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SINGULAR INITIAL-VALUE AND BOUNDARY-VALUE PROBLEMS FOR INTEGRODIFFERENTIAL EQUATIONS IN DYNAMICAL INSURANCE MODELS WITH INVESTMENTS

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Abstract. We investigate two insurance mathematical models of the following behavior of an insurance company in the insurance market: the company invests a constant part of the capital in a risk asset (shares) and invests the remaining part in a risk-free asset (a bank account). Changing parameters (characteristics of shares), this strategy is reduced to the case where all the capital is invested in a risk asset. The first model is based on the classical Cramér–Lundberg risk process for the exponential distribution of values of insurance demands (claims). The second one is based on a modification of the classical risk process (the so-called stochastic premium risk process) where both demand values and insurance premium values are assumed to be exponentially distributed. For the infinite-time nonruin probability of an insurance company as a function of its initial capital, singular problems for linear second-order integrodifferential equations arise. These equations are defined on a semiinfinite interval and they have nonintegrable singularities at the origin and at infinity. The first model yields a singular initial-value problem for integrodifferential equations with a Volterra integral operator with constraints. The second one yields more complicated problem for integrodifferential equations with a non-Volterra integral operator with constraints and a nonlocal condition at the origin. We reduce the problems for integrodifferential equations to equivalent singular problems for ordinary differential equations, provide existence and uniqueness theorems for the solutions, describe their properties and long-time behavior, and provide asymptotic representation of solutions in neighborhoods of singular points. We propose efficient algorithms to find numerical solutions and provide the computational results and their economics interpretation.

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1. Introduction

In this paper, a brief review of the main authors' results (as well as related background results) is provided: we compare two insurance mathematical models under the same behavior of an insurance company in the insurance market, assuming that the company invests its circulating capital in a risk asset, which is shares with prices modelled by a geometrical Brownian motion. The considered models cover a more general case of a permanent investment structure, i.e., the case where a constant part

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of the circulating capital is invested in a risk asset (shares) while the remaining part is invested in a risk-free asset (a bank account with a constant interest rate). To reduce the latter case to the former, we have to change the characteristics of the shares, which are parameters of the problem (this is briefly discussed below).

Model I is based on the classical Cramér–Lundberg risk process for the exponential distribution of values of insurance demands (claims). In the Cramér–Lundberg model, the process describing the capital change is the sum of a determinate process of the insurance premium income and a compound Poisson process of insurance payments (see, e.g., [26, 34]). In [21, 22, 33, 37], a modified Cramér–Lundberg model is described: the premium income process is a compound Poisson process as well and its parameters are different from the ones for the insurance payment process; following [21, 33, 37], we call the corresponding model the Cramér–Lundberg model with stochastic (random) premiums. Model II is based on the Cramér–Lundberg model with stochastic premiums and exponentially distributed values of claims and premiums.

For dynamical insurance models, it is very important to estimate the nonruin probability, which is a traditional determinate solvency characteristic for insurance companies. In the majority of mathematical models, the dynamics of the capital of the insurance company is described by a homogeneous continuous-time Markov process. In particular, if the capital is invested in risk assets, then the specified process is described by a stochastic differential equation. For such processes, treating the nonruin probability as a function of the initial capital, one can find assumptions for the properties of that function such that the integrodifferential equation for them can be found by means of the generator technique (see, e.g., [13, 25] and references therein). The integrodifferential equations are defined on $\mathbb{R}_+$. If their solutions exist such that they are nonnegative on $\mathbb{R}_+$, do not exceed 1, and tend to 1 at infinity, then those solutions determine the sought probability. This can be proved by means of probabilistic methods (for details, see [11] and references therein). In particular, in [11], problems for Models I and II of the present paper are justified in the above sense.

For each of those models, the infinite-time nonruin probability of an insurance company, treated as a function of its initial capital, is a solution of a singular problem for a linear second-order integrodifferential equation defined on a semiinfinite interval and having nonintegrable singularities at the origin and at infinity (see [14–16] and references therein). Model I yields a singular initial-value problem for an integrodifferential equation with a Volterra integral operator with constraints; in [14, 15], this problem is studied in detail. Model II yields a more complicated problem for integrodifferential equations with Volterra and non-Volterra integral operators with constraints and a nonlocal condition at the origin; it is posed and studied in [16].

In this paper, the main results of [14–16] are presented taking into account earlier results obtained by other authors: singular problems for integrodifferential equations are posed, methods to reduce them to equivalent singular problems for ordinary differential equations are described, existence and uniqueness theorems are presented for their solutions, asymptotical representations of solutions in neighborhoods of singular points are given, the long-time behavior of solutions of the posed problems is described, efficient algorithms of finding numerical solutions are presented, and computational results are provided and interpreted (some misprints and errors found in [14–16] are corrected here). For important assertions, we present brief schemes or their proofs.

In the sequel, we use the following notation: $\mathbf{P}(A)$ is the probability of an event A and $\mathbf{E}X$ is the mathematical expectation of a random variable X. The other notation is introduced below when needed.

2. Singular Problems for Second-Order Integrodifferential Equations with Constraints

First, we pose two singular problems for integrodifferential equations related to Model I and Model II. The models itself leading to the above problems are described in the next section.

2.1. Problem 1: singular problem for Model I. In Model I, a second-order integrodifferential equation with a Volterra integral operator, arises for the nonruin probability $\varphi(u)$ (considered as a function of the initial capital u). That equation is defined on $\mathbb{R}_+$ and has singularities as $u \to +0$ and $u \to \infty$. The following singular problem with constraints for the range of the solution is posed for that integrodifferential equation:

$$(b^2/2)u^2\varphi''(u) + (au + c)\varphi'(u) - \lambda\,[\varphi(u) - (J_m\varphi)(u)] = 0, \quad 0 < u < \infty, \tag{2.1}$$

$$\{|\lim_{u\to+0}\varphi(u)|,\, |\lim_{u\to+0}\varphi'(u)|\} < \infty, \quad \lim_{u\to+0}[c\,\varphi'(u) - \lambda\,\varphi(u)] = 0, \tag{2.2}$$

$$0 \le \varphi(u) \le 1, \qquad u \in \mathbb{R}_+, \tag{2.3}$$

$$\lim_{u\to\infty}\varphi(u) = 1, \qquad \lim_{u\to\infty}\varphi'(u) = 0. \tag{2.4}$$

Unless the opposite is stated, the parameters a, b, c, λ, and m are real and positive and J_m is the following Volterra integral operator:

$$(J_m\varphi)(u) = \frac{1}{m}\int_0^u \varphi(u - x)\exp(-x/m)\,dx = \frac{1}{m}\int_0^u \varphi(s)\exp\big(-(u - s)/m\big)\,ds, \quad u \in \mathbb{R}_+, \tag{2.5}$$

where $J_m : C[0,\infty) \to C[0,\infty)$ and $C[0,\infty)$ is the real linear space of functions continuous and bounded on $\mathbb{R}_+$.

The second limit condition as $u \to 0$ follows from the first one and from the integrodifferential equation itself: conditions (2.2) imply the limit relation $\lim_{u\to+0}[u^2\varphi''(u)] = 0$ in (2.1), ensuring the degeneration of Eq. (2.1) as $u \to +0$ (any solution $\varphi(u)$ of problem (2.1), (2.2), which is a singular problem without initial-value data, satisfies Eq. (2.1) till the singular point $u = 0$).

The "truncated" problem (2.1)–(2.3), which is a singular problem with constraints, always has the trivial solution $\varphi(u) \equiv 0$. To select a nontrivial solution, condition (2.4) is imposed.

2.2. Problem 2: singular problem for Model II. In Model II, a second-order integrodifferential equation with Volterra and non-Volterra integral operators arises for the nonruin probability $\varphi(u)$ (considered as a function of the initial capital u). That equation is defined on $\mathbb{R}_+$ and has singularities as $u \to +0$ and $u \to \infty$. The following singular problem with constraints for the range of the solution is posed for that integrodifferential equation:

$$(b^2/2)u^2\varphi''(u) + au\varphi'(u) - \lambda[\varphi(u) - (J_m\varphi)(u)] - \lambda_1[\varphi(u) - (J_{1,n}\varphi)(u)] = 0, \quad 0 < u < \infty, \tag{2.6}$$

$$|\lim_{u\to+0}\varphi(u)| < \infty, \qquad \lim_{u\to+0}[u\varphi'(u)] = 0, \tag{2.7}$$

$$(\lambda + \lambda_1)\lim_{u\to+0}\varphi(u) = \frac{\lambda_1}{n}\int_0^\infty \varphi(y)\exp(-y/n)\,dy, \tag{2.8}$$

$$0 \le \varphi(u) \le 1, \qquad u \in \mathbb{R}_+, \tag{2.9}$$

$$\lim_{u\to+\infty}\varphi(u) = 1, \qquad \lim_{u\to+\infty}\varphi'(u) = 0. \tag{2.10}$$

Unless the opposite is stated, the parameters a, b, λ, λ_1, m, and n are real and positive, J_m is the Volterra integral operator defined by (2.5), and $J_{1,n}$ is the following non-Volterra integral operator:

$$(J_{1,n}\varphi)(u) = \frac{1}{n}\int_0^\infty \varphi(u + y)\exp(-y/n)\,dy = \frac{1}{n}\int_u^\infty \varphi(s)\exp(-(s - u)/n)\,ds, \quad u \in \mathbb{R}_+, \tag{2.11}$$

where $J_{1,n} : C[0,\infty) \to C[0,\infty)$.

The second identity in (2.11) can be treated as a transformation of a non-Volterra integral operator with deviating argument into a singular Volterra operator. In any case, the sum $J_{m,n} : (J_{m,n}\varphi)(u) = \lambda\,(J_m\varphi)(u) + \lambda_1\,(J_{1,n}\varphi)(u)$, $u \in \mathbb{R}_+$, is a singular non-Volterra[1] operator.

Condition (2.7) at the origin and relation (2.8) imply the limit relation $\lim\limits_{u\to+0}[u^2\varphi''(u)] = 0$ in (2.6). This implies that Eq. (2.6) degenerates as $u \to +0$ (if there exists a solution $\varphi(u)$ of the singular problem (2.6)–(2.10), then it satisfies the integrodifferential equation (2.6) till the singular point $u = 0$). Also, it follows from conditions (2.7) that the first derivative of the solution is integrable at the origin though its boundedness is not guaranteed.

The "truncated" problem (2.6)–(2.9), which is a singular problem with constraints and a nonlocal condition at the origin, always has a trivial solution $\varphi(u) \equiv 0$; to select a nontrivial one, we impose condition (2.10).

3. Genesis od Problems: Dynamical Insurance Models with Investments in Risk Assets and Nonruin Probability for Insurance Companies

3.1. Classical Cramér–Lundberg risk theory model. The classical continuous-time risk process (where the premium income is a determinate process with constant positive intensity $c > 0$) has the following form (see, e.g., [26, 34]):

$$R_t = u + ct - \sum_{k=1}^{N(t)} Z_k, \qquad t \geq 0. \tag{3.1}$$

Here R_t is the amount of the capital of the insurance company at the time t, u is the value of the initial capital, c is the income velocity of insurance payments (c is the premium value per time period and ct is the total insurance payment received by the time t), the sum on the right-hand side is the total insurance premium, $N(t)$ is the homogeneous Poisson process with positive intensity λ ($\mathbf{E}N(t) = \lambda t$, $N(0) = 0$) determining the number of client claims during the time interval $(0, t]$ for any positive t, and $Z_1, Z_2, \ldots$ are independent random variables with the same distribution function $F(x)$ ($F(0) = 0$, $\mathbf{E}Z_1 = m < \infty$) that express values of the claims and do not depend on the process $N(t)$ (so that Z_j is the payment for the jth claim initiated at a random moment generating the jth jump of the process $N(t)$).

For the risk process (3.1), we provide the classical definition of the relative "security load" characterizing the expected "specific income" of an insurance company per time period (see, e.g., [26, 34]).

Definition 3.1. The *security load* (*coefficient*) for the risk process (3.1) is

$$\rho_1 = (c - \lambda m)/(m\lambda) \tag{3.2}$$

and the condition

$$c - \lambda m > 0 \tag{3.3}$$

provides the positive of the expected net revenue.

Let $\tau = \inf\{t : R_t < 0\}$ denote the ruin moment; then $\mathbf{P}(\tau < \infty)$ is the infinite-time ruin probability.

The following assertion is a classical result from the Cramér–Lundberg risk theory (see, e.g., [26]).

Assertion 3.1. *Suppose that condition* (3.3) *is satisfied and there exists a positive constant* R (*"the Lundberg coefficient"*) *such that*

$$\int_0^\infty [1 - F(x)] \exp(Rx)dx = c/\lambda. \tag{3.4}$$

[1] Definitions and investigation of Volterra and non-Volterra operators for classes of systems of functional-differential equations (including integrodifferential, nonlinear, and singular equations) can be found, e.g., in [8, 31, 32] (see also references therein).

Then the ruin probability $\xi(u)$ (*considered as a function of the initial capital*) *satisfies the estimate*

$$\xi(u) = \boldsymbol{P}(\tau < \infty) \le \exp(-Ru), \qquad u \ge 0. \tag{3.5}$$

Moreover, if the distribution of insurance payments is exponential, i.e.,

$$F(x) = 1 - \exp(-x/m), \qquad m > 0, \qquad x \ge 0, \tag{3.6}$$

then $R = (c - \lambda m)/(mc) > 0$ *and the nonruin probability* $\varphi(u) = 1 - \xi(u)$ *is expressed as follows*:

$$\varphi(u) = \varphi_1(u) = 1 - \frac{\lambda m}{c} \exp\left(-\frac{c - \lambda m}{mc} u\right), \qquad 0 \le u < \infty. \tag{3.7}$$

Remark 3.1. In the Cramér–Lundberg model, Eq. (3.4) is called the *characteristic equation.* It can be written as follows:

$$\lambda[\mathbf{E} \exp(RZ_i) - 1] - cR = 0. \tag{3.8}$$

If there exists a positive solution R of this equation, then the process $\exp(-RR_t)$ is a martingale and $\mathbf{E} \exp(-RR_t) = \exp(-Ru)$ (taking into account (3.8), one can verify that directly). Using this fact, one can easily obtain ruin probability estimates such as (3.5) and more general estimates (see, e.g., [26]).

Remark 3.2. The history of creation of the collective risk theory and the stochastic process theory is quite interesting. We provide a very brief review (for details, see, e.g., [33, 35] and `www.hse.ru/news/avant/85659995.html`).

A recognized founder of the collective risk theory is Ernst Filip Oskar Lundberg (1876–1965). The mathematical collective risk theory is established by Carl Harald Cramér (1893–1985). References to main publications of those Swedish scientists and a description of their results can be found, e.g., in [26].

It is generally accepted that the birthday of the financial mathematics is March 29, 1900. On that day, Louis Bachelier, follower of Henri Poincaré, defended his thesis “Théorie de la spéculation” [9]. In particular, he proved that the evolution of asset market prices can be described by a stochastic process with continuous trajectories (called Wiener process or Brownian motion nowadays). It happened that Bachelier was recognized as the founder of financial mathematics only many decades later.

Along with the Wiener process, the Poisson process introduced by Lundberg for the collective risk theory became the basis of contemporary stochastic process theory.

3.2. Cramér–Lundberg model with stochastic premiums. Described in [21, 22] (see also [33, Sec. 9.5]), this model treats the continuous-time risk process in the following form:

$$R_t = u + \sum_{i=1}^{N_1(t)} C_i - \sum_{j=1}^{N(t)} Z_j, \qquad t \ge 0. \tag{3.9}$$

Here R_t is the amount of the capital of the insurance company at the time t, u is the value of the initial capital, the first sum on the right-hand side is the total insurance premium by the time t, $N_1(t)$ is the Poisson process with positive intensity λ_1, ($\mathbf{E}N_1(t) = \lambda_1 t$, $N_1(0) = 0$) determining the number of premiums paid by clients during the time interval $(0, t]$ for any positive t, $C_1, C_2, \dots$ are independent stochastic variables with the same distribution function $G(y)$ ($G(0) = 0$, $\mathbf{E}C_1 = n < \infty$) defining the values of premiums (those stochastic variables are assumed to be independent of the process $N_1(t)$ and each C_i is the ith payment occurring at a random time determining the time of the ith jump of the process $N_1(t)$), and the second sum is equal to the total insurance premium, i.e., is the same as in (3.1). The total premium process and total claim process are assumed to be independent as well.

For the risk process (3.9), we introduce a definition of the relative security load similar to Definition 3.1 introduced for the classical Cramér–Lundberg model.

Definition 3.2. The value

$$\rho_2 = (\lambda_1 n - \lambda m)/(\lambda m) \tag{3.10}$$

is called the *security load* for the risk process (3.9) and the condition

$$\lambda_1 n - \lambda m > 0 \tag{3.11}$$

yields the positivity of the expected net revenue.

If the values of claims and premiums are exponentially distributed, i.e.,

$$F(x) = 1 - \exp(-x/m), \quad G(y) = 1 - \exp(-y/n), \quad m, n > 0, \quad x, y \geq 0, \tag{3.12}$$

then the nonruin probability $\varphi(u)$ satisfies the integral equation

$$(\lambda + \lambda_1)\varphi(u) = \lambda (J_m\varphi)(u) + \lambda_1 (J_{1,n}\varphi)(u), \qquad u \in \mathbb{R}_+. \tag{3.13}$$

If condition (3.11) is satisfied, then the nonruin probability $\varphi(u)$ is expressed by the relation

$$\varphi(u) = \varphi_2(u) = 1 - \frac{\lambda (n+m)}{n(\lambda + \lambda_1)} \exp\left(-\frac{\lambda_1 n - \lambda m}{mn(\lambda + \lambda_1)} u\right), \quad u \in \mathbb{R}_+, \tag{3.14}$$

because it is a solution of Eq. (3.13) that is positive on $\mathbb{R}_+$ and it does not exceed 1 (see [21]).

Remark 3.3. In the Cramér–Lundberg model with stochastic premium, the process $\exp(-RR_t)$ is a martingale and estimates for the nonruin probability (including estimates of type (3.5)) are obtained due to the characteristic equation having the following form (see [21, 37]):

$$\lambda[\mathbf{E}\exp(RZ_i) - 1] + \lambda_1[\mathbf{E}\exp(-RC_i) - 1] = 0. \tag{3.15}$$

It is equivalent to the equation

$$\lambda \int_0^\infty \exp(Rx)dF(x) + \lambda_1 \int_0^\infty \exp(-Ry)dG(y) = \lambda + \lambda_1. \tag{3.16}$$

3.3. Insurance models with investments in risk assets. Now, consider the case where the capital is continuously invested in shares such that the dynamics of their prices is described by the model of the geometric Brownian motion:

$$dS_t = S_t(a\,dt + b\,dw_t), \qquad t \geq 0. \tag{3.17}$$

Here S_t is the price of a share at the time t, a is the expected profitability of a share, $b > 0$ is the volatility, and $\{w_t\}$ is the standard Wiener process.

Denoting the company capital at the time t by X_t, we see that $X_t = \theta_t S_t$, where θ_t is the number of shares in the portfolio. Then the capital changes as a function of time according to the relation

$$dX_t = \theta_t dS_t + dR_t,$$

where R_t is the original risk process.

Taking into account (3.17), we see that

$$dX_t = aX_t\,dt + bX_t\,dw_t + dR_t, \qquad t \geq 0. \tag{3.18}$$

Note that, unlike initial risk processes, we do not assume the positivity of the "security load" in models with investments in risk assets, i.e., we do not suppose that condition (3.3) for the classical model or condition (3.11) for the Cramér–Lundberg model with stochastic premium holds.

For the dynamical process (3.18) with the initial risk process (3.1), the nonruin probability function $\varphi(u)$ satisfies the linear integrodifferential equation (see, e.g., [13, 25] and references therein)

$$\lambda \int_0^u \varphi(u-z)\,dF(z) - \lambda\,\varphi(u) + (au + c)\,\varphi'(u) + (b^2/2)\,u^2\,\varphi''(u) = 0, \quad u \in \mathbb{R}_+, \tag{3.19}$$

and the following limit relation providing the degeneration of Eq. (3.19) as $u \to +0$ is satisfied at the singular point $u = 0$:

$$\lim_{u \to +0} [c\varphi'(u) - \lambda\varphi(u)] = 0. \tag{3.20}$$

Assuming that the payments are distributed exponentially according to (3.6), from (3.19) we obtain the integrodifferential equation (2.1) for Model I.

For the dynamical process (3.18) with the initial risk process (3.9), the nonruin probability function $\varphi(u)$ satisfies the linear integrodifferential equation

$$(b^2/2)u^2\varphi''(u) + au\varphi'(u) = \lambda \left[\varphi(u) - \int_0^u \varphi(u-x)\,dF(x)\right] + \lambda_1 \left[\varphi(u) - \int_0^\infty \varphi(u+y)dG(y)\right] \tag{3.21}$$

on $\mathbb{R}_+$ (see [22]) and the following limit nonlocal condition providing the degeneration of Eq. (3.21) as $u \to +0$ is satisfied at the singular point $u = 0$:

$$(\lambda + \lambda_1) \lim_{u \to +0} \varphi(u) = \lambda_1 \int_0^\infty \varphi(y)\,dG(y) \tag{3.22}$$

(cf. the local condition (3.20) for Model I). Conditions of type (3.22) do not arise in [22] for (3.21); we discuss the necessity of such conditions below (see also [16]).

Assuming that the payments are distributed exponentially according to (3.12), from (3.21) we obtain the integrodifferential equation (2.6) for Model II.

Remark 3.4. Models under consideration cover more general investment strategies with constant structure where instead of all the circulating capital only its fixed part α ($0 < \alpha < 1$) is invested in shares (with expected profitability μ and volatility σ) while the remaining (positive) part $1 - \alpha$ is invested in a risk-free asset (a bank account with a constant interest rate $r > 0$). To reduce the case where $0 < \alpha < 1$ to the case where $\alpha = 1$, one can change the parameters of the problem (characteristics of the shares): $a = \alpha\mu + (1 - \alpha)r$ and $b = \alpha\sigma$ (for details, see [14–16]). Thus, it remains to consider only models where all the circulating capital is invested in risk assets; this is done in [14–16] and the present paper.

Singular problems and corresponding models become different if some parameters of the considered integrodifferential equations vanish: for $b = 0$, we obtain degenerated problems for insurance models where all the circulating capital is invested in risk-free assets; for $c = 0$ or $\lambda_1 = 0$, we obtain problems for noninsurance models like charity foundations (no insurance payments in the model). The passages (along the parameters) from the original problems to the new ones are singular for small (and/or large) values of the initial capital. Such problems and corresponding models are interesting from the mathematical point of view and from the point of view of the collective risk theory, but we do not consider them in this paper (apart from the degenerated problems obtained for $a = b = 0$ corresponding to investment-free models). For Model I, the most comprehensive investigation of problem (2.1)–(2.4) including all "degenerated" cases is done in [17].

Remark 3.5. For Model I, the singular problem for Eq. (2.1) considered on the positive semiaxis (problem 1) was posed and completely investigated in [14, 15, 17]. However, the first result for the nonruin probability $\varphi(u)$ considered as a solution of Eq. (2.1) under the assumption of its existence was obtained in [25]: it referred to the asymptotic representation of $\varphi(u)$ for large values of the initial capital u.

Theorem 3.1 (see [25]). *Let $b > 0$ and the payment value be exponentially distributed, i.e., Eq.* (3.6) *be satisfied. Then*:

(1) *If the "reliability condition" for shares holds, i.e., the inequality*

$$\rho = 2a/b^2 > 1 \tag{3.23}$$

is satisfied, then the asymptotic representation

$$\varphi(u) = 1 - Ku^{1-\rho}[1 + o(1)], \quad u \to \infty, \tag{3.24}$$

is true for some constant $K > 0$.

(2) *If $\rho < 1$, then $\varphi(u) \equiv 0$, $u \in \mathbb{R}_+$.*

4. Problem 1: Main Results

The singular problem (2.1)–(2.4) with constraints for the range of solution can be represented in the following equivalent parametric form:

$$(b^2/2)u^2\varphi''(u) + (au + c)\varphi'(u) - \lambda[\varphi(u) - (J_m\varphi)(u)] = 0, \quad u \in \mathbb{R}_+, \tag{4.1}$$

$$\lim_{u\to+0} \varphi(u) = C_0, \qquad \lim_{u\to+0} \varphi'(u) = \lambda C_0/c, \tag{4.2}$$

$$\lim_{u\to\infty} \varphi(u) = 1, \qquad \lim_{u\to\infty} \varphi'(u) = 0, \tag{4.3}$$

$$0 \le \varphi(u) \le 1, \qquad u \in \mathbb{R}_+, \tag{4.4}$$

where the value of the unknown parameter C_0 is to be defined.

Main corollaries from results of [14, 15] are provided below.

4.1. Existence, uniqueness, and behavior of solutions. Auxiliary singular Cauchy problem for ordinary differential equations.

Lemma 4.1. *Let all the parameters of Eq. (4.1) be fixed real numbers such that $c > 0$, $\lambda > 0$, and $m > 0$. Let there exist a solution $\varphi_1(u) = \varphi(u, C_0)$ of problem (4.1)–(4.3) (which is a problem without constraints for the range of the solution) for some $C_0 \in \mathbb{R}$. Then such a constant C_0 is unique, $0 < C_0 < 1$, the function $\varphi(u) = \varphi_1(u)$ satisfies condition (4.4) and is a unique solution of problem (4.1)–(4.4) (which is a problem with constraints for the range of the solution), and $\varphi'(u) > 0$ for any finite $u \in \mathbb{R}_+$, i.e., $\varphi(u)$ strictly increases on $\mathbb{R}_+$.*

Proof.

1. To prove the uniqueness, suppose, to the contrary, that $\varphi_2(u)$ is another solution of problem (4.1)–(4.3). Then either

$$\lim_{u\to+0} \varphi_2(u) = \lim_{u\to+0} \varphi_1(u) \tag{4.5}$$

or

$$\lim_{u\to+0} \varphi_2(u) \ne \lim_{u\to+0} \varphi_1(u). \tag{4.6}$$

If (4.5) is satisfied, then consider the difference $\widetilde{\varphi}(u) = \varphi_2(u) - \varphi_1(u)$. This function is a solution of Eq. (4.1) such that

$$\lim_{u\to+0} \widetilde{\varphi}(u) = \lim_{u\to\infty} \widetilde{\varphi}(u) = 0. \tag{4.7}$$

If there exists a nontrivial solution of problem (4.1), (4.7) that is positive on $\mathbb{R}_+$, then it has a positive maximum on $\mathbb{R}_+$ (if the function $\widetilde{\varphi}(u)$ is nonpositive on $\mathbb{R}_+$, then consider the function $-\widetilde{\varphi}(u)$ instead). Suppose that $\widetilde{u} > 0$ is a maximum point of such a solution: $\widetilde{\varphi}(\widetilde{u}) = \max_{u\in[0,\infty)} \widetilde{\varphi}(u) > 0$. Then $\widetilde{\varphi}'(\widetilde{u}) = 0$ and $\widetilde{\varphi}''(\widetilde{u}) \le 0$. However, a contradiction follows from (4.1) and (2.5):

$$(b^2/2)\widetilde{u}^2\widetilde{\varphi}''(\widetilde{u}) = \lambda\widetilde{\varphi}(\widetilde{u}) - \lambda m^{-1} \exp(-\widetilde{u}/m) \int_0^{\widetilde{u}} \widetilde{\varphi}(s) \exp(s/m)\, ds$$

$$\ge \lambda\widetilde{\varphi}(\widetilde{u}) \left[1 - m^{-1} \exp(-\widetilde{u}/m) \int_0^{\widetilde{u}} \exp(s/m)\, ds\right] = \lambda\widetilde{\varphi}(\widetilde{u}) \exp(-\widetilde{u}/m) > 0. \tag{4.8}$$

Therefore, $\widetilde{\varphi}(u) \equiv 0$.

If (4.6) is satisfied, then it is easy to see that there exists a linear combination $\widehat{\varphi}(u) = c_1\varphi_1(u) + c_2\varphi_2(u)$ of solutions such that $\widehat{\varphi}(u) \not\equiv 1$ and it is a solution of Eq. (4.1) satisfying the limit conditions

$$\lim_{u\to+0} \widehat{\varphi}(u) = \lim_{u\to\infty} \widehat{\varphi}(u) = 1.$$

If there exists $\widehat{u}$ such that $\widehat{\varphi}(\widehat{u}) > 1$, then the line of reasoning is the same as in the first case. The assumption that $\widehat{\varphi}(u) \le 1$ for any nonnegative u contradicts conditions (4.2) because those conditions imply that $\lim\limits_{u\to+0} \widehat{\varphi}'(u) = \lambda/c > 0$. Hence, there is no other solution of problem (4.1)–(4.3) satisfying condition (4.6).

2. The remaining assertions are proved by contradiction in the same way. □

The second-order integrodifferential equation (4.1) can be reduced to a third-order ordinary differential equation. This is an important reduction. To do this, we differentiate (4.1) and, taking into account the relation

$$(J_m\varphi)'(u) = [\varphi(u) - (J_m\varphi)(u)]/m \tag{4.9}$$

and using the original integrodifferential equation (4.1), remove the integral $(J_m\varphi)(u)$ from the obtained third-order integrodifferential equation.

This transforms the singular initial-value problem for an integrodifferential equation (4.1), (4.2) into the singular Cauchy problem for an ordinary differential equation.

Lemma 4.2. *Let all parameters of Eq.* (4.1) *be fixed real numbers such that* $c > 0$, $b \ne 0$, $\lambda \ne 0$, *and* $m > 0$.

Then for any fixed value of the real parameter C_0, *the singular "integrodifferential" initial-value problem* (4.1), (4.2) *is equivalent to the singular "differential" Cauchy problem*

$$(b^2/2)u^2\varphi'''(u) + \left[c + (b^2 + a)u + b^2u^2/(2m)\right]\varphi''(u) + (a - \lambda + c/m + au/m)\,\varphi'(u) = 0, \quad u > 0, \tag{4.10}$$

$$\lim_{u\to+0} \varphi(u) = C_0, \quad \lim_{u\to+0} \varphi'(u) = \lambda\, C_0/c, \quad \lim_{u\to+0} \varphi''(u) = [m(\lambda - a) - c]\lambda\, C_0/(mc^2). \tag{4.11}$$

Proof. The passage from Eq. (4.1) to Eq. (4.10) obviously follows from the above construction of Eq. (4.10). The third limit relation in (4.11) follows from the first and the second ones and from Eq. (4.10) itself, providing its degeneration as $u \to +0$.

Now, let $\widetilde{\varphi}(u) = \widetilde{\varphi}(u, C_0)$ be a solution of the singular Cauchy problem (4.10), (4.11). We need to prove that $\widetilde{\varphi}(u)$ satisfies the integrodifferential equation (4.1).

By $g(u)$ denote the left-hand side of Eq. (4.1) with the function $\widetilde{\varphi}(u)$. We claim that $g(u) \equiv 0$. Indeed, taking into account the method of derivation of Eq. (4.10), it is easy to see that $g(u)$ satisfies the following first-order ordinary differential equation [15, 17]:

$$g'(u) + g(u)/m = 0, \quad u > 0.$$

Its general solution is

$$g(u) = \widetilde{C}\,\exp(-u/m), \quad u \ge 0,$$

where $\widetilde{C}$ is an arbitrary constant. However, taking into account that $\widetilde{\varphi}(u)$ satisfies condition (4.11), we deduce the relation $g(0) = 0$ from the integrodifferential equation (4.1). This implies that $\widetilde{C} = 0$, i.e., $g(u) \equiv 0$. □

The third-order linear ordinary differential equation (4.10) has irregular (strong) singularities[2] of rank 1 as $u \to +0$ and as $u \to \infty$. In [14, 15], results of [19, 20, 29, 30] are used to investigate singular Cauchy problems in neighborhoods of singular points of the obtained third-order ordinary differential equation.

Lemma 4.3. *Let the conditions of Lemma* 4.2 *be fulfilled. Then*:

[2]The classification of pole-like singularities for systems of linear ordinary differential equation can be found, e.g., in [18, 23, 24, 28, 36].

(1) *The solution* $\varphi(u, C_0)$ *of the singular Cauchy problem* (4.10), (4.11) *for the ordinary differential equation* (*respectively, of the equivalent initial-value problem* (4.1), (4.2) *for the integrodifferential equation*) *exists, is unique, and is represented by the asymptotic series*

$$\varphi(u, C_0) \sim C_0 \left[1 + \frac{\lambda}{c}\left(u + \sum_{k=2}^{\infty} D_k u^k/k\right)\right], \qquad u \sim +0, \tag{4.12}$$

where the constant coefficients D_k *do not depend on* C_0 *and are defined by formal substitution of series* (4.12) *in Eq.* (4.10):

$$D_2 = -[(a-\lambda)/c + 1/m], \tag{4.13}$$

$$D_3 = -[D_2(b^2 + 2a - \lambda + c/m) + a/m]/(2c), \tag{4.14}$$

and

$$D_k = -\{D_{k-1}[(k-1)(k-2)b^2/2 + (k-1)a - \lambda + c/m] + D_{k-2}[(k-3)b^2/2 + a]/m\}/[c(k-1)], \tag{4.15}$$

$$k = 4, 5, \dots.$$

(2) *All solutions of the ordinary differential equation* (4.10) (*the singular initial-value problem* (4.1), (4.2) *for the integrodifferential equation*) *have finite limits as* $u \to \infty$ *if and only if inequality* (3.23), *which is the "reliability condition" for shares, is satisfied.*

Theorem 4.1. *Let the parameters* a, b^2, c, λ, *and* m *in Eq.* (4.1) *be fixed positive numbers. Let condition* (3.23) *be satisfied. Then the following assertions are valid*:

(1) *Problem* (4.1)–(4.4) *has a unique solution* $\varphi(u)$. *It is infinitely differentiable on* $(0, \infty)$ *and strictly increasing on* $\mathbb{R}_+$. *If all the parameters of Eq.* (4.1) *are positive, then inequality* (3.23) *is a necessary and sufficient condition for the solvability of problem* (4.1)–(4.4).

(2) *The solution* $\varphi(u)$ *can be obtained as a solution of problem* (4.1), (4.2) *for the integrodifferential equation* (*it is a solution of the equivalent problem* (4.10), (4.11) *for the ordinary differential equation as well*), *where the value of the positive parameter* C_0 *is selected to satisfy condition* (4.3) *normalizing the solution at infinity. Condition* (4.4) *always holds for such a solution.*

(3) *For small* $u > 0$ *the asymptotic representation* (4.12) *with* $0 < C_0 < 1$ *is valid for* $\varphi(u)$.

(4) *The asymptotic representation*

$$\varphi(u) = 1 - K\, u^{1-2a/b^2}[1 + o(1)], \qquad u \to \infty, \tag{4.16}$$

holds, where $K = C_0\widetilde{K}$ (C_0 *and* $\widetilde{K}$ *are positive and cannot be found by local analysis*).

(5) *If* $i_{\mathrm{r,I}} \geq 0$, *where*

$$i_{\mathrm{r,I}} = m(a - \lambda) + c \tag{4.17}$$

defines the "risk factor" for Model I, *then* $\varphi(u)$ *is concave on* $\mathbb{R}_+$; *if* $i_{\mathrm{r,I}} < 0$, *then there exists a positive point of inflection* $\widehat{u}$ *such that* $\varphi(u)$ *is convex on* $[0, \widehat{u}]$.

Remark 4.1. Assertion 5 of Theorem 4.1 is more exact than the one from [14, 15, 17], which states (without a proof) that the (stronger) inequality $i_{r,\mathrm{I}} > 0$ implies the concavity on $\mathbb{R}_+$ of the solution $\varphi(u)$ of problem (4.1)–(4.4). For small positive u, the latter assertion follows from expansion (4.12) and relation (4.13): $\varphi''(u) < 0$ for small nonnegative u and $\varphi''(u)$ does not change sign as u increases. Indeed, let there exist a positive $\widetilde{u}$ such that $\varphi''(\widetilde{u}) = 0$. Then (4.10) implies the inequality $\varphi'''(\widetilde{u}) < 0$, but $\varphi''(u) > 0$ for $u > \widetilde{u}$; this yields a contradiction.

We add one more new assertion: for $i_{r,\mathrm{I}} = 0$ the solution $\varphi(u)$ remains concave on $\mathbb{R}_+$. To prove this, we note that relations (4.12)–(4.14) imply that $\varphi''(0) = 0$ and $\varphi'''(0) = -am/(2c) < 0$. Thus, there exists a neighborhood of the origin where $\varphi''(u)$ is negative and, as above, $\varphi''(u)$ does not change sign as u increases. Relations (4.12)–(4.14) imply that there exists a positive point of inflection $\widehat{u}$ such that the function $\varphi(u)$ is convex on $[0, \widehat{u}]$ provided that $i_{r,\mathrm{I}} < 0$ (note that the nonstrict inequality $i_{r,\mathrm{I}} \leq 0$ is claimed in [14, 15, 17] by mistake).

Remark 4.2. For $a < 0$, Lemma 4.2 and assertion 1 of Lemma 4.3 are applied to investigate the optimal control of investments for the model "with constraints on loans admitting short positions" (see [12]). In general, this model is nonlinear. The behavior of its Bellman function for small values of the independent variable is described by a singular linear Cauchy problem of kind (4.10), (4.11), while condition (3.23) is not imposed.

Remark 4.3. If condition (3.3) is satisfied, then function (3.7) is an exact solution of the degenerated problem obtained from the original singular problem (2.1)–(2.4) by assigning $a = b = 0$, i.e., a solution of the following problem for an integrodifferential equation:

$$c\varphi'(u) - \lambda[\varphi(u) - (J_m\varphi)(u)] = 0, \quad u \in \mathbb{R}_+, \quad c\varphi'(0) - \lambda\varphi(0) = 0, \quad \lim_{u\to\infty} \varphi(u) = 1. \tag{4.18}$$

This problem is equivalent to the Cauchy problem with parameter and normalization condition at infinity:

$$c\varphi''(u) + (c/m - \lambda)\, \varphi'(u) = 0, \qquad u \in \mathbb{R}_+, \tag{4.19}$$

$$\varphi(0) = C_0, \qquad \varphi'(0) = \lambda C_0/c, \qquad \lim_{u\to\infty} \varphi(u) = 1. \tag{4.20}$$

This implies that $C_0 = 1 - \lambda\, m/c$, $0 < C_0 < 1$, $\varphi'(0) > 0$, $\varphi''(0) < 0$, and relation (3.7) holds determining the exact classical solution for the Cramér–Lundberg model with positive security load.

Remark 4.4. For problems (4.18) and (4.19), (4.20), the critical value of the bifurcation parameter is equal to $c = \lambda m$: for $c \le \lambda m$, those problems have no solutions.[3] For Model I, due to the presence of investments, the inequality $\varphi(u) > 0$ is satisfied on $\mathbb{R}_+$ even if the risk factor is negative, i.e., if $i_{r,\mathrm{I}} < 0$ (where $i_{r,\mathrm{I}}$ is defined by (4.17)).

4.2. Algorithm of a numerical solution. Theorem 4.1 allows us to find the solution of problem (4.1)–(4.4) numerically, using the solution of the auxiliary singular Cauchy problem (4.10), (4.11) with parameter C_0 such that its value is defined by the normalization conditions (4.3) at infinity. In [16], the following improved algorithm is presented. In (4.10), we assign $\psi(u) = \varphi'(u)$ and, taking into account (4.11), consider the following auxiliary singular Cauchy problem:

$$(b^2/2)u^2\psi''(u) + [c + (b^2 + a)u + b^2u^2/(2m)]\psi'(u) + [a - \lambda + c/m + au/m]\psi(u) = 0, \;\; u > 0, \tag{4.21}$$

$$\lim_{u\to+0} \psi(u) = 1, \qquad \lim_{u\to+0} \psi'(u) = [m(\lambda - a) - c]/(mc) = -i_{r,\mathrm{I}}/(mc). \tag{4.22}$$

There exists a solution $\psi(u)$ of this problem; it is unique and is represented by the asymptotic series

$$\psi(u) \sim 1 + \sum_{k=2}^{\infty} D_k u^{k-1}, \quad \psi'(u) \sim \sum_{k=2}^{\infty} (k-1) D_k u^{k-2}, \quad u \sim +0, \tag{4.23}$$

where the coefficients D_k, $k \ge 2$, are defined by (4.13)–(4.15). Expansions (4.23) are used to approximately move the limit conditions (4.22) from the singular point $u = 0$ to a close regular point $u_0 > 0$. To find the solution $\varphi(u)$ of the original problem (4.1)–(4.4), we use the relation

$$\varphi(u) = \left[1 + \frac{\lambda}{c}\int_0^u \psi(s)ds\right]\left[1 + \frac{\lambda}{c}\int_0^\infty \psi(s)ds\right]^{-1}, \tag{4.24}$$

where $\psi(u)$ is the solution of problem (4.21), (4.22).

The computational results for Model I are presented in [14, 15, 17] and in this paper for various values of its parameters.

[3]In the classical Cramér–Lundberg model, the nonruin probability $\varphi(u)$ is the identical zero for $c \le \lambda m$ (see, e.g., [26]), which corresponds to the trivial solution of the initial-value problem for the integrodifferential equation from (4.18) (respectively, for the ordinary differential equation (4.19)), while the normalization condition at infinity refers only to nontrivial solutions.

In conclusion of this section, we note that investigation of the auxiliary singular Cauchy problem (4.1), (4.2) for the ordinary differential equation is a part of the solution of the optimal control problem for investments of an insurance company (see [14] for solution of such an optimal control problem in the case of Model I).

5. Problem 2: Main Results

Problem (2.6)–(2.10) can be written in the equivalent parametrized form:

$$(b^2/2)u^2\varphi''(u) + au\varphi'(u) - \lambda[\varphi(u) - (J_m\varphi)(u)] - \lambda_1[\varphi(u) - (J_{1,n}\varphi)(u)] = 0, \quad u \in \mathbb{R}_+, \tag{5.1}$$

$$\lim_{u\to+0}\varphi(u) = C_0, \qquad \lim_{u\to+0}[u\varphi'(u)] = 0, \tag{5.2}$$

$$\lim_{u\to+0}\varphi(u) = C_0 = \frac{\lambda_1}{n(\lambda+\lambda_1)}\int_0^\infty \varphi(y)\exp(-y/n)\,dy, \tag{5.3}$$

$$\lim_{u\to+\infty}\varphi(u) = 1, \qquad \lim_{u\to+\infty}\varphi'(u) = 0, \tag{5.4}$$

$$0 \le \varphi(u) \le 1, \qquad u \in \mathbb{R}_+. \tag{5.5}$$

Here C_0 is the parameter satisfying nonlocal condition (5.3).

Below, we formulate the main corollaries from results of [16] with refinements and remarks (we also fix some inaccuracies and misprints).

5.1. The uniqueness of the solution and the concomitant singular nonlocal problem for a fourth-order ordinary differential equation.

Lemma 5.1. *Let in integrodifferential equation* (5.1) *all parameters be fixed real numbers such that* $b^2 > 0$, $\lambda > 0$, $\lambda_1 > 0$, $m > 0$, *and* $n > 0$, *while* $a \in \mathbb{R}$ *has any sign. Then the following assertions are valid*:

(1) *If there exists a solution* $\varphi_1(u) = \varphi(u, C_0)$ *of problem* (5.1)–(5.5) *for some real* C_0, *then this solution is unique.*
(2) *If there exists a solution* $\varphi_1(u) = \varphi(u, C_0)$ *of problem* (5.1)–(5.4) *for some real* C_0, *then this solution satisfies condition* (5.5) *and* $0 < C_0 < 1$.

Proof.

1. *Uniqueness.* Suppose, to the contrary, that $\varphi_2(u)$ is another solution of problem (5.1)–(5.5). Then either

$$\lim_{u\to+0}\varphi_1(u) = \lim_{u\to+0}\varphi_2(u) \tag{5.6}$$

or

$$\lim_{u\to+0}\varphi_1(u) \ne \lim_{u\to+0}\varphi_2(u). \tag{5.7}$$

If (5.6) is valid, then the difference $\widetilde{\varphi}(u) = \varphi_2(u) - \varphi_1(u)$ is a solution of Eq. (5.1) such that

$$\lim_{u\to+0}\widetilde{\varphi}(u) = \lim_{u\to\infty}\widetilde{\varphi}(u) = 0. \tag{5.8}$$

If a nontrivial solution of problem (5.1), (5.8) exists and has positive values on $\mathbb{R}_+$, then it has a positive maximum on $\mathbb{R}_+$ (if $\widetilde{\varphi}(u)$ is nonpositive on $\mathbb{R}_+$, then consider the function $-\widetilde{\varphi}(u)$ instead). Suppose that $\widetilde{u}$ is a positive point of maximum of that solution: $\widetilde{\varphi}(\widetilde{u}) = \max_{u\in[0,\infty)}\widetilde{\varphi}(u) > 0$. Then $\widetilde{\varphi}'(\widetilde{u}) = 0$ and $\widetilde{\varphi}''(\widetilde{u}) \le 0$. However, it follows from (5.1), (2.5), and (2.11) that

$$(b^2/2)\widetilde{u}^2\widetilde{\varphi}''(\widetilde{u}) = \lambda\,[\widetilde{\varphi}(\widetilde{u}) - (J_m\widetilde{\varphi})(\widetilde{u})] + \lambda_1\,[\widetilde{\varphi}(\widetilde{u}) - (J_{1,n}\widetilde{\varphi})(\widetilde{u})] \ge \lambda\widetilde{\varphi}(\widetilde{u})\exp(-\widetilde{u}/m) > 0. \tag{5.9}$$

We arrive to a contradiction. Hence, $\widetilde{\varphi}(u) \equiv 0$.

Consider the case where (5.7) holds. Then one can easily check that there exists a linear combination $\widehat{\varphi}(u) = c_1\varphi_1(u) + c_2\varphi_2(u)$ of solutions such that $\widehat{\varphi}(u) \not\equiv 1$ and $\widehat{\varphi}(u)$ satisfies Eq. (5.1) and the limit conditions

$$\lim_{u\to+0} \widehat{\varphi}(u) = \lim_{u\to\infty} \widehat{\varphi}(u) = 1.$$

If there exists $u \in \mathbb{R}_+$ such that $\widehat{\varphi}(u) > 1$, then the line of reasoning is the same as for the first case. If we assume that $\widehat{\varphi}(u) \le 1$ for any positive u, then we obtain the inequality

$$(J_{1,n}\widehat{\varphi})(0) = \frac{1}{n}\int_0^\infty \widehat{\varphi}(y)\exp(-y/n)\,dy < 1.$$

Then, taking into account the limit relation $\lim\limits_{u\to+0} \widehat{\varphi}(u) = 1$, we obtain a contradiction: condition (5.3) is not satisfied. Hence, there is no other solution of problem (5.1)–(5.5) satisfying condition (5.7).

2. Now, let $\varphi(u) = \varphi(u, C_0)$ be a solution of problem (5.1)–(5.4) without constraints for some $C_0 \in \mathbb{R}$. We claim that $\varphi(u) < 1$ for any finite positive u. Indeed, first we show that the greatest positive value of the function $\varphi(u)$ is not achieved as $u \to +0$. Suppose, to the contrary, that $\varphi(u) \le C_0$ for any positive u, where $\lim\limits_{u\to+0} \varphi(u) = C_0 > 0$. However, (5.3) implies a contradiction: $(\lambda+\lambda_1)C_0 \le \lambda_1 C_0$, which implies that $C_0 \le 0$.

Now, let $\varphi(u) \ge 1$ for some finite positive u. Then $\varphi(u)$ has a maximum on $\mathbb{R}_+$, exceeding 1. However, similarly to the proof of inequality (5.9), we have a contradiction at that point of maximum.

In the same way, we prove that the least negative value of $\varphi(u)$ cannot be achieved as $u \to +0$ and there is no negative minimum of $\varphi(u)$ on $\mathbb{R}_+$. $\square$

Further, the second-order integrodifferential equation (5.1) can be reduced to a fourth-order ordinary differential equation. As well as in model 1, this is an important reduction. To do that, we differentiate (5.1) twice and, taking into account (4.9) and the relation

$$(J_{1,n}\varphi)'(u) = [(J_{1,n}\varphi)(u) - \varphi(u)]/n, \tag{5.10}$$

remove the integrals $(J_m\varphi)(u)$ and $(J_{1,n}\varphi)(u)$ from the obtained integrodifferential equation, using the original integrodifferential equation (5.1) and an intermediate auxiliary third-order integrodifferential equation.

Thus, the following Lemma 5.2 is valid with the following notation:

$$\begin{gathered} a_1 = 2(2+a/b^2), \quad a_2 = (n-m)/(mn), \quad a_3 = 2\left[1 + (2a-\lambda-\lambda_1)/b^2\right], \\ a_4 = 2(1+a/b^2)(n-m)/(mn), \qquad a_5 = -1/(mn), \\ a_6 = 2\left[a(n-m) + \lambda m - \lambda_1 n\right]/(b^2 mn), \qquad a_7 = -2a/(b^2 mn). \end{gathered} \tag{5.11}$$

Lemma 5.2. *Let the parameters of Eq.* (5.1) *satisfy the conditions of Lemma* 5.1. *Then problem* (5.1)–(5.4) *is equivalent to the following linear singular boundary-value problem for an ordinary differential equation with nonlocal condition at the origin*:

$$u^2\varphi''''(u) + (a_1 + a_2 u)u\varphi'''(u) + (a_3 + a_4 u + a_5 u^2)\varphi''(u) + (a_6 + a_7 u)\varphi'(u) = 0, \quad u \in \mathbb{R}_+, \tag{5.12}$$

$$\lim_{u\to+0} \varphi(u) = C_0, \quad \lim_{u\to+0}[u\varphi'(u)] = \lim_{u\to+0}[u^2\varphi''(u)] = \lim_{u\to+0}[u^3\varphi'''(u)] = 0, \tag{5.13}$$

$$\lim_{u\to+0} \varphi(u) = C_0 = \frac{\lambda_1}{n(\lambda+\lambda_1)}\int_0^\infty \varphi(y)\exp(-y/n)\,dy, \tag{5.14}$$

$$\lim_{u\to\infty} \varphi(u) = 1, \quad \lim_{u\to\infty} \varphi'(u) = \lim_{u\to\infty} \varphi''(u) = \lim_{u\to\infty} \varphi'''(u) = 0. \tag{5.15}$$

Here C_0 is a parameter and the values a_j, $j = \overline{1,7}$, are defined by (5.11).

Proof. The passage from Eq. (5.1) to Eq. (5.12) is obvious: it follows from the construction of Eq. (5.12) explained above. Now, let $\widetilde{\varphi}(u) = \widetilde{\varphi}(u, C_0)$ be a solution of problem (5.12)–(5.15). We have to prove that $\widetilde{\varphi}(u)$ satisfies Eq. (5.1).

Substitute function $\widetilde{\varphi}(u)$ in Eq. (5.1) and denote the left-hand side by $g(u)$. We claim that $g(u) \equiv 0$. Indeed, taking into account the derivation of Eq. (5.12) (see [16] for details), one can see that $g(u)$ satisfies the ordinary differential equation

$$g''(u) + \frac{(n-m)}{mn}\, g'(u) - \frac{1}{mn}\, g(u) = 0, \qquad 0 < u < \infty. \tag{5.16}$$

The general solution of Eq. (5.16) is

$$g(u) = c_1 \exp(-u/m) + c_2 \exp(u/n), \qquad u \geq 0, \tag{5.17}$$

where c_1 and c_2 are arbitrary constants.

Taking into account the definition of $g(u)$ and limit conditions (5.15), we see that $\lim\limits_{u\to\infty} [g(u)/u^2] = 0$, which implies that $c_2 = 0$ in (5.17). Finally, taking into account conditions (5.13) and (5.14), we obtain the relation $g(0) = 0$, which implies that $c_1 = 0$ in (5.17). □

5.2. Auxiliary singular boundary-value problem for third-order ordinary differential equations. Assigning $\varphi'(u) = \psi(u)$, we reduce the order of Eq. (5.12) and obtain the following auxiliary linear singular boundary-value problem for the function $\psi(u)$:

$$u^3\psi'''(u) + (a_1 + a_2 u)u^2\psi''(u) + (a_3 + a_4 u + a_5 u^2)u\psi'(u) + (a_6 u + a_7 u^2)\psi(u) = 0, \quad u \in \mathbb{R}_+, \tag{5.18}$$

$$\lim_{u\to+0}[u\psi(u)] = \lim_{u\to+0}[u^2\psi'(u)] = \lim_{u\to+0}[u^3\psi''(u)] = 0, \tag{5.19}$$

$$\lim_{u\to+\infty} \psi(u) = \lim_{u\to+\infty} \psi'(u) = \lim_{u\to+\infty} \psi''(u) = 0. \tag{5.20}$$

Equation (5.18) has a regular singularity as $u \to +0$ and an irregular singularity of rank 1 as $u \to \infty$. In particular, results of [19, 20, 29, 30] are used to study singular Cauchy problems in neighborhoods of singular points of this ordinary differential equation (for details, see [16]).

The singular boundary-value problem (5.18)–(5.20) is always satisfied by the trivial solution $\psi \equiv 0$. We investigate the existence of its nontrivial solution selected by the normalization condition presented in Lemma 5.3 below.

Lemma 5.3. *Let assumptions of Lemma* 5.2 *be satisfied, and let $\psi(u)$ be a nontrivial solution of the auxiliary linear singular boundary-value problem* (5.18)–(5.20) *such that $\psi(u)$ belongs to $L_1(\mathbb{R}_+)$ and is normalized by the requirement*

$$\int_0^\infty [1 + (\lambda_1/\lambda)\exp(-s/n)]\ \psi(s)\, ds = 1. \tag{5.21}$$

Then the function $\varphi(u)$ defined by the relation

$$\varphi(u) = (\lambda_1/\lambda) \int_0^\infty \psi(s)\exp(-s/n)\, ds + \int_0^u \psi(s)\, ds, \quad u \geq 0, \tag{5.22}$$

satisfies both linear singular boundary-value problem (5.12)–(5.15) *and main linear nonlocal singular boundary-value problem* (5.1)–(5.5) *for an integrodifferential equation with constraints.*

Thus, it suffices to prove the existence of a nontrivial solution $\psi(u)$ of the auxiliary linear singular boundary-value problem (5.18)–(5.20) satisfying conditions of Lemma 5.3 and to find such a solution.

To solve that auxiliary boundary-value problem, we have to investigate singular points of Eq. (5.18) and reduce the singular boundary-value problem (5.18)–(5.20) to an equivalent boundary-value problem without singularities on a finite interval. To move the limit boundary-value conditions from singular points, we use results from the theory of stable initial manifolds of solutions or, which is the same, conditional stability manifolds in the Lyapunov sense (see, e.g., [2, 3, 7]). The notion of admissible boundary-value conditions at pole-type singular points (see, e.g., [5, 6]) is taken into account as well.

5.2.1. Transfer of limit boundary-value conditions from the singular point $u = 0$. At $u = 0$, Eq. (5.18) has a regular (weak) singularity with characteristic parameters μ_j, $j = 0, 1, 2$:

$$\mu_0 = 0, \qquad \mu_1 = -1/2 - a/b^2 + \sqrt{(1/2 + a/b^2)^2 + 2(\lambda + \lambda_1 - a)/b^2}, \tag{5.23}$$

and

$$\mu_2 = -1/2 - a/b^2 - \sqrt{(1/2 + a/b^2)^2 + 2(\lambda + \lambda_1 - a)/b^2}. \tag{5.24}$$

The relations for $\mu_{1,2}$ imply that

$$\mu_1 + 1 = 1/2 - a/b^2 + \sqrt{(1/2 - a/b^2)^2 + 2(\lambda + \lambda_1)/b^2},$$

$$\mu_2 + 1 = 1/2 - a/b^2 - \sqrt{(1/2 - a/b^2)^2 + 2(\lambda + \lambda_1)/b^2}.$$

Thus, at least for $\lambda + \lambda_1 > 0$ we have

$$\mu_0 = 0, \qquad \mu_1 > -1, \qquad \mu_2 < -1. \tag{5.25}$$

In particular, this means that the singular Cauchy problem (5.18), (5.19) has a two-parameter family of solutions and their values generate a two-dimensional linear subspace in the three-dimensional phase space (of variables ψ, ψ', and ψ'') of the ordinary differential equation (5.18). The said phase space is the plane containing the origin. This plane depends on parameter u and is defined by a single linear relation in $\mathbb{R}^3$.

Further, using the theory of transfer of boundary-value conditions from singular points for systems of linear ordinary differential equations and results for singular Cauchy problems for systems of nonlinear ordinary differential equations, we obtain the following assertion (its simple proof can be found in [16]):

Lemma 5.4. *Let the coefficients a_j, $j = \overline{1,7}$, of Eq.* (5.18) *be defined by relations* (5.11)*, where a, b, λ, λ_1, m, and n satisfy the assumptions of Lemma* 5.1*. Then there exists a sufficiently small u_0 such that conditions* (5.19) *for solutions of Eq.* (5.18) *are equivalent to the linear relation*

$$u^2\psi''(u) = \alpha(u)u\psi'(u) + \beta(u)\psi(u), \qquad 0 < u \le u_0. \tag{5.26}$$

Here the pair of functions $\{\alpha(u), \beta(u)\}$ is a solution of the nonlinear singular Cauchy problem

$$u\alpha' = (1 - a_1 - a_2u)\alpha - \alpha^2 - \beta - (a_3 + a_4u + a_5u^2), \tag{5.27}$$

$$u\beta' = (2 - a_1 - a_2u)\beta - \alpha\beta - (a_6u + a_7u^2), \qquad u > 0, \tag{5.28}$$

$$\lim_{u\to+0} \alpha(u) = \alpha_0 = \mu_1 - 1, \quad \lim_{u\to+0} \beta(u) = 0, \tag{5.29}$$

where μ_1 is defined by (5.23)*. For sufficiently small positive u, problem* (5.27)*–*(5.29) *has a solution $\{\alpha(u), \beta(u)\}$ and this solution is unique and is a holomorphic function at the point $u = 0$:*

$$\alpha(u) = \sum_{k=0}^{\infty} \alpha_k u^k, \qquad \beta(u) = \sum_{k=1}^{\infty} \beta_k u^k, \qquad |u| \le u_0, \qquad u_0 > 0. \tag{5.30}$$

Here α_0 is defined by (5.29)*, while the coefficients α_k and β_k, $k \ge 1$, are defined by the formal substitution of expansions* (5.30) *in* (5.27) *and* (5.28)*, which yields the following recurrent relations*:

$$\beta_1 = -\frac{a_6}{\alpha_0 + a_1 - 1}, \qquad \alpha_1 = -\frac{\beta_1 + a_2\alpha_0 + a_4}{2\alpha_0 + a_1}; \tag{5.31}$$

$$\beta_2 = -\frac{a_7 + a_2\beta_1 + \alpha_1\beta_1}{\alpha_0 + a_1}, \qquad \alpha_2 = -\frac{\beta_2 + \alpha_1^2 + a_2\alpha_1 + a_5}{2\alpha_0 + a_1 + 1}; \tag{5.32}$$

$$\beta_k = -\frac{a_2\beta_{k-1} + \sum\limits_{l=1}^{k-1} \alpha_l \beta_{k-l}}{\alpha_0 + a_1 + k - 2}, \qquad \alpha_k = -\frac{\beta_k + \sum\limits_{l=1}^{k-1} \alpha_l \alpha_{k-l}}{2\alpha_0 + a_1 + k - 1}, \qquad k = 3, 4, \dots. \tag{5.33}$$

This allows us to transfer the boundary-value conditions (5.19) from the singular point $u = 0$ to a close regular point $u = u_0 > 0$ by means of relation (5.26): at the point $u = u_0$ we have the boundary-value condition

$$u_0^2 \psi''(u_0) = \alpha(u_0) u_0 \psi'(u_0) + \beta(u_0)\psi(u_0), \tag{5.34}$$

where approximate values of $\alpha(u_0)$ and $\beta(u_0)$ can be found by means of expansions (5.30)–(5.33) with any prescribed precision. It is important for the computations that in a neighborhood of the singular point $u = 0$ condition (5.34) is stably transferred from the left to the right, i.e., in the direction from the singular point.[4]

Remark 5.1. It was not noted in [16] that Lemma 5.4 remains valid for $a \le 0$ as well, which can be interesting for other models of insurance and financial mathematics (cf. Remark 4.2).

Remark 5.2. For completeness, taking into account the results of general theory of linear ordinary differential equations with regular (weak) singular points (see, e.g., [24, § 2]), let us establish the following representation for the two-parameter family of solutions $\psi(u, q_0, q_1)$ of problem (5.18), (5.19):

$$\psi(u, q_0, q_1) = q_0\{1 + \psi_1(u) + Au^{\mu_1} \log(u)\,[1 + \psi_2(u)]\} + q_1 u^{\mu_1}\,[1 + \psi_2(u)]\,. \tag{5.35}$$

(Under the assumptions of Lemma 5.4, it can be derived from relation (5.26) for small positive u.) Here q_0 and q_1 are arbitrary constants, μ_1 is defined by (5.23), $\psi_j(u)$ are functions holomorphic at the origin such that $\psi_j(0) = 0$, $j = 1, 2$, and the constant A depends on the parameters of Eq. (5.18) such that $A = 0$ if μ_1 is not an integer. To obtain the coefficients of the convergent series (with respect to powers of u) for the functions $\psi_j(u)$, $j = 1, 2$, one can formally substitute all the expansions in Eq. (5.26).

5.2.2. Transfer of limit conditions from the singular point at infinity. As $u \to \infty$, Eq. (5.18) has an irregular (strong) singularity of rank 1.

Singular boundary-value problem (5.18), (5.20) can be represented as follows:

$$\psi'''(u) + \left[a_2 + \frac{a_1}{u}\right]\psi''(u) + \left[a_5 + \frac{a_4}{u} + \frac{a_3}{u^2}\right]\psi'(u) + \left[\frac{a_7}{u} + \frac{a_6}{u^2}\right]\psi(u) = 0, \quad 0 < u < \infty, \tag{5.36}$$

$$\lim_{u\to+\infty} \psi(u) = \lim_{u\to+\infty} \psi'(u) = \lim_{u\to+\infty} \psi''(u) = 0. \tag{5.37}$$

The characteristic parameters of Eq. (5.36), defining the behavior of solutions for large u, are as follows:

$$\nu_0 = 0, \qquad \nu_1 = -1/m < 0, \qquad \nu_2 = 1/n > 0. \tag{5.38}$$

To completely investigate the behavior of solutions of Eq. (5.36) as $u \to \infty$, one has to find the first correction (for large u) for the parameter $\nu_0 = 0$ in the sense of perturbation theory. It is done in detail in [16] (results of [19, 20, 29, 30] are used as well). These results combined with the general theory of linear ordinary differential equations with irregular singular points yield the following assertion.

Lemma 5.5. *Let the coefficients a_j, $j = \overline{1,7}$, of Eq.* (5.18) *be defined by relations* (5.11)*, where the constants a, b^2, n, and m are positive, while the constants λ and λ_1 are real. Then singular Cauchy problem* (5.18), (5.20) *has a two-parameter family of solutions $\psi(u, p_1, p_2)$ and the following representation is valid for large u*:

$$\psi(u, p_1, p_2) = p_1 u^{-2a/b^2}[1 + \xi_1(u)/u] + p_2 u^{-2}\exp(-u/m)[1 + \xi_2(u)/u]. \tag{5.39}$$

[4]See [16]; also, in [4], this is studied for the case of transfer of stable manifolds of solutions for quite general systems of ordinary differential equations with pole-type singularities at boundary points.

Here p_1 and p_2 are arbitrary constants, the functions $\xi_j(u)$, $j=1,2$, have finite limits as $u\to\infty$, and for large u those functions are represented by the asymptotic series

$$\xi_j(u) \backsim \sum_{k=0}^{\infty} \xi_k^{(j)}/u^k, \qquad j=1,2. \tag{5.40}$$

The coefficients of the series can be obtained by formal substitutions of all expansions in Eq. (5.18)*. All solutions of the family $\psi(u,p_1,p_2)$ are integrable at infinity if and only if inequality* (3.23) *is satisfied.*

The values of solutions (5.39) generate a two-dimensional linear subspace in the three-dimensional phase space (of variables ψ, ψ', and ψ'') of the ordinary differential equation (5.18). The said phase space is the plane containing the origin. This plane depends on parameter u and is defined by a single linear relation in $\mathbb{R}^3$.

More exactly, using the theory of transfer of boundary-value conditions from singular points of systems of linear ordinary differential equations and results for singular Cauchy problems for systems of nonlinear ordinary differential equations, we obtain the following assertion (see its proof in [16]).

Lemma 5.6. *Let the assumptions of Lemma* 5.5 *be satisfied. Then, for sufficiently large u, conditions* (5.20) *for solutions of Eq.* (5.18) *are equivalent to the linear relation*

$$\psi''(u) = \gamma(u)\psi'(u) + \varkappa(u)\psi(u), \qquad u \ge u_\infty, \tag{5.41}$$

where the pair of functions $\{\gamma(u), \varkappa(u)\}$ is a solution of the singular nonlinear Cauchy problem

$$\gamma' = -\left(a_2 + a_1/u\right)\gamma - \gamma^2 - \varkappa - \left(a_5 + a_4/u + a_3/u^2\right), \tag{5.42}$$

$$\varkappa' = -\left(a_2 + a_1/u\right)\varkappa - \gamma\varkappa - \left(a_7/u + a_6/u^2\right), \qquad u_\infty \le u < \infty, \tag{5.43}$$

$$\lim_{u\to+\infty} \gamma(u) = \gamma_0 = -1/m, \qquad \lim_{u\to+\infty} \varkappa(u) = 0. \tag{5.44}$$

For sufficiently large u, there exists a unique solution $\{\gamma(u), \varkappa(u)\}$ of problem (5.42)–(5.44) *that can be represented by the asymptotic series*:

$$\gamma(u) \sim \gamma_0 + \sum_{k=1}^{\infty} \gamma_k/u^k, \qquad \varkappa(u) \sim \sum_{k=1}^{\infty} \varkappa_k/u^k, \qquad u \gg 1. \tag{5.45}$$

Here γ_0 is defined by (5.44) *and the coefficients γ_k and $\varkappa_k$ for $k\ge 1$ can be obtained by formal substitution of expansions* (5.45) *in* (5.42) *and* (5.43)*, which yields the recurrent relations*

$$\varkappa_1 = -a_7/(a_2+\gamma_0), \qquad \gamma_1 = -(a_1\gamma_0 + \varkappa_1 + a_4)/(a_2 + 2\gamma_0), \tag{5.46}$$

$$\varkappa_2 = [\varkappa_1(1-a_1-\gamma_1) - a_6]/(a_2+\gamma_0), \quad \gamma_2 = [\gamma_1(1-a_1-\gamma_1) - \varkappa_2 - a_3]/(a_2+2\gamma_0), \tag{5.47}$$

$$\varkappa_k = \left[(k-1-a_1)\varkappa_{k-1} - \sum_{l=1}^{k-1}\gamma_l\varkappa_{k-l}\right] \Big/ (a_2+\gamma_0), \tag{5.48}$$

$$\gamma_k = \left[\gamma_{k-1}(k-1-a_1) - \varkappa_k - \sum_{l=1}^{k-1}\gamma_l\gamma_{k-l}\right] \Big/ (a_2+2\gamma_0), \qquad k = 3,4,\ldots. \tag{5.49}$$

In particular, then representation (5.39), (5.40) *for the two-parameter family $\psi(u,p_1,p_2)$ of solutions of singular Cauchy problem* (5.36), (5.37) *can be obtained by formal substitution of all expansions in relation* (5.41)*.*

Thus, to transfer the boundary-value conditions (5.20) from infinity to a finite point $u = u_\infty \gg 1$, we use relation (5.41): the boundary-value condition

$$\psi''(u_\infty) = \gamma(u_\infty)\psi'(u_\infty) + \varkappa(u_\infty)\psi(u_\infty) \tag{5.50}$$

holds at the point $u = u_\infty$, where the approximate values of $\gamma(u_\infty)$ and $\varkappa(u_\infty)$ can be found by means of expansions (5.46)–(5.49). It is important for computations that if u is large, then the condition (5.50) is stably transferred from the right to the left, i.e., from the singular point $u=\infty$.

5.2.3. Equivalent homogeneous regular boundary-value problem with underdetermined boundary-value conditions. Existence of a nontrivial solution and its properties.

Lemma 5.7. *Assume that the coefficients* a_j, $j = \overline{1,7}$, *of Eq.* (5.18) *are defined by formulas* (5.11) *and* a, b^2, n, m, λ, *and* λ_1 *are positive. Then problem* (5.18)–(5.20) *on* $\mathbb{R}_+$ *is equivalent to the following homogeneous linear boundary-value problem on the finite interval* $0 < u_0 \le u \le u_\infty$ *without singularities*:

$$u^3\psi'''(u)+(a_1+a_2u)u^2\psi''(u)+(a_3+a_4u+a_5u^2)u\psi'(u)+(a_6u+a_7u^2)\psi(u)=0,\ u_0\le u\le u_\infty, \tag{5.51}$$

$$u_0^2\psi''(u_0)=\alpha(u_0)u_0\psi'(u_0)+\beta(u_0)\psi(u_0), \tag{5.52}$$

$$\psi''(u_\infty)=\gamma(u_\infty)\psi'(u_\infty)+\varkappa(u_\infty)\psi(u_\infty). \tag{5.53}$$

Here $\alpha(u)$ *and* $\beta(u)$ *are defined in Lemma* 5.4, $\gamma(u)$ *and* $\varkappa(u)$ *are defined in Lemma* 5.6 *while the ranges of values of* u_0 *and* u_∞ $(0 < u_0 \ll 1$ *and* $u_\infty \gg 1)$ *depend on the parameters of the problem* (*moving boundaries*). *Problem* (5.51)–(5.53) *is underdetermined with respect to the number of boundary-value conditions* (*the ordinary differential equation has the third order, but only two separated boundary-value conditions are given*) *and always has a nontrivial solution.*

Taking into account relations (5.23) and (5.24) as well as representations (5.35) and (5.39) for the two-parameter families of solutions of problems (5.18), (5.19) and (5.18), (5.20) respectively, we deduce the following assertion.

Theorem 5.1. *Let Eq.* (5.18) *satisfy assumptions of Lemma* 5.7 *with fixed parameters* a, b^2, n, m, λ, *and* λ_1, *and let inequality* (3.23) *be satisfied* (*which is the "reliability condition for the portfolio of assets"*).

Then the singular boundary-value problem (5.18)–(5.20) *has a unique* (*up to a normalizing factor*) *nontrivial solution* $\psi(u)$, $\psi(u) \in L_1(0,\infty)$ *and the following assertions hold*:

(1) *If*

$$0 < a < \lambda + \lambda_1, \tag{5.54}$$

then $\mu_1 > 0$, $\lim\limits_{u\to+0}\psi(u) = D_1 > 0$, *and the inequality* $|\lim\limits_{u\to+0}\psi'(u)| < \infty$ *is valid if and only if, moreover,*

$$\lambda + \lambda_1 > b^2 + 2a. \tag{5.55}$$

More exactly, in this case $\mu_1 > 1$ *and*

$$\lim_{u\to+0}\psi'(u) = D_1D_2 = D_1\,[a(m-n)+\lambda_1 n-\lambda m]\,/\,[mn(b^2+2a-\lambda-\lambda_1], \tag{5.56}$$

whence $D_2 \le 0$ *provided that* $i_{\mathrm{r,II}} \ge 0$ *and* $D_2 > 0$ *provided that* $i_{\mathrm{r,II}} < 0$, *where*

$$i_{\mathrm{r,II}} = a(m-n)+\lambda_1 n-\lambda m \tag{5.57}$$

determines the "risk factor" for Model II.

If condition (5.55) *is violated, i.e.,*

$$\lambda+\lambda_1 \le b^2+2a, \tag{5.58}$$

then $0 < \mu_1 \le 1$ *and the function* $\psi'(u)$ *is unbounded, but integrable as* $u \to +0$.

(2) *If*

$$a \ge \lambda+\lambda_1 > 0, \tag{5.59}$$

then $-1 < \mu_1 \le 0$ *and the function* $\psi(u)$ *is unbounded, but integrable as* $u \to +0$.

(3) *The behavior of* $\psi(u)$ *for large* u *follows from Lemma* 5.5: *namely, the asymptotic representation*

$$\psi(u) = q_1u^{-2a/b^2}[1+o(1)], \qquad u\to\infty, \tag{5.60}$$

holds with $q_1 \ne 0$ *and* $\psi(u)$ *is integrable as* $u \to \infty$ *by virtue of* (3.23).

5.3. Existence, uniqueness, and behavior of solutions of the original singular nonlocal problem for integrodifferential equation with constraints.

Theorem 5.2. *Let the assumptions of Theorem* 5.1 *hold, and let* $\psi(u)$ *be a nontrivial solution for the auxiliary singular Cauchy problem* (5.18)–(5.20) *normed by condition* (5.21). *Then the following assertions are valid*:

(1) *The function* $\psi(u)$ *is positive for any positive* u *while the function* $\varphi(u)$ *defined by relation* (5.22) *is the unique solution of the original singular boundary-value problem* (5.1)–(5.5) *for integrodifferential equation with constraints and a nonlocal condition at the origin. Function* $\varphi(u)$ *strictly increases on* $\mathbb{R}_+$.

(2) *If inequalities* (5.54) *are satisfied, then the derivative* $\varphi'(u)$ *has a finite limit as* $u \to +0$. *If, moreover, condition* (5.55) *is satisfied, then* $\varphi''(u)$ *has a finite limit as* $u \to +0$, *which is nonnegative if* $i_{\mathrm{r,II}} \geq 0$ *and negative if* $i_{\mathrm{r,II}} < 0$; *if inequality* (5.58) *satisfied, then* $\varphi''(u)$ *is unbounded but integrable at the origin. If inequality* (5.59) *is satisfied, then* $\varphi'(u)$ *is unbounded but integrable at the origin.*

(3) *For large* u, *the following representation holds*:

$$\varphi(u) = 1 - K\,u^{1-2a/b^2}\,[1 + o(1)], \qquad u \to \infty, \tag{5.61}$$

where K *is a positive constant that cannot be found by methods of local analysis.*

(4) *If conditions* (5.54) *and* (5.55) *and the inequality* $i_{\mathrm{r,II}} < 0$ *are satisfied, then there exists a positive* $\widetilde{u}$ *such that the function* $\psi(u) = \varphi'(u)$ *attains a positive maximum at the point* $\widetilde{u}$ *and the function* $\varphi(u)$ *has an inflection at the point* $\widetilde{u}$.

Remark 5.3. If condition (3.11) is satisfied, then function (3.14) is a solution of the degenerate problem obtained from the main singular problem (2.6)–(2.10) by formally assigning $a = b = 0$, i.e., of the following singular nonlocal problem for an integral equation:

$$(\lambda + \lambda_1)\varphi(u) = \lambda\,(J_m\varphi)(u) + \lambda_1\,(J_{1,n}\varphi)(u), \qquad u \in \mathbb{R}_+, \tag{5.62}$$

$$\varphi(0) = C_0 = \frac{\lambda_1}{\lambda + \lambda_1}(J_{1,n}\varphi)(0), \quad \lim_{u\to+\infty}\varphi(u) = 1. \tag{5.63}$$

This problem is equivalent to the following linear boundary-value problem on a half-axis for a second-order ordinary differential equation with nonlocal constraint at the origin:

$$\varphi''(u) = \frac{\lambda m - \lambda_1 n}{mn(\lambda + \lambda_1)}\,\varphi'(u), \quad u \in \mathbb{R}_+, \quad \varphi(0) = C_0 = \frac{\lambda_1}{\lambda + \lambda_1}(J_{1,n}\varphi)(0), \quad \lim_{u\to\infty}\varphi(u) = 1. \tag{5.64}$$

Then it is easy to see that $C_0 = 1 - \lambda\,(n+m)/[n\,(\lambda + \lambda_1)]$, $0 < C_0 < 1$,

$$\varphi'(0) = D_1 = \lambda\,(n+m)(\lambda_1\,n - \lambda\,m)/[mn^2\,(\lambda + \lambda_1)^2] > 0,$$

$$\varphi''(0) = D_1 D_2 = -D_1(\lambda_1\,n - \lambda\,m)/[mn\,(\lambda + \lambda_1)] < 0,$$

and relation (3.14) determines the exact solution of the Cramér–Lundberg model with stochastic premium and positive security load.

The critical value of the bifurcation parameter is equal to $\lambda_1 = \lambda m/n$: if $\lambda_1 \leq \lambda m/n$, then problem (5.62), (5.63) (problem (5.64) respectively) has no solutions.

5.4. On algorithms of numerical simulation. It follows from the above that to solve the original singular boundary-value problem (2.6)–(2.10) for an integrodifferential equation with constraints and nonlocal condition at the origin, one has to find nontrivial solutions of the auxiliary homogeneous boundary-value problem (5.51)–(5.53) for a differential equation, which is set on a finite segment $[u_0, u_\infty]$ without singularities and with underdetermined boundary-value conditions.

It is well known that differential sweep methods are efficient to solve linear boundary-value problems on finite intervals without singularities. Results of [4] are important for nontrivial solutions of problem (5.51)–(5.53); in that paper, a brief review of variants of the differential sweep is provided and the stability of computations is studied in neighborhoods of singular points for boundary-value problems (including spectral ones) obtained from singular boundary-value problems by transfer of boundary-value conditions from singular points. In particular, boundary-value problem (5.51)–(5.53) is obtained from singular boundary-value problem (5.18)–(5.20) by means of the above methods of local transfer of boundary-value conditions from singular points. To find nontrivial solutions of problem (5.51)–(5.53), methods of [4] are important to stably find eigenfunctions in spectral problems. However, as far as we know, the said methods were not used for homogeneous underdetermined boundary-value problems until the paper [16].

In [16], numerical algorithms of solution for problem (5.51)–(5.53) are proposed and implemented. They are efficient with respect to the number of sweep equations and with respect to the number of computational operations. Here, we do not focus on this. We only mention that one of those algorithms is based on a combination of two variants of the globally stable method of the differential sweep: the sweep variant of [1] and its modification proposed in [10]; this approach might be interesting for other boundary-value problems as well. Another algorithm is more efficient: we apply the direct sweep and, instead of applying the method of [1], we solve Cauchy problems (5.27)–(5.29) and (5.42)–(5.44) (respectively, from the left to the right from the point $u = u_0 > 0$ to the point $u = \widehat{u}$ and from the right to the left from the point $u = u_\infty < \infty$ to the point $u = \widehat{u}$, $u_0 < \widehat{u} < u_\infty$), but still use the method of [10] for the inverse sweep.

Further, due to Theorem 5.2, once a nontrivial solution $\psi(u)$ of problem (5.18)–(5.20) normalized by condition (5.21) is found, the solution $\varphi(u)$ of problem (5.1)–(5.5) can be found by means of relation (5.22).

6. Numerical Simulation: Computation Comparison for Models I and II

For computations, the program environment Maple[5] 14.01 was used with a prescribed precision of computations and additional tools to control the number of valid digits.

First, we formulate *comparison conditions for computational results for Models I and II* (as well as their comparison with the data for the initial risk models 1 and 2, i.e., models without investments):

(1) the values of the parameters a, b^2, λ, and m in Model II (model 2) are the same as in Model I (model 1);

(2) the value of c in Model I (model 1) is related to λ_1 and n in Model II (model 2) so that $\lambda_1 n = c$, i.e., the expected premium values per time period are the same for both models.

If $a > 0$ and $b \neq 0$, the "reliability condition" for shares is valid for all computational examples, i. e., $2a/b^2 > 1$; the risk factors $i_{\mathrm{r,I}}$ and $i_{\mathrm{r,II}}$ are defined in (4.17) and (5.57) respectively.

Further, we present the plots of dependence of the nonruin probability on the initial capital provided that the comparison conditions hold for Models I and II (models 1 and 2).

First, we compare solutions (3.7) and (3.14) for models 1 and 2. It is easy to see that the following assertion is valid.

Assertion 6.1. *Let λ, m, c, λ_1, and n be positive, and let $c = \lambda_1 n > \lambda m$; in particular, this implies the relation $\rho_1 = \rho_2 > 0$ for the security load.*

Then $\varphi_2(u) < \varphi_1(u)$ for any finite positive u and $\varphi_2(u) \to \varphi_1(u)$ for any positive u ($\varphi_2(u) \to 0$ for any finite positive u) provided that there exists positive c such that $\lambda_1 = c/n \to \infty$ as $n \to 0$ (respectively, $\lambda_1 = c/n \to 0$ as $n \to \infty$).

This assertion is illustrated by Fig. 1a and Fig. 1b. In the sequel, the digit 1 marks the plots of exact solutions of (3.7) and the digit 2 marks the plots of exact solutions of (3.14). The values of the parameters for Fig. 1a and Fig. 1b (exact solutions of degenerated problems for integrodifferential equations are for models without investments) are as follows: $a = b = 0$, $m = 0,5$, $\lambda = 0,09$; 1: $c = 0,1$; 2 (Fig. 1a): $n = 0,1$, $\lambda_1 = 1$; 2 (Fig. 1b): $n = 0,4$, and $\lambda_1 = 0,25$ ($c = \lambda_1 n > \lambda m$).

[5] The license of CS RAS.

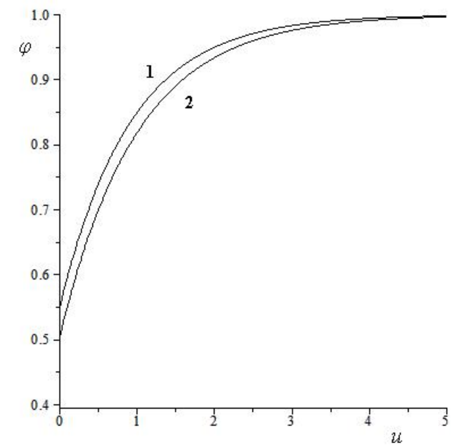

Fig. 1a.

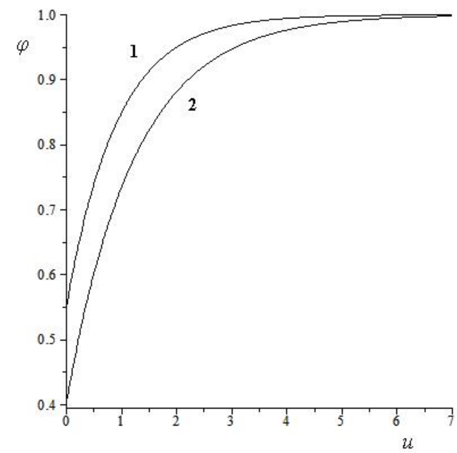

Fig. 1b.

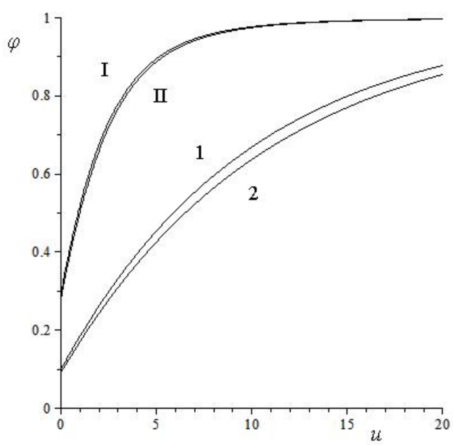

Fig. 2a.

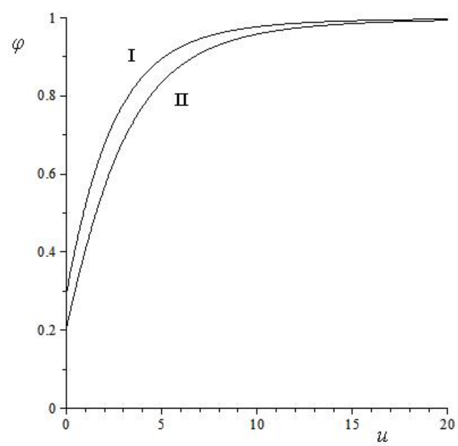

Fig. 2b.

Theorems 4.1 and 5.2 are illustrated by Figs. 2–5, where the digit I marks the plots for Model I and the digit II marks the plots for Model II. The values of the parameters for Fig. 2a and Fig. 2b are as follows: $m = 1$, $\lambda = 0,09$; 1,2: $a = b = 0$; 1: $c = 0,1$; 2: $n = 0,1$, $\lambda_1 = 1$; I, II: $a = 0,02$, $b = 0,1$; I: $c = 0,1$; II (Fig. 2a): $n = 0,1$, $\lambda_1 = 1$; II (Fig. 2b): $n = 0,9$, $\lambda_1 = 1/9$ ($c = \lambda_1 n > \lambda m$; I: $0 < a < \lambda$, $i_{\mathrm{r,I}} > 0$; II: $0 < a < \lambda + \lambda_1$, $\lambda + \lambda_1 > b^2 + 2a$, and $i_{\mathrm{r,II}} > 0$). The values of the parameters for Fig. 3 are as follows: $b = 0,1$, $m = 1$, $\lambda = 0,09$; I: $c = 0,02$; II: $n = 0,2$, $\lambda_1 = 0,1$ ($c = \lambda_1 n < \lambda m$); (a) for the upper plot, we have $a = 0,1$ (I: $a > \lambda$, $i_{\mathrm{r,I}} > 0$; II: $a < \lambda + \lambda_1$, $\lambda + \lambda_1 < b^2 + 2a$, and $i_{\mathrm{r,II}} > 0$); (b) for the lower plot, we have $a = 0,02$ (I: $a < \lambda$, $i_{\mathrm{r,I}} < 0$; II: $a < \lambda + \lambda_1$, $\lambda + \lambda_1 > b^2 + 2a$, and $i_{\mathrm{r,II}} < 0$); both for Models I and II, the results of numerical simulation coincide up to the graphical precision (for the difference in another scale, see Fig. 4a and Fig. 4b). In Fig. 4a and Fig. 4b, we present the plots of the differences $\varphi_{\mathrm{I}}(u) - \varphi_{\mathrm{II}}(u)$ for the graphically coinciding differences in Fig. 3: Fig. 4a (Fig. 4b) corresponds to the upper plot (lower plot) of Fig. 3. The values of the parameters for Fig. 5a and Fig. 5b are as follows: $a = 0,2$, $b = 0,1$, $m = 1$, and $\lambda = 0,05$; Fig. 5a: I: $c = 0,02$ ($c < \lambda m$ and $i_{\mathrm{r,I}} > 0$); II: $n = 0,2$ and $\lambda_1 = 0,1$ ($\lambda_1 n < \lambda m$, $a > \lambda + \lambda_1$, and $i_{\mathrm{r,II}} > 0$); Fig. 5b: I: $c = 0,08$ ($c > \lambda m$ and $i_{\mathrm{r,I}} > 0$); II: $n = 0,8$ and $\lambda_1 = 0,1$ ($\lambda_1 n > \lambda m$, $a > \lambda + \lambda_1$, and $i_{\mathrm{r,II}} > 0$).

Detailed data for the computations (including $\varphi(0)$, $\varphi'(0)$, $\varphi''(0)$, and other values) for Models I and II (models 1 and 2) are provided in [16].

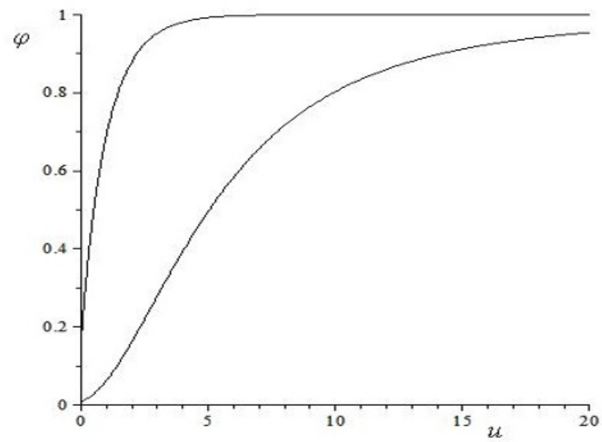

Fig. 3.

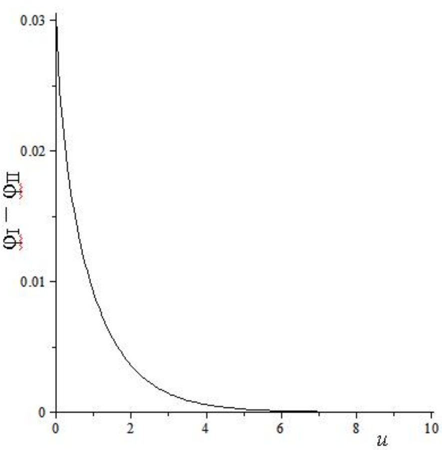

Fig. 4a.

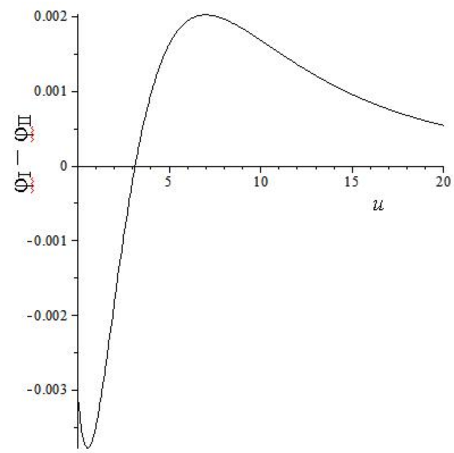

Fig. 4b.

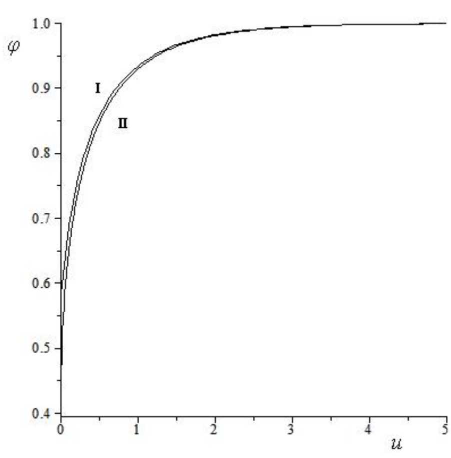

Fig. 5a.

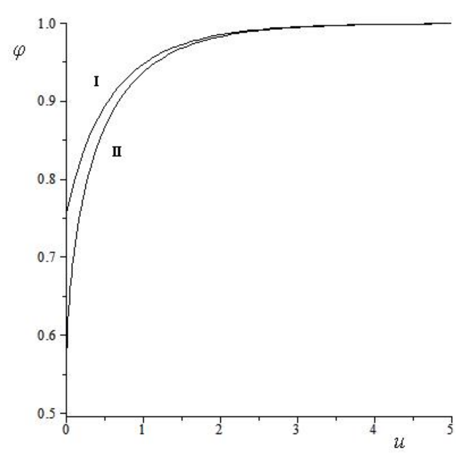

Fig. 5b.

7. Final Remarks and Economic Interpretation of Results

1. Investigating Model I, we see that if the initial capital is large and the structure of the investment portfolio is constant, then the use of risk assets is not a good strategy from the nonruin point of view. However, if the initial capital is small, especially if the security load is nonpositive, then risk assets are efficient to minimize the total risk: if there are no investments, then the ruin is inevitable; otherwise, the nonruin probability rapidly grows as the initial capital u grows (this is valid even for negative values of the risk factor provided that the second derivative of the nonruin probability for small u is positive).

In [13, 14], the optimal strategy is investigated for the case where claims are distributed exponentially; it shows that the share of risk investments should be an infinitesimal variable of order $O(1/x)$ as the circulating capital x tends to infinity.

2. In [16], we investigated singular boundary-value problem (2.6)–(2.10) with constraints for an integrodifferential equation (Model II) in order to find the nonruin probability in the insurance model with stochastic premium and investments of the capital in risk assets. We also constructed an algorithm to compute the nonruin probability as a function of the initial capital and computed it numerically. We note that one has to prove the existence theorem for the posed boundary-value problem in order to justify theoretically the kind of function defining the nonruin probability in the considered model.

In [22], natural heuristic reasoning combined with direct application of the generator technique for Markov processes results in the equation for the nonruin probability treated as a function of the initial capital (in this case, a linear integrodifferential equation is obtained) provided that the nonruin probability is a twice continuously differentiable function of the initial capital. Then, to use the obtained integrodifferential equation, e. g., for investigation of the asymptotics of the nonruin probability for large values of the initial capital, one has to prove that the nonruin probability is a twice continuously differentiable function indeed and, on the other hand, to justify the limit condition at infinity for the corresponding solution of that integrodifferential equation (more exactly, the condition for that solution to tend to unity). To do that, one can use, e. g., upper estimates for the ruin probability similar to the Lundberg estimate for the classical model, proving that the ruin probability tends to zero as the initial capital tends to infinity provided that the security load is positive (such an estimate for the Cramér–Lundberg model with stochastic premium is provided, e. g., in [37]). However, this was not proved in [22], where the model with investments of the capital in risk assets was investigated. As a result, the function of the asymptotics of the solution of the integrodifferential equation, obtained in [22], contains an indeterminate constant. Thus, in fact it remains unproved that the obtained function determines the asymptotics of the nonruin probability at least for some value of the above constant.

The approach used in [16] and in the present paper is based on the investigation of the well-posed problem (2.6)–(2.10) for an integrodifferential equation on the whole nonnegative semiaxis and on the proof of the existence of its solution. Using that approach and taking into account results of [11], we avoid the problems mentioned above. In particular, we do not need to prove that the nonruin probability is twice continuously differentiable and to obtain its upper estimates for large values of the initial capital. (For Model I, estimates from [27] are used and refined to justify the asymptotics obtained in [25] for the nonruin probability for large values of the initial capital; however, no similar estimates for Model II were obtained in [22].) Moreover, we do not need to prove lower estimates for the ruin probability (for Model I, such estimates are given in [25] and [27]).

3. Additionally, the approach used allows us to compute the nonruin probability numerically, to compare computational results for Model I and Model II, and to provide their economic interpretation.

In particular, the adequacy of the constructed solutions and computations is confirmed by the closeness of the plots of the functions of nonruin probability in Model I and Model II for frequent small-size premiums in Model II provided that the expected premium per time unit coincides for both models. Also, this shows that the premium income can be approximately considered as a determinate process under the assumption that the premium frequency substantially exceeds the claim frequency (the classical Cramér–Lundberg model is based on the same assumption).

At the same time, the results of computations allow us to analyze cases where the use of the classical Cramér–Lundberg model as the initial risk process might overstate or understate the nonruin probability compared with its values resulting from the model based on the risk process with stochastic premium. In particular, if the security load in the original model is positive, then the nonruin probability computed according to Model I is overstated for any nonnegative initial capital. For both models, application of investments substantially increases the nonruin probability for small values of the initial capital compared with the corresponding models without investments (the classical Cramér–Lundberg model and the Cramér–Lundberg model with stochastic premium). If the security load is negative, then the ruin probability is equal to 1 in the initial risk models, but application of investments with constant portfolio structure always provides positive values of the nonruin probability provided that the portfolio is reliable.

Thus, application of investments efficiently compensates the proper risk of insurers when this risk is high. This conclusion is based on study of solutions of the problems on the whole nonnegative semiaxis, but it cannot be made only based on comparison of their asymptotic behavior for large values of the initial capital as done in [25] claiming that "in insurance, investments in risk assets are dangerous." It turns out that if the initial capital is small, then the conclusion is different: Investments in risk assets are not dangerous but are necessary to increase solvency. More information is provided by investigation of the optimal control of investments minimizing the ruin probability in the Cramér–Lundberg model with constraints; in particular, this is based on the investigation of Model I (see [12]).

Comparing the computational results for the nonruin probability in Model I and Model II for nonpositive security loads in the initial risk models and for equal expected premiums, we see that the conclusions depend on the risk factors of models $i_{r,\mathrm{I}}$ and $i_{r,\mathrm{II}}$ defined in (4.17) and (5.57) respectively. The greatest risk occurs if the risk factor is negative: then the plot of the nonruin probability has an inflection.

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