Enumeration of Chord Diagrams without Loops and Parallel Chords

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Abstract

We enumerate chord diagrams without loops and without both loops and parallel chords. For labelled diagrams we obtain generating functions, for unlabelled ones we derive recurrence relations.

Keywords: chord diagrams; Hamiltonian cycles; n-dimensional octahedron; loopless diagrams; simple diagrams; unlabelled enumeration; generating functions

1 Introduction

A chord diagram consists of $2n$ points on a circle labelled with the numbers $1, 2, \ldots, 2n$ in a circular order, joined pairwise by chords (Figure 1). Depending on the notion of isomorphism used, two diagrams are said to be isomorphic if one could be obtained from the other either by a rotation or by a combination of rotations and reflections of the circle. Isomorphism classes of labelled chord diagrams are said to be unlabelled ones.

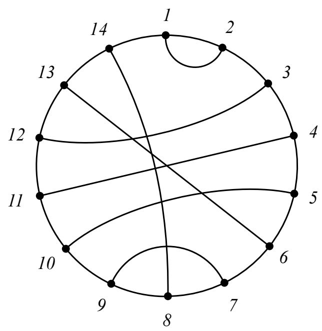


Figure 1: A chord diagram

In different combinatorial problems some special classes of chord diagrams appear [1, 8]. In this paper we study two of them, which we will call loopless diagrams and simple diagrams. A chord is said to be a loop if it connects two neighboring points (chord $\{1, 2\}$ on figure 1). Two chords are called parallel if they connect two pairs of neighboring points and don't intersect. For example, chords $\{3, 12\}$ and $\{4, 11\}$ on figure 1 are parallel, as well as chords $\{4, 11\}$ and $\{5, 10\}$, but chords $\{3, 12\}$ and $\{5, 10\}$ are not.

A loopless chord diagram is a diagram without loops. By a simple diagram we mean a diagram that has neither loops nor pairs of parallel chords. Both types of diagrams appear quite often in problems related to generating knots and links [2, 5]. Besides that, such diagrams arise in many other combinatorial problems [5]. In particular, loopless diagrams correspond to Hamiltonian cycles in n-dimensional octahedrons. Simple diagrams correspond to so-called shapes, one-face maps without vertices of degree 1 or 2 [6]. They are important to the analysis of arbitrary one-face maps, as they describe their basic structure in the following sense: any one-face map can be obtained from a unique shape by subdividing its edges and attaching trees to it [6].

The sequence of the numbers of labelled loopless chord diagrams (`http://oeis.org/A003436`) appeared in [7] as a result of enumerating labelled Hamiltonian cycles in n-dimensional octahedrons. A recurrence relation for these numbers was obtained in [4] using a bijective approach. For the unlabelled case the corresponding numbers (`http://oeis.org/A003437`) were found with the help of a computer in [7] for $n \leqslant 8$, but no analytic expression was given. Despite the variety of applications, the analytic expressions enumerating simple chord diagrams did not exist for neither labelled nor unlabelled case.

In this paper we obtain explicit generating functions for loopless and simple labelled chord diagrams. We also derive a multivariate generating function that classifies chord diagrams according to the numbers of loops and pairs of parallel chords. As an intermediate result, we provide generating functions for the corresponding classes of linear diagrams, which are not circles, but segments with points identified pairwise.

For unlabelled objects we derive systems of recurrences that can be used to efficiently compute the numbers of loopless and simple chord diagrams. In both cases we give the answers for two kinds of symmetries: only rotations, as well as rotations and reflections. In particular, we obtain the recurrence enumerating unlabelled Hamiltonian cycles in unlabelled octahedrons.

2 Labelled loopless chord diagrams

The number of all chord diagrams having n chords is easy to calculate. Indeed, to obtain a chord diagram we can connect the point 1 to any of $2n - 1$ other points, the next free point with $2n - 3$ remaining ones and so on. Thus the number of chord diagrams is equal to $(2n-1)!!$. The exponential generating function $b(t)$ for these numbers has the following form:

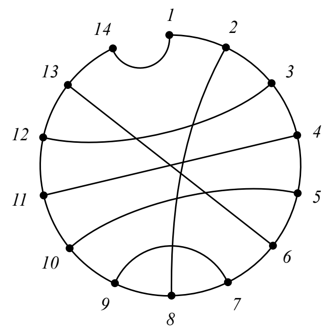
$$b(t) = \frac{1}{\sqrt{1-2t}} = \sum_{n=0}^{\infty} (2n-1)!! \frac{t^n}{n!}.$$

For loopless and simple diagrams it will be convenient to study so-called linear diagrams together with chord ones. Imagine we cut the circle of some chord diagram between points 1 and $2n$. As a result we obtain a linear diagram, which is different from the original chord diagram: points 1 and $2n$ are no longer considered to be neighboring (see Figure 2).

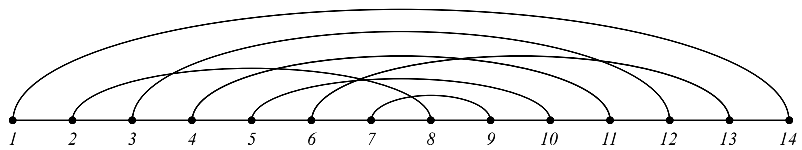
Lemma 2.1. *For loopless diagrams with n chords the numbers of linear (a_n) and chord (b_n) diagrams are related as*

$$b_n = a_n - a_{n-1}, \qquad n \geqslant 2; \qquad b_1 = 0. \tag{1}$$

Proof. Indeed, if we glue a loopless linear diagram with $n \geqslant 2$ chords into a chord diagram, that is, start considering points 1 and $2n$ as neighbors, we get a diagram with a loop if and only if these points were initially connected with a chord. The number of linear diagrams with this chord is obviously equal to the number a_{n-1} of linear loopless diagrams with $n-1$ chords, as the operation of removing the chord $(1, 2n)$ together with its endpoints establishes a bijection between the corresponding sets. □



(a) Cutting a chord diagram



(b) The same diagram drawn on a line

Figure 2: A linear diagram

So we reduced the problem of enumerating loopless chord diagrams to the enumeration of loopless linear ones. Instead of enumerating them directly, it will be convenient to derive a recurrence for the numbers $a_{n,k}$ of linear diagrams with n chords, k of which are loops.

Lemma 2.2. *The numbers $a_{n,k}$ satisfy the recurrence relation*

$$a_{n+1,k} = a_{n,k-1} + (2n-k)a_{n,k} + (k+1)a_{n,k+1}, \tag{2}$$

$$a_{n,k} = 0 \qquad \textit{if } k > n \textit{ or } k < 0, \qquad\qquad a_{0,0} = 1.$$

Proof. Consider a chord $(i, 2n+2)$ in a diagram with $n+1$ chords and k loops. Three different cases are possible. If $i = 2n+1$, removing this chord yields a linear diagram with $k-1$ loops (Figure 3(a)). This gives us the summand $a_{n,k-1}$. Diagrams corresponding to the summand $(k+1)a_{n,k-1}$ are built from diagrams with $k+1$ loops by inserting a point in the middle of one of these loops and connecting it with another new point added to the

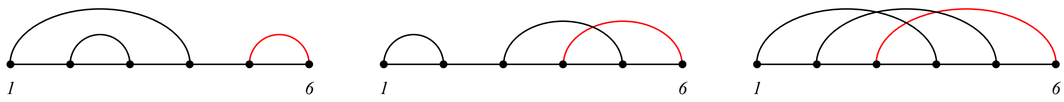


Figure 3: Counting linear diagrams

right of the rightmost point of the initial diagram (Figure 3(b)). The multiplier $(k+1)$ is explained by the fact that the loop could be chosen in $k+1$ ways.

It remains to explain the summand $(2n-k)a_{n,k}$ in (2). The corresponding diagrams are built in such a way: take a diagram with k loops and n chords and add a new point so that it does not break any loops (Figure 3(c)). Then connect it with another new point added to the right of the diagram. The multiplier $2n-k$ appears because the initial diagram had $2n-1$ intervals between consecutive points and one more position to the left of it. However, k of these positions are forbidden for the first new point, because they correspond to loops. □

For solving the recurrence (2) we introduce the polynomials

$$P_n(z) = a_{n,0} + a_{n,1}z + a_{n,2}z^2 + \ldots + a_{n,n}z^n,$$

multiply (2) by z^k and sum over k from 0 to $n+1$:

$$\sum_{k=0}^{n+1} a_{n+1,k}z^k = \sum_{k=0}^{n+1} a_{n,k-1}z^k + 2n\sum_{k=0}^{n+1} a_{n,k}z^k - \sum_{k=0}^{n+1} ka_{n,k}z^k + \sum_{k=0}^{n+1}(k+1)a_{n,k+1}z^k.$$

Taking into account the boundary conditions $a_{n,k} = 0$ if $k > n$ or $k < 0$, we get the following equation for the function $P_n(z)$:

$$P_{n+1}(z) = zP_n(z) + 2nP_n(z) - zP_n'(z) + P_n'(z) = zP_n(z) + 2nP_n(z) + (1-z)P_n'(z). \quad (3)$$

The initial condition $a_{0,0} = 1$ in terms of $P_n(z)$ can be rewritten as $P_0(z) = 1$. To solve the infinite system of equations (3) it is convenient to define the following two-variable generating function

$$w(z,t) = \sum_{n=0}^{+\infty} P_n(z)\frac{t^n}{n!}.$$

Multiplying (3) by $t^n/n!$ and summing over n from 0 to $+\infty$, we get

$$\sum_{n=0}^{+\infty} P_{n+1}(z)\frac{t^n}{n!} = z\sum_{n=0}^{+\infty} P_n(z)\frac{t^n}{n!} + 2\sum_{n=1}^{+\infty} P_n(z)\frac{t^n}{(n-1)!} + (1-z)\sum_{n=0}^{+\infty} P_n'(z)\frac{t^n}{n!}, \quad (4)$$

which can be rewritten in terms of $w(z,t)$ as

$$\frac{\partial w}{\partial t} = zw(z,t) + 2t\frac{\partial w}{\partial t} + (1-z)\frac{\partial w}{\partial z}.$$

The condition $P_0(z) = 1$ takes the form $w(z, 0) = 1$. Solving this Cauchy problem (see for example [3]) yields

$$w(z,t) = \frac{e^{(-1+\sqrt{1-2t})(1-z)}}{\sqrt{1-2t}}.$$

Now the generating function $\varphi(t)$ for loopless linear diagrams is easily obtained by setting $z = 0$ in $w(z,t)$:

$$\varphi(t) = w(0,t) = \frac{e^{-1+\sqrt{1-2t}}}{\sqrt{1-2t}} = 1 + 0 \cdot t + 1 \cdot \frac{t^2}{2!} + 5 \cdot \frac{t^3}{3!} + 36 \cdot \frac{t^4}{4!} + 329 \cdot \frac{t^5}{5!} + \dots \quad (5)$$

This function allows us to derive some useful recurrences for the numbers a_n. As an example, we express the derivative

$$\varphi'(t) = \frac{e^{-1+\sqrt{1-2t}}}{(1-2t)^{3/2}} \left[1 - \sqrt{1-2t}\right]$$

through the function $\varphi(t)$:

$$\varphi'(t) = \varphi(t) \left[\frac{1-\sqrt{1-2t}}{1-2t}\right] \qquad \Longleftrightarrow \qquad (1-2t)\varphi'(t) = \varphi(t)\left[1 - \sqrt{1-2t}\right].$$

This yields the following recurrence:

$$a_{n+1} = 2na_n + \sum_{k=1}^{n} \binom{n}{k} (2k-3)!!a_{n-k}.$$

Using the second derivative

$$\varphi''(t) = \frac{e^{-1+\sqrt{1-2t}}}{\sqrt{1-2t}} \left[\frac{1-2t+3[1-\sqrt{1-2t}]}{(1-2t)^2}\right] = \frac{\varphi(t) + 3\varphi'(t)}{1-2t},$$

we can get from the equation

$$\varphi''(t)[1-2t] = \varphi(t) + 3\varphi'(t)$$

the following second-order recurrence:

$$a_{n+2} = (2n+3)a_{n+1} + a_n \qquad \Longleftrightarrow \qquad a_{n+1} = (2n+1)a_n + a_{n-1}; \qquad a_0 = 1, \quad a_1 = 0. \quad (6)$$

According to [4], it was first guessed by Jean Betramas of LABRI, Bordeaux, from the numerical computations and then combinatorially proved by Michiel Hazewinkel and V. V. Kalashnikov [4].

So we have proved the following result.

Theorem 2.1. *The numbers a_n of loopless linear diagrams with $n \geqslant 2$ chords are described by the generating function*

$$\varphi(t) = \sum_{n=0}^{\infty} a_n t^n = \frac{e^{-1+\sqrt{1-2t}}}{\sqrt{1-2t}}$$

and satisfy the recurrence relation

$$a_{n+1} = (2n+1)a_n + a_{n-1}; \qquad a_0 = 1, \quad a_1 = 0.$$

Now we consider the equality (1) which connects the numbers b_n and a_n. Let $\psi(t) = = \sum_{n=1}^{+\infty} b_n t^n/n!$ be the exponential generating function for the numbers b_n. Multiplying (1) by $t^n/n!$ and summing the result over n, we get

$$\sum_{n=2}^{\infty} b_n \frac{t^n}{n!} = \sum_{n=2}^{\infty} a_n \frac{t^n}{n!} - \sum_{n=2}^{\infty} a_{n-1} \frac{t^n}{n!} \qquad \Longleftrightarrow \qquad \psi(t) = \varphi(t) - a_0 - a_1 t - \chi(t), \tag{7}$$

where $\chi(t)$ is the generating function for the numbers a_{n-1}. This function can be expressed through the integral of $\varphi(t)$:

$$\chi(t) = \sum_{n=2}^{\infty} a_{n-1} \frac{t^n}{n!} = \sum_{n=1}^{\infty} a_n \frac{t^{n+1}}{(n+1)!} = \int \varphi(t)\, dt - a_0 t = 1 - t - e^{-1+\sqrt{1-2t}}.$$

Consequently we have proved the following result.

Theorem 2.2. *The generating function for the number b_n of labelled loopless chord diagrams is equal to*

$$\psi(t) = \sum_{n=0}^{\infty} b_n t^n = \varphi(t) - 2 + t + e^{-1+\sqrt{1-2t}} = e^{-1+\sqrt{1-2t}} \left(1 + \frac{1}{\sqrt{1-2t}}\right) - 2 + t =$$

$$= 0 \cdot t + 1 \cdot \frac{t^2}{2!} + 4 \cdot \frac{t^3}{3!} + 31 \cdot \frac{t^4}{4!} + 293 \cdot \frac{t^5}{5!} + 3326 \cdot \frac{t^6}{6!} + \dots$$

(sequence A003436 *in oeis.org).*

3 Unlabelled loopless chord diagrams

To count the numbers $\widetilde{b}_n$ of unlabelled loopless chord diagrams with n chords we will use the Burnside's lemma

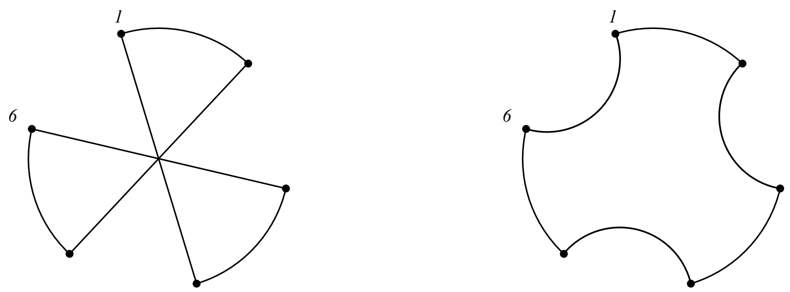
$$\widetilde{b}_n = \frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|. \tag{8}$$

Here $|\operatorname{Fix}(g)|$ is the number of labelled diagrams fixed by the action of the element g of some group G that defines the isomorphism relation between diagrams. In our case G

will be either the cyclic group C_{2n} of diagram's rotations or the dihedral group D_{2n} of rotations and reflections.

We start with the simpler case of the cyclic group C_{2n}. Consider the action of the group C_{2n} on the set of loopless chord diagrams with $2n$ points and n chords. Let d be a divisor of $2n$, $\varphi(d)$ be the Euler function of it. There are $\varphi(d)$ elements of order d in C_{2n}. Any such element fixes the same number $f(2n, d)$ of diagrams. These diagrams will be called d-symmetric. So, (8) can be rewritten as

$$\widetilde{b}_n = \frac{1}{2n} \sum_{d \,|\, 2n} \varphi(d)\, f(2n, d). \tag{9}$$



(a) (b)

Figure 4: 3-symmetric linear diagrams

Instead of enumerating d-symmetric chord diagrams directly, we start with counting so-called d-symmetric linear diagrams (Figure 4(a)). Any such a diagram with $2n$ points is obtained by cutting the circle of a d-symmetric chord diagram into d sectors, each of them having $m := 2n/d$ points, between points m and $m+1$, $2m$ and $2m+1$, …, $2n$ and 1. By cutting between points i and $i+1$ we mean again that these points are no longer considered to be neighbors. Note that 1-symmetric linear diagrams are just linear diagrams studied in the previous section. If we cut a d-symmetric loopless chord diagram, we get a loopless d-symmetric linear diagram. The converse is not true: if points m and $m+1$, $2m$ and $2m+1$, …, $2n$ and 1 are connected by chords in a loopless linear d-symmetric diagram, then gluing this diagram back into a chord diagram results in d loops in it (Figure 4(b)).

Denote by $a_m^{(d)}$ the number of d-symmetric linear diagrams having $m \cdot d$ points. Our next goal is to derive a recurrence relation for these numbers. First consider the case of odd d.

Lemma 3.1. *The numbers $a_m^{(d)}$ in the case of odd d can be calculated with the help of the following recurrence relation:*

$$a_m^{(d)} = d(m-1)a_{m-2}^{(d)} + a_{m-4}^{(d)}. \tag{10}$$

Proof. Remove the chord $(1, i)$ together with its endpoints as well all the other chords on its orbit (under a rotation with a period d) in a d-symmetric linear diagram. In the

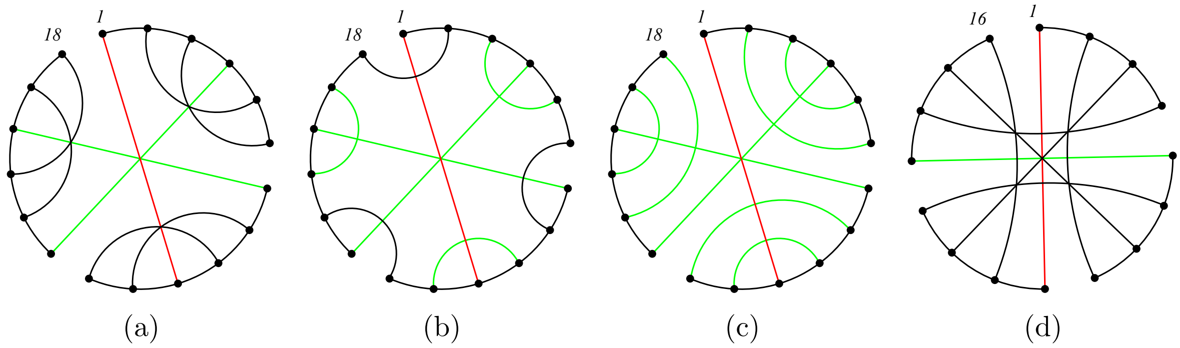


Figure 5: d-symmetric loopless linear diagrams

simple case the result of this operation is a loopless d-symmetric linear diagram with $(m - 2) \cdot d$ points. This is illustrated on figure 5(a). Hereafter, the red chord is the one that we remove first, and the green ones are those that we are obliged to remove subsequently to preserve some diagram's property like symmetry or looplessness.

In a more complex case we get d loops in the resulting diagram (Figure 5(b)). If that happens, we remove these loops too. However we can get d new loops again (Figure 5(c)) and so on. As a result we obtain the following recurrence for the numbers $a_m^{(d)}$:

$$a_m^{(d)} = [d(m-1)-1] \cdot a_{m-2}^{(d)} + \sum_{i=1}^{m/2-1} d \cdot (m-1-2i) \cdot a_{m-2-2i}^{(d)}. \tag{11}$$

The multiplier $d(m-1)-1$ in (11) is explained by the fact that the second endpoint i of the chord $(1, i)$ can be any point in $\{1, \ldots, 2n\}$ except for the points $1, 2, m, 2m, 3m, \ldots, (d - 1)m$. The multipliers $d \cdot [m-1-2i]$ can be explained by similar arguments.

Rewriting the equality (11) for $a_{m-2}^{(d)}$ and subtracting it from (11) we obtain (10). Observe that by setting $d = 1$ we prove (6) combinatorially in slightly different terms. $\square$

Note that in the case of odd d we always delete a multiplier of d chords at a time. In the case of even d it could happen that we should delete only $d/2$ chords. This corresponds to the case of a chord $(1, n+1)$ connecting the opposite points of the diagram (Figure 5(d)). As a consequence, one more summand $a_{m-1}^{(d)}$ is added to the recurrence relation:

$$a_m^{(d)} = a_{m-1}^{(d)} + [d(m-1)-1] \cdot a_{m-2}^{(d)} + \sum_{i=1}^{m/2-1} d \cdot (m-1-2i) \cdot a_{m-2-2i}^{(d)}. \tag{12}$$

Rewriting (12) for $a_{m-2}^{(d)}$ and subtracting from (12) as before, we get the following result.

Lemma 3.2. *The numbers $a_m^{(d)}$ of d-symmetric linear diagrams having $m \cdot d$ points in the case of even d are calculated using the following recurrence relation:*

$$a_m^{(d)} = a_{m-1}^{(d)} + d(m-1)a_{m-2}^{(d)} - a_{m-3}^{(d)} + a_{m-4}^{(d)}. \tag{13}$$

The initial conditions for (10) and (13) are the following:

$$a_m^{(d)} = 0, \quad m < 0; \qquad a_0^{(d)} = 1;$$
$$a_1^{(2k)} = 1; \qquad a_1^{(2k+1)} = 0; \qquad a_2^{(2k)} = d; \qquad a_2^{(2k+1)} = d - 1.$$

Lemma 3.3. *The numbers $f(m \cdot d, d)$ of d-symmetric loopless chord diagrams with $m \cdot d > 2$ points are equal to*

$$f(m \cdot d, d) = a_m^{(d)} - a_{m-2}^{(d)}.$$

Proof. Indeed, any loopless d-symmetric linear diagram with $m \cdot d > 2$ points corresponds to a loopless d-symmetric chord diagram if and only if it has no chord $(m, m+1)$. The number of diagrams having such a chord is clearly equal to $a_{m-2}^{(d)}$ (Figure 6). If $m \cdot d = 2$ the number $b_m^{(d)}$ is equal to zero. □

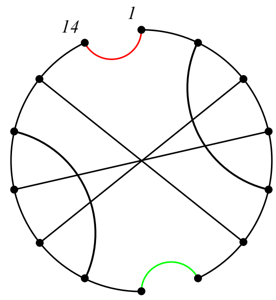


Figure 6: Special case of a d-symmetric loopless linear diagram

Substituting the last expression into (9) we obtain the following result.

Theorem 3.1. *The numbers $\widetilde{b}_n$ of loopless chord diagrams with $n \geqslant 2$ chords not isomorphic under rotations are calculated by the formula*

$$\widetilde{b}_n = \frac{1}{2n} \sum_{d \,|\, 2n} \varphi(d) \left[a_{2n/d}^{(d)} - a_{2n/d-2}^{(d)} \right], \tag{14}$$

where the numbers $a_m^{(d)}$ are defined by the recurrence relations (10) for odd d and (13) for even d. The sequence $\widetilde{b}_n$ starts with the terms $0, 1, 2, 7, 36, 300, 3218, \ldots$ (see Table 1).

Next we enumerate non-isomorphic chord diagrams under the action of the dihedral group D_{2n}. The Burnside's lemma can be rewritten for this case as

$$\widetilde{c}_n = \frac{1}{4n} \left[\sum_{d \,|\, 2n} \varphi(d)\, f(2n, d) + n \cdot K(n) + n \cdot H(n) \right], \tag{15}$$

where $K(n)$ denotes the number of chord diagrams with n chords symmetric under reflection about the axis passing through two opposite points of the diagram (Figure 7(a)), $H(n)$ is the number of diagrams that are symmetric under the reflection about the axis passing through the midpoints of two arcs connecting neighboring points (Figure 7(b)).

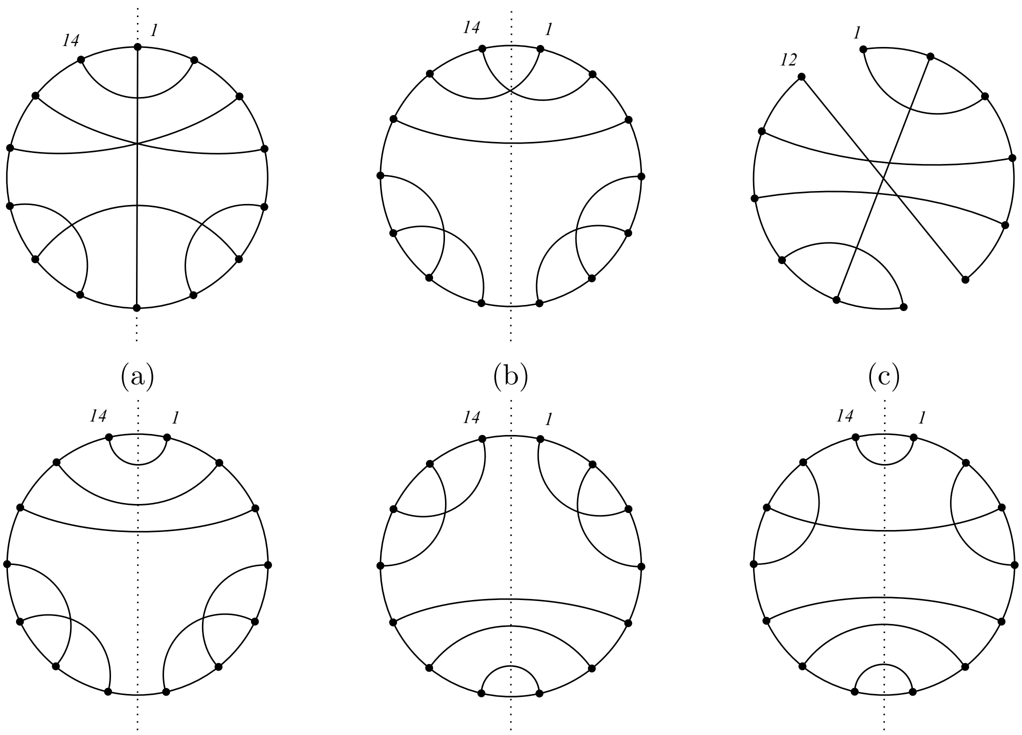


Figure 7: Diagrams with reflectional symmetry

Lemma 3.4. *The numbers $K(n)$ are equal to $a^{(2)}_{n-1}$.*

Proof. Indeed, any diagram of the corresponding kind can be obtained by taking a 2-symmetric loopless linear diagram, flipping its right half, and adding one more chord lying of the axis of symmetry (Figure 7(a) can be obtained from figure 7(c) in such a way). □

Lemma 3.5. *For the numbers $H(n)$ the following equality holds:*

$$H(n) = a^{(2)}_n - 2a^{(2)}_{n-1} + a^{(2)}_{n-2}, \qquad n \geqslant 2; \qquad\qquad H(2) = 0.$$

Proof. Take any 2-symmetric loopless linear diagram with n chords and flip its right half as before. Gluing it into a chord diagram results in not more than 2 loops. The number of loopless diagrams can be obtained with the help of inclusion-exclusion principle: take the number $a^{(2)}_n$ of all diagrams, subtract twice the number $a^{(2)}_{n-1}$ of diagrams with at least 1 loop (Figure 7(d) and (e)) and add the number $a^{(2)}_{n-2}$ of diagrams with exactly 2 loops (Figure 7(f)). □

Substituting the obtained expressions into the formula (15) we prove the following result.

Theorem 3.2. *The numbers $\widetilde{c}_n$ of non-isomorphic chord diagrams with n chords under the action of the dihedral group D_{2n} are calculated by the formula*

$$\widetilde{c}_n = \frac{\widetilde{b}_n}{2} + \frac{a^{(2)}_n - a^{(2)}_{n-1} + a^{(2)}_{n-2}}{4}, \qquad n \geqslant 2; \qquad\qquad \widetilde{c}_1 = 0, \quad \widetilde{c}_2 = 1,$$

where the numbers $\widetilde{b}_n$ *are defined by (14) and the numbers* $a_n^{(2)}$ *are calculated by the formula (13).*

The first terms of the corresponding sequence $0, 1, 2, 7, 29, 1788, \ldots$ (see Table 1) were first obtained by D. Singmaster in [7] as a result of numerical computation of non-isomorphic Hamiltonian cycles in an n-dimensional octahedron (the sequence $A003437$ in oeis.org), that is, in n-partite graphs $K_{2,2,\ldots,2}$ having $2n$ vertices. We show that there indeed exists a bijection between Hamiltonian cycles in octahedrons and loopless chord diagrams.

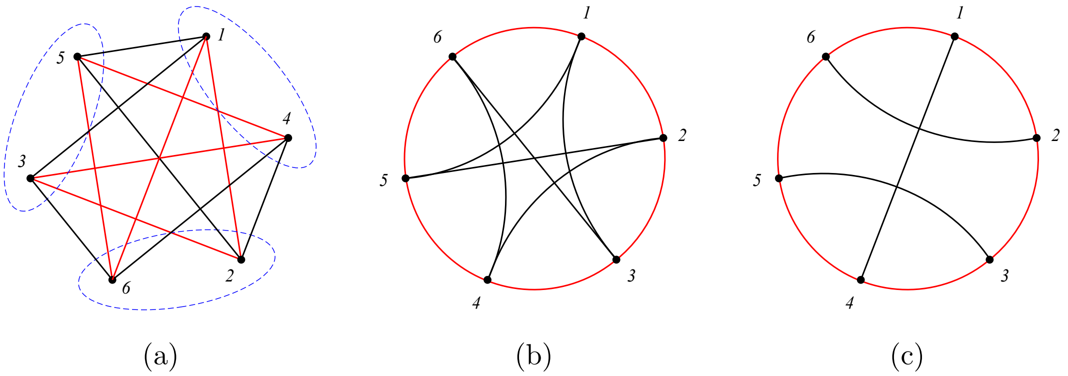


Figure 8: Correspondence between Hamiltonian cycles in octahedrons and chord diagrams

Take an n-dimensional octahedron with a distinguished Hamiltonian cycle (Figure 8(a)) and draw it in such a way that this cycle forms a circle on a plane (Figure 8(b)). Then remove all of its edges that don't belong to the Hamiltonian cycle and add chords between those vertices that weren't connected by an edge before (Figure 8(c)). The resulting object is a chord diagram which is necessarily loopless: traversing a Hamiltonian cycle in $K_{2,2,\ldots,2}$ we can't visit two vertices of the same part one after another. Clearly, this transformation is invertible.

Labels on figure 8 are added just to visualize the correspondence between vertices. Any unlabelled octahedron with a distinguished unlabelled and undirected Hamiltonian cycle corresponds to an unlabelled loopless chord diagram considered with respect to both rotational and reflectional symmetries in exactly the same way.

4 Labelled simple chord diagrams

Let $a_{n,k,l}$ be the number of linear diagrams with $n+1$ chords in total, k loops, and l pairs of parallel chords.

Theorem 4.1. *The numbers* $a_{n,k,l}$ *satisfy the following recurrence relation:*

$$a_{n+1,k,l} = a_{n,k-1,l} + (2n+1-k-2l)a_{n,k,l} + (k+1)a_{n,k+1,l} + 2(l+1)a_{n,k,l+1} + a_{n,k,l-1}. \quad (16)$$

Proof. Take some simple linear diagram with $n+2$ chords and remove a chord ending in the point $2n+4$. Five cases are possible. The first summand corresponds to the case of removing a single loop $(2n+3, 2n+4)$ (figure 9(a)). In the second case the numbers of loops and parallel chords don't change (figure 9(b), summand $(2n+1-k-2l)a_{n,k,l}$). The third and the fourth cases are the situations when a new loop (figure 9(c)) or a new pair of parallel chords (figure 9(d)), correspondingly, are introduced. Finally, in the fifth case the chord being removed was parallel to some other chord (figure 9(e)). □

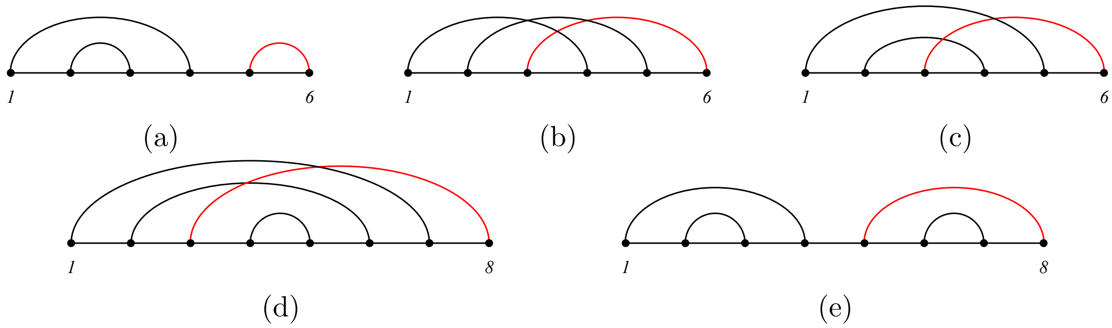


Figure 9: Counting linear diagrams with counting parallel edges

The recurrence (16) has to be solved with the boundary conditions $a_{n,k,l} = 0$ when $l > n$, $k > n+1$, $k < 0$, or $l < 0$, and with the initial conditions $a_{0,k,l} = 0$ for all k and l, except for $a_{0,1,0} = 1$, that corresponds to a diagram with a single chord that is a loop.

The recurrence (16) can be rewritten as

$$Q_{n+1,k}(x) = Q_{n,k-1}(x) + (2n+1-k)Q_{n,k}(x) - 2x\frac{dQ_{n,k}(x)}{dx} +$$

$$(k+1)Q_{n,k+1}(x) + xQ_{n,k}(x) + 2\frac{dQ_{n,k}(x)}{dx}$$

for the polynomials

$$Q_{n,k}(x) = a_{n,k,0} + a_{n,k,1}x + a_{n,k,2}x^2 + \ldots + a_{n,k,n}x^n.$$

Introducing the polynomials $P_n(z,x) = \sum_{k=0}^{n} Q_{n,k}(x)z^k$ we get the following equality:

$$P_{n+1}(z,x) = (z+x+2n-1)P_n(z,x) - (z-1)\frac{\partial P_n}{\partial z} - 2(x-1)\frac{\partial P_n}{\partial x}.$$

Finally, multiplying this equalitiy by $t^n/n!$ and summing over n from 0 to $+\infty$ yields the following partial differential equation for the generating function

$$w(t,z,x) = \sum_{n=0}^{+\infty} P_n(z,x)\frac{t^n}{n!}:$$

$$\frac{\partial w}{\partial t} = (z+x+1)w(z,t) + 2t\frac{\partial w}{\partial t} - (z-1)\frac{\partial w}{\partial z} - 2(x-1)\frac{\partial w}{\partial x}, \qquad w(0,z,x) = z.$$

It can be shown (see [3]) that it has the following analytic solution:

$$w(t,z,x) = \frac{(z-1)\sqrt{1-2t}+1}{(1-2t)^{3/2}} \exp\left((z-1)(1-\sqrt{1-2t}) - t(1-x)\right).$$

This function classifies linear diagrams by the number of chords, loops and parallel chords. In particular, setting $z = 1$ results in the generating function

$$\widetilde{w}(t,x) = w(t,1,x) = \frac{1}{(1-2t)^{3/2}} \exp\left(t(x-1)\right) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} \sum_{l=0}^{n} \widehat{a}_{n,l}\, x^l,$$

that classifies linear diagrams by the numbers of chords and pairs of parallel chords only. Its coefficients $\widehat{a}_{n,l}$ satisfy the following recurrence relation:

$$\widehat{a}_{n+1,l} = \widehat{a}_{n,l-1} + (2n+1-2l)\widehat{a}_{n,l} + 2(l+1)\widehat{a}_{n,l+1}.$$

Moreover, substituting $z = x = 0$ into $w(t,z,x)$, we obtain the generating function $W(t)$ for the numbers $a_{n,0,0} \equiv \bar{a}_n$.

Theorem 4.2. *The generating function $W(t)$ for the numbers $a_{n,0,0} \equiv \bar{a}_n$ enumerating simple linear diagrams with $n+1$ chords is equal to*

$$W(t) = w(t,0,0) = \frac{1-\sqrt{1-2t}}{(1-2t)^{3/2}} \exp\left(-1 - t + \sqrt{1-2t}\right) =$$

$$1 \cdot t + 3 \cdot \frac{t^2}{2!} + 24 \cdot \frac{t^3}{3!} + 211 \cdot \frac{t^4}{4!} + 2325 \cdot \frac{t^5}{5!} + \ldots$$

Consequence 4.1. *The numbers $\bar{a}_n$ satisfy the following recurrence relation:*

$$\bar{a}_n = (2n-1) \cdot \bar{a}_{n-1} + (4n-3) \cdot \bar{a}_{n-2} + (2n-4) \cdot \bar{a}_{n-3}, \qquad n \geqslant 2;$$

$$\bar{a}_n = 0 \qquad \textit{for} \quad n \leqslant 0; \qquad\qquad \bar{a}_1 = 1.$$

Using these results we can enumerate simple chord diagrams. As a result of gluing a simple linear diagram with n chords we get a simple chord diagram with n chords if the original linear diagram had neither a chord $(1, 2n)$ nor a pair of chords $(1, i)$, $(i + 1, 2n)$ for some i. Let $\bar{q}_n$ be the number of diagrams with n chords and no chord $(1, 2n)$.

Lemma 4.2. *The numbers $\bar{q}_n$ satisfy the following recurrence relation:*

$$\bar{q}_n = \bar{a}_{n-1} - \bar{q}_{n-1}.$$

Proof. Consider a simple diagram with a chord $(1, 2n)$. Removing this chord, we obtain a simple diagram with $n-1$ chords and no chord $(1, 2n-2)$. Indeed, the existence of such a chord would mean that the original diagram had two parallel chords, but that is impossible by definition of a simple diagram. Thus we conclude that the number of simple diagrams with the chord $(1, 2n)$ is equal to $\bar{q}_{n-1}$. □

Lemma 4.3. *The numbers of simple chord diagrams $\bar{b}_n$ with n chords satisfy the recurrence*

$$\bar{b}_n = \bar{q}_n - \bar{b}_{n-1}.$$

Proof. Indeed, gluing any linear diagram without a chord $(1, 2n)$ into a chord diagram we get a pair of parallel chords if and only if the original diagram had chords $(1, i)$ and $(i + 1, 2n)$ for some i. Removing the chord $(i + 1, 2n)$ we get a linear diagram without such pairs and without the chord $(1, 2n-2)$. The number of such linear diagrams is equal to $\bar{b}_{n-1}$. □

Theorem 4.3. *The exponential generating function $U(t)$ for labelled simple chord diagrams is equal to*

$$U(t) = \frac{e^{-1-t+\sqrt{1-2t}}}{\sqrt{1-2t}}(1 + \sqrt{1-2t}) - (2 - t)e^{-t} = \tag{17}$$

$$1 \cdot \frac{t^2}{2!} + 1 \cdot \frac{t^3}{3!} + 21 \cdot \frac{t^4}{4!} + 168 \cdot \frac{t^5}{5!} + 1968 \cdot \frac{t^6}{6!} + \dots$$

Proof. In terms of exponential generating functions $W(t)$, $V(t)$, and $U(t)$ for the numbers $\bar{a}_n$, $\bar{q}_n$ and $\bar{b}_n$ we get the system

$$V'(t) = W(t) - V(t),$$
$$U'(t) = V'(t) - U(t).$$

Solving this system proves the formula (17). □

5 Unlabelled simple chord diagrams

The technique of enumerating unlabelled simple diagrams doesn't differ much from the one used for loopless diagrams. However, for these diagrams we have to consider more cases to obtain the recurrence relations. In order not to overload the article, we will still describe all of them, but omit some details about proving the exact forms of the coefficients and initial conditions. It should be straightforward to check them.

Let $\bar{a}_m^{(d)}$ be the number of simple d-symmetric linear diagrams with $m \cdot d$ points.

Lemma 5.1. *In the case of odd d the numbers $\bar{a}_m^{(d)}$ satisfy the following recurrence:*

$$\bar{a}_m^{(d)} = [(m-1) \cdot d - 2] \cdot \bar{a}_{m-2}^{(d)} + (2m - 7) \cdot d \cdot \bar{a}_{m-4}^{(d)} + (m - 6) \cdot d \cdot \bar{a}_{m-6}^{(d)} \quad \textit{for } m > 2; \tag{18}$$

$$\bar{a}_m^{(d)} = 0 \quad \textit{for } m < 0 \textit{ or } m = 1; \quad \bar{a}_0^{(d)} = 1; \quad \bar{a}_2^{(d)} = d - 1.$$

Proof. Consider four cases that are possible for the chord $\{1, i\}$:

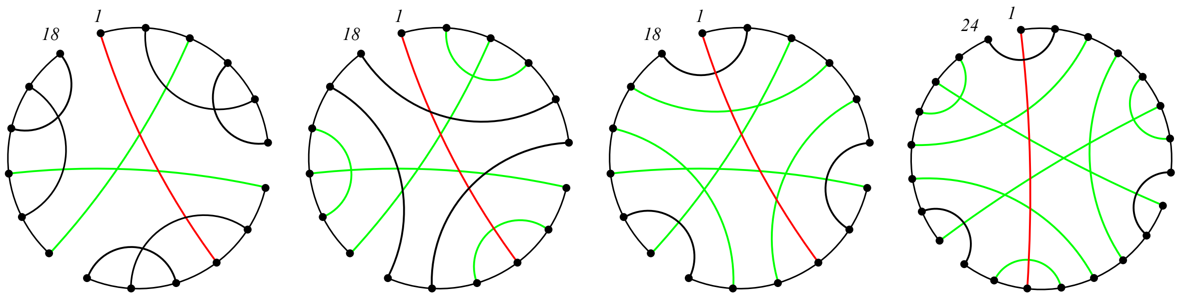
- After removing this chord the diagram is still simple (figure 10(a)). The number of such diagrams is $[(m-1) \cdot d - 2] \cdot \bar{a}_{m-2}^{(d)}$.

– Removing this chord creates a loop, but after removing this loop the diagram becomes simple (figure 10(b)).

– Removing the chord $\{1, i\}$ creates a pair of parallel chords. If that happens, we remove one (arbitrary) chord from this pair too (figure 10(c)).

– We remove the chord $\{1, i\}$, a loop appears, but removing this loop results in a pair of parallel chords (figure 10(d)). In this case we remove one chord from that pair too.

Counting the diagrams in each case and simplifying we get (18). □



(a) (b) (c) (d)

Figure 10: Counting simple d-symmetric linear diagrams, d odd

In the case of even d the situation is a bit more complicated. Namely, we have to introduce a new parameter k. Let $\bar{a}^{(d)}_{m,k}$ be the number of d-symmetric linear diagrams with $m \cdot d$ chords, $k \cdot d/2$ of which connect the opposite points of the diagram.

Lemma 5.2. *The numbers $\bar{a}^{(d)}_{m,k}$ satisfy the recurrence*

$$\bar{a}^{(d)}_{m,k} = \bar{a}^{(d)}_{m-1,k-1} + [(m-1) \cdot d - 3 + \delta_{m,2}] \cdot \bar{a}^{(d)}_{m-2,k} + (k+1) \cdot d \cdot \bar{a}^{(d)}_{m-3,k+1} +$$

$$+ (2m + k - 7) \cdot \bar{a}^{(d)}_{m-4,k} + (k+1) \cdot d \cdot \bar{a}^{(d)}_{m-5,k+1} + (m + k - 6) \cdot d \cdot \bar{a}^{(d)}_{m-6,k},$$

$$\bar{a}^{(d)}_{m,k} = 0 \quad \text{for } m < 0 \text{ or } k < 0 \text{ or } k > m, \qquad \bar{a}^{(d)}_{0,0} = 1.$$

Here $\delta_{m,2}$ is the Kronecker delta that equals 1 if $m = 2$ and 0 otherwise.

Proof. Three types of diagrams that are new compared to the case of odd d are shown on figure 11. The summand $\bar{a}^{(d)}_{m-1,k-1}$ enumerates diagrams with a chord $\{1, i\}$ that is a diameter (figure 11(a)). The summand $(k+1) \cdot d \cdot \bar{a}^{(d)}_{m-3,k+1}$ corresponds to the case when removing the chord $\{1, i\}$ creates $d/2$ pairs of parallel chords, and we remove one chord from each pair, so that the other one becomes a diameter (figure 11(b)). The summand $(k+1) \cdot d \cdot \bar{a}^{(d)}_{m-5,k+1}$ describes a similar case with an intermediate step of removing d loops (figure 11(c)). □

For both odd and even cases the numbers $\bar{a}^{(d)}_m$ are obviously equal to $\sum_{k=0}^{m} \bar{a}^{(d)}_{m,k}$.

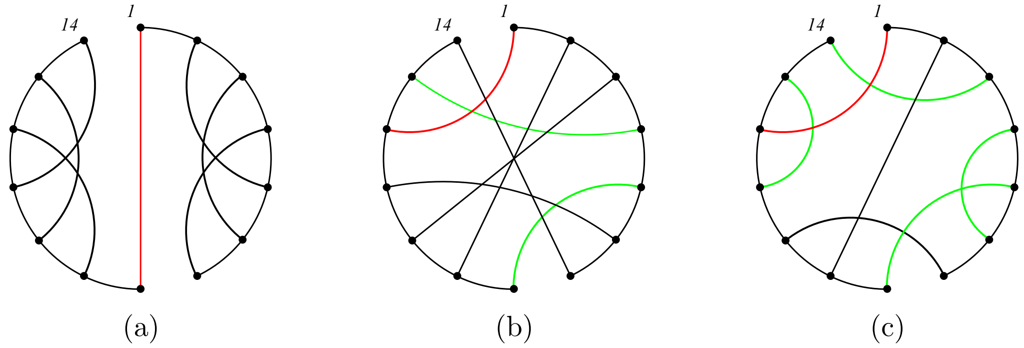


Figure 11: Counting simple d-symmetric linear diagrams, d even

In order to express the number of simple d-symmetric chord diagrams through the numbers of simple d-symmetric linear diagrams we need to introduce some special classes of chord diagrams which can be obtained by gluing linear ones.

Lemma 5.3. *The numbers $\bar{q}_m^{(d)}$ of loopless d-symmetric chord diagrams with $m \cdot d$ points that can be obtained by gluing simple d-symmetric linear diagrams satisfy the following recurrence relation:*

$$\bar{q}_m^{(d)} = \bar{a}_m^{(d)} - \bar{q}_{m-2}^{(d)} \qquad \textit{for } m \geqslant 2; \qquad \bar{q}_0^{(d)} = \bar{q}_1^{(d)} = \begin{cases} 0, & d \leqslant 2, \\ 1, & d > 2; \end{cases} \qquad \bar{q}_m^{(d)} = 0,\ \ m < 0.$$

Proof. Indeed, a loop can appear after gluing only if the vertices 1 and $m \cdot d$ were joined by a chord (figure 12(a)). If a diagram has such a chord, removing it and all the chords on its orbit results in a diagram having $(m-2) \cdot d$ points that cannot have chords of the same type anymore, as that would mean that the original diagram had parallel chords. By induction, such diagrams are enumerated by $\bar{q}_{m-2}^{(d)}$. □

Remark 5.4. We will also need simple diagrams with $m \cdot d$ points, $d = 2k$, $k \geqslant 2$, that don't have a chord $\{1, m \cdot d/2\}$. Such diagrams can also be enumerated by $\bar{q}_m^{(d)}$. This statement can be proved in the same way as the Lemma 5.3 using the diagrams that have such chord (figure 12(b)).

Finally, we also need 2-symmetric diagrams that have neither a chord $\{1, m\}$ nor a chord $\{1, 2m\}$. Let $\bar{p}_m$ be the numbers of such diagrams.

Lemma 5.5. *The numbers $\bar{p}_m$ are given by*

$$\bar{p}_m = \bar{q}_m^{(2)} - \bar{p}_{m-2} - \bar{q}_{m-4}^{(2)}, \qquad \textit{for } m \geqslant 2; \qquad \bar{p}_0 = 0, \quad \bar{p}_1 = 1.$$

Proof. Indeed, the summand $\bar{q}_m^{(2)}$ enumerates those diagrams that don't have a chord $\{1, 2m\}$. A subset of those diagrams that have a chord $\{1, m\}$ is enumerated by $\bar{p}_{m-2} + \bar{q}_{m-4}^{(2)}$: there are $\bar{p}_{m-2}$ diagrams without a chord $\{2, 2m-1\}$ and $\bar{q}_{m-4}^{(2)}$ diagrams with such a chord (figure 12(c)). □

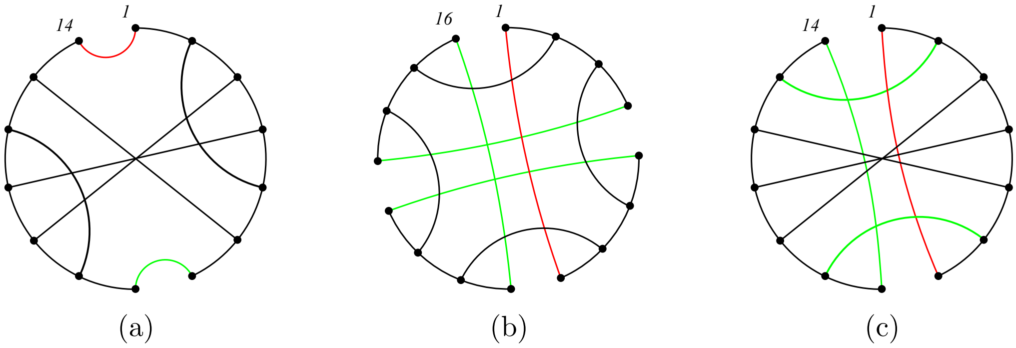


Figure 12: Special kinds of d-symmetric linear diagrams

To calculate the number of unlabelled simple chord diagrams we rewrite the Burnside's lemma (8) as

$$\bar{b}_n = \frac{1}{2n} \sum_{d \mid 2n} \varphi(d)\, \bar{f}(2n, d), \tag{19}$$

where $\bar{f}(2n, d)$ is the number of d-symmetric simple chord diagrams with $2n$ points. Our immediate goal is to express the numbers $\bar{f}(2n, d)$ in terms of sequences obtained above. It turns out that the case $d = 2$ is special.

Lemma 5.6. *The numbers $\bar{f}(2m, 2)$ of 2-symmetric simple chord diagrams with $2m$ points satisfy the following recurrence relation:*

$$\bar{f}(2m, 2) = \bar{p}_m - \bar{f}(2m-4, 2) - \bar{q}^{(2)}_{m-1} + \bar{p}_{m-1} \qquad \textit{for } m \geqslant 2; \tag{20}$$

$$\bar{f}(0, 2) = 1, \qquad\qquad \bar{f}(2, 2) = 0.$$

Proof. We start with the number $\bar{p}_m$ of diagrams that are both loopless and free of chords $\{1, m\}$ and $\{m+1, 2m\}$, and subtract the number of those diagrams that have other kinds of parallel chords. Observe that the diagrams enumerated by $\bar{p}_m$ are obtained from simple linear diagrams, so if they have parallel chords, one pair of them has the form $(\{1, i\}, \{2m, i+1\})$ and the other pair is obtained from the first one by a rotation by 180°.

Two cases are possible: either the chord $\{1, i\}$ is a diameter $(i = m+1)$ or not one. In the first case (figure 13(a)) removing the diameter yields a simple 2-symmetric diagram with $2m-2$ points and a pair of chords $(\{1, m-1\}, \{m, 2m-2\})$. These diagrams are enumerated by $\bar{q}^{(2)}_{m-1} - \bar{p}_{m-1}$. In the second case (figure 13(b)) removing the chord $\{1, i\}$ together with the one symmetric to it yields a linear diagram that can be glued into a simple chord diagram. The corresponding summand is $\bar{f}(2m-4, 2)$. □

Lemma 5.7. *For d-symmetric simple chord diagrams with $d > 2$ we have the following enumeration formulas:*

$$\bar{f}((2i+1)\cdot m, 2i+1) = \bar{q}^{(2i+1)}_m - \bar{f}((2i+1)\cdot(m-2), 2i+1) \quad \textit{for } m \geqslant 1; \tag{21}$$

$$\bar{f}(0, 2i+1) = 0; \qquad\qquad i \geqslant 0;$$

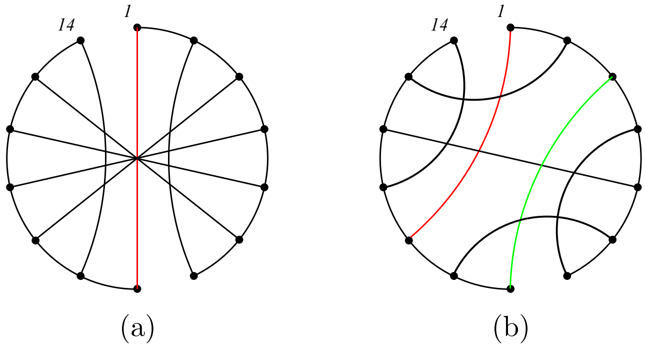


Figure 13: Counting 2-symmetric chord diagrams

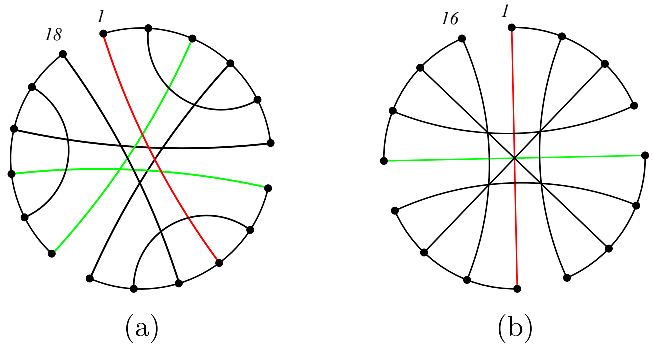


Figure 14: Counting d-symmetric chord diagrams, $d > 2$

$$\bar{f}(2i \cdot m, 2i) = \bar{q}_m^{(2i)} - \bar{f}(2i \cdot (m-2), 2i) - \bar{q}_{m-2}^{(2i)} - \bar{q}_{m-3}^{(2i)} \qquad \textit{for } m \geqslant 1; \qquad (22)$$
$$\bar{f}(0, 2i) = 0; \qquad\qquad i > 1.$$

Proof. We start with proving (21). In addition to the possibility of loops which was already excluded in $\bar{q}_m^{(2i+1)}$, we also exclude those diagrams that contain d pairs of parallel chords after gluing (figure 14(a)). Clearly, such diagrams are enumerated by $\bar{f}(2i \cdot (m - 2), 2i)$: removing a chord $\{1, j\}$ establishes a bijection with the corresponding type of diagrams.

Consider now the equation (22) which describes $2i$-symmetric diagrams. In this case removing a chord $\{1, j\}$ does not necessarily lead to an arbitrary $2i$-symmetric simple chord diagram. Consequently, two additional summands appear in (22). The summand $\bar{q}_{m-2}^{(2i)}$ describes the case that was already shown on figure 12(b). The second summand corresponds to the case of the chord $\{1, i \cdot d\}$ being parallel to two different chords (figure 14(b)). Removing this chord transforms this case into the first one, so the corresponding number of diagrams is $\bar{q}_{m-3}^{(2i)}$. □

Summarizing, we state the following result.

Theorem 5.1. *The numbers $\bar{b}_n$ of simple chord diagrams with n chords not isomorphic under rotations are given by the formula (19) where the numbers $\bar{f}(2n, d)$ are calculated by the formulas (20), (21) and (22). The sequence $\bar{b}_n$ starts with the terms* $0, 1, 1, 4, 21, 176, 1893, \ldots$ *(see Table 2).*

Finally, we will enumerate unlabelled simple chord diagrams for the case when our notion of isomorphism includes reflection symmetries as well. As before, we begin with enumerating some objects analogous to linear diagrams, as it is more convenient to establish a recurrence for them.

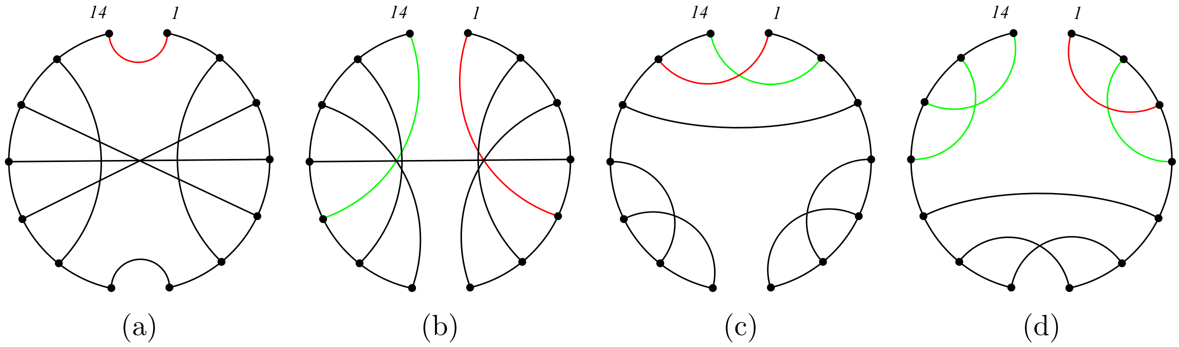


Figure 15: Simple linear diagrams with reflectional symmetry

Let $\bar{r}_{n,k}$ be the number of simple diagrams with the following properties: they have $2n$ points, points 1 and $2n$ as well as n and $n+1$ are not thought to be neighboring, reflection across the line passing through the center of a diagram and the center of the segment connecting points 1 and $2n$ transforms the diagram into itself, the number of chords that are transformed into themselves under such a reflection is equal to k. For example the diagram on figure 15(a) is the one counted by $\bar{r}_{7,3}$. For the number of such diagrams with an additional property that they have the chord $\{1, 2n\}$ we introduce a separate sequence $\bar{s}_{n,k}$.

Lemma 5.8. *The numbers $\bar{s}_{n,k}$ satisfy the following recurrence relation:*

$$\bar{s}_{n,k} = \bar{r}_{n-1,k-1} - \bar{s}_{n-1,k-1} \text{ for } n > 0; \qquad \bar{s}_{0,k} = 0.$$

Proof. Indeed, any corresponding diagram consists of a chord $\{1, 2n\}$ and the remaining part which is a diagram with $2n-2$ points that does not have the same type of chord, thus counted by $\bar{r}_{n-1,k-1} - \bar{s}_{n-1,k-1}$. □

The recurrence for $\bar{r}_{n,k}$ is more complicated.

Lemma 5.9. *For the numbers $\bar{r}_{n,k}$ the following relations hold:*

$$\bar{r}_{n,k} = 0 \quad \text{if} \quad n < 0 \text{ or } k < 0 \text{ or } k > n. \qquad \bar{r}_{0,0} = \bar{r}_{2,0} = 1.$$

$$\bar{r}_{n,k} = \bar{s}_{n,k} + 2(n-2)\bar{r}_{n-2,k} + \bar{s}_{n-2,k} + 2(2n-k-7)\bar{r}_{n-4,k} + 2(k-1)\bar{r}_{n-3,k-1} +$$
$$+ 2(k-1)\bar{r}_{n-5,k-1} + 2(n-k-6)\bar{r}_{n-6,k} \quad \text{otherwise.}$$

Proof. Classify all the corresponding diagrams according to the properties of the chord $\{1, i\}$. In the simplest case $i = 2n$ we have $\bar{s}_{n,k}$. If $i \neq 2n$ there are 5 possible cases.

1) Removing the chord $\{1, i\}$ together with the one symmetric to it yields a simple diagram (figure 15(b)). This case is enumerated by $2(n-2)\bar{r}_{n-2,k} + \bar{s}_{n-2,k}$, because to reconstruct an initial diagram one could take a diagram with $2n-4$ points and

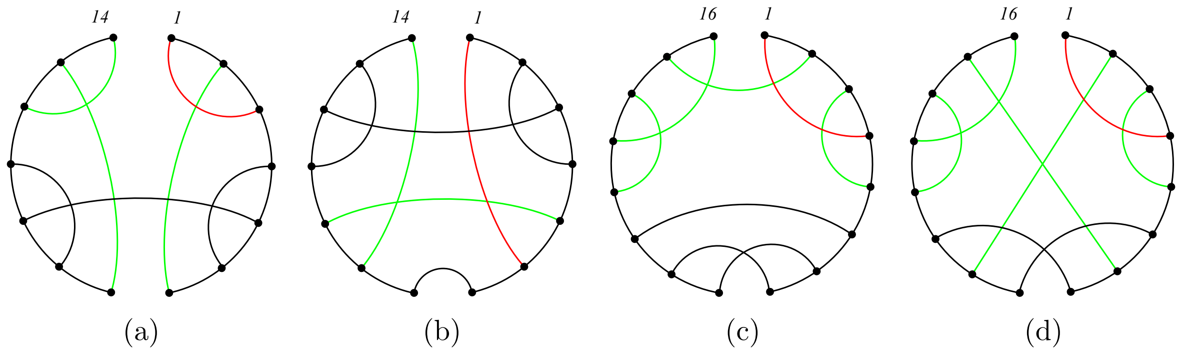


Figure 16: Counting simple diagrams with reflectional symmetry

find a place for the new chord. $2n-4$ options are always available, and one more option becomes available if the chord $\{1, 2n-4\}$ is present, because it cannot be parallel to the newly added chord (result is shown on figure 15(c)).

2) Removing the chord $\{1, i\}$ together with the one symmetric to it yields a diagram with a pair of loops (figure 15(d)). Removing these loops yields a simple diagram. This situation corresponds to the summand $2(n-3)\bar{r}_{n-4,k}$.

3) Removing the chord $\{1, i\}$ together with the one symmetric to it yields a diagram with two pairs of parallel chords. Removing one chord from each pair yields a simple diagram (figure 16(a), summand $2(n-k-4)\bar{r}_{n-4,k}$).

4) Removing the chord $\{1, i\}$ together with the one symmetric to it yields a diagram with one pair of parallel chords. Removing one chord from it yields a simple diagram (figure 16(b), summand $2(k-1)\bar{r}_{n-3,k-1}$).

5) Removing the chord $\{1, i\}$ together with the one symmetric to it yields a diagram with two pairs of loops. There are two subcases. In the first subcase removing these pairs creates one pair of parallel chords. Removing a chord from this pair yields a simple diagram (figure 16(c), summand $2(k-1)\bar{r}_{n-5,k-1}$). In the second subcase two pairs of parallel chords appear. Removing a chord from each pair yields a simple diagram (figure 16(d), summand $2(n-k-6)\bar{r}_{n-6,k}$). $\square$

Denote by $\bar{r}_n$ the sum of $\bar{r}_{n,k}$ over k and by $\bar{s}_n$ the corresponding sum of $\bar{s}_{n,k}$. Chord diagrams that are transformed into themselves by reflections can be of two types. The axis of symmetry can pass either through two opposite points or through the centers of two segments. Let the number of diagrams of the first type with n chords be $\bar{K}(n)$, and of the second type — $\bar{H}(n)$.

Lemma 5.10. *The numbers $\bar{K}(n)$ are equal to*

$$\bar{K}(n) = \bar{r}_{n-1} - \bar{K}(n-2) \qquad \text{for } n > 1; \qquad \bar{K}(0) = \bar{K}(1) = 0.$$

Proof. Indeed, any diagram enumerated by K_n has a chord on its axis of symmetry. Removing this chord yields a diagram that can be almost arbitrary, except that it can't have chords $\{1, n-1\}$ and $\{n, 2n-2\}$ like the one on figure 17(a). But this forbidden type of diagrams is enumerated by $\bar{K}(n-2)$, as these two chords can be viewed as a single chord that lies on the axis of symmetry. □

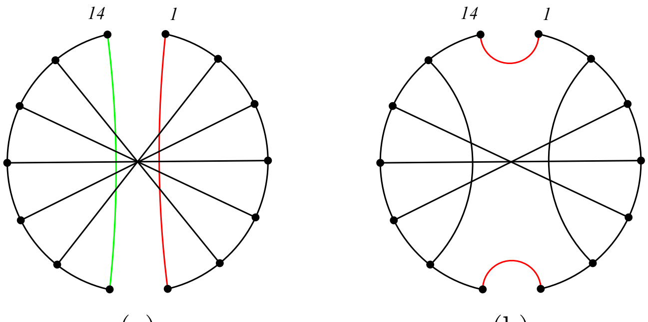


Figure 17: Counting chord diagrams with reflectional symmetry

For counting symmetric simple chord diagrams described by $\bar{H}_n$ we introduce the numbers $\bar{L}_n$ which count symmetric diagrams

1) whose axis of symmetry goes through the segments $(1, 2n)$ and $(n, n+1)$;

2) that have no parallel chords and loops except maybe the pair of parallel chords $(1, n)$ and $(n+1, 2n)$

(see fig.17(a)).

Lemma 5.11. *The numbers $\bar{L}_n$ satisfy the recurrence relation*

$$\bar{L}(n) = \bar{r}_n - 2\bar{s}_n + \bar{L}(n-2) \text{ for } n > 1, \qquad \bar{L}(0) = \bar{L}(1) = 0.$$

Proof. Take all $\bar{r}_n$ diagrams, subtract those with the chord $\{1, 2n\}$, then those with the chord $\{n, n+1\}$, and finally add the number of diagrams that have both of these chords. The last number is $\bar{L}(n-2)$, as deleting these chords (figure 17(b)) yields a diagram that doesn't have chords of this type anymore. □

Lemma 5.12. *The numbers $\bar{H}(n)$ are equal to*

$$\bar{H}(n) = \bar{L}(n) - \bar{K}(n-1) \text{ for } n > 0, \qquad \bar{H}(0) = 0.$$

Proof. Indeed, take all diagrams described by $\bar{L}(n)$ and exclude those that have the pair of parallel chords $(1, n)$ and $(n+1, 2n)$. The number of such diagrams is equal to $\bar{K}(n-1)$. □

Theorem 5.2. *The numbers $\bar{c}_n$ of simple chord diagrams enumerated up to both rotations and reflections are calculated by the formula*

$$\bar{c}_n = \frac{1}{4n}\left[\sum_{d \mid 2n} \varphi(d)\, \bar{f}(2n, d) + n \cdot \bar{K}(n) + n \cdot \bar{H}(n)\right]. \tag{23}$$

The sequence obtained from it starts with the numbers $0, 1, 1, 4, 18, 116, 1060$ *(see Table 2).*

The proof follows from the Burnside lemma (15) rewritten in terms of the sequences that we derived above.

Conclusion

In the first part of this paper we enumerated labelled and unlabelled loopless chord diagrams. Consequently, we obtained the expressions for the numbers of non-isomorphic hamiltonian cycles in unlabelled n-dimensional octahedrons. In the second part we solved a technically more complex problem of enumerating simple diagrams. We obtained a generating function for labelled diagrams classified by the numbers of loops and parallel chords. As a special case, we derived a generating function for simple chord diagrams. For unlabelled simple chord diagrams we gave a solution in the form of several recurrence relations. The following tables list the numbers of different kinds of diagrams described above.

n	Linear	Chord labelled	Under rotations	Under all symmetries
1	0	0	0	0
2	1	1	1	1
3	5	4	2	2
4	36	31	7	7
5	329	293	36	29
6	3655	3326	300	196
7	47844	44189	3218	1788
8	721315	673471	42335	21994
9	12310199	11588884	644808	326115
10	234615096	222304897	11119515	5578431
11	4939227215	4704612119	213865382	107026037
12	113836841041	108897613826	4537496680	2269254616
13	2850860253240	2737023412199	105270612952	52638064494
14	77087063678521	74236203425281	2651295555949	1325663757897
15	2238375706930349	2161288643251828	72042968876506	36021577975918
16	69466733978519340	67228358271588991	2100886276796969	1050443713185782
17	2294640596998068569	2225173863019549229	65446290562491916	32723148860301935
18	80381887628910919255	78087247031912850686	2169090198219290966	1084545122297249077
19	2976424482866702081004	2896042595237791161749	76211647261082309466	38105823782987999742

Table 1: Loopless diagrams by number of chords

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n	Linear	Chord labelled	Under rotations	Under all symmetries
1	0	0	0	0
2	1	1	1	1
3	3	1	1	1
4	24	21	4	4
5	211	168	21	18
6	2325	1968	176	116
7	30198	26094	1893	1060
8	452809	398653	25030	13019
9	7695777	6872377	382272	193425
10	146193678	132050271	6604535	3313522
11	3069668575	2798695656	127222636	63667788
12	70595504859	64866063276	2702798537	1351700744
13	1764755571192	1632224748984	62778105236	31390695708
14	47645601726541	44316286165297	1582725739329	791372281393
15	1381657584006399	1291392786926821	43046433007765	21523271532811
16	42829752879449400	40202651019430461	1256332883208474	628166776833181
17	1413337528735664887	1331640833909877144	39165907107963273	19582955637428422
18	49465522112961344241	46762037794122159492	1298945495674093932	649472761243051940
19	1830184115528550306438	1735328399106396110310	45666536827274985585	22833268501579122332

Table 2: Simple diagrams by number of chords

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