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# The weighted coloring problem for two graph classes characterized by small forbidden induced structures

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ABSTRACT

We show that the weighted coloring problem can be solved for $\{P_5, banner\}$-free graphs and for $\{P_5, dart\}$-free graphs in polynomial time on the sum of vertex weights.


## 1. Introduction

A *proper coloring of a graph G* is a mapping $c : V(G) \longrightarrow \mathbb{N}$ such that $c(u) \neq c(v)$ for any two adjacent vertices $u$ and $v$ of $G$. Elements of the set $\{c(v) | \; v \in V(G)\}$ are said to be *colors*. The *chromatic number of a graph G* denoted by $\chi(G)$ is the minimum number $k$ such that $G$ can be properly colored in $k$ colors. For a given graph $G$ and a number $k$, the *coloring problem* (the COL *problem*, for short) is to decide whether $\chi(G) \leq k$ or not.

For a given graph $G$ and a function $w : V(G) \longrightarrow \mathbb{N}$, a pair $(G, w)$ is called a *weighted graph*. For a weighted graph $(G, w)$, the *weighted coloring problem* (the WCOL *problem*, for short) is to find the smallest number $k$ such that there is a function $c : V(G) \longrightarrow 2^{\{1,2,\ldots,k\}}$, where $|c(v)| = w(v)$ for any $v \in V(G)$ and $c(v_1) \cap c(v_2) = \emptyset$ for any edge $v_1v_2$ of $G$. The WCOL problem becomes the COL problem for the all-ones vector of vertex weights.

A class of simple graphs is called *hereditary* if it is closed under deletion of vertices. Any hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{S}$. We write $\mathcal{X} = Free(\mathcal{S})$, and the graphs in $\mathcal{X}$ are said to be $\mathcal{S}$*-free*. If $\mathcal{S} = \{G\}$, then we write "$G$-free" instead of "$\{G\}$-free".

The computational complexity of the COL problem was intensively studied for families of hereditary classes defined by small graphs only or by a small number of forbidden induced structures. We would mention the papers [2–6,8–12,14] in this field. The computational complexity of the COL problem was completely determined for all classes of the form $Free(\{G\})$ [9]. Namely, if $\subseteq_i$ is the induced subgraph relation, then the problem is polynomial-time solvable for $Free(\{G\})$ whenever $G \subseteq_i P_4$ or $G \subseteq_i P_3 + K_1$, otherwise it is NP-complete. A study of forbidden pairs was also initiated in [9]. The following result shows some recent advances in classification of the complexity of the COL problem for $\{G_1, G_2\}$-free graphs. Note that by symmetry the graphs $G_1$ and $G_2$ may be swapped in each of the subcases of the theorem.

**Theorem 1** ([5]). *Let $G_1$ and $G_2$ be two fixed graphs. The coloring problem is NP-complete for $Free(\{G_1, G_2\})$ if:*

1. $C_p \subseteq_i G_1$ *for* $p \geq 3$, *and* $C_q \subseteq_i G_2$ *for* $q \geq 3$
2. $K_{1,3} \subseteq_i G_1$, *and* $K_{1,3} \subseteq_i G_2$ *or* $\overline{K_2 + O_2} \subseteq_i G_2$ *or* $C_r \subseteq_i G_2$ *for* $r \geq 4$ *or* $K_4 \subseteq_i G_2$

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3. *$G_1$ and $G_2$ contain a spanning subgraph of a $2K_2$ as an induced subgraph*
4. *bull $\subseteq_i G_1$, and $K_{1,4} \subseteq_i G_2$ or $\overline{C_4 + K_1} \subseteq_i G_2$*
5. *$C_3 \subseteq_i G_1$, and $K_{1,p} \subseteq_i G_2$ for $p \geq 5$*
6. *$C_3 \subseteq_i G_1$ and $P_{22} \subseteq_i G_2$*
7. *$C_p \subseteq_i G_1$ for $p \geq 5$, and $G_2$ contains a spanning subgraph of a $2K_2$ as an induced subgraph*
8. *$C_p + K_1 \subseteq_i G_1$ for $p \in \{3, 4\}$ or $\overline{C_q} \subseteq_i G_1$ for $q \geq 6$, and $G_2$ contains a spanning subgraph of a $2K_2$ as an induced subgraph*
9. *$K_5 \subseteq_i G_1$ and $P_7 \subseteq_i G_2$*
10. *$K_6 \subseteq_i G_1$ and $P_6 \subseteq_i G_2$.*

*It is polynomial-time solvable for Free($\{G_1, G_2\}$) if:*

1. *$G_1$ is an induced subgraph of a $P_4$ or a $P_3 + K_1$*
2. *$G_1 \subseteq_i K_{1,3}$, and $G_2 \subseteq_i$ hammer or $G_2 \subseteq_i$ bull or $G_2 \subseteq_i P_5$*
3. *$G_1 \neq K_{1,5}$ is a forest on at most six vertices or $G_1 = K_{1,3} + 3K_1$, and $G_2 \subseteq_i$ paw*
4. *$G_1 \subseteq_i sK_2$ or $G_1 \subseteq_i P_5 + O_s$ for $s > 0$, and $G_2$ is a complete graph or $G_2 \subseteq_i$ hammer*
5. *$G_1 \subseteq_i P_4 + K_1$ or $G_1 \subseteq_i P_5$, and $G_2 \subseteq_i \overline{P_4 + K_1}$ or $G_2 \subseteq_i \overline{P_5}$*
6. *$G_1 \subseteq_i K_2 + O_2$, and $G_2 \subseteq_i \overline{2K_2 + K_1}$ or $G_2 \subseteq_i \overline{P_3 + O_2}$ or $G_2 \subseteq_i \overline{P_3 + K_2}$*
7. *$G_1 \subseteq_i \overline{K_2 + O_2}$, and $G_2 \subseteq_i 2K_2 + K_1$ or $G_2 \subseteq_i P_3 + O_2$ or $G_2 \subseteq_i P_3 + P_2$*
8. *$G_1 \subseteq_i K_2 + O_s$ for $s > 0$ or $G_1 = P_5$, and $G_2 \subseteq_i \overline{K_2 + O_t}$ for $t > 0$*
9. *$G_1 \subseteq O_4$ and $G_2 \subseteq \overline{P_3 + O_2}$*
10. *$G_1 \subseteq P_5$, and $G_2 \subseteq_i C_4$ or $G_2 \subseteq \overline{P_3 + O_2}$.*

For all but three cases either NP-completeness or polynomial-time solvability was shown for the COL problem in the family of all hereditary classes defined by four-vertex forbidden induced structures [10]. A similar result was obtained in [11] for two connected five-vertex forbidden induced fragments, where the number of open cases was 13. A list of the open cases is presented below, where the numbers in parentheses show the number of sets of the given type.

1. $\{K_{1,3}, G\}$, where $G \in \{$*bull*, *butterfly*$\}$ (2)
2. $\{$*fork*, *bull*$\}$ (1)
3. $\{P_5, G\}$, where $G$ is an arbitrary connected five-vertex complement graph of the line graph of a forest with at most 3 leaves in each connected component and $G \notin \{K_5, gem\}$ (10).

Recently, the number of the open cases was reduced to 10 by showing that the COL problem can be solved in polynomial time for *Free*($\{P_5, \overline{P_5}\}$), *Free*($\{K_{1,3}, bull\}$), *Free*($\{P_5, \overline{P_3 + O_2}\}$) [8,12]. Next, the number of the remaining open cases was reduced to eight by showing that the COL problem can be polynomially solved for $\{P_5, \overline{P_3 + P_2}\}$-free graphs and for $\{P_5, K_p - e\}$-free graphs [14]. In the present paper we also narrow the set of the open cases by proving that the WCOL problem can be solved for $\{P_5, banner\}$-free graphs and for $\{P_5, dart\}$-free graphs in polynomial time on the sum of vertex weights. As a corollary, this result gives polynomial-time solvability of the COL problem for $\{P_5, banner\}$-free graphs and for $\{P_5, dart\}$-free graphs. The main result relies on the Strong Perfect Graph Theorem and on the polynomial-time algorithm to solve the WCOL problem for perfect graphs.

## 2. Notation

As usual, $P_n, C_n, O_n, K_n$ stand for a simple path, a chordless cycle, an edgeless graph, and a complete graph on $n$ vertices, respectively. A graph $K_{p,q}$ is a complete bipartite graph with $p$ vertices in the first part and $q$ vertices in the second one. A graph $K_p - e$ can be obtained from a $K_p$ by deleting an arbitrary edge.

The graphs *paw*, *bull*, *hammer*, *fork*, *gem*, *butterfly*, *banner*, *dart* are depicted in Fig. 1.

For a vertex $x$ of a graph, $N(x)$ is its neighborhood. For a graph $G$ and a subset $V' \subseteq V(G)$, $G(V')$ is the subgraph of $G$ induced by $V'$. A graph $G_1 + G_2$ is the disjoint union of graphs $G_1$ and $G_2$ having disjoint sets of vertices. A graph $kG$ is the disjoint union of $k$ copies of a graph $G$. A graph $\overline{G}$ is the complement graph of a graph $G$.

Let $A$ and $B$ be disjoint subsets of vertices of a given graph. If all possible edges are present between the sets $A$ and $B$, then $A$ is said to be *complete* to $B$. If no edges between $A$ and $B$ are present, then $A$ is said to be *anti-complete* to $B$.

The symbol "$\triangleq$" means the equality by definition.

## 3. Auxiliary results

### *3.1. Prime graphs and their application to the weighted coloring problem*

Let $G$ be a graph. A non-empty set $M \subseteq V(G)$ is a *module* in $G$ if either $x$ is adjacent to all elements of $M$ or to none of them for each $x \in V(G) \setminus M$. A module in a graph is *trivial* if it contains only one vertex or all vertices of the graph, otherwise it is *non-trivial*. A graph is *prime* if all its modules are trivial.

Let $[\mathcal{X}]_P$ be the class of all graphs whose every prime induced subgraph belongs to $\mathcal{X}$.

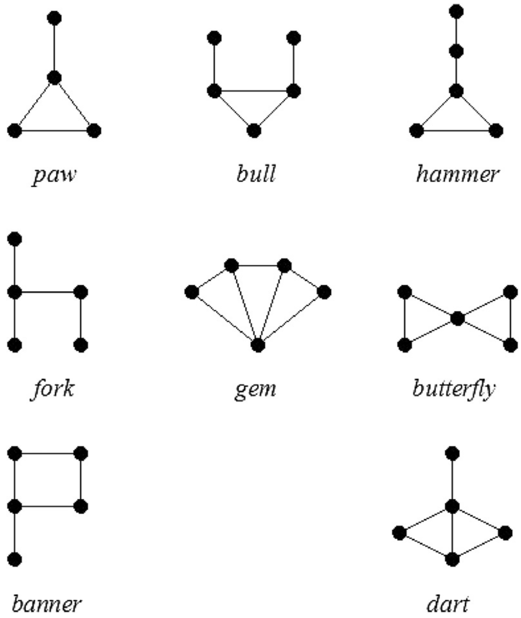


**Fig. 1.** The graphs *paw*, *bull*, *hammer*, *fork*, *gem*, *butterfly*, *banner*, *dart*.

**Lemma 1** ([8]). *If the* WCOL *problem can be solved for a hereditary class $\mathcal{X}$ in polynomial time on the sum of weights, then it can be solved for $[\mathcal{X}]_P$ in polynomial time on the sum of vertex weights as well.*

*3.2. Properties of prime $\{\overline{P_5}, \overline{banner}\}$-free graphs and $\{\overline{P_5}, \overline{dart}\}$-free graphs containing long induced odd cycles*

**Lemma 2.** *No prime $\{\overline{P_5}, \overline{banner}\}$-free graph containing an induced odd cycle of length at least 5 contains a triangle.*

**Proof.** Assume the opposite, i.e. there is a prime $\{\overline{P_5}, \overline{banner}\}$-free graph $G \triangleq (V, E)$ containing an induced odd cycle $C$ of length at least 5 and a triangle. Then $G$ is connected, as it is prime and it contains at least 5 vertices. We will show that there exists a vertex adjacent to all vertices of $C$. To this end, it is sufficient to show that there is a vertex adjacent to two consecutive vertices of $C$. Indeed, if a vertex $x$ has two adjacent neighbors on $C$, then $\{x\}$ is complete to $V(C)$. Let us consider a longest path $P \triangleq (v_1, \ldots, v_k)$ of vertices of $C$ each adjacent to $x$. Suppose that $k \neq |V(C)|$. Then $2 \leq k \leq |V(C)| - 2$, as $G$ is $\overline{P_5}$-free. Let $v_{k+1}$ be the neighbor of $v_k$ on $C$ distinct from $v_{k-1}$, and let $v_{k+2}$ be the neighbor of $v_{k+1}$ on $C$ distinct from $v_k$. As $P$ is longest, $xv_{k+1} \notin E$. As $G$ is $\overline{P_5}$-free, $xv_{k+2} \notin E$. Then $v_{k-1}, v_k, v_{k+1}, v_{k+2}$, and $x$ induce a $\overline{banner}$.

The *distance between a vertex $v$ and the cycle $C$* is the minimum among usual distances between $v$ and vertices of $C$. Let $N_i$ be the set of vertices $v$ such that the distance between $v$ and $C$ is equal to $i$. Clearly, $N_0 = V(C)$. As $G$ is $\overline{banner}$-free, any triangle of $G$ has a vertex in $N_0 \cup N_1 \cup N_2$. Similarly, some triangle has a vertex in $N_0 \cup N_1$. Suppose that a triangle $T$ of $G$ and the cycle $C$ have a common vertex $u$. We may assume that $V(T) \cap V(C) = \{u\}$, otherwise some vertex of $T$ has two consecutive neighbors on $C$ and this case has already been considered in the first paragraph. As $G$ is $\{\overline{P_5}, \overline{banner}\}$-free, either a vertex of $V(T) \setminus \{u\}$ is adjacent to a neighbor of $u$ on $C$ or $V(T) \setminus \{u\}$ is complete to $\{w_1, w_2\}$, where $w_1$ and $w_2$ lie at distance 2 from $u$ in $C$. Indeed, if $V(T) \setminus \{u\} = \{u_1, u_2\}$, $\{u_1, u_2\}$ is not complete to $\{w_1, w_2\}$, and $\{v_1, v_2\}$ is anti-complete to $\{u_1, u_2\}$, where $v_i$ is the common neighbor of $u$ and $w_i$ on $C$, then $u_1, u_2, u$, and either $v_1, w_1$ or $v_2, w_2$ induce a $\overline{P_5}$ or a $\overline{banner}$. As $C$ is *odd*, the last observation leads to the fact that some element of $V(T) \setminus \{u\}$ has two consecutive neighbors on $C$. Now suppose that there is a triangle $T'$ of $G$ such that $V(T') \cap N_0 = \emptyset$ and $V(T') \cap N_1 \neq \emptyset$. We may assume that no two vertices of $T'$ have a common neighbor on $C$, for otherwise we are back in the previous case. If $a_1 \in V(T') \cap N_1$, then $a_1$ must have two adjacent neighbors on $C$. Otherwise, all vertices of $T'$, a neighbor $a_2$ of $a_1$ on $C$, and a neighbor of $a_2$ on $C$ induce a $\overline{banner}$, as $G$ is $\overline{P_5}$-free.

In all possible cases we obtained that there is a vertex of $G$, which is adjacent to all vertices of $C$.

Let $V'$ be the set of all vertices of $G$ each adjacent to all vertices of $C$. This set is not empty. Let $V''$ be the set of all vertices of the connected component of $G \setminus V'$ containing $C$. We will show that $V''$ is a module in $G$. It is obvious whenever $V' = N_1$. Therefore, we will suppose that $V' \neq N_1$. Any element of $N_1 \setminus V'$ has no two consecutive neighbors on $C$. Hence, if $|V(C)| = 5$, then any element of $N_1 \setminus V'$ has exactly one neighbor on $C$ or exactly two non-adjacent neighbors on it, as $G$ is $\{\overline{P_5}, \overline{banner}\}$-free. As $G$ is $\{\overline{P_5}, \overline{banner}\}$-free, $V'$ is complete to $N_1 \setminus V'$ whenever $C$ has exactly 5 vertices. Suppose that $|V(C)| \geq 7$. Let $v \in V'$ and $u \in N_1 \setminus V'$. There are adjacent vertices $v_1$ and $v_2$ on $C$, a vertex $v_3 \in V(C)$ such that $v_1u \notin E$, $v_2u \notin E$, $v_3u \in E$ and $v_1v_3 \notin E$, $v_2v_3 \notin E$, as $C$ is an odd cycle of length at least 7 and none of the elements of $N_1 \setminus V'$ has two consecutive neighbors on $C$. The vertices $v$ and $u$ are adjacent, otherwise $v_1, v_2, v, v_3, u$ induce a $\overline{banner}$. Hence, $V'$ is complete to $N_1 \setminus V'$ whenever $|V(C)| \geq 7$.

Let $a \in V'' \cap N_2$. Hence, there is a vertex $b \in V'' \cap N_1$ such that $ab \in E$. Clearly, $b \in N_1 \setminus V'$. As $C$ is odd and $b \in N_1 \setminus V'$, the vertex $b$ has two adjacent non-neighbors $c'$ and $c''$ on $C$. The vertex $a$ is adjacent to all vertices of $V'$, for otherwise, $c'$, $c''$, $a$, $b$, and any vertex of $V'$ induce a $\overline{banner}$. Hence, $V'$ is complete to $N_2 \cap V''$. As $V' \subset N_1$, none of the elements of $V'$ has a neighbor in $N_3$. Hence, $N_3 \cap V'' = \emptyset$, to avoid an induced $\overline{banner}$. Therefore, $V''$ is non-trivial module in $G$. ■

Recall that a *dominating set* of a graph $G$ is a subset $D \subseteq V(G)$ such that any element of $V(G) \setminus D$ has a neighbor in $D$.

**Lemma 3.** *No prime $\{\overline{P_5}, \overline{dart}\}$-free graph containing an induced odd cycle of length at least 7 contains a triangle. If a prime $\{\overline{P_5}, \overline{dart}\}$-free graph contains an induced 5-cycle that is not a dominating set, then it contains no triangles.*

**Proof.** Assume that there is a prime $\{\overline{P_5}, \overline{dart}\}$-free graph $G \triangleq (V, E)$ containing a triangle. Additionally, assume that $C$ is an induced odd cycle of $G$ of length at least 5, and that if $C = C_5$, then $V(C)$ is not a dominating set of $G$. Therefore $G$ is connected.

Let $N_i$ be the set of vertices of $G$ lying at distance $i$ from $C$. Clearly, $N_2 \neq \emptyset$, if $C$ is of length five. Let a vertex $x$ be adjacent to at least two consecutive vertices of $C$. If $C$ has exactly 5 vertices and $x$ is not adjacent to all its vertices, then $x$ has exactly two or three consecutive neighbors on $C$, as $G$ is $\overline{P_5}$-free. Hence, $\{x\}$ is complete to $\bigcup_{i\geq 2} N_i$, as $G$ is $\overline{dart}$-free. Hence, $G$ contains an induced $\overline{dart}$. Therefore, $x$ must be adjacent to all vertices of $C$.

Suppose that $|V(C)| \geq 7$. Let us consider a longest path $P \triangleq (v_1, \ldots, v_k)$ of vertices of $C$ each adjacent to $x$. Suppose that $k \neq |V(C)|$. Then $2 \leq k \leq |V(C)| - 2$, as $G$ is $\overline{P_5}$-free. Let $v_0$ be the neighbor of $v_1$ on $C$ distinct from $v_2$, and let $v_{k+1}$ be the neighbor of $v_k$ on $C$ distinct from $v_{k-1}$. As $P$ is longest, $xv_0 \notin E$ and $xv_{k+1} \notin E$. If $k \geq 5$, then $v_0, v_2, v_3, x, v_5$ induce a $\overline{dart}$. If $3 \leq k \leq 4$, then $v_0v_{k+1} \notin E$ and $v_0, v_1, v_2, x, v_{k+1}$ induce a $\overline{dart}$. If $k = 2$, then $x, v_0, v_1, v_2, v_4$ induce a $\overline{dart}$. Hence, $k = |V(C)|$, i.e. $x$ must be adjacent to all vertices of $C$.

So, any vertex having at least two adjacent neighbors on $C$ must be adjacent to all vertices of $C$.

Let $V'$ be the set of all vertices in $N_1$ each adjacent to all vertices of $C$. Let $V''$ be the set of all vertices of $N_1$ each having not a pair of adjacent neighbors on $C$. Clearly, $N_1 = V' \cup V''$. As $G$ is $\overline{dart}$-free, $V'$ is complete to $V(C) \cup V'' \cup \bigcup_{i\geq 2} N_i$. Hence, $V(C) \cup V'' \cup \bigcup_{i\geq 2} N_i$ is a non-trivial module in $G$ whenever $V' \neq \emptyset$.

Let us justify that $V' \neq \emptyset$. Suppose that $V' = \emptyset$. Hence, none of the elements of $N_1$ has two adjacent neighbors on $C$. If there is a triangle $(a, b, c)$, where $a \in V(C)$, then, to avoid an induced $\overline{P_5}$ or $\overline{dart}$, each of the vertices $b$ and $c$ is simultaneously adjacent to both vertices $w_1$ and $w_2$ that are at distance two from $a$ in $C$. Hence, $w_1', w_2', w_1, b, c$ induce a $\overline{dart}$, where $w_i'$ is the common neighbor of $a$ and $w_i$ on $C$, $i = 1, 2$. We just proved that any two adjacent vertices both lying in $N_1$ have not a common neighbor on $C$. If $(a', b', c')$ is a triangle of $G$ such that $\{a', b', c'\} \cap V(C) = \emptyset$ and $a' \in N_1$, then, as $C$ is odd, there are pairwise distinct vertices $a'', a_1'', a'''$ on $C$ such that $a'a'' \in E$, $a'a_1'' \notin E$, $a'a''' \notin E$, $a''a_1'' \in E$, $a_1''a''' \in E$. Let $a_2''$ be the neighbor of $a$ on $C$ distinct from $a_1''$. As $G$ is $\overline{P_5}$-free, none of the vertices $a_1''$ and $a_2''$ has a neighbor in $\{b', c'\}$. To avoid a $\overline{dart}$ induced by $a''', a'', a', b', c'$, the vertex $a'''$ must have a neighbor in $\{b', c'\}$. Then $a''', a', b', c', a_2''$ induce a $\overline{dart}$. Hence, we may suppose that any triangle of $G$ has not a vertex in $N_0 \cup N_1$. Let $i^* \triangleq \min\{i |\ N_i$ contains a vertex of some triangle$\}$, and let $T$ be a triangle having a vertex of $N_{i^*}$. Clearly, $i^* > 1$. Then all vertices of $T$, some vertex in $N_{i^*-1}$, and a vertex of $C$ induce a $\overline{dart}$. We have a contradiction with the initial assumption. ■

### 3.3. *Some properties of $P_5$-free graphs and $\{P_5, dart\}$-free graphs containing an induced 5-cycle*

Let $G \triangleq (V, E)$ be a connected $P_5$-free graph containing an induced $C_5 \triangleq (v_1, v_2, v_3, v_4, v_5)$. We associate the following notation with $G$ taking the indices modulo 5 throughout this subsection:

- $V_i \triangleq \{x \notin V(C_5) |\ N(x) \cap V(C_5) = \{v_{i-1}, v_{i+1}\}\}$,
- $V_i' \triangleq \{x \notin V(C_5) |\ N(x) \cap V(C_5) = \{v_{i-1}, v_i, v_{i+1}\}\}$,
- $V_i'' \triangleq \{x \notin V(C_5) |\ N(x) \cap V(C_5) = V(C_5) \setminus \{v_i\}\}$,
- $V_i''' \triangleq \{x \notin V(C_5) |\ N(x) \cap V(C_5) = \{v_{i-2}, v_i, v_{i+2}\}\}$,
- $V''''$ be the set of all vertices adjacent to all vertices of the 5-cycle.

**Lemma 4** ([14]). *Every element of $V \setminus V(C_5)$ having a neighbor on the 5-cycle belongs to*

$$\bigcup_{j=1}^{5} (V_j \cup V_j' \cup V_j'' \cup V_j''') \cup V''''.$$

*For each i, none of the elements of $V_i \cup V_i'$ has a neighbor outside $\bigcup_{j=1}^{5} N(v_j)$.*

**Lemma 5.** *For each i, every of the following statements is true:*

(1) *The set $V_i$ is complete to*

$$\bigcup_{j\in\{i-1,i+1\}} (V_j \cup V_j' \cup V_j'') \cup V_i'''$$

*and $V_i$ is anti-complete to $\bigcup_{j\in\{i-2,i+2\}} V_j'$.*

(2) *The set $V_i'$ is complete to*

$$\bigcup_{j\in\{i-1,i+1\}} (V_j \cup V_j') \cup \bigcup_{j\in\{i-2,i+2\}} V_j'' \cup \bigcup_{j\in\{i-2,i,i+2\}} V_j'''$$

*and $V_i'$ is anti-complete to $\bigcup_{j\in\{i-2,i+2\}} V_j \cup \bigcup_{j\in\{i-1,i+1\}} V_j'''$.*

(3) *The set $V_i''$ is complete to*

$$\bigcup_{j\in\{i-1,i+1\}} V_j \cup \bigcup_{j\in\{i-2,i+2\}} V_j' \cup \bigcup_{j\in\{i-2,i+2\}} V_j'''.$$

(4) *The set $V_i'''$ is complete to*

$$V_i \cup \bigcup_{j\in\{i-2,i,i+2\}} V_j' \cup \bigcup_{j\in\{i-2,i+2\}} V_j'' \cup \bigcup_{j\in\{i-1,i+1\}} V_j'''$$

*and $V_i'''$ is anti-complete to $\bigcup_{j\in\{i-1,i+1\}} V_j'$.*

**Proof.** (1) Let $a \in V_i$ and

$$b \in \bigcup_{j\in\{i-1,i+1\}} (V_j \cup V_j' \cup V_j'') \cup V_i'''.$$

Assume that $ab \notin E$. If $b \in \bigcup_{j\in\{i-1,i+1\}}(V_j \cup V_j')$, then $a, b$, and either $v_{i-1}, v_{i-2}, v_{i+2}$ or $v_{i-2}, v_{i+1}, v_{i+2}$ induce a $P_5$. If $b \in \bigcup_{j\in\{i-1,i+1\}} V_j''$, then either $v_{i+2}, b, v_i, v_{i-1}, a$ or $v_{i-2}, b, v_i, v_{i+1}, a$ induce a $P_5$. If $b \in V_i'''$, then $v_{i-2}, b, v_i, v_{i+1}, a$ induce a $P_5$. Let $c \in \bigcup_{j\in\{i-2,i+2\}} V_j'$. Then $ac \notin E$, otherwise either $v_{i+2}, b, a, v_{i-1}, v_i$ or $v_{i-2}, b, a, v_{i+1}, v_i$ induce a $P_5$.

(2) Let $a \in V_i'$ and

$$b \in \bigcup_{j\in\{i-1,i+1\}} (V_j \cup V_j') \cup \bigcup_{j\in\{i-2,i+2\}} (V_j'' \cup V_j''') \cup V_i'''.$$

Assume that $ab \notin E$. By the previous part,

$$b \in \bigcup_{j\in\{i-1,i+1\}} V_j' \cup \bigcup_{j\in\{i-2,i+2\}} (V_j'' \cup V_j''') \cup V_i'''.$$

If $b \in \bigcup_{j\in\{i-1,i+1\}} V_j'$, then $a, b$, and either $v_{i-2}, v_{i+1}, v_{i+2}$ or $v_{i-1}, v_{i-2}, v_{i+2}$ induce a $P_5$. If $b \in \bigcup_{j\in\{i-2,i+2\}} V_j''$, then $a, v_i, b, v_{i-2}, v_{i+2}$ induce a $P_5$. If $b \in V_i'''$, then $b, v_{i-2}, v_{i-1}, a, v_{i+1}$ induce a $P_5$. Let $c \in \bigcup_{j\in\{i-2,i+2\}} V_j \cup \bigcup_{j\in\{i-1,i+1\}} V_j'''$. By the previous part, one may assume that $c \in \bigcup_{j\in\{i-1,i+1\}} V_j'''$. Then $ac \notin E$, otherwise $v_i, a, c, v_{i+2}, v_{i-2}$ induce a $P_5$.

(3) Let $a \in V_i''$ and

$$b \in \bigcup_{j\in\{i-1,i+1\}} V_j \cup \bigcup_{j\in\{i-2,i+2\}} V_j' \cup \bigcup_{j\in\{i-2,i+2\}} V_j'''.$$

If $b \in \bigcup_{j\in\{i-1,i+1\}} V_j \cup \bigcup_{j\in\{i-2,i+2\}} V_j'$, then $ab \in E$, by the previous parts. If $b \in \bigcup_{j\in\{i-2,i+2\}} V_j'''$, then $ab \in E$, otherwise either $v_{i+2}, a, v_{i-1}, v_i, c$ or $v_{i-2}, a, v_{i+1}, v_i, b$ induce a $P_5$.

(4) The set $V_i'''$ is complete to

$$V_i \cup \bigcup_{j\in\{i-2,i,i+2\}} V_j' \cup \bigcup_{j\in\{i-2,i+2\}} V_j''$$

and anti-complete to $\bigcup_{j\in\{i-1,i+1\}} V_j'$, by the previous parts. Let $a \in V_i'''$ and $b \in \bigcup_{j\in\{i-1,i+1\}} V_j'''$. Then $ab \in E$, otherwise $v_{i+2}, a, v_i, v_{i-1}, b$ or $v_{i-2}, a, v_i, v_{i+1}, b$ induce a $P_5$. ■

Recall that an *independent set* and a *clique* in a graph are subsets of its pairwise non-adjacent and pairwise adjacent vertices, respectively. In all the next lemmas from this subsection we additionally assume that $G$ is a prime *dart*-free graph.

**Lemma 6.** *For each i, every of the following statements is true:*

(1) *Each of the sets $V_i', V_i'', V_i'''$ is a clique. The set $V_i'$ is complete to $V_i'''$.*

(2) *If $V_i \neq \emptyset$, then $\bigcup_{j=1,j\neq i}^{5} V_j'' \cup V'''' = \emptyset$. If $V_i''' \neq \emptyset$, then the set*

$$\bigcup_{j\in\{i-1,i+1\}} V_j''' \cup \bigcup_{j=1,j\neq i}^{5} V_j'' \cup V''''$$

*is empty.*

(3) *The set* $V_i$ *is anti-complete to* $V_i' \cup V_i''$, *and the set* $V_i'''$ *is anti-complete to*

$$\bigcup_{j=1, j\neq i}^{5} V_j \cup \bigcup_{j\in\{i-2,i+2\}} V_j''' \cup V_i''.$$

(4) *None of the elements of* $V_i'' \cup V''''$ *has a neighbor outside* $\bigcup_{j=1}^{5} N(v_j)$.

**Proof.** (1) Let $a, b \in V_i'$ or $a, b \in V_i''$ or $a, b \in V_i'''$. Then $ab \in E$, otherwise either $a, v_i, v_{i-1}, b, v_{i-2}$ or $a, v_{i-2}, v_{i+2}, b, v_{i-1}$ induce a *dart*. Let $a \in V_i'$ and $b \in V_i'''$. Then $ab \in E$, otherwise $a, v_{i-1}, v_i, v_{i+1}, b$ induce a *dart*.

(2) Let $a \in V_i$ and $b \in \bigcup_{j=1, j\neq i}^{5} V_j'' \cup V''''$. Then $ab \in E$, otherwise either $a, b, v_{i-2}, v_{i-1}, v_i$ or $a, b, v_{i+2}, v_{i+1}, v_i$ induce a *dart*. Hence, $a, b, v_i$, and either $v_{i+1}, v_{i-2}$ or $v_{i-1}, v_{i+2}$ induce a *dart*. Let $a \in V_i'''$ and

$$b \in \bigcup_{j\in\{i-1,i+1\}} V_j''' \cup \bigcup_{j=1, j\neq i}^{5} V_j'' \cup V''''.$$

If $b \in \bigcup_{j\in\{i-2,i+2\}} V_j'' \cup \bigcup_{j\in\{i-1,i+1\}} V_j'''$, then $ab \in E$, by Lemma 5 (part 4). Then either $a, b, v_{i-1}, v_{i-2}, v_{i+1}$ or $a, b, v_{i+1}, v_{i+2}, v_{i-1}$ induce a *dart*. If $b \in \bigcup_{j\in\{i-1,i+1\}} V_j''$, then $ab \in E$, otherwise $a, v_{i-2}, v_{i+2}, b$, and either $v_{i-1}$ or $v_{i+1}$ induce a *dart*. Then $a, b, v_{i-1}, v_i, v_{i+1}$ induce a *dart*. If $b \in V''''$, then $ab \in E$, otherwise, $b, v_{i-1}, v_i, v_{i+1}, a$ induce a *dart*. Then $a, v_{i-2}, v_{i-1}, b, v_{i+1}$ induce a *dart*.

(3) Let $a \in V_i$ and $b \in V_i' \cup V_i''$. Then $ab \notin E$, otherwise either $a, b, v_i, v_{i+1}, v_{i+2}$ or $a, b, v_i, v_{i-1}, v_{i+1}$ induce a *dart*. Let $a \in V_i'''$ and

$$b \in \bigcup_{j=1, j\neq i}^{5} V_j \cup \bigcup_{j\in\{i-2,i+2\}} V_j''' \cup V_i''.$$

If $b \in \bigcup_{j\in\{i-2,i+2\}} (V_j \cup V_j''')$, then $ab \notin E$, otherwise $v_{i-2}, v_{i+2}, a, b$, and either $v_{i+1}$ or $v_{i-1}$ induce a *dart*. If $b \in \bigcup_{j\in\{i-1,i+1\}} V_j \cup V_i''$, then $ab \notin E$, otherwise either $v_{i+1}, v_{i+2}, a, b, v_{i-1}$ or $v_{i-1}, v_{i-2}, a, b, v_{i+1}$ induce a *dart*.

(4) If an element $a \in V_i'' \cup V''''$ has a neighbor $b \notin \bigcup_{j=1}^{5} N(v_j)$, then $v_{i+1}, a, v_{i+2}, v_{i-2}, b$ induce a *dart*. ■

**Lemma 7.** *If* $\bigcup_{j=1}^{5} V_j = \bigcup_{j=1}^{5} V_j''' = \emptyset$, *then* $G$ *is* $O_3$*-free.*

**Proof.** Recall that $G$ is connected. By this fact and by Lemmas 4, 6 (part 4), $V = \bigcup_{j=1}^{5} N(v_j)$. Let $a$ and $b$ be non-adjacent elements of $V''''$. Then any element of $\tilde{V} \triangleq \bigcup_{j=1}^{5} V_j' \cup \bigcup_{j=1}^{5} V_j''$ has a neighbor in $\{a, b\}$, otherwise $a, b$, some two consecutive vertices of the 5-cycle, and an element of $\tilde{V}$ induce a *dart*. Similarly, if $c \in V''''$ and $ac \notin E$, $bc \notin E$, then every element of $\tilde{V}$ is adjacent to all the vertices $a, b, c$. Indeed, every vertex $x \in \tilde{V}$ has at least two neighbors in $\{a, b, c\}$. If some $x \in \tilde{V}$ is adjacent to precisely two of $a, b, c$, then $a, b, c, x$ and any neighbor $v_i$ of $x$ induce a *dart* in $G$, a contradiction. Moreover, any element of $V'''' \setminus \{a, b, c\}$ has three or at most one neighbor in the set $\{a, b, c\}$.

Suppose that $\hat{V}$ be a maximum independent set of $G(V'''')$ and $|\hat{V}| \geq 3$. Let $V^*$ be the union of $\hat{V}$ and the set of elements of $V'''' \setminus \hat{V}$ having the only one neighbor in $\hat{V}$. Hence, $|V^*| \geq 3$. By the reasonings from the first paragraph, $V^*$ is complete to $\tilde{V}$. Similarly, $V'''' \setminus V^*$ is complete to $\hat{V}$. If a vertex $d_1 \in V'''' \setminus V^*$ is not adjacent to an element $d_2$ of $V^* \setminus \hat{V}$, then any vertex of the 5-cycle, any two non-neighbors of $d_2$ in $\hat{V}$, $d_1$ and $d_2$ induce a *dart*. Hence, $V^*$ is complete to $V'''' \setminus V^*$. Therefore, $V^*$ is a non-trivial module in $G$. Hence, $G(V'''')$ is $O_3$-free.

Suppose that $\{x, y, z\}$ be independent. By the reasonings from the previous paragraphs, $V'''' \cap \{x, y, z\}$ has at most one element. If $x = v_i$ for some $i$, then $y$ and $z$ must belong to $\{v_{i-2}, v_{i+2}\} \cup V_{i-2}' \cup V_{i+2}' \cup V_i''$. Hence, $y$ and $z$ are adjacent, by Lemmas 5 (part 2) and 6 (part 1). If at least two of the vertices $x, y, z$ belong to $\bigcup_{j=1}^{5} V_j'$, then we may consider that $x \in V_i'$ and $y \in V_{i+2}'$, by Lemmas 5 (part 2) and 6 (part 1). Similarly, $z \in V_{i+1}'' \cup V''''$. Then $x, v_i, z, v_{i+2}, y$ induce a $P_5$. If at least two of the vertices $x, y, z$ belong to $\bigcup_{j=1}^{5} V_j''$, then we may consider that either $x \in V_i''$, $y \in V_{i+1}''$ or $x \in V_i''$, $y \in V_{i+2}''$, by Lemma 6 (part 1). If $z \in V''''$, then $v_{i+1}, v_{i+2}, x, y, z$ induce a *dart* or $x, v_i, z, v_{i+2}, y$ induce a $P_5$. Otherwise, by Lemmas 5 (part 2) and 6 (part 1), the vertex $z$ belongs to

$$V_i' \cup V_{i+1}' \cup V_{i-2}'' \cup V_{i-1}'' \cup V_{i+2}''$$

in the first case and to

$$V_{i+1}'' \cup V_{i-2}'' \cup V_{i-1}'' \cup V_{i+1}'$$

in the second one. Then $x, y, z$, and two non-adjacent vertices of the 5-cycle induce a $P_5$. If $x \in V''''$, $y \in V_i'$, and $z \in \bigcup_{j=1}^{5} V_j''$, then $z \in V_{i-1}'' \cup V_i'' \cup V_{i+1}''$, by Lemma 5 (part 2). Then $x, y, z$, and some two non-adjacent vertices of the 5-cycle induce a $P_5$. ■

**Lemma 8.** *For each i, every of the following statements is true:*

(1) *Any two vertices of $V_i$ have the same sets of neighbors in*

$$\bigcup_{j\in\{i-1,i+1\}} V_j \cup \bigcup_{j=1}^{5}(V_j' \cup V_j'' \cup V_j''') \cup V'''' \cup (V \setminus \bigcup_{j=1}^{5} N(v_j)).$$

(2) *The vertex $v_i$ and any element $v_i' \in V_i'$ have the same sets of neighbors in*

$$(\bigcup_{j=1}^{5} V_j \cup \bigcup_{j\in\{i-1,i,i+1\}} V_j' \cup \bigcup_{j\in\{i-2,i+2\}} V_j'' \cup \bigcup_{j=1}^{5} V_j''' \cup (V \setminus \bigcup_{j=1}^{5} N(v_j))) \setminus \{v_i, v_i'\}.$$

(3) *The vertex $v_i$ and any element of $V_i$ have the same sets of neighbors in*

$$\bigcup_{j\in\{i-1,i+1\}} V_j \cup \bigcup_{j=1, j\neq i}^{5} V_j' \cup \bigcup_{j=1}^{5} V_j'' \cup V_i''' \cup V'''' \cup (V \setminus \bigcup_{j=1}^{5} N(v_j)).$$

(4) *Any two elements of $V_i'$ have the same sets of neighbors in*

$$\bigcup_{j=1}^{5} V_j \cup \bigcup_{j\in\{i-1,i+1\}} V_j' \cup \bigcup_{j\in\{i-2,i+2\}} V_j'' \cup \bigcup_{j=1}^{5} V_j''' \cup (V \setminus \bigcup_{j=1}^{5} N(v_j)).$$

*Any two elements of $V_i''$ have the same sets of neighbors in*

$$\bigcup_{j\in\{i-1,i,i+1\}} V_j \cup \bigcup_{j\in\{i-2,i+2\}} V_j' \cup \bigcup_{j\in\{i-2,i+2\}} V_j'''.$$

(5) *The set $V_i'''$ has at most one element.*

(6) *The set $V \setminus \bigcup_{j=1}^{5} N(v_j)$ has at most one element.*

**Proof.** (1) The fact follows from Lemmas 4, 5 (part 1), 6 (parts 2 and 3).

(2) The fact follows from Lemmas 4, 5 (part 2), 6 (parts 1 and 3).

(3) The fact follows from Lemmas 4, 5 (part 1), 6 (parts 2 and 3).

(4) The fact follows from Lemmas 4, 5 (parts 2 and 3), 6 (part 3).

(5) By Lemma 6 (part 1) and as $G$ is *dart*-free, any neighbor of a vertex in $V_i'''$ that lies outside $\bigcup_{j=1}^{5} N(v_j)$ must be adjacent to all elements of $V_i'''$. By this fact, Lemmas 5 (part 4), and 6 (parts 2 and 3), $V_i'''$ is a module in $G$. Hence, $|V_i'''| \leq 1$.

(6) By Lemmas 4 and 5 (part 4), $V \setminus \bigcup_{j=1}^{5} N(v_j)$ is anti-complete to

$$\bigcup_{j=1}^{5}(V_j \cup V_j' \cup V_j'') \cup V''''.$$

By this fact and as $G$ is a connected $P_5$-free graph, each vertex in $V \setminus \bigcup_{j=1}^{5} N(v_j)$ has a neighbor in $\bigcup_{j=1}^{5} V_j'''$. Let a vertex $a \in V_i'''$ be adjacent to a vertex $b \in V \setminus \bigcup_{j=1}^{5} N(v_j)$, and let $c$ be an arbitrary element of $V_{i-2}''' \cup V_{i+2}'''$. By Lemma 6 (part 2), $V_{i-1}''' \cup V_{i+1}''' = \emptyset$. By Lemma 6 (part 3), $ac \notin E$. Then $bc \in E$, otherwise $b, a, v_{i+2}, c, v_{i-1}$ or $b, a, v_{i-2}, c, v_{i+1}$ induce a $P_5$. Hence, by Lemma 8 (part 5), $b$ is adjacent to all vertices in $\bigcup_{j=1}^{5} V_j'''$. Hence, $V \setminus \bigcup_{j=1}^{5} N(v_j)$ is a module in $G$. Therefore, it has at most one element. ■

**Lemma 9.** *For each i, the set $V_i$ is independent and $|V_i| \leq 3$.*

**Proof.** To avoid an induced *dart*, for any $i$, $G(V_i)$ must be $P_3$-free, i.e. this graph is the disjoint union of complete graphs. By Lemma 8 (part 1), any two vertices in $V_i$ have the same sets of neighbors in $V \setminus (V_{i-2} \cup V_i \cup V_{i+2})$. As $G$ is *dart*-free, any two vertices of any connected component of $G(V_i)$ have the same sets of neighbors in $V_{i-2} \cup V_{i+2}$. Hence, all vertices of any connected component of $G(V_i)$ form a module in $G$. Therefore, $V_i$ must be independent.

Suppose that $V_i$ has at least four elements. None of the elements of $V_{i-2} \cup V_{i+2}$ have two neighbors in $V_i$, to avoid an induced *dart*. Let $a, b \in V_i$ be distinct. By Lemma 8 (part 1), there is a vertex $c \in V_{i-2} \cup V_{i+2}$ adjacent to exactly one element of $\{a, b\}$, otherwise $\{a, b\}$ is a module in $G$. Hence, there are vertices $a', b' \in V_i$, a number $j \in \{i-2, i+2\}$, vertices $a'', b'' \in V_j$ such that $a'a'' \in E$, $b'b'' \in E$, $a'b'' \notin E$, $a''b' \notin E$. Then $b'', b'$, and either $v_{i-1}$ or $v_{i+1}$, $a', a''$ induce a $P_5$. ■

In the next lemmas we will also address to Lemmas 5, 6, 8. To simplify readability of the text, we repeat some of their results rephrasing statements of Lemma 8.

**Lemma 5.** *For each i, every of the following statements is true:*
Part 1: *The set $V_i$ is complete to*

$$\bigcup_{j\in\{i-1,i+1\}} (V_j \cup V'_j \cup V''_j) \cup V'''_i$$

*and $V_i$ is anti-complete to $\bigcup_{j\in\{i-2,i+2\}} V'_j$.*
Part 2: *The set $V'_i$ is complete to*

$$\bigcup_{j\in\{i-1,i+1\}} (V_j \cup V'_j) \cup \bigcup_{j\in\{i-2,i+2\}} V''_j \cup \bigcup_{j\in\{i-2,i,i+2\}} V'''_j$$

*and $V'_i$ is anti-complete to $\bigcup_{j\in\{i-2,i+2\}} V_j \cup \bigcup_{j\in\{i-1,i+1\}} V'''_j$.*
Part 4: *The set $V'''_i$ is complete to*

$$V_i \cup \bigcup_{j\in\{i-2,i,i+2\}} V'_j \cup \bigcup_{j\in\{i-2,i+2\}} V''_j \cup \bigcup_{j\in\{i-1,i+1\}} V'''_j$$

*and $V'''_i$ is anti-complete to $\bigcup_{j\in\{i-1,i+1\}} V'_j$.*

**Lemma 6.** *For each i, every of the following statements is true:*
Part 1: *Each of the sets $V'_i$, $V''_i$, $V'''_i$ is a clique. The set $V'_i$ is complete to $V'''_i$.*
Part 2: *If $V_i \neq \emptyset$, then $\bigcup_{j=1,j\neq i}^{5} V''_j \cup V'''' = \emptyset$. If $V'''_i \neq \emptyset$, then the set*

$$\bigcup_{j\in\{i-1,i+1\}} V'''_j \cup \bigcup_{j=1,j\neq i}^{5} V''_j \cup V''''$$

*is empty.*
Part 3: *The set $V_i$ is anti-complete to $V'_i \cup V''_i$, and the set $V'''_i$ is anti-complete to*

$$\bigcup_{j=1,j\neq i}^{5} V_j \cup \bigcup_{j\in\{i-2,i+2\}} V'''_j \cup V''_i.$$

**Lemma 8.** *For each i, every of the following statements is true:*
Part 2: *The vertex $v_i$ and any element $v'_i \in V'_i$ have the same sets of neighbors in*

$$V \setminus (\{v_i, v'_i\} \cup \bigcup_{j\in\{i-2,i+2\}} V'_j \cup \bigcup_{j\in\{i-1,i,i+1\}} V''_j \cup V'''').$$

Part 3: *The vertex $v_i$ and any element of $V_i$ have the same sets of neighbors in*

$$V \setminus ( \bigcup_{j\in\{i-2,i,i+2\}} V_j \cup V'_i \cup \bigcup_{j=1,j\neq i}^{5} V'''_j).$$

Part 4: *Any two elements of $V'_i$ have the same sets of neighbors in*

$$V \setminus ( \bigcup_{j\in\{i-2,i,i+2\}} V'_j \cup \bigcup_{j\in\{i-1,i,i+1\}} V''_j \cup V'''').$$

*Any two elements of $V''_i$ have the same sets of neighbors in*

$$V \setminus ( \bigcup_{j\in\{i-2,i+2\}} V_j \cup \bigcup_{j\in\{i-1,i,i+1\}} V'_j \cup \bigcup_{j=1}^{5} V''_j \cup \bigcup_{j\in\{i-1,i,i+1\}} V'''_j \cup V'''').$$

**Lemma 10.** *The following properties are true:*
(1) *If $V_i \neq \emptyset$, then $\bigcup_{j=1}^{5} V'_j \cup V''_i = \emptyset$.*
(2) *If $\bigcup_{j=1}^{5} V_j = \emptyset$ and $V'''_i \neq \emptyset$, then*

$$V'_{i-1} \cup V'_{i+1} = \emptyset \text{ and } |V'_i \cup V'_{i-2} \cup V'_{i+2} \cup V''_i| \leq 4.$$

**Proof.** (1) Assume that $a \in V_i$. Hence, by Lemma 6 (part 2), the set $\bigcup_{j=1,j\neq i}^{5} V''_j \cup V''''$ is empty. Suppose that $b \in V'_i$. By Lemma 6 (part 3), $ab \notin E$. By Lemma 8 (part 2), $\{b, v_i\}$ is not a module in $G$ iff there is a vertex $c \in V'_{i-2} \cup V'_{i+2} \cup V''_i$ adjacent to $b$. By Lemmas 5 (part 1) and 6 (part 3), $ac \notin E$. Hence, either $v_i, b, c, v_{i+1}, a$ or $v_i, b, c, v_{i-1}, a$ induce a *dart*.

Suppose that $b \in V'_{i+1}$. The case when $b \in V'_{i-1}$ can be considered in a similar way. By Lemma 5 (part 1), $ab \in E$. By Lemma 8 (part 2), $\{b, v_{i+1}\}$ is not a module in $G$ iff there is a vertex $c \in V'_{i-1} \cup V'_{i-2} \cup V''_i$ adjacent to $b$. If $c \in V'_{i-1}$, then $ac \in E$, by Lemma 5 (part 1), and $a, b, c, v_i, v_{i-2}$ induce a *dart*. If $c \in V'_{i-2}$, then $ac \notin E$, by Lemma 5 (part 1), and $a, b, c, v_i, v_{i+1}$ induce a *dart*. By Lemma 6 (part 3), we have $ac \notin E$. If $c \in V''_i$, then $bc \notin E$, and $a, b, c, v_i, v_{i+1}$ induce a *dart*.

Now suppose that $b \in V'_{i-2}$. The case when $b \in V'_{i+2}$ can be considered in a similar way. By Lemma 5 (part 1), $ab \notin E$. By the previous reasonings, we may assume that $V'_i = V'_{i+1} = \emptyset$. Then, by Lemma 8 (part 2), $\{b, v_{i-2}\}$ is a module in $G$.

Suppose that $b \in V''_i$. Hence, $\bigcup_{j=1, j\neq i}^{5} V'''_j \cup V'_i = \emptyset$, by Lemma 6 (part 2) and by the previous reasonings. By this fact, Lemmas 8 (part 3) and 9, $\{a, v_i\}$ is not a module in $G$ iff there is a vertex $V_{i-2} \cup V_{i+2}$ adjacent to $a$. We have a contradiction with Lemma 6 (part 2), as if $V''_i \neq \emptyset$, then $V_{i-2} \cup V_{i+2} = \emptyset$.

(2) Assume that $a \in V'''_i$. Hence,

$$\bigcup_{j\in\{i-1,i+1\}} V'''_j \cup \bigcup_{j=1, j\neq i}^{5} V''_j \cup V''''$$

is empty, by Lemma 6 (part 2). Suppose that $b \in V'_{i-1}$. The case when $b \in V'_{i+1}$ can be considered in a similar way. By Lemma 5 (part 2), $ab \notin E$. By Lemma 8 (part 2), $\{b, v_{i-1}\}$ is not a module in $G$ iff there is a vertex $c \in V'_{i+1} \cup V'_{i+2} \cup V''_i$ adjacent to exactly one element of $\{b, v_{i-1}\}$. If $c \in V'_{i+1} \cup V'_{i+2}$, then $bc \in E$. If $c \in V''_i$, then $bc \notin E$ and $ac \notin E$ (by Lemma 6, part 3), and $b, v_{i-1}, c, v_{i+2}, a$ induce a $P_5$. If $c \in V'_{i+1}$, then $ac \notin E$, by Lemma 5 (part 2), and $a, v_{i-2}, b, c, v_{i+1}$ induce a $P_5$. If $c \in V'_{i+2}$, then $ac \in E$, by Lemma 5 (part 2), and $a, c, v_{i+2}, v_{i+1}, b$ induce a *dart*.

By Lemmas 5 (part 4) and 6 (part 3), $V'''_i$ is complete to $V'_{i-2} \cup V'_i \cup V'_{i+2}$ and is anti-complete to $V''_i$. Hence, $V''_i$ is anti-complete to $V'_i$, otherwise a vertex of $V''_i$, a vertex of $V'_i$, $v_{i-1}, v_{i+1}, a$ induce a *dart*. Similarly, $V'_{i-2} \cup V'_{i+2}$ is anti-complete to $V'_i$, otherwise $a$, a vertex in $V'_{i-2} \cup V'_{i+2}$, a vertex in $V'_i$, $v_{i-1}, v_{i+1}$ induce a *dart*. Recall that

$$V'_{i-1} \cup V'_{i+1} \cup \bigcup_{j\in\{i-1,i+1\}} V'''_j \cup \bigcup_{j=1, j\neq i}^{5} V''_j \cup V'''' = \emptyset.$$

Hence, by Lemmas 6 (part 1) and 8 (part 4), each of the sets $V'_i, V'_{i-2}, V'_{i+2}, V''_i$ is a module in $G$. Hence, $|V'_i \cup V'_{i-2} \cup V'_{i+2} \cup V''_i| \leq 4$. ■

**Lemma 11.** *If $\bigcup_{j=1}^{5}(V_j \cup V'''_j) \neq \emptyset$, then $|V| \leq 26$.*

**Proof.** If $\bigcup_{j=1}^{5} V_j \neq \emptyset$, then $\bigcup_{j=1}^{5}(V'_j \cup V''_j) \cup V'''' = \emptyset$, by Lemmas 10 (part 1) and 6 (part 2). Hence,

$$|V| \leq |V(C_5)| + \sum_{j=1}^{5} |V_j| + \sum_{j=1}^{5} |V'''_j| + |V \setminus \bigcup_{j=1}^{5} N(v_j)| \leq 26,$$

by Lemmas 8 (parts 5 and 6) and 9. If $\bigcup_{j=1}^{5} V_j = \emptyset$ and $V'''_i \neq \emptyset$ for some $i$, then

$$|V| \leq |V(C_5)| + |V'_i \cup V'_{i-2} \cup V'_{i+2} \cup V''_i| + |V'''_i \cup V'''_{i-2} \cup V'''_{i+2}| + |V \setminus \bigcup_{j=1}^{5} N(v_j)| \leq 13,$$

by Lemmas 6 (part 2), 8 (parts 5 and 6), 10 (part 2). ■

### 3.4. Some complexity results for the weighted coloring problem

**Lemma 12** ([14]). *The* WCOL *problem for any $O_3$-free graph $(G, w)$ can be solved in $O((\sum_{v\in V(G)} w(v))^3)$ time.*

**Lemma 13** ([14]). *For each fixed $C$, the* WCOL *problem can be solved in polynomial time on the sum of vertex weights in the class of all graphs having at most $C$ vertices.*

## 4. Main result

A graph is said to be *Berge* if it belongs to the class $Free(\{C_{2i+1} | \, i > 1\} \cup \{\overline{C_{2i+1}} | \, i > 1\})$. A graph is said to be *perfect* if for every its induced subgraph $G$ the chromatic number of $G$ equals the size of a maximum clique of $G$. The Strong Perfect Graph Theorem (see [1]) states that a graph is perfect iff it is Berge. The WCOL problem can be solved in polynomial time for perfect graphs [7].

**Theorem 2.** *The* WCOL *problem can be solved for $\{P_5, banner\}$-free graphs and for $\{P_5, dart\}$-free graphs in polynomial time on the sum of vertex weights.*

**Proof.** Note that a graph is prime if and only if its complement is prime. Note furthermore that $\overline{C_5} = C_5$, and that every $P_5$-free graph is $\{C_{2i+1} | i \geq 3\}$-free. Lemma 2 now implies that if a graph is prime and $\{P_5, banner\}$-free, then either it is Berge (and therefore perfect [1]) or it is $O_3$-free. There is a trivial polynomial-time algorithm of verification whether a given graph is $O_3$-free. Hence, by these facts, results of [7], Lemmas 1 and 12, the WCOL problem can be solved for $\{P_5, banner\}$-free graphs in polynomial time on the sum of vertex weights.

By Lemmas 3, 7 and 11, if a prime $\{P_5, dart\}$-free graph on at least 27 vertices is not $O_3$-free, then it is Berge (and therefore prefect [1]). By this fact, results of [7], Lemmas 1, 12, 13, the WCOL problem can be solved for $\{P_5, dart\}$-free graphs in polynomial time on the sum of vertex weights. ■

A straightforward corollary from Theorem 2 is the fact that the COL problem can be solved in polynomial time for $\{P_5, banner\}$-free graphs and for $\{P_5, dart\}$-free graphs.

## 5. Conclusions and problems for future work

There are many gaps in understanding the complexity of the COL problem for hereditary classes. For example, the complexity of the COL problem is known for all but three classes in the family of hereditary classes defined by forbidden induced subgraphs each on at most four vertices [10]. The remaining three classes are the classes of $\{C_4, O_4\}$-free graphs, $\{K_{1,3}, O_4\}$-free graphs, $\{K_{1,3}, O_4, K_2 + O_2\}$-free graphs [10]. Determining the complexity of the COL problem for these three classes is an interesting problem for future research. There is known an approximation polynomial-time algorithm for the COL problem and the three classes. More specific, there exists a polynomial-time algorithm computing a number $p(G)$ for a graph $G$ such that $\chi(G) \leq p(G) \leq r \cdot \chi(G) + O(1)$, where $r = \frac{3}{2}$ if $G$ is $\{O_4, K_{3,3}\}$-free and $r = \frac{4}{3}$ if $G$ is $\{K_{1,3}, O_4, K_2 + 2K_1\}$-free (see [13]).

In this paper we considered the complexity of the COL problem for $\{G_1, G_2\}$-free graphs, where $G_1$ and $G_2$ are both connected graphs each on at most five vertices. Prior to our study, the complexity of the COL problem was open for each of the eight pairs $\{G_1, G_2\}$ described below (see [8,11,12,14]):

1. $\{K_{1,3}, G\}$, where $G \in \{bull, butterfly\}$
2. $\{for\ k, bull\}$
3. $\{P_5, G\}$, where $G \in \{banner, dart, bull, \overline{K_3 + O_2}, \overline{K_3 + K_2}, \overline{2K_2 + K_1}\}$.

In this paper we showed that the (W)COL problem can be solved in polynomial time (on the sum of vertex weights) for $\{P_5, banner\}$-free graphs and for $\{P_5, dart\}$-free graphs. Clarification of the complexity of the (W)COL problem for the remaining six pairs is an interesting research problem for future work.

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## References

[1] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas, The strong perfect graph theorem, Ann. of Math. 164 (2006) 51–229.
[2] K. Dabrowski, F. Dross, D. Paulusma, Colouring diamond-free graphs, J. Comput. System Sci. 89 (2017) 410–431.
[3] K. Dabrowski, P. Golovach, D. Paulusma, Colouring of graphs with Ramsey-type forbidden subgraphs, Theoret. Comput. Sci. 522 (2014) 34–43.
[4] K. Dabrowski, V. Lozin, R. Raman, B. Ries, Colouring vertices of triangle-free graphs without forests, Discrete Math. 312 (2012) 1372–1385.
[5] P. Golovach, M. Johnson, D. Paulusma, J. Song, A survey on the computational complexity of coloring graphs with forbidden subgraphs, J. Graph Theory 84 (2017) 331–363.
[6] P. Golovach, D. Paulusma, B. Ries, Coloring graphs characterized by a forbidden subgraph, Discrete Appl. Math. 180 (2015) 101–110.
[7] M. Grötschel, L. Lovász, A. Schrijver, Polynomial algorithms for perfect graphs, Ann. Discrete Math. 21 (1984) 325–356.
[8] C. Hoàng, D. Lazzarato, Polynomial-time algorithms for minimum weighted colorings of $(P_5, \overline{P_5})$-free graphs and similar graph classes, Discrete Appl. Math. 186 (2015) 106–111.
[9] D. Král', J. Kratochvíl, Z. Tuza, G. Woeginger, Complexity of coloring graphs without forbidden induced subgraphs, Lecture Notes in Comput. Sci. 2204 (2001) 254–262.
[10] V. Lozin, D. Malyshev, Vertex coloring of graphs with few obstructions, Discrete Appl. Math. 216 (2017) 273–280.
[11] D. Malyshev, The coloring problem for classes with two small obstructions, Optim. Lett. 8 (2014) 2261–2270.
[12] D. Malyshev, Two cases of polynomial-time solvability for the coloring problem, J. Comb. Optim. 31 (2016) 833–845.
[13] D. Malyshev, Polynomial-time approximation algorithms for the coloring problem in some cases, J. Comb. Optim. 33 (2017) 809–813.
[14] D. Malyshev, O. Lobanova, Two complexity results for the vertex coloring problem, Discrete Appl. Math. 219 (2017) 158–166.