Technical Physics Letters, Vol. 28, No. 10, 2002, pp. 877–880. Translated from Pis'ma v Zhurnal Tekhnicheskoĭ Fiziki, Vol. 28, No. 20, 2002, pp. 87–94.

Interaction of a Simple Prandtl–Meyer Wave with a Weakly Vortexed Layer

A. V. Omel'chenko

St. Petersburg State University, St. Petersburg, Russia

e-mail: vmu@peterlink.ru

Received May 23, 2002

Abstract—The problem of interaction of a centered wave of rarefaction with a shear layer is solved in the case of a small flow vorticity in the shear layer. The solution is found in the form of an asymptotic series with respect to a small parameter of the problem. A system of equations derived in the zero approximation describes the flow in a simple wave. A uniformly applicable first-order expansion is constructed using the method of deformed coordinates. © *2002 MAIK "Nauka/Interperiodica".*

Let us consider the interaction of a centered rarefaction wave *1* with a vortex (shear) layer *2* of finite thickness (Fig. 1). The region of interaction is bounded from the left by a weak discontinuity $A_1A_2A_3$ (which is a continuation of the weak discontinuity OA_1 separating the uniform flow and the Prandtl–Meyer wave) and from below by a weak discontinuity $A_1F_1C_1$ originating from point A_1 (the point of intersection of weak discontinuities OA_1 and QA_1). The Mach numbers M_1 and M_2 (in the regions below and above the shear layer, respectively), the distribution $M(y)$ of this number across the layer, and the slope angle φ_2 of a weak discontinuity OB_1 terminating the simple wave are considered to be preset.

The interaction of a simple wave with a shear layer is encountered in descriptions of supersonic streams [2] and shock waves interacting with simple waves [3] and in the problems of external aerodynamics [4, 5]. A small level of vorticity of the shear layers involved in these problems allows this factor to be ignored and the flow to be considered as potential. However, this simplification leads to physically incorrect consequences (see, e.g., [4]) and hinders adequate description of the flow pattern [2].

The main purpose of this paper is to obtain an analytical solution, with allowance for the flow vorticity, based on an asymptotic expansion of the gasdynamic functions with respect to a small parameter $\delta = \max_y |M(y) - M_1|/M_1$ characterizing the flow vorticity in the shear layer *2* (Fig. 1).

A flow in the region of interaction is described by a system of Euler equations. It is convenient to pass from this system to an extended system of equations [1]. For a flat supersonic flow, the new system is as follows:

$$\frac{\partial P_{1,2}}{\partial x} + \tan(\vartheta \pm \alpha)\frac{\partial P_{1,2}}{\partial y} = \pm\frac{\psi}{\cos^2(\vartheta \pm \alpha)}$$
$$\times[-M^2P_{1,2}^2 - (2\mu - M^2)P_1P_2 + 2\mu P_{1,2}P_3]$$
$$\pm Z\left[\frac{-2P_1P_2\cos\alpha}{\cos(\vartheta-\alpha)\cos(\vartheta+\alpha)}\right.$$
$$\left. + \frac{P_1P_3}{\cos(\vartheta+\alpha)\cos\vartheta} + \frac{P_2P_3}{\cos(\vartheta-\alpha)\cos\vartheta}\right],$$

$$\frac{\partial P_3}{\partial x} + \tan\vartheta\frac{\partial P_3}{\partial y} = \frac{P_3}{2\Gamma(M)\cos^2\vartheta}(P_2 - P_1),$$
$$\Gamma(M) = \frac{\gamma M^2}{A}, \quad A = \sqrt{M^2-1}, \tag{1}$$
$$\frac{\partial\vartheta}{\partial x} = \frac{P_2\tan(\vartheta-\alpha) - P_1\tan(\vartheta+\alpha)}{2\Gamma(M)},$$
$$\frac{\partial M}{\partial x} = \nu\left[-P_3\tan\vartheta + \frac{P_2\tan(\vartheta-\alpha) - P_1\tan(\vartheta+\alpha)}{2}\right],$$
$$\frac{\partial\vartheta}{\partial y} = \frac{P_1 - P_2}{2\Gamma(M)}, \quad \frac{\partial M}{\partial y} = \nu\left[P_3 - \frac{P_1+P_2}{2}\right],$$
$$\nu = \frac{\mu}{(1+\varepsilon)M}.$$

Here, ϑ is the slope of the flow velocity vector relative to the abscissa axis, $\alpha = \arcsin(1/M)$, γ is the adi-

1063-7850/02/2810-0877$22.00 © 2002 MAIK "Nauka/Interperiodica"

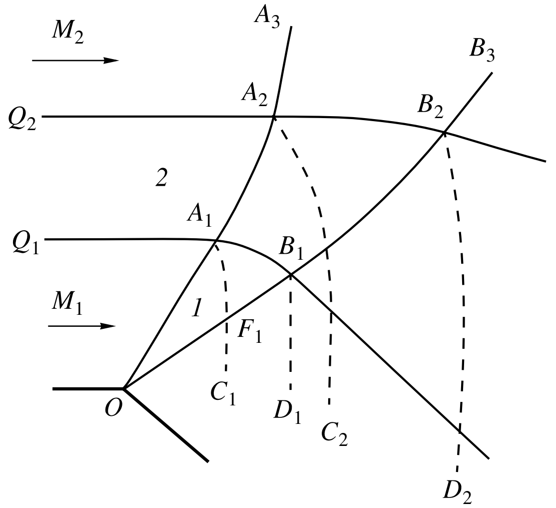

Fig. 1. A schematic diagram illustrating the interaction of a centered wave of rarefaction with a shear layer.

abate index, and

$$Z = \frac{(M^2-2)}{2(M^2-1)}\frac{\mu}{(1+\varepsilon)M^3}, \quad \psi = \frac{1}{2(1+\varepsilon)M^2\sqrt{M^2-1}},$$

$$\mu = 1+\varepsilon(M^2-1), \quad \varepsilon = \frac{\gamma-1}{\gamma+1}.$$

The functions

$$P_{1,2} = \frac{\partial \ln p}{\partial y} \pm \Gamma(M)\frac{\partial \vartheta}{\partial y},$$

$$P_3 = \frac{\partial \ln p}{\partial y} + \frac{(1+\varepsilon)M^2}{\mu}\frac{\partial \ln M}{\partial y}$$

characterize the intensity of small perturbations propagating along characteristics of the first (P_1) and second (P_2) families and along the current lines (P_3). In particular, for a simple Prandtl–Meyer wave, $P_2 = P_3 = 0$ and

$$P_1 = \frac{2(1+\varepsilon)\sqrt{M^2-1}\cos^2(\vartheta+\alpha)}{x-c}. \qquad (2)$$

Here, the constant c changes upon passage from one to another characteristic of the first family. For a centered wave, this value is constant over the wave and equal to the abscissa x_0 of the wave center (x_0, y_0). For wave *1* in Fig. 1, $x_0 = y_0 = 0$. In the shear layer *2*, which is parallel to the abscissa axis, $P_1 = P_2 = 0$ and P_3 is proportional to the vorticity:

$$P_3(y) = \frac{-1}{(\gamma-1)}\frac{dS(y)}{dy} = \frac{(1+\varepsilon)M(y)}{\mu(y)}\frac{dM(y)}{dy}, \qquad (3)$$

where $M(y)$ and $S(y)$ are distributions of the Mach number and the entropy in the shear layer.

Let us seek a solution in the region of interaction in the following form:

$$f = \sum_{k=0}^{\infty} \delta^k f^{(k)}, \;\; \delta \longrightarrow 0, \;\; f \in \{A, \vartheta, P_1, P_2, P_3\}. \qquad (4)$$

In the zero approximation, system (1) describes the flow in a centered Prandtl–Meyer wave with $P_2^{(0)} = P_3^{(0)} = 0$ and $P_1^{(0)}$ determined by formula (2). The quantities $\nu^{(0)}$ and $A^{(0)}$ are related to the Mach number M_1 of the uniform flow ahead of wave *1* and to the polar angle φ by the relations

$$\vartheta^{(0)} + \omega(M^{(0)}) = \omega(M_1),$$

$$A^{(0)} = \frac{1}{\sqrt{\varepsilon}}\tan(\sqrt{\varepsilon}[g(M_1)-\varphi]).$$

Now let us pass to a polar coordinate system in Eqs. (1), substitute series (4) into this system, and equate the terms at equal powers of δ. As a result, we obtain for the functions $f^{(1)}$

$$\frac{\partial P_3^{(1)}}{\partial \varphi} - r\cot(\varphi-\vartheta^{(0)})\frac{\partial P_3^{(1)}}{\partial r} = \alpha_3(\varphi)P_3^{(1)},$$

$$\frac{\partial P_2^{(1)}}{\partial \varphi} - r\cot(\varphi+\alpha^{(0)}-\vartheta^{(0)})\frac{\partial P_2^{(1)}}{\partial r}$$

$$= \alpha_{22}(\varphi)P_2^{(1)} + \alpha_{23}(\varphi)P_3^{(1)},$$

$$\frac{D^{(1)}}{r}\frac{\partial \vartheta^{(0)}}{\partial \varphi} + \frac{\partial \vartheta^{(1)}}{\partial r} = \alpha_{\vartheta 2}(\varphi)P_2^{(1)}, \;\; D^{(1)} = \vartheta^{(1)} - \frac{A^{(1)}}{(M^{(0)})^2}, \qquad (5)$$

$$\frac{D^{(1)}}{r}\frac{\partial A^{(0)}}{\partial \varphi} + \frac{\partial A^{(1)}}{\partial r} = \alpha_{A2}(\varphi)P_2^{(1)} + \alpha_{A3}(\varphi)P_3^{(1)},$$

$$\frac{D^{(1)}}{r}\frac{\partial P_1^{(0)}}{\partial \varphi} + \frac{\partial P_1^{(1)}}{\partial r} = \alpha_{11}(\varphi) + \alpha_{12}(\varphi)P_2^{(1)}$$

$$+ \alpha_{13}(\varphi)P_3^{(1)} + \alpha_{1\vartheta}(\varphi)\vartheta^{(1)} + \alpha_{1A}(\varphi)A^{(1)}.$$

System (5), as well as the initial system (1), is written in the invariant form [1]. In addition, the matrix in the right-hand part of (5) is triangular. These circumstances allow us to obtain an analytical solution to system (5) by sequentially solving the linear inhomogeneous first-order equations in partial derivatives with respect to functions $P_3^{(1)}$, $P_2^{(1)}$, $\vartheta^{(1)}$, $A^{(1)}$, and $P_1^{(1)}$.

It should be noted that the region of interaction is infinite in *r*. This circumstance leads to nonuniformity of the asymptotic expansion. Indeed, integration of the last three equations of system (5) along the characteristics of the first family leads to the appearance of secular terms of the type $r\alpha(\varphi)$ in the expressions for $\vartheta^{(1)}$, $A^{(1)}$,

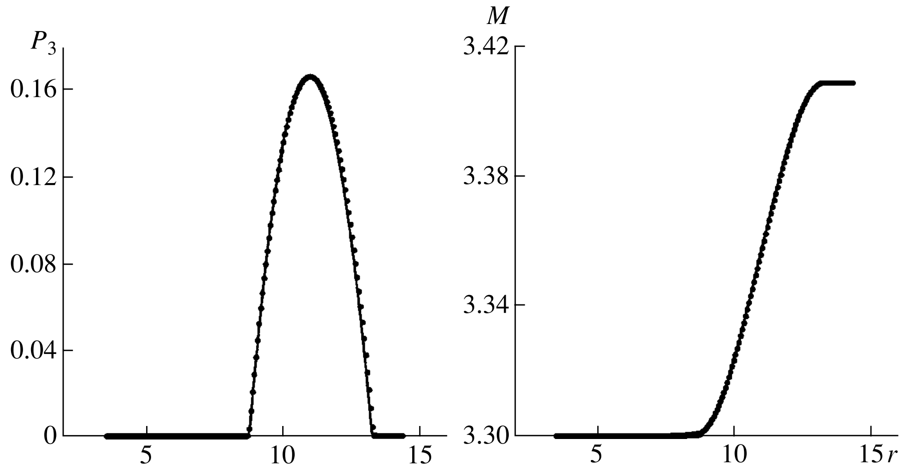

Fig. 2. A comparison of the results of analytical and numerical calculations (see the text for explanations).

and $P_1^{(1)}$, which makes the expansion inapplicable at large distances from the wall.

In order to obtain a uniformly applicable first approximation, let us employ the method of deformed coordinates [6, 7] and pass from (r, φ) to the new variables (s, t) using the formulas

$$\varphi = s + \delta\varphi_2(s, t) + \ldots, \quad r = t.$$

As a result, the new equation for $\vartheta^{(1)}$ is

$$\frac{\partial\vartheta^{(1)}}{\partial t} - \frac{\partial\varphi_2}{\partial t}\frac{\partial\vartheta^{(0)}}{\partial s} - \frac{1}{t}\frac{\partial\vartheta^{(0)}}{\partial s}\left(\varphi_2 - \vartheta^{(1)} + \frac{A^{(1)}}{(M^{(0)})^2}\right) = \alpha_{\vartheta 2}(\varphi)P_2^{(1)}.$$

By selecting φ_2 from the condition

$$\frac{\partial\varphi_2}{\partial t} + \frac{1}{t}\left(\varphi_2 - \vartheta^{(1)} + \frac{A^{(1)}}{(M^{(0)})^2}\right) = 0, \tag{6}$$

we arrive at

$$\frac{\partial\vartheta^{(1)}}{\partial t} = 0 \longrightarrow \vartheta^{(1)} = \vartheta^{(1)}(s),$$

where the function $\vartheta^{(0)} + \delta\vartheta^{(1)}$ remains constant in the region above the vortex layer. Equation (6) can be rewritten as

$$\frac{\partial}{\partial t}(t\varphi_2) = \vartheta^{(1)} - \frac{A^{(1)}}{(M^{(0)})^2} \longrightarrow \varphi_2(s, t)$$

$$= \frac{1}{t}\tilde{\varphi}(s) + \frac{1}{t}\int_{t_0}^{t}\left[\vartheta^{(1)} - \frac{A^{(1)}}{(M^{(0)})^2}\right]dt.$$

At $t = t_0$, we obtain the flow in a simple wave with $\varphi = s$. For this reason, the arbitrary function $\tilde{\varphi}(s)$ in the last equation should be taken equal to zero. With this choice of $\tilde{\varphi}(s)$, the parametric variable s is determined by the implicit relation

$$\varphi = s + \frac{\delta}{t}\int_{t_0}^{t}\left[\vartheta^{(1)} - \frac{A^{(1)}}{(M^{(0)})^2}\right]dt + O(\delta^2). \tag{7}$$

Figure 2 presents the results of calculations of the distribution of vorticity P_3 and the Mach number M on the terminal wave characteristic OB_3. These results were obtained for $M_1 = 3$, $M_2 = 3.1$, $M_{w1} = 3.3$, and a velocity profile in the vortex layer described by a cubic parabola. For these initial conditions, $\delta = 0.033$. Solid curves in Fig. 2 correspond to the data obtained using an asymptotic expansion, while dotted curves represent the values calculated by the method of characteristics. As can be seen from this figure, even the former approximation provides for a good coincidence with the exact calculation: the maximum relative error of determination of the Mach number was about 10^{-4}. It is interesting to note that an increase in the level of vorticity does not lead to a catastrophic growth in the error. Indeed, for a Mach number of $M_2 = 4$, the parameter δ is 0.33 and the maximum relative error of determination of the Mach number is on the order of 10^{-2}.

Conclusion. We have demonstrated that the problem under consideration belongs to the class of singularly perturbed problems of vortex gasdynamics. A uniformly applicable first approximation was obtained using the method of deformed coordinates.

Acknowledgments. The author is grateful to E.A. Tropp for critical remarks which inspired this study and to V.R. Meshkov for his help in numerical calculations.

This study was supported in part by the INTAS Foundation (grant no. 99-785) and the Russian Foundation for Basic Research (project no. 00-15-96106).

REFERENCES

1. B. L. Rozhdestvenskiĭ and N. N. Yanenko, *Systems of Quasilinear Equations and Their Applications to Gasdynamics* (Nauka, Moscow, 1968).
2. A. E. Medvedev and V. M. Fomin, Prikl. Mekh. Tekh. Fiz. **39** (3), 52 (1998).
3. H. Li and G. Ben-Dor, AIAA J. **34** (2), 418 (1996).
4. W. D. Hayes and R. F. Probstein, *Hypersonic Flow Theory* (Academic, New York, 1959; Inostrannaya Literatura, Moscow, 1962).
5. G. G. Chernyĭ, *Gas Flows with Fair Supersonic Speed* (Fizmatgiz, Moscow, 1959).
6. A. H. Nayfeh, *Introduction to Perturbation Techniques* (Wiley, New York, 1981; Mir, Moscow, 1984).
7. M. Van Dyke, *Perturbation Methods in Fluid Mechanics* (Academic, New York, 1964; Mir, Moscow, 1967).

Translated by P. Pozdeev