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Dynamics of a Wave Packet on the Surface of an Inhomogeneously Vortical Fluid (Lagrangian Description)

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Abstract—A nonlinear Schrödinger equation (NSE) describing packets of weakly nonlinear waves in an inhomogeneously vortical infinitely deep fluid has been derived. The vorticity is assumed to be an arbitrary function of Lagrangian coordinates and quadratic in the small parameter proportional to the wave steepness. It is shown that the modulational instability criteria for the weakly vortical waves and potential Stokes waves on deep water coincide. The effect of vorticity manifests itself in a shift of the wavenumber of high-frequency filling. A special case of Gerstner waves with a zero coefficient at the nonlinear term in the NSE is noted.

Keywords: nonlinear Schrödinger equation, vorticity, Gerstner wave

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The nonlinear Schrödinger equation (NSE) is an effective model for studying the propagation of wave packets on the surface of a fluid. For waves on deep water, the NSE was derived for the first time by Zakharov using Hamiltonian formalism [1]. Hasimoto and Ono [2] and Davey [3] obtained the same result independently by multiple-scale expansions in Eulerian coordinates. Yuen and Lake, in turn, derived the NSE on the basis of the averaged Lagrangian [4]. In all those works, the wave motion was assumed to be potential.

At the same time, the propagation of waves in the ocean rather often occurs against the background of vortical flows. Using the method of multiple-scale expansions, Johnson [5] studied the slow modulation of a harmonic wave running along a shear flow with an arbitrary profile $U(y)$, where y is the vertical coordinate. He obtained the NSE with coefficients depending in a complicated way on the shear flow. Baumstein [6] studied modulational instability of a train of Stokes waves on a flow with a homogeneous velocity shift when $dU/dy = \mathrm{const}$. Thomas et al. [7] generalized their results to the case of a finite depth of the fluid and confirmed that a shear flow with a linear velocity profile can have a significant effect on the stability of weakly linear Stokes waves. In particular, for waves propagating in the direction of the flow, the Benjamin–Feir instability is absent in the case of positive vorticity for any depth.

In the traditional consideration of the propagation of weakly nonlinear waves on a flow, the shear flow determines the vorticity of the zero approximation [5]. Depending on the shape of the flow profile, the vorticity can be to a certain extent arbitrary. At the same time, the vorticity in the first and subsequent approximations in the wave steepness parameter already depends on the profile shape and is quite definite. In this approach, the form of vorticity perturbations is very difficult to predict in this case. They are no longer functions only of y; they also depend on the variables x and t. From this point of view, there is the possibility of developing a somewhat different approach, according to which the unperturbed shear flow is absent and vorticities of wave perturbations are specified as some arbitrary functions. The NSE form in this case depends on the form of these functions. In particular, Hjelmervik and Trulsen [8] obtained the NSE for a vorticity distribution of the following form: $\Omega_y/\omega = O(\varepsilon^2)$; $(\Omega_x, \Omega_z)/\omega = O(\varepsilon^3)$, where x and z are the horizontal coordinates, ω is the wave frequency, and ε is the small parameter of wave steepness; the last is equal to the product of the characteristic wave amplitude and wavenumber. The vertical vorticity component exceeds horizontal components by an order of magnitude. This vorticity distribution corresponds to a weak (on the order of ε) horizontally inhomogeneous shear flow.

Below, in contrast to [8], two-dimensional flows in the (x, y) plane, when only the z component of vorticity is different from zero, are studied. The analysis was carried out in Lagrangian variables a and b (the first of them is the horizontal coordinate; the second one, the vertical coordinate). In the absence of an unperturbed shear flow, the general expression for the vorticity of a plane flow is written as follows:

$$\Omega_z = \Omega(a,b) = \sum_{n\geq 1} \varepsilon^n \Omega_n(a,b).$$

Earlier, one of the authors of this paper has solved the problem about the propagation of a wave packet in the case $\Omega_1 = \Omega_1(b)$; $\Omega_2 = \Omega_2(b)$ [9]. For the complex amplitude of the wave packet envelope, an evolutionary equation was constructed and reduced to the NSE by a simple substitution. In this work, it is assumed that the vorticity of waves depends on both the Lagrangian variables. The vorticity distribution is specified in the following form: $\Omega_1 = 0$; $\Omega_2 = \Omega_2(a,b)$, where Ω_2 is an arbitrary function. The hydrodynamics equations in the Lagrange form are solved by multiple-scale expansions. For the envelope amplitude, the nonlinear Schrödinger equation with an additional term has been obtained. Different ways to reduce it to integrable equations are considered. The case of a weakly linear Gerstner wave for which, as is shown below, the coefficient at the nonlinear NSE term vanishes is distinguished.

1. FORMULATION OF THE PROBLEM

Let us consider the propagation of a packet of surface waves in a vortical infinitely deep fluid. The equations of two-dimensional hydrodynamics of an ideal incompressible fluid are written in the following form [10, 11]:

$$\frac{D(X,Y)}{D(a,b)} = [X,Y] = 1, \tag{1}$$

$$X_{tt}X_a + (Y_{tt} + g)Y_a = -\frac{1}{\rho}p_a, \tag{2}$$

$$X_{tt}X_b + (Y_{tt} + g)Y_b = -\frac{1}{\rho}p_b, \tag{3}$$

where X and Y are the horizontal and vertical coordinates of the trajectory of the liquid particle, t is time, ρ is the density, p is the pressure, g is the acceleration of gravity, and subscripts denote differentiation with respect to the corresponding variable. The square brackets denote the Jacobian. The b axis is directed upwards and $b = 0$ corresponds to the free surface. Equation (1) is the equation of continuity and Eqs. (2) and (3) are the momentum equations. The flow region is associated with the condition $b \leq 0$ (Fig. 1).

Using cross-differentiation, one can exclude the pressure from system (2), (3) and obtain the condition of vorticity conservation along the trajectory [10]:

$$X_{ta}X_b + Y_{ta}Y_b - X_{tb}X_a - Y_{tb}Y_a = \Omega(a,b). \tag{4}$$

This equation is equivalent to equations of fluid motion (2), (3), but explicitly includes the vorticity of

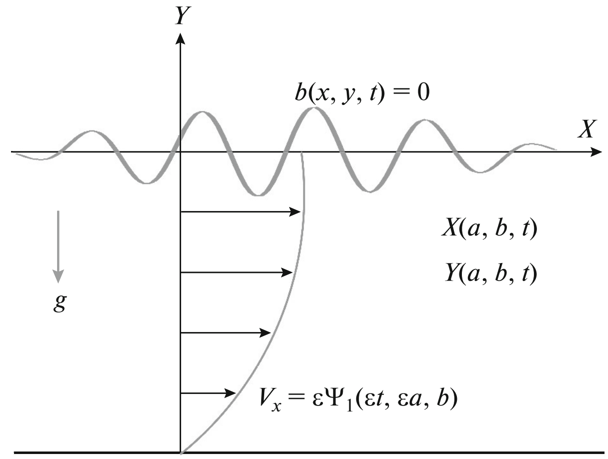

Fig. 1. Geometry of the problem: V_x is the average horizontal flow.

liquid particles Ω, which is a function of only Lagrangian coordinates in the case of two-dimensional flows.

Let us introduce a complex coordinate of the liquid particle trajectory $W = X + iY$ $(\overline{W} = X - iY)$; the bar is the sign of complex conjugation. In the new variables, Eqs. (1) and (4) take the following form:

$$\lfloor W, \overline{W} \rfloor = -2i, \tag{5}$$

$$\mathrm{Re}\left[W_t, \overline{W}\right] = \Omega(a,b), \tag{6}$$

and system of equations (2), (3) after easy algebraic transformations is reduced to a single equation

$$W_{tt} = -ig + i\rho^{-1}[p,W]. \tag{7}$$

Below, Eqs. (5), (6) are used for finding the complex coordinate of liquid particle trajectories and Eq. (7) is used for determining the pressure in the fluid. The nonflow condition at the bottom ($Y_t \to 0$ as $b \to -\infty$) and condition of pressure constancy on the free surface (at $b = 0$) serve as the boundary conditions.

2. METHOD OF SOLUTION

Let us use the method of multiple scales. Represent function W as follows:

$$\begin{gathered} W = a_0 + ib + w(a_l, b, t_l), \quad a_l = \varepsilon^l a, \\ t_l = \varepsilon^l t; \quad l = 0,1,2, \end{gathered} \tag{8}$$

where ε is the small wave steepness parameter. Let us represent the unknown functions p and w as a series in this parameter:

$$w = \sum_{n=1} \varepsilon^n w_n; \quad p = p_0 - \rho g b + \sum_{n=1} \varepsilon^n p_n. \tag{9}$$

In the expression for the pressure, the term with hydrostatic pressure is written separately and p_0 is the constant atmospheric pressure on the fluid surface; the atmospheric pressure can be immediately set to zero. Let us substitute representations (8) and (9) into Eqs. (5)–(7). In the first approximation, the solution has the following form:

$$w_1 = A(a_1,a_2,t_1,t_2)\exp[i(ka_0+\omega t_0)+kb] + \psi_1(a_1,a_2,b,t_1,t_2). \quad (10)$$

Here and below, A is the complex amplitude of the wave running to the left (to write expressions for a wave running to the right, one should change the sign of the frequency in them). The function ψ_1 is real and its form is determined when considering the next approximation. Expression (10) describes the wave motion in the laboratory frame of reference. The motion consists of the vibrational motion of liquid particles along a circumference and average motion.

We do not present the calculations in detail, but restrict ourselves by mentioning the main results in higher approximations. An analysis of the second approximation yields a pair of equations

$$A_{t_1} - c_g A_{a_1} = 0, \quad (11)$$

$$\psi_{1t_1b} = -2k^2\omega|A|^2 e^{2kb} - \Omega_2(a_1,a_2,b), \quad (12)$$

here, $c_g = g/(2\omega)$ is the group velocity of linear waves on the water. Using Eqs. (11) and (12), we come in the third approximation to the following evolution equation:

$$i\frac{\partial A}{\partial t_2} + \frac{\omega}{8k^2}\frac{\partial^2 A}{\partial a_1^2} - 2k^2 A\int_{-\infty}^{0}\psi_{1t_1}e^{2kb}db = 0. \quad (13)$$

This equation is written in a frame of reference moving with the group velocity to the left. It has an inhomogeneous summand containing the function ψ_{1t_1}. The function is obtained by the integration of Eq. (12) with respect to b:

$$\psi_{1t_1} = -k\omega|A|^2 e^{2kb} - \int_{-\infty}^{b}\Omega_2(a,b')db' - U(a,t_1). \quad (14)$$

From now on, the variable a denotes the "slow" coordinate a_1 or a_2 (the choice is made for the ease of convenience when solving Eq. (13)). Function $U(a,t_1)$ describes horizontally inhomogeneous and vertically homogeneous (not depending on coordinate b) unsteady potential flow. Substituting relationship (14) into Eq. (13), we come to the final form of the equation for the wave packet envelope amplitude:

$$i\frac{\partial A}{\partial t} + \frac{\omega}{8k^2}\frac{\partial^2 A}{\partial a^2} + k\left[\frac{\omega k^2}{2}|A|^2 + 2k\int_{-\infty}^{0}e^{2kb} \times\left(\int_{-\infty}^{b}\Omega_2(a,b')db'\right)db + U(a,t)\right]A = 0. \quad (15)$$

Writing this equation, we used the substitutions $A \to \frac{A}{\varepsilon}$, $t_2 \to \varepsilon^2 t$, $a_1 \to \varepsilon a$. This is a modified NSE for weakly vortical waves. Different versions of its possible analytical solutions are considered below.

3. EXAMPLES OF WAVES

Potential Stokes waves. In the case $\Omega_2 = 0$, $U = 0$, Eq. (15) goes over into the classical nonlinear Schrödinger equation for waves on deep water

$$i\frac{\partial A}{\partial t} + \frac{\omega}{8k^2}\frac{\partial^2 A}{\partial a^2} + \frac{1}{2}\omega k^2|A|^2 A = 0. \quad (16)$$

As applied to waves in the ocean, three analytical forms of NSE solutions are discussed most frequently—the Peregrine breather localized both in space and time [12], the so-called Akhmediev breather (a solution which is periodic in space and localized in time [13]), and the Kuznetsov breather (a solution which is periodic in time and localized in space [14]). The problem of the formation and propagation of abnormal oceanic waves (rogue waves) in the context of the classical NSE has been discussed many times (see the survey [15], papers [16, 17], and references therein), and we will not dwell on it.

Gerstner waves: $\Omega_2 = -2k^2\omega|A|^2 e^{2kb}$, $U = 0$. Gerstner's exact solution for waves on the surface of a fluid in the complex form is written as follows [10]:

$$W = a_0 + ib + iA\exp[i(ka_0+\omega t_0)+kb]. \quad (17)$$

It describes stationary running vortex waves with a trochoidal profile. Their dispersion characteristic coincides with the dispersion law for linear waves on deep water $\omega^2 = gk$. Liquid particles move along circumferences and the drift flow is absent. In representation (9), the Gerstner wave has the following form:

$$w = \sum_{n\geq 1}\varepsilon^n iA\exp[i(ka_0+\omega t_0)+kb].$$

Substituting it into the expression for vorticity (6), we obtain that Gerstner waves are not vorticial in the linear approximation ($\Omega_1 = 0$) but have a vorticity $\Omega_2 = -2k^2\omega|A|^2 e^{2kb}$ in the quadratic approximation. For this vorticity distribution, the two first summands in the square brackets of Eq. (15) annihilate each other. From the physical point of view, this is related to the fact that the flow induced by vorticity exactly compensates the Stokes drift. Therefore, in this approximation, the packet of weakly nonlinear Gerstner waves does not undergo the action of nonlinearity and the effect of modulational instability is absent for Gerstner waves.

Waves with a nonuniform vorticity distribution: $\Omega = \varepsilon^2\Omega_2(a_2,b)$, $U = 0$. In the general case, one can assume that the vorticity function depends on three coordinates: a_1, a_2, and b. Under this assumption, Eq. (13) is an NSE with an inhomogeneous coefficient at the additional linear term. For simplicity, let us consider the case where the function Ω_2 depends only on the coordinates a_2 and b. Then it is convenient to pass in Eq. (13) to the coordinates t_1 and a_2 and, after the substitutions $A \to \frac{A}{\varepsilon}$, $t_1 \to \varepsilon t$, $a_2 \to \varepsilon^2 a$, it takes the form

$$i\frac{\partial A}{\partial a} - \frac{k}{\omega^2}\frac{\partial^2 A}{\partial t^2} - k\left(k^2|A|^2 + 2\beta(a)\right)A = 0;$$
$$\beta(a) = 2k^2\omega^{-1}\int\limits_{-\infty}^{0} e^{2kb}\left(\int\limits_{-\infty}^{b}\Omega_2(a,b')\,db'\right)db. \tag{18}$$

Using the substitution $A = A'\exp\left(-2ik\int_{-\infty}^{a}\beta(a)da\right)$, it is reduced to the NSE:

$$i\frac{\partial A'}{\partial a} - \frac{k}{\omega^2}\frac{\partial^2 A'}{\partial t^2} - k^3|A'|^2 A' = 0. \tag{19}$$

This equation has the same coefficients as the NSE for potential waves on deep water. It follows that conditions of modulational instability for the considered waves with a nonuniform vorticity distribution are the same as for potential waves and all known analytical and numerical calculations of the formation of rogue waves for potential waves can be also extended to them.

We present here a modification of the solution in the form of the Peregrine breather, which is an analytical model of rogue waves [12, 15]:

$$A(t,a) = A_0\left[-1 + 4\frac{1 + 2ik^3A_0^2a}{1 + 2k^2\omega^2A_0^2t^2 + 4k^6A_0^4a^2}\right]$$
$$\times\exp\left[ik\left(k^2A_0^2a - 2\int\limits_{-\infty}^{a}\beta(a)da\right)\right],$$

where A_0 is the amplitude of the unperturbed monochromatic wave. As is seen, the vorticity has an effect only on the shift of the spatial wavenumber—by decreasing it when compared to the Stokes wave. Compared to the Peregrine breather in an ideal fluid, vorticity leads to changes in the carrier wavelength, which has an effect on the number of individual waves in the wave packet.

Waves in a weakly vortical fluid in the presence of an additional potential flow: $\Omega = \varepsilon^2\Omega_2(a_2,b)$, $U \neq 0$. In this case, Eq. (13) can be rewritten as follows:

$$i\frac{\partial A}{\partial a} - \frac{k}{\omega^2}\frac{\partial^2 A}{\partial t^2}$$
$$- k\left(k^2|A|^2 + 2\beta(a) + \frac{2k}{\omega}U(a,t)\right)A = 0. \tag{20}$$

Equation (20) is one version of the variable coefficient nonlinear Schrödinger equation (VCNSE) being actively studied in optics and hydrodynamics. Under certain conditions, it has solutions in the form of breathers and demonstrates the possibility of rogue wave formation in the inhomogeneous nonlinear Schrödinger equation. As applied to optical problems, a survey of cases where the VCNSE can be reduced to an NSE with constant coefficients was presented in [18]; however, linear damping of the wave packet was also taken into account in that paper. It is clear that large amplitude waves can also be generated in more general cases where the breather solution cannot be obtained. We mention, for example, an important case where the function U is a linear function only of time. Introducing dimensionless variables

$$E = \frac{1}{\sqrt{2}}k\bar{A}\exp\left(-2ik\int\limits_{-\infty}^{a}\beta(a)da\right), \quad \tau = \omega t,$$
$$q = ka, \quad U(t) = -\alpha\frac{\omega}{k}\tau,$$

where the bar is the sign of complex conjugation and α is a constant, we reduce Eq. (20) to the following equation:

$$i\frac{\partial E}{\partial q} + \frac{\partial^2 E}{\partial \tau^2} + \left(-2\alpha\tau + 2|E|^2\right)E = 0,$$

which has an exact one-soliton solution [19]:

$$E = E_0(q,\tau)\exp i\varphi(q,\tau);$$
$$E_0 = 2\eta\,\mathrm{sech}\,2\eta\left(\tau + 2\alpha q^2 - 4\xi q - \tau_0\right);$$
$$\varphi = 2(\xi - \alpha q)\tau - 4\left[\frac{1}{3}\alpha^2q^3 - \alpha\xi q^2 + \left(\xi^2 - \eta^2\right)q\right],$$

where ξ and η are constants and τ_0 is the initial time instant. It describes a nonuniformly moving soliton of the envelope with an amplitude of 2η. The parameter ξ specifies a point where the soliton velocity $dq/d\tau$ changes the sign (blocking point). The existence of a soliton with a constant amplitude is caused by the competition of two effects: the dispersion compression of the wave momentum due to the frequency modulation and a spread in the inhomogeneous medium. The existence of the soliton of the envelope is typical for a focusing nonlinear Schrödinger equation, which points to the possible manifestation of modulational instability and rogue waves.

4. ON THE RELATION BETWEEN THE LAGRANGIAN AND EULERIAN DESCRIPTIONS

All solutions of the equations presented in this work were obtained in Lagrangian coordinates. In connection with this, a natural question arises: are they different from NSE solutions in Eulerian variables? We consider it using the example of calculating the shift of the free boundary.

In our description, it is determined by the formula

$$Y_L = \operatorname{Im} A(a,t) \exp i(ka_0 + \omega t_0),$$

here, $A(a,t)$ is a solution of any of Eqs. (15), (17), (18), or (20). This expression defines the wave profile in Lagrangian coordinates (subscript "L" at quantity Y). To write it in Eulerian coordinates, it is necessary to express Lagrangian coordinates in terms of Eulerian variables. It follows from relations (8) and (9) that

$$X = a + \varepsilon \operatorname{Re}\left(w_1 + \sum_{n=2} \varepsilon^{n-1} w_n \right) = a + O(\varepsilon),$$

and, therefore, the shift of the free surface in Eulerian coordinates is

$$Y_E = \operatorname{Im} A(X,t) \exp i(kX_0 + \omega t_0) + O\left(\varepsilon^2\right).$$

Thus, coordinate a plays the part of X. This result can be called the principle of correspondence between the Lagrangian and Eulerian descriptions.

5. CONCLUSIONS

In this work, the dynamics of wave packets propagating on the surface of an inhomogeneously vortical fluid has been studied using the method of multiple-scale expansions in Lagrangian variables. The fluid vorticity Ω was specified as an arbitrary function quadratic in the small wave steepness parameter of a function of Lagrangian coordinates. The calculations were carried out by introducing a complex coordinate of the liquid particle trajectory.

For wave packets, an evolutionary equation for the envelope has been obtained. From the mathematical point of view, the novelty of the equation is related to the appearance of a new summand proportional to the envelope amplitude with a factor depending on the spatial coordinate. It determines the average flow related to the presence of vorticity in the fluid. Cases in which the equation can be reduced by a simple substitution to the NSE with the same coefficients as for potential waves on deep water are mentioned. It has been shown that the effect of vorticity is related to the shift of the high-frequency filling wavenumber. The criteria of modulational instability for the considered weakly vortical waves and potential waves on deep water coincide. All known analytical and numerical solutions of the NSE are also applicable to these weakly vortical waves. The special case of Gerstner waves for which the coefficient at the nonlinear NSE term vanishes has been distinguished.

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