemma 3.7 provides us with the set of functions $U_0, U_1, U_2, \dots$ such that U_i and U_j coincide on $P_i \cap P_j$ if $P_i \cap P_j$ is a $(d-1)$-face. For all such pairs (i, j), let $U_j = U_i + \mathbf{a}_{ij}^T\mathbf{x} + b_{ij}$ for vector argument $\mathbf{x}$ and some vector parameter $\mathbf{a}_{ij}$.

Assume that P_i and P_j share a common facet F_{ij}. The difference $U_j - U_i$ is zero on F_{ij} and is not zero outside the affine hull of F_{ij}. Thus $U_j - U_i$ is not a constant, and therefore $\mathbf{a}_{ij} \neq \mathbf{0}$.

Hence, the affine hull of F_{ij} is a hyperplane represented by the equation $\mathbf{a}_{ij}^T\mathbf{x} + b_{ij} = 0$. Since the vector $\mathbf{a}_{ij}$ is normal to the hyperplane, we conclude that $\mathbf{a}_{ij}$ and F_{ij} are orthogonal.

If P_0, P_i, and P_j share a common $(d-2)$-face F_{0ij}, then, since F_{0ij} is primitive and

$$\mathbf{a}_{0i} + \mathbf{a}_{ij} - \mathbf{a}_{0j} = \mathbf{0},$$

we have $\mathfrak{g}(F_{0i}, F_{0j}) = |\mathbf{a}_{0i}|/|\mathbf{a}_{0j}|$.

Now, if $\gamma = [F_{01}, F_{02}, \dots F_{0n}, F_{01}]$ is a cycle around F, then

$$\mathfrak{g}(\gamma) = \frac{|\mathbf{a}_{01}|}{|\mathbf{a}_{02}|} \cdot \frac{|\mathbf{a}_{02}|}{|\mathbf{a}_{03}|} \cdot \dots \cdot \frac{|\mathbf{a}_{0n}|}{|\mathbf{a}_{01}|} = 1.$$

□

4 Voronoi's Conjecture for Parallelohedra with Simply Connected δ-Surfaces

Lemma 4.1 *If two generic cycles γ_1 and γ_2 on the δ-surface of a parallelohedron P are homotopy equivalent, then $\mathfrak{g}(\gamma_1) = \mathfrak{g}(\gamma_2)$.*

Remark By *homotopy* here and henceforth we mean the relation of continuous homotopy.

Proof Consider an arbitrary homotopy $F(t)$ between γ_1 and γ_2 such that $F(0) = \gamma_1$ and $F(1) = \gamma_2$. With a small perturbation of the homotopy F we can obtain another homotopy $G(t)$ such that:

(1) $G(0) = \gamma_1$ and $G(1) = \gamma_2$;
(2) At any moment t cycle $G(t)$ does not intersect any face of P with dimension less than $d - 3$;
(3) At any moment t cycle $G(t)$ does not have more than one point of intersection with the set of all $(d-3)$-faces of P;
(4) There is only a finite number of moments $t_1, \ldots, t_n$ such that each cycle $G(t_i)$ intersects some $(d-3)$-face F_i^{d-3} of P;
(5) There is only a finite number of moments $\tau_1, \ldots, \tau_k$ $(t_i \neq \tau_j)$ such that each cycle $G(\tau_j)$ has exactly one nontransversal intersection of $G(\tau_j)$ with some $(d-2)$-face F_j^{d-2}. For all other $t \neq \tau_j$, each intersection of $G(t)$ with $(d-2)$-faces of P is transversal.

Thus, in other words, we can find a perturbation of the homotopy $F(t)$ that will be in a general position.

For any $t \in (0, 1)$ equal neither to any t_i nor to any τ_j, the cycle $G(t)$ is generic, so we can evaluate the gain function $\mathfrak{g}(G(t))$. We show that $\mathfrak{g}(G(t))$ does not depend on t.

For any segment $[a, b] \subset [0, 1]$ that does not contain points t_i and τ_j, the function $\mathfrak{g}(G(t))$ is constant because the primitive path γ_t' that supports $G(t)$ does not depend on t while $t \in [a, b]$. Thus we just need to show that the gain function $\mathfrak{g}(G(t))$ does not change when t passes across t_i or τ_j.

If t passes across τ_j, then either the supporting primitive path does not change or one of its facets F^{d-1} could be replaced by a copy of a sequence $[F^{d-1}, G^{d-1}, F^{d-1}]$ (or vice versa) for some facets F^{d-1} and G^{d-1} with a common primitive $(d-2)$-face. In the latter case, the gain function $\mathfrak{g}$ does not change because $\mathfrak{g}([F^{d-1}, G^{d-1}, F^{d-1}]) = 1$.

If t passes across t_i, then either the supporting primitive path does not change or some subpath $[F_{i,1}^{d-1}, \ldots, F_{i,2}^{d-1}]$ with each facet containing F_i^{d-3} changes into a subpath with the same startpoint and endpoint and again all facets of this new subpath containing F_i^{d-3}. In this case, the gain function $\mathfrak{g}(G(t))$ will not change due to Lemma 3.6.

Therefore, the gain function $\mathfrak{g}(F(t))$ is constant and $\mathfrak{g}(\gamma_1) = \mathfrak{g}(F(0)) = \mathfrak{g}(F(1)) = \mathfrak{g}(\gamma_2)$. □

Corollary 4.2 *The gain function $\mathfrak{g}$ is a homomorphism of the fundamental group $\pi_1(P_\delta)$ into $\mathbb{R}_+$.*

Since $\mathbb{R}_+$ is a commutative group, then we trivially get that $\mathfrak{g}$ also gives us a homomorphism of the group $\pi_1(P_\delta)/[\pi_1(P_\delta)]$ (here $[G]$ denotes the commutant of a group G) to $\mathbb{R}_+$. This group is isomorphic to a group of homologies $H_1(P_\delta, \mathbb{Z})$ (see [10]).

Theorem 4.3 *Given a parallelohedron P with a connected δ-surface, if the fundamental group $\pi_1(P_\delta)$ or the group of homologies $H_1(P_\delta, \mathbb{Z})$ is trivial, then Voronoi's conjecture 1 is true for P.*

Proof In both cases, an arbitrary generic cycle γ can be represented as a product $\gamma = \gamma_1, \ldots, \gamma_k$, where $\gamma_i = a_i b_i a_i^{-1} b_i^{-1}$ is a cycle from the commutant $[\pi_1(P_\delta)]$. This is true because in both cases $[\pi_1(P_\delta)] = \pi_1(P_\delta)$ (in addition, this is the trivial group in the first case). It is clear that $\mathfrak{g}(\gamma_i) = 1$ and, hence, $\mathfrak{g}(\gamma) = 1$. Thus, by Lemma 2.6, Voronoi's conjecture is true for P. □

Now we show how to generalize this theorem in terms of a π-surface of parallelohedron P. This theorem can be generalized in two directions. First we can consider polytopes with nonconnected δ- or π-surfaces. In that case, the polytope P can be represented as a direct sum of parallelohedra of smaller dimensions, as was proved in Ordine and Magazinov [16], so if for both polytopal summands Theorem 4.3 (and, therefore, Voronoi's conjecture) is true, then it is true for P.

The second way to generalize the theorem is to introduce new cycles that have gain function 1. We can use a special family of *half-belt* cycles on the π-surface of P for these purposes.

Definition 4.4 The cycle γ on P_π is called a *half-belt cycle* if its support is a combinatorial path $\gamma' = [F_1, F_2, F_3, F_1]$ such that all three facets F_i belong to the same belt of length 6.

On the δ-surface P_δ, this cycle corresponds to a path that starts on the facet F_1 and ends on the opposite facet F_1' of P and crosses only three parallel primitive faces of codimension 2.

Lemma 4.5 *For every half-belt cycle γ, we have $\mathfrak{g}(\gamma) = 1$.*

Proof Let $\alpha_1\mathbf{n}_1+\alpha_2\mathbf{n}_2+\alpha_3\mathbf{n}_3 = \mathbf{0}$ be the unique linear dependence of normal vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ to facets F_1, F_2, and F_3. Then, by definition of the gain function $\mathfrak{g}$, we have

$$\mathfrak{g}(\gamma) = \mathfrak{g}(F_1, F_2) \cdot \mathfrak{g}(F_2, F_3) \cdot \mathfrak{g}(F_3, F_1') = \frac{|\alpha_2|}{|\alpha_1|} \cdot \frac{|\alpha_3|}{|\alpha_2|} \cdot \frac{|\alpha_1|}{|\alpha_3|} = 1.$$

□

Theorem 4.6 *If the homology group $H_1(P_\pi, \mathbb{Q})$ is generated by half-belt cycles of a parallelohedron P, then Voronoi's conjecture is true for P.*

Proof First we show that $\mathfrak{g}$ is well-defined on $H_1(P_\pi, \mathbb{Q})$. As earlier, we have that the gain function $\mathfrak{g}$ is a homomorphism of $\pi_1(P_\pi)$ to $(\mathbb{R}_+, \cdot)$. The group $(\mathbb{R}_+, \cdot)$ is commutative, so the action $\mathfrak{g}$ is well-defined on the commutant of $\pi_1(P_\pi)$ and, therefore, on $H_1(P_\pi, \mathbb{Z})$. Since $(\mathbb{R}_+, \cdot)$ has no torsion, $\mathfrak{g}$ is trivial on the torsion subgroup of $H_1(P_\pi, \mathbb{Z})$. Thus $\mathfrak{g}$ is well-defined on $H_1(P_\pi, \mathbb{Q})$.

Further, because $\mathfrak{g}$ acts trivially on all generators of $H_1(P_\pi, \mathbb{Q})$, it is a trivial action on the group as well. Furthermore, because $\mathfrak{g}$ also acts trivially on the torsion subgroup of $H_1(P_\pi, \mathbb{Z})$, it acts trivially on the group $H_1(P_\pi, \mathbb{Z})$. Clearly, the standard gluing map $P_\delta \longrightarrow P_\pi$ holds the action of $\mathfrak{g}$ on cycles of P_δ. Thus $\mathfrak{g}$ acts trivially on the group $H_1(P_\delta, \mathbb{Z})$, and the application of Lemma 2.6 completes the proof. □

5 Three-Dimensional Case

In this section we will show that the conditions of Theorem 4.6 hold for all three-dimensional parallelohedra.

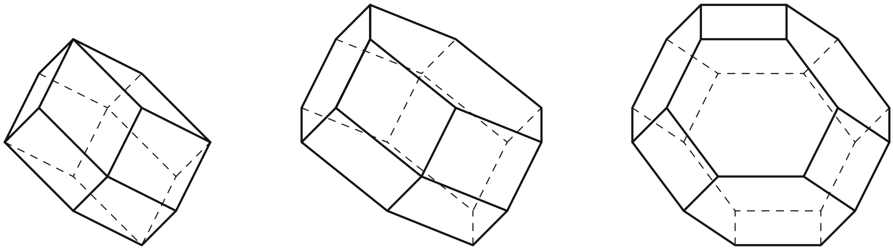


Fig. 3 Irreducible parallelohedra in $\mathbb{R}^3$

Example 5.1 There are five combinatorial types of three-dimensional parallelohedra. Two types of reducible parallelohedra are a cube $\mathcal{C}$ and a hexagonal prism $\mathcal{P}$. It easy to see that $\mathcal{C}_\delta$ is a collection of six disjoint open disks (corresponding to facets of $\mathcal{C}$) and $\mathcal{C}_\pi$ is a collection of three disjoint open disks. In both cases it is easy to see that both fundamental groups $\pi_1(\mathcal{C}_\delta)$ and $\pi_1(\mathcal{C}_\pi)$ as well as homology groups $H_1(\mathcal{C}_\delta)$ and $H_1(\mathcal{C}_\pi)$ (over $\mathbb{Z}$ or $\mathbb{Q}$) are trivial, and the conditions of both Theorems 4.3 (for nonconnected cases) and 4.6 are true. For $\mathcal{P}$, the situation is a bit more interesting. The surface $\mathcal{P}_\delta$ is a collection of two open disks (bases of a prism) and an open strip (the side surface) and $\mathcal{P}_\pi$ is a collection of a disk and a Möbius strip. In this case,

$$\pi_1(\mathcal{P}_\delta) \cong \pi_1(\mathcal{P}_\pi) \cong H_1(\mathcal{P}_\delta, \mathbb{Z}) \cong H_1(\mathcal{P}_\pi, \mathbb{Z}) \cong \mathbb{Z}.$$

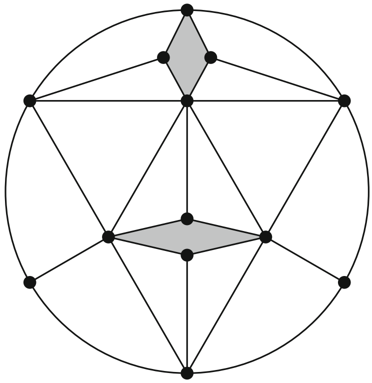
It is easy to find generators for all of these groups. For both fundamental and homology groups of δ-surfaces the generator is the cycle represented by a single belt cycle on the side surface of $\mathcal{P}$. For a π-surface the generator is represented by a unique half-belt cycle of $\mathcal{P}$.

Three types of irreducible parallelohedra are a rhombic dodecahedron $\mathcal{R}$, an elongated dodecahedron $\mathcal{E}$, and a truncated octahedron $\mathcal{O}$ (left to right in Fig. 3). The rhombic dodecahedron and the truncated octahedron are 2-primitive (the Zhitomirskii case [20]) since all edges of these polytopes generate belts of length 6. For any such parallelohedron its δ-surface is just the surface of the parallelohedron itself. Thus, $\mathcal{R}_\delta$ and $\mathcal{O}_\delta$ are homeomorphic to the sphere $\mathbb{S}^2$, and $\mathcal{R}_\pi$ and $\mathcal{O}_\pi$ are homeomorphic to the projective plane $\mathbb{RP}^2$. Hence, the groups $\pi_1(\mathcal{R}_\delta)$, $H_1(\mathcal{R}_\delta)$, $\pi_1(\mathcal{O}_\delta)$, and $H_1(\mathcal{O}_\delta)$ are trivial, and we can apply Theorem 4.3 for these polytopes. The groups $H_1(\mathcal{R}_\pi, \mathbb{Q})$ and $H_1(\mathcal{O}_\pi, \mathbb{Q})$ are trivial. Thus, for this case we can also apply Theorem 4.6.

The most interesting case is the elongated dodecahedron $\mathcal{E}$. The manifold $\mathcal{E}_\delta$ is a sphere with four cuts. (The cuts correspond to the four vertical edges of the middle polytope in Fig. 3.) The fundamental group $\pi_1(\mathcal{E}_\delta)$ is the free group with three generators. We can take three cycles consisting of two half-belts each going around some deleted edge as independent generators. Thus, the homology group $H_1(\mathcal{E}_\delta, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^3$.

The manifold $\mathcal{E}_\pi$ is the real projective plane with two cuts or, equivalently, a Möbius strip with one cut. One possible triangulation (closure) of this surface is shown in Fig. 4, the cuts are shaded in gray, and opposite points of the boundary circle are glued together as usual.

Fig. 4 Triangulation of $\mathcal{E}_\pi$



The fundamental group $\pi_1(\mathcal{E}_\pi)$ is generated by two cycles around cuts and arbitrary half-belt cycle (one can use a cycle represented by a half-circle from Fig. 4 for that), but these three cycles are not independent. The group of one-dimensional homologies can also be found easily using the fact that the Betti number h_1 of $\mathcal{E}_\pi$ is 2 (to check this, one can calculate the Euler characteristic of $\mathcal{E}_\pi$ using Fig. 4). For homology groups, we have $H_1(\mathcal{E}_\pi, \mathbb{Z}) = \mathbb{Z}^2 \times T$, where T is the torsion part of the homology group and does not affect the existence of canonical scaling, and $H_1(\mathcal{E}_\pi, \mathbb{Q})$ is just $\mathbb{Q}^2$. In both cases, the generators of the nontorsion part can be chosen as one half-belt cycle and the composition of two half-belt cycles around one of the cuts (or the generating cycle of the corresponding Möbius strip and a cycle around cut on the Möbius strip). Thus Theorem 4.6 can be applied to this case as well.

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