# Differential Characteristic of the Flow behind a Shock Wave

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**Abstract**—Relationships between the derivatives on both sides of a discontinuity in a nonstationary shock wave moving with acceleration in a one-dimensional vortex flow of perfect gas are deduced. The problem of interaction between the shock wave and a weak discontinuity is solved based on these relationships. © 2002 *MAIK "Nauka/Interperiodica"*.

#### INTRODUCTION

The need for deriving relationships between the properties of a strong discontinuity, such as the acceleration or the curvature of a shock wave, and the derivatives of gasdynamic variables on both sides of the strong discontinuity has been associated largely with two problems. The first is the study of flows behind curved shock waves, and the second is the analysis of strong-weak discontinuity interaction. Early results in this field, which date back to late in the 1940s [1, 2], were concerned with the special case of a planar or an axisymmetric stationary curved shock wave. Later, they were extended for higher dimensionality problems [3, 4]. However, most relationships that related the derivatives on both sides of a strong discontinuity were awkward. Therefore, the problem of strong-weak discontinuity interaction in gas dynamics either was solved by the perturbation method or was considered as a special case of strong-strong discontinuity interaction [5].

In this work, we derive simple relationships between the derivatives on both sides of a nonstationary onedimensional shock wave. Based on them, we attack the problem of shock wave interaction with counter and weak cocurrent discontinuities. The application of the results obtained is exemplified by the propagation of a shock wave in a duct of variable cross section.

#### STATEMENT OF THE PROBLEM

We consider the accelerated motion of a nonstationary shock wave (SW) in a one-dimensional vortex nonisobaric flow of perfect gas. In terms of the Lagrange variables, the set of equations for this flow has the form [6]

$$\frac{\partial \ln \rho}{\partial \tau} + \frac{\gamma^2 p x^{\delta} \partial v}{a^2} = -\frac{\delta \gamma v}{x},$$
$$\frac{\partial v}{\partial \tau} + p x^{\delta} \frac{\ln p}{\partial q} = 0, \qquad (1)$$

$$\frac{\partial S}{\partial \tau} = 0.$$

Here,  $\rho$  and *a* are the pressure and the sound velocity in the flow, respectively; *v* is the velocity of the flow; *S* is the entropy, which is related to *p* and *a* as

$$S = 2c_p \left( \ln a + \frac{\gamma - 1}{2\gamma} \ln p \right) + \text{const}, \qquad (2)$$

where  $\gamma$  is the adiabatic exponent; q and  $\tau$  are Lagrangean coordinates; and  $\delta = 0$ , 1, and 2 for planar, axisymmetric, and spherically symmetric flows, respectively.

The Eulerian coordinate x = x(q, t) is considered as a solution of the differential equation  $\partial x/\partial t = v(q, t)$ . Introducing the vector  $u = [\ln p, v, S]$ , we come to the set of equations in the matrix form

$$\frac{\partial u}{\partial \tau} + A \frac{\partial u}{\partial q} = b, \qquad (3)$$

where the matrix *A* has a rank of 2 and A[1...2, 3] = 0[1...2, 3]. This set can be represented in the characteristic form [6]:

$$L^{(k)}U + \lambda_k L^{(k)}V = L^{(k)}b; \quad k = 1, ..., 3.$$
 (4)

Here, 
$$L^{(1)} = [1, -\gamma/a, 0], L^{(2)} = [1, \gamma/a, 0], L^{(3)} = [0, 0, 1],$$

$$U = \left(\frac{\partial \ln p}{\partial \tau}, \frac{\partial v}{\partial \tau}, \frac{\partial S}{\partial \tau}\right), \quad V = \left(\frac{\partial \ln p}{\partial q}, \frac{\partial v}{\partial q}, \frac{\partial S}{\partial q}\right),$$
$$\lambda_{1,2} = \mp \frac{\gamma p x^{\delta}}{a}, \quad \lambda_3 = 0, \quad b = \left(-\frac{\delta \gamma v}{x}, 0, 0\right).$$

In the fixed coordinate system, the discontinuity  $[f] = f_2 - f_1$  of the gasdynamic variables  $f \in \{\ln p, v, v\}$ 

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Fig. 1. Interaction of a shock wave with weak discontinuities. Parenthesized figures indicate the continuity regions of the derivatives.

 $\ln a$ , S} in an SW and the SW velocity D = dx/dt relate by the Hugoniot relationships

$$\Lambda = \ln p_2 - \ln p_1 = \ln \left[ (1+\varepsilon) \left( \frac{D-v_1}{a_1} \right)^2 - \varepsilon \right],$$
  

$$\varepsilon = \frac{\gamma - 1}{\gamma + 1},$$
  

$$v_2 - v_1 = (D - v_1) \frac{(1-\varepsilon)(J-1)}{(J+\varepsilon)}, \quad p = \frac{p_2}{p_1}, \quad (5)$$
  

$$\frac{\ln a_2 - \ln a_1}{2} = \ln \frac{J(1+\varepsilon J)}{(J+\varepsilon)},$$
  

$$\frac{\ln S_2 - \ln S_1}{c_p} = \ln \frac{J(1+\varepsilon J)J^{1/\gamma}}{(J+\varepsilon)}.$$

In passing over a shock, not only the gasdynamic variables but also their variables,  $\partial f/\partial \tau$  and  $\partial f/\partial q$ , have discontinuities. The basic goal of the first part of the article is to express the derivatives of the main gasdynamic variables behind the shock through the main gasdynamic variables and shock parameters in front of it. The second part of the particle deals with the interaction of an SW with a weak discontinuity. As has been shown (see, e.g., [6]), the line of a weak discontinuity is necessarily coincident with one of the characteristics of set (4); in other words, there exists  $k \in (1, ..., 3)$  such that  $dq/d\tau = \lambda_k$ . Hereafter, such a line will be referred to as the line of the weak discontinuity of the subscript *k*.

For definiteness, we assume that the SW direction coincides with that of the characteristic of the first family (Fig. 1a). Then, the SW may interfere with a weak counter discontinuity of the subscript k (k = 1, ..., 3) or with a weak cocurrent discontinuity of subscript 1. As a result, a weak discontinuity of subscript 2 and that of subscript 3 arise (reflected weak discontinuities of k = 2and 3; Fig. 1a). Moreover, the SW acceleration Wchanges stepwise. The problem of strong–weak discontinuity interaction is stated as follows: given relation-

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ships (5) for a strong discontinuity and the discontinuities of the derivatives  $\partial f/\partial \tau$  and  $\partial f/\partial q$  for a counter weak discontinuity, it is necessary to find the jump in the derivatives of weak discontinuities outgoing from the point of interaction.

### RELATIONSHIPS BETWEEN THE DERIVATIVES BEHIND A SHOCK WAVE AND THOSE ALONG THE SHOCK PATH

Let  $w(\tau)$  be the SW path. The derivative of the gasdynamic function  $f_2$  behind the SW with respect to  $\tau$  in the direction of  $w(\tau)$ , the SW velocity *D*, and the derivatives  $\partial f_2/\partial \tau_2$  and  $\partial f_2/\partial q_2$  are related as follows:

$$\frac{df_2}{d\tau} = \frac{\partial f_2}{\partial \tau_2} + (D - v_2) \frac{\gamma p_2 x^2}{a_2^2} \frac{\partial f_2}{\partial q_2}.$$
 (6)

As was noted, the rank of the matrix of set (3) equals two; that is, the third row of the matrix is the linear combination of the first two. In addition, the first two rows of the set involve only the derivatives of  $\ln p$  and v, while the third one contains only the derivative of the function S. This means that, from (1), we can separate the subset

$$\frac{\partial \ln p}{\partial \tau} + \frac{\gamma^2 p x^{\delta}}{a^2} \frac{\partial v}{\partial q} = -\frac{\delta \gamma v}{x},$$

$$\frac{\partial v}{\partial \tau} + p x^{\delta} \frac{\partial \ln p}{\partial q} = 0,$$
(7)

as well as the equation

$$\frac{\partial S}{\partial s} = 0. \tag{8}$$

Then, using formulas (5), we can separately find a relation between the derivatives of  $\ln p$  and v, as well as between the derivatives of *S*.

Let us express the derivatives of  $\ln p$  and v behind the SW through the derivatives  $\partial f_2 / \partial \tau$  along the SW path. To do this, we substitute  $p = p_2$  and  $v = v_2$  into (7). In view of relationships (6) written for  $f_2 = \ln p_2$  and  $f_2 = v_2$ , we have the linear system for the derivatives of  $\ln p$  and v behind the SW. Solving this system, we obtain

$$\frac{\partial \ln p_2}{\partial q_2} = \frac{1}{z} \left[ \frac{1}{\gamma} \frac{d \ln p_2}{\partial \tau} + \frac{1}{[D - v_2]} \frac{d v_2}{\partial \tau} + \frac{\delta v_2}{x} \right] \times \left( -\frac{[D - v_2]}{p_2 x^{\delta}} \right),$$

$$\frac{\partial \mathbf{v}_2}{\partial q_2} = \frac{1}{z} \left[ \frac{1}{\gamma} \frac{d \ln p_2}{\partial \tau} + \frac{[D - \mathbf{v}_2]}{a_2^2} \frac{d \mathbf{v}_2}{d \tau} + \frac{\delta \mathbf{v}_2}{x} \right] \left( -\frac{a_2^2}{\gamma p_2 x^\delta} \right),$$

$$\frac{\partial \ln p_2}{\partial \tau_2} = \frac{1}{z} \left[ \frac{1}{\gamma} \frac{a_2^2}{[D - v_2]} \frac{d \ln p_2}{d \tau} + \frac{d v_2}{d \tau} + [D - v_2] \frac{\delta v_2}{x} \right] (9)$$
$$\times \left( \frac{\gamma [D - v_2]}{a_2^2} \right),$$

$$\frac{\partial v_2}{\partial \tau_2} = \frac{1}{z} \left[ \frac{1}{\gamma} [D - v_2] \frac{d \ln p_2}{d \tau} + \frac{d v_2}{d \tau} + [D - v_2] \frac{\delta v_2}{x} \right],$$
$$z = 1 - \frac{[D - v_2]^2}{a_2^2} = \frac{J - 1}{J(1 + \varepsilon)}.$$

Using relationships (5) for the discontinuities, we can express the derivatives  $df_2/d\tau$  through the derivatives  $df_1/d\tau$  of the gasdynamic variables before the SW:

$$\frac{d\ln p_2}{d\tau} = \frac{d\ln p_1}{d\tau} + 2\frac{(J+\varepsilon)}{J} \left[ \frac{1}{D-v_1} \frac{d(D-v_1)}{d\tau} - \frac{d\ln a_1}{d\tau} \right]$$
$$\frac{dv_2}{d\tau} = \frac{dv_1}{d\tau} + \frac{(1-\varepsilon)}{(J+\varepsilon)} \frac{d(D-v_1)}{d\tau} \left[ (J+\varepsilon) + (1+\varepsilon) \right]$$
$$-2(D-v_1) \frac{(1-\varepsilon^2)}{(J+\varepsilon)} \frac{d\ln a_1}{d\tau}.$$

Introducing the designations

$$N_{p} = \frac{d \ln p_{1}}{d\tau}, \quad N_{u} = \frac{d v_{1}}{d\tau},$$
$$N_{a} = \frac{d \ln a_{1}}{d\tau}, \quad N_{\delta} = \frac{\delta}{x}, \quad N_{D} = \frac{dD}{d\tau}$$

and taking into account the expressions for the derivatives along the SW, we easily come to the desired relationships for  $\ln p$  and v:

$$T_{i}^{(2)} = \frac{-1}{z} [\psi_{p}^{(i)} N_{p} + \psi_{v}^{(i)} N_{v} + \psi_{a}^{(i)} N_{a} + \psi_{\delta}^{(i)} N_{\delta} + \psi_{D}^{(i)} N_{D}],$$

$$N_{i}^{(2)} = \frac{1}{z} [\phi_{p}^{(i)} N_{p} + \phi_{v}^{(i)} N_{v} + \phi_{a}^{(i)} N_{a} + \phi_{\delta}^{(i)} N_{\delta} + \phi_{D}^{(i)} N_{D}],$$
(10)

where i = 1 and 2,

$$\begin{split} T_{1}^{(2)} &= \frac{\partial \ln p_{2}}{\partial q_{2}}, \ T_{2}^{(2)} &= \frac{\partial v_{2}}{\partial q_{2}}, \ N_{1}^{(2)} &= \frac{\partial \ln p_{2}}{\partial \tau_{2}}, \ N_{2}^{(2)} &= \frac{\partial v_{2}}{\partial \tau_{2}} \\ \psi_{p}^{(1)} &= \tilde{\psi}_{v}^{(2)} &= \frac{D - v_{2}}{\gamma p_{2} x^{\delta}}, \ \tilde{\psi}_{v}^{(1)} &= \frac{1}{p_{2} x^{d}}, \ \psi_{2}^{(2)} &= \frac{a_{2}^{2}}{\gamma^{2} p_{2} x^{d}} \\ \phi_{p}^{(1)} &= \tilde{\phi}_{v}^{(2)} &= 1, \ \tilde{\phi}_{v}^{(1)} &= \frac{(D - v_{2})\gamma}{a_{2}^{2}}, \ \phi_{p}^{(2)} &= \frac{D - v_{2}}{\gamma}, \\ \psi_{a}^{(i)} &= -d_{1}(D - v_{1})(\psi_{p}^{(i)} + \tilde{\psi}_{v}^{(i)}g_{2}), \\ \phi_{a}^{(i)} &= -d_{1}(D - v_{1})(\phi_{p}^{(i)} + \tilde{\phi}_{v}^{(i)}g_{2}), \\ \psi_{D}^{(i)} &= d_{1}(\psi_{p}^{(i)} + g_{2}\tilde{\psi}_{v}^{(i)}) + \tilde{\psi}_{v}^{(i)}d_{2}, \\ \psi_{D}^{(i)} &= d_{1}(\phi_{p}^{(i)} + g_{2}\tilde{\phi}_{v}^{(i)}) + \tilde{\phi}_{v}^{(i)}d_{2}, \\ \psi_{\delta}^{(i)} &= \psi_{p}^{(i)}v_{2}\gamma, \ \phi_{\delta}^{(1)} &= \phi_{p}^{(1)}v_{2}\gamma, \\ \phi_{\delta}^{(2)} &= (\phi_{p}^{(2)} - 1)v_{2}\gamma, \\ \psi_{v}^{(i)} &= \tilde{\psi}_{v}^{(i)} - \psi_{D}^{(i)}, \ \phi_{v}^{(i)} &= \tilde{\phi}_{v}^{(i)} - \phi_{D}^{(i)}. \end{split}$$

Here,

$$d_{1} = \frac{\partial \Lambda}{\partial D} = \frac{2(J+\varepsilon)}{J(D-v_{1})},$$
  

$$d_{2} = \frac{\partial [v]}{\partial D} = \frac{(1-\varepsilon)(J-1)}{(J+\varepsilon)},$$
  

$$g_{2} = \frac{\partial [v]}{\partial \Lambda} = (D-v_{1})\frac{(1-\varepsilon^{2})J}{(J+\varepsilon)^{2}}.$$

In practice, it is sometimes convenient to replace  $N_a$ by the function  $N_S = dS/d\tau$ , which characterizes the vorticity of the flow.  $N_S$ ,  $N_a$ , and  $N_p$  are related as [2]

$$N_a = \frac{d\ln a}{d\tau} = \frac{\gamma - 1}{2\gamma} \frac{d\ln p}{d\tau} + \frac{1}{2c_p} \frac{dS}{d\tau} = \frac{\gamma - 1}{2\gamma} N_p + \frac{1}{2c_p} N_s.$$

It is clear that the coefficients before  $N_s$  differ from the associated coefficients before  $N_a$  by a factor of  $1/(2c_p)$ ,

$$\tilde{\Psi}_{S}^{(i)} = \Psi_{a}^{(i)}/(2c_{p}), \quad \tilde{\varphi}_{S}^{(i)} = \varphi_{a}^{(i)}/(2c_{p}),$$

and the new and old coefficients before  $N_p$  are related as

$$\tilde{\Psi}_p^{(i)} = \Psi_p^{(i)} - \frac{\gamma - 1}{2\gamma} \Psi_a^{(i)}, \quad \tilde{\varphi}_p^{(i)} = \varphi_p^{(i)} - \frac{\gamma - 1}{2\gamma} \varphi_a^{(i)}.$$

To set a relationship between the derivatives of the function *S*, we note that the derivatives  $\partial S/\partial \tau$  vanish on both sides of the discontinuity by virtue of (8). Hence,

$$N_3^{(2)} = \frac{\partial \ln S_2}{\partial q_2} = \frac{a_2^2}{\gamma p_2 x^d (D - v_2)} \left[ \frac{d \ln S}{d\tau} + \frac{d[S]}{d\tau} \right].$$

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Using the last relationship in (5), we rearrange the derivative on the right-hand side to obtain (in terms of the above designations)

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$$N_{3}^{(2)} = \frac{a_{2}^{2}}{\gamma p_{2} x^{d} (D - v_{2})} \bigg[ N_{S} - \frac{2c_{p} (1 - \varepsilon)\varepsilon (J - 1)^{2}}{(1 + \varepsilon)J(1 + \varepsilon J)} + \sum_{k=1}^{\infty} \frac{N_{k} - v_{k}}{N_{a} + \frac{1}{D - v_{1}} (N_{v} - N_{D})} \bigg].$$
(11)

## **RELATIONSHIPS BETWEEN THE DERIVATIVES** BEHIND THE SHOCK AND BASIC FLOW NONUNIFORMITIES BEFORE THE SHOCK

Now let us express the derivatives behind the SW through the discontinuity acceleration  $N_D$ , as well as through the functions

$$N_1 = \frac{\partial \ln p_1}{\partial \tau_1}, \quad N_2 = \frac{\partial v_1}{\partial \tau_1}, \quad N_3 = \frac{\partial S_1}{\partial q_1},$$

which are, respectively, flow nonisobaricity, flow acceleration, and flow vorticity (so-called flow nonuniformities) before the SW.

The derivative  $df_1/d\tau$  of the gasdynamic function  $f_1$ before the SW is related to the derivatives  $df_1/d\tau_1$  and  $df_1/dq_1$  as

$$\frac{df_1}{d\tau} = \frac{\partial f_1}{\partial \tau_1} + (D - v_1) \frac{\gamma p_1 x^{\delta}}{a_1^2} \frac{\partial f_1}{\partial q_1}.$$
 (12)

Using set (1) written for  $f = f_1$ , as well as relationships (12), one can easily express  $N_p$ ,  $N_v$ , and  $N_s$ through the basic flow nonuniformities:

$$N_{p} = N_{1} - (D - v_{1}) \frac{\gamma}{a_{1}^{2}} N_{2}, \quad N_{S} = (D - v_{1}) \frac{\gamma p_{1} x^{\delta}}{a_{1}^{2}} N_{3},$$
$$N_{v} = N_{2} = \frac{1}{\gamma} (D - v_{1}) [N_{1} + \gamma v_{1} N_{\delta}].$$

Substituting these expressions into formulas (10) and (11) in place of  $N_p$ ,  $N_v$ , and  $N_s$  and designating  $N_4 = N_8$  and  $N_5 = N_D$  yields the desired relation between the derivatives behind the SW and the basic flow nonuniformities before it:

$$N_{1}^{(2)} = \frac{\Gamma(a_{2})}{z} \sum_{k=1}^{5} A_{1k} N_{k}, \quad T_{1}^{(2)} = \frac{-1}{z p_{2} x^{\delta}} \sum_{k=1}^{5} A_{2k} N_{k},$$

$$N_{2}^{(2)} = \frac{1}{z} \sum_{k=1}^{5} A_{2k} N_{k},$$

$$T_{2}^{(2)} = \frac{-a_{2}}{z p_{2} x^{\delta}} \sum_{k=1}^{5} A_{1k} N_{k} - \frac{a_{2}^{2} v_{2}}{p_{2} x^{\delta}} N_{4},$$
(13)

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$$N_3^{(2)} = N_3 - \sigma (J-1)^2 \sum_{k=1}^5 A_{3k} N_k.$$

Here,

F

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$$A_{15} = d_1 f_1 + d_2 \tilde{s}_2, \quad A_{25} = d_1 f_2 + d_2 \tilde{c}_2,$$
  

$$A_{35} = f_3 / (D - v_1),$$
  

$$f_1 = g_1 \tilde{c}_2 + g_2 \tilde{s}_2, \quad f_2 = g_1 \tilde{s}_2 + g_2 \tilde{c}_2, \quad f_3 = 1,$$
  

$$g_1 = \Gamma^{-1}(a_2), \quad \Gamma(a) = \frac{\gamma}{a}, \quad \tilde{c}_2 = 1, \quad \tilde{s}_2 = \frac{D - u_2}{a_2},$$
  

$$A_{14} = A_{15} v_1 (D - v_1) + \tilde{s}_2 (v_2 (D - v_2) - v_1 (D - v_1)),$$
  

$$A_{24} = A_{25} v_1 (D - v_1) + \tilde{c}_2 (v_2 (D - v_2) - v_1 (D - v_1)),$$
  

$$A_{34} = A_{35} v_1 (D - v_1),$$
  

$$A_{13} = \alpha d_1 (D - v_1) f_1, \quad A_{23} = \alpha d_1 (D - v_1) f_2,$$
  

$$A_{33} = \alpha f_3,$$

$$\begin{split} &= -\frac{\gamma p_1 x^{\delta}}{2c_p a_1^2} (D - u_1), \quad \mathfrak{S} = \frac{a_2^2}{\gamma^2 p_2 x^{\delta} (D - u_2)} \frac{2c_p \varepsilon}{J(1 + \varepsilon J)}, \\ &A_{12} = -A_{15} \tilde{c}_1 + \frac{\varepsilon}{1 + \varepsilon} d_1 (D - v_1) f_1 \Gamma(a_1) \tilde{s}_1 \\ &\quad + \tilde{c}_1 \tilde{s}_2 - \frac{\Gamma(a_1)}{\Gamma(a_2)} \tilde{s}_1 \tilde{c}_2, \\ &A_{22} = -A_{25} \tilde{c}_1 + \frac{\varepsilon}{1 + \varepsilon} d_1 (D - v_1) f_2 \Gamma(a_1) \tilde{s}_1 \\ &\quad + \tilde{c}_1 \tilde{c}_2 - \frac{\Gamma(a_1)}{\Gamma(a_2)} \tilde{s}_1 \tilde{s}_2, \\ &A_{32} = -A_{35} \tilde{c}_1 + \frac{\varepsilon}{1 + \varepsilon} f_3 \Gamma(a_1) \tilde{s}_1, \\ &\quad \tilde{c}_1 = 1, \quad \tilde{s} = \frac{D - u_1}{a_1}, \\ &A_{11} = \frac{1}{\Gamma(a_1)} A_{15} \tilde{s}_1 - \frac{\varepsilon}{1 + \varepsilon} d_1 (D - v_1) f_1 \tilde{c}_1 \\ &\quad - \frac{1}{\Gamma(a_1)} \tilde{s}_1 \tilde{s}_2 + \frac{1}{\Gamma(a_2)} \tilde{c}_1 \tilde{c}_2, \\ &A_{21} = \frac{1}{\Gamma(a_1)} A_{25} \tilde{s}_1 - \frac{\varepsilon}{1 + \varepsilon} d_1 (D - v_1) f_2 \tilde{c}_1 \\ &\quad - \frac{1}{\Gamma(a_1)} \tilde{s}_1 \tilde{c}_2 + \frac{1}{\Gamma(a_2)} \tilde{c}_1 \tilde{s}_2, \\ &A_{31} = \frac{1}{\Gamma(a_1)} A_{25} \tilde{s}_1 - \frac{\varepsilon}{1 + \varepsilon} f_3 \tilde{c}_1. \end{split}$$

## RELATIONSHIPS FOR A WEAK DISCONTINUITY

Before passing to the problem of strong–weak discontinuity interaction, it would be well to establish a number of relationships concerning a weak discontinuity of the subscript k. Let  $q(\tau)$  be the line of a weak discontinuity of the subscript m that is defined by the equation  $dq/d\tau = \lambda_m$ . From the continuity condition for the gasdynamic functions in the vicinity of a weak discontinuity, it follows that the derivatives of the vector functions  $u = (\ln p, v, S)$  are equal in the direction  $q(\tau)$  of the weak discontinuity:

$$U_1 + \lambda_m V_1 = U_2 + \lambda_m V_2. \tag{14}$$

Subscripts 1 and 2 refer to the derivatives on the opposite sides of the discontinuity. Since u satisfies characteristic set (4), we have

$$L^{(k)}U_1 + \lambda_k L^{(k)}V_1 = L^{(k)}b,$$
  

$$L^{(k)}U_2 + \lambda_k L^{(k)}V_2 = L^{(k)}b, \quad k = 1, ..., 3$$
(15)

at any point in the line of this discontinuity.

Subtracting the second expression from the first one yields

$$(L^{(k)}U_1 - L^{(k)}U_2) + \lambda_k (L^{(k)}V_1 - L^{(k)}V_2) = 0;$$
  

$$k = 1, ..., 3.$$
(16)

Eliminating the difference of the derivatives with respect to  $\tau$  with (14), we can eventually write

$$(\lambda_k - \lambda_m)(L^{(k)}V_1 - L^{(k)}V_2) = 0; \quad k = 1, ..., 3.$$
 (17)

It follows from (17) that the equalities

$$L^{(k)}V_1 - L^{(k)}V_2 = 0, \quad k \neq m$$
(18)

hold at the weak discontinuity of the subscript *m* for any  $k \neq m$ .

By virtue of (16), the equalities

$$L^{(k)}U_1 - L^{(k)}U_2 = 0, \quad k \neq m$$
(19)

are also valid.

Introducing the designation  $[f] = f_2 - f_1$ , we can obtain from the last two formulas

$$\left[\frac{\partial \ln p}{\partial q} + \frac{\gamma \partial u}{a \partial q}\right] = 0, \quad \left[\frac{\partial S}{\partial q}\right] = 0, \quad (20)$$

$$\left[\frac{\partial \ln p}{\partial \tau} + \frac{\gamma \partial u}{a \partial \tau}\right] = 0, \quad \left[\frac{\partial S}{\partial \tau}\right] = 0 \tag{21}$$

for a weak discontinuity of the subscript k = 1.

Similarly, for a weak discontinuity of the subscript k = 2, we have

$$\left[\frac{\partial \ln p}{\partial q} + \frac{\gamma \partial u}{a \partial q}\right] = 0, \quad \left[\frac{\partial S}{\partial q}\right] = 0, \quad (22)$$

$$\left[\frac{\partial \ln p}{\partial \tau} + \frac{\gamma \partial u}{a \partial \tau}\right] = 0, \quad \left[\frac{\partial S}{\partial \tau}\right] = 0. \tag{23}$$

Finally, for a weak discontinuity of the subscript k = 3 (weak contact discontinuity), the differential conditions

$$\left[\frac{\partial \ln p}{\partial q}\right] = 0, \quad \left[\frac{\partial v}{\partial q}\right] = 0, \quad (24)$$

$$\left[\frac{\partial \ln p}{\partial \tau}\right] = 0, \quad \left[\frac{\partial v}{\partial \tau}\right] = 0 \tag{25}$$

for dynamic compatibility follow from (18) and (19).

### INTERACTION OF A SHOCK WAVE WITH A COUNTER WEAK DISCONTINUITY

The above relationships, which relate the derivatives of the basic gasdynamic functions at weak and strong discontinuities, allow one to effectively solve the problem of SW–weak discontinuity interaction. In this section, we will consider the interaction of SW 1 with counter weak discontinuity 2 of the subscript k (k = 1, ..., 3) (Fig. 1b).

At the point of interaction, SW 3 with an acceleration  $W_3$  and weak discontinuities 4 and  $\tau$  of subscripts 2 and 3, respectively, originate. Let us introduce the vectors of discontinuity of the derivatives behind the SW:  $[V]_w = V^{(4)} - V^{(2)}$  and  $[U]_w = U^{(4)} - U^{(2)}$ .

**Theorem 1.** If an SW whose direction coincides with that of the characteristic of the first family interacts with a counter weak discontinuity, the vectors  $[V]_w$ and  $[U]_w$  of derivative discontinuity behind the SW (strong discontinuity) are orthogonal to the left eigenvector  $L^{(1)}$ ; that is,

$$L^{(1)}[V]_{w} = 0, \quad L^{(1)}[U]_{w} = 0.$$
 (26)

**Proof.** As follows from Fig. 1b, the differences of the derivatives  $V^{(4)}$ ,  $U^{(4)}$  and  $V^{(2)}$ ,  $U^{(2)}$  in the regions in front of and behind the point of interaction are related to the vectors  $[V]_w$  and  $[U]_w$  of derivative discontinuity at originating weak discontinuities  $\tau$  and 4 by the obvious expressions

$$[V]_{w} = V^{(4)} - V^{(2)} = (V^{(4)} - V^{(3)}) + (V^{(3)} - V^{(2)}),$$
  

$$[U]_{w} = U^{(4)} - U^{(2)} = (U^{(4)} - U^{(3)}) + (U^{(3)} - U^{(2)}).$$
(27)

Weak discontinuities 4 and  $\tau$  have subscripts 2 and 3, respectively. Multiplying by the left eigenvector  $L^{(1)}$  and taking into account formulas (18) and (19), we prove the theorem.

**Consequence 1.** When an SW interacts with a counter weak discontinuity, the product of the left eigenvector  $L^{(1)}$  by the derivative  $du_2/d\tau$  of the vector function  $u_2$  along the SW path remains unchanged and

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equals

$$L^{(1)}\frac{du^4}{d\tau} = L^{(1)}\frac{du^{(2)}}{d\tau} =: L^{(1)}\frac{du_2}{d\tau} = L^{(1)}b.$$
(28)

**Proof.** Consider the discontinuity line  $w(\tau)$  for the vector function u(x, t). Behind the SW, the derivative of an arbitrary gasdynamic function  $f_2$  with respect to  $\tau$  is related to the derivatives  $\partial f_2/\partial \tau_2$  and  $\partial f_2/\partial q_2$  through (6). We multiply the second equality in (26) by  $(D - v_2)\gamma p_2 x^{\delta}/a_2^2$  and add the result to the first equality. In view of (6), we obtain

$$L^{(1)}\frac{du^{(4)}}{d\tau} - L^{(1)}\frac{du^{(2)}}{d\tau} = 0.$$

This equality means that, when multiplied by  $L^{(1)}$  on the left, the derivative of u in the direction of the strong discontinuity does not change when the strong discontinuity interacts with any arbitrary weak discontinuity:

$$L^{(1)}\frac{du^{(4)}}{d\tau} = L^{(1)}\frac{du^{(2)}}{d\tau} = \text{const} = C.$$
 (29)

It only remains to find the constant on the right of (29). To do this, consider the characteristic of subscript 1 arriving at the point of interaction. Since it lies in region (4), the conditions on this characteristic have the form

$$L^{(1)}U^{(4)} + \lambda_1 L^{(1)}V^{(4)} = L^{(1)}b.$$
(30)

Subtracting (30) from (29), we obtain on the left-hand side

$$\left((D-\mathbf{v}_2)\frac{\gamma p_2 x^{\delta}}{a_2^2}-\lambda_1\right)(L^{(1)}V^{(4)}-L^{(1)}V^{(4)})=0.$$

Hence,  $C = L^{(1)}b$ , which is the required result.

**Consequence 2.** The discontinuity  $[W] = W_3 - W_1$  of the SW acceleration is linearly related with the discontinuities  $N_i^{(1)} - N_i^{(0)}$  (i = 1, ..., 3) of the basic flow non-uniformities near the counter weak discontinuity of the subscript *k*.

**Proof.** Consider, for example, the second relationship in (26). It can be recast as

$$(N_1^{(2)} - N_1^{(4)}) - \Gamma(a_2)(N_2^{(2)} - N_2^{(4)}) = 0.$$

The derivatives  $N_1^{(2)}$ ,  $N_2^{(2)}$  and  $N_1^{(4)}$ ,  $N_2^{(4)}$  refer to the regions immediately behind the SW. Expressing them through the derivatives in front of the wave with (13), we obtain the equality that linearly relates the discontinuity [W] of the SW acceleration to the discontinuities [N<sub>i</sub>] of the basic flow nonuniformities near the *k*th counter weak discontinuity:

$$(A_{15} - A_{25})[W] + \sum_{k=1}^{5} (A_{1k} - A_{2k})(N_k^{(1)} - N_k^{(0)}) = 0.$$
(31)

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# Now let us consider specific cases of the problem.

(i) k = 1. As follows from (21), the function  $N_3$  remains continuous near the counter weak discontinuity of the subscript k = 1. At the same time, the discontinuity  $[N_1]$  of the derivative  $\partial \ln p / \partial \tau$  and the discontinuity  $[N_2]$  relate as

$$N_1^{(1)} - N_1^{(0)} = -\Gamma(a_1)(N_2^{(1)} - N_2^{(0)}).$$
(32)

Then, (31) can be recast as

$$\frac{W_3 - W_1}{N_2^{(1)} - N_2^{(0)}} = \frac{\Gamma(a_1)(A_{11} - A_{21}) - (A_{12} - A_{22})}{A_{15} - A_{25}}.$$

(ii) k = 2. From (23), it follows, as before, that  $[N_3] = 0$  and the functions  $[N_1]$  and  $[N_2]$  relate as

$$N_1^{(1)} - N_1^{(0)} = \Gamma(a_1)(N_2^{(1)} - N_2^{(0)}).$$
(33)

Substituting this into (31) yields

$$\frac{W_3 - W_1}{N_2^{(1)} - N_2^{(0)}} = -\frac{\Gamma(a_1)(A_{11} - A_{21}) + (A_{12} - A_{22})}{A_{15} - A_{25}}.$$

(iii) k = 3. From the differential conditions for dynamic compatibility at a weak contact discontinuity [see (25)], we have  $[N_1] = [N_2] = 0$ ; hence, formula (31) is reduced to

$$\frac{W_3 - W_1}{N_3^{(1)} - N_3^{(0)}} = -\frac{A_{13} - A_{23}}{A_{15} - A_{25}}.$$

**Theorem 2.** A weak discontinuity of the subscript *m* does not originate at the point where the SW interacts with the counter weak discontinuity of the subscript *k* if the vectors  $[V]_w$  and  $[U]_w$  of discontinuity of the derivatives behind the SW and the left eigenvector  $L^{(m)}$  are orthogonal to each other; that is, if

$$L^{(m)}[V]_{w} = 0, \quad L^{(m)}[U]_{w} = 0; \quad m = 2, 3.$$
 (34)

**Proof.** We will prove the statement for m = 2; for m = 3, it is proved in a similar way. Multiplying equalities (27) by the eigenvector  $L^{(2)}$  and taking into account formulas (18) and (19) yields

$$L^{(2)}[V]_{w} = L^{(2)}(V^{(4)} - V^{(3)}),$$
  

$$L^{(2)}[U]_{w} = L^{(2)}(U^{(4)} - U^{(3)}).$$
(35)

If the rights of these equalities are zero, this means that the vectors of derivative discontinuity at weak discontinuity 4 are orthogonal to all three eigenvectors. Since the latter are linearly independent, the orthogonality takes place only if  $V^{(4)} - V^{(3)} = U^{(4)} - U^{(3)} = 0$ , that is, if discontinuities near characteristic 4 are absent.

Let us examine the criteria that weak discontinuity 4 does not originate. The second relationship in (35) can then be recast as

$$(N_1^{(2)} - N_1^{(4)}) + \Gamma(a_2)(N_2^{(2)} - N_2^{(4)}) = 0.$$

Expressing the derivatives  $N_1^{(2)}$ ,  $N_2^{(2)}$  and  $N_1^{(4)}$ ,  $N_2^{(4)}$  through the derivatives in front of the SW with (13), we come to

$$(A_{15} + A_{25})[W] + \sum_{k=1}^{5} (A_{1k} + A_{2k})(N_k^{(1)} - N_k^{(0)}) = 0.$$

This relationship along with (31) forms the set of linear homogeneous equations in the variables [W] and  $(N_k^{(1)} - N_k^{(0)})$  (k = 1, ..., 3) that is conveniently recast as

$$A_{15}[W] + \sum_{k=1}^{3} A_{1k} (N_k^{(1)} - N_k^{(0)}) = 0,$$
  

$$A_{25}[W] + \sum_{k=1}^{3} A_{2k} (N_k^{(1)} - N_k^{(0)}) = 0.$$
(36)

Using formulas (21), (23), and (25), one can express the discontinuities  $(N_k^{(1)} - N_k^{(0)})$  (k = 1, ..., 3) through the discontinuity of one of the nonuniformities. The nontrivial solutions of the thus-obtained set of linear homogeneous equations will serve as criteria for the absence of weak reflected discontinuity 4.

(i) k = 1, m = 1. In this case,  $N_3^{(1)} = N_3^{(0)}$ ; hence, by virtue of (32), set (36) takes the form

$$A_{15}[W] - (\Gamma(a_1)A_{11} - A_{12})(N_2^{(1)} - N_2^{(0)}) = 0,$$
  

$$A_{25}[W] - (\Gamma(a_1)A_{21} - A_{22})(N_2^{(1)} - N_2^{(0)}) = 0.$$

This set has nontrivial solutions if

$$A_{15}(\Gamma(a_1)A_{21} - A_{22}) = A_{25}(\Gamma(a_1)A_{11} - A_{12}).$$
(37)

(ii) k = 2, m = 1. With such k, expression (33) and the equality  $N_3^{(1)} - N_3^{(0)}$  are valid, so that the nontrivial solutions of set (36) are found if

$$A_{15}(\Gamma(a_1)A_{21} + A_{22}) = A_{25}(\Gamma(a_1)A_{11} + A_{12}).$$
(38)

With (13) it is easy to check that Eqs. (37) and (38) has the same analytic solution

$$J = 2\varepsilon^{3/2} \left(\frac{1+\sqrt{\varepsilon}}{1-\varepsilon}\right)^2, \qquad (39)$$

which implies that the interaction without a reflected discontinuity is possible only if  $\varepsilon > 1/4$ ,  $\gamma > 5/3$ , and k = 1.

(iii) k = 3, m = 1. Substituting the differential conditions for dynamic compatibility at a weak contact discontinuity [see (25)] into set (36), one can easily find the following criterion for the absence of a weak discontinuity:

$$A_{15}A_{23} - A_{25}A_{13} = 0.$$

With the expressions for the associated coefficients, this equality is reduced to the form

$$2(1+\varepsilon)(J+\varepsilon)p_1x^{\delta}/(D-v_1) = 0$$

from which it follows that the interaction of the SW with a weak contact discontinuity without generating reflected weak discontinuity 4 is impossible.

### INTERACTION OF A SHOCK WAVE WITH A COCURRENT WEAK DISCONTINUITY

As was noted above, if the direction of SW 1 coincides with that of characteristic 2 of the second family, the SW may interact with the cocurrent weak discontinuity of subscript 1 (Fig. 1c). The result of the interaction is the discontinuity  $[W] = W_4 - W_1$  of the SW acceleration, as well as weak discontinuities 3 and  $\tau$  of subscripts 2 and 3, respectively.

**Theorem 1.** If the SW interacts with the weak cocurrent discontinuity of subscript 1, the eigenvector  $L^{(k)}$  (k = 1, ..., 3) is orthogonal to the differences  $[V]_w - [U]_w$  and  $[V]_k - [U]_k$ , where the former is the difference of the derivatives behind the SW and the latter is that near a weak discontinuity of the subscript k; that is,

$$L^{(k)}([V]_{w} - [V]_{k}) = 0,$$
  
$$L^{(k)}([U]_{w} - [U]_{k}) = 0; \quad k = 1, ..., 3.$$

**Consequence 1.** The discontinuity  $[W] = W_4 - W_1$  of the SW acceleration is linearly related to the discontinuity of the path curvature  $N_2^{(2)} - N_2^{(1)}$  at the weak cocurrent discontinuity of subscript 1; that is,

$$(A_{15} - A_{25})[W] = -2\Gamma(a_2)(N_2^{(2)} - N_2^{(1)}).$$

**Consequence 2.** Weak discontinuity 3 does not originate if

$$A_{15} = A_{25}. (40)$$

All the three statements are proved as those in the previous section.

Using expressions (13) for the coefficients  $A_{ij}$ , one can show that equality (40) holds if

$$J = \frac{4\varepsilon^2}{1-3\varepsilon}.$$

Thus, the interaction without the reflected discontinuity is possible if  $\varepsilon > 1/4$  and  $\gamma > 5/3$ .

#### THE CHESTER–WHITHAM FORMULA

Let us turn back to the interaction of an SW with a counter weak discontinuity. For one specific case of great importance, formula (28) is likely to be first derived by Whitham [7, 8]. He analyzed the results reported in [9–11], where a shock wave propagated in a stationary gas through a duct with a small cross-sec-

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a gas in a variable-section duct.

In [9], the relation between a small variation of the relative velocity M = D/a of the shock and the variation of the cross-sectional area A of the duct was derived based on the linearization of the relationships at the cross-sectional discontinuity:

tional discontinuity. This problem is a specific case of

the more general problem of discontinuity breakdown

in a variable-section duct [6]. In essence, the general problem involves two subproblems: the propagation of

$$d\ln A = f(M)dM.$$

Whitham noticed that the same result can be obtained if one writes the condition for the characteristic of the second family in the flow behind the shock and, instead of  $p_2$ ,  $u_2$ , and  $a_2$ , substitutes their associated expressions (in terms of M) for the shock wave [see (5)]. If the wave propagates in a stationary gas with the parameters  $p_1$ ,  $u_1$ , and  $a_1$ , we have

$$\frac{p_2}{p_1} = (1+\varepsilon)M^2 - \varepsilon, \quad \frac{u_2}{a_1} = (1-\varepsilon)\left(M - \frac{1}{M}\right),$$
$$\frac{a_2}{a_1} = \sqrt{\left[(1+\varepsilon) - \frac{\varepsilon}{M^2}\right]\left[(1-\varepsilon) + \varepsilon M^2\right]}.$$

Whitham called such an expedient the rule of characteristics and assumed that it can also apply to other cases [7]. Formula (28) proved in our work generalizes this rule for the interaction of a shock wave propagating in a vortex nonisobaric one-dimensional flow with a counter discontinuity of an arbitrary subscript. Also, this formula allows one to derive approximate analytical solutions for the interaction of a shock wave with a Riemann wave, a shear layer, etc.

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