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MATHEMATICS

Asymptotic Analysis of Solutions to a Riccati Equation

V. S. Samovol

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Abstract—A Riccati equation with coefficients expandable into convergent power series in a neighborhood of infinity is considered. Continuable solutions to equations of this type are studied. Conditions for the expansion of these solutions into convergent series in a neighborhood of infinity are obtained by methods of power geometry.

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INTRODUCTION

We consider the Riccati equation

$$y' + F(x,y) = 0, \quad F(x,y) = \sum_{i=0}^{2} f_i(x)y^i = 0, \quad y = y(x), \quad x, y \in R^1. \tag{1}$$

In [1] Riccati studied the equation

$$y' + ay^2 + bx^p = 0, \quad y = y(x), \quad x, y \in R^1, \quad a, b, p = \text{const}. \tag{2}$$

It is well known that Eq. (2) admits the separation of variables and, hence, can be integrated by quadratures in the case when $p = 4n(1-2n)^{-1}$, where n is an integer (see [2]). This equation is also integrable by quadratures when $p = -2$. Liouville proved that this equation is not integrable by quadratures for all other values of the parameter p [2, 3]. The change of variables $y = (az)^{-1}z'$ reduces Eq. (2) to the second-order equation

$$z'' + abzx^p = 0, \tag{3}$$

whose solutions can be expressed in terms of cylinder functions (see, for example, [4]).

The main method traditionally used to study Eq. (1) lies in transforming it by means of a suitable change of variables into a form as simple as possible, so that the equation can be integrated using elementary methods. However, this approach is restrictive and efficiently helps in analysis only in the cases indicated above.

This study is devoted to an asymptotic analysis of solutions to Eq. (1) in a neighborhood of the point $x = +\infty$ based on two-dimensional power geometry methods [5, 6]. We will determine conditions under which solutions to this equation are representable in the form of convergent functional series. We will consider the case when the functions $f_i(x)$ are expandable into convergent power series in a neighborhood of infinity. Precise definitions are given below. This work is a continuation of the studies begun in [7], where we presented the possibilities of using power geometry methods in analyzing solutions to Eq. (1) of the form

$$y' + y^2 + c(x) = 0, \quad c(x) = \sum_{k=1}^{n} c_k x^{p_k},$$
$$p_{i+1} < p_i, \quad i \in \{1, 2, \dots, n-1\}, \quad p_1 \neq -1.$$

Definition. A solution $y(x)$ to Eq. (1) is called *continuable to the right* if it is defined in a neighborhood of the point $x = +\infty$.

From now on, for brevity, solutions continuable to the right are referred to as continuable.

We study continuable solutions to Eq. (1) under the assumption that the functions $f_i(x)$, $i \in \{0, 1, 2\}$, can be represented as series of the form

$$f_i(x) = \sum_{j=1}^{\infty} c_{ij} x^{p_{ij}}, \quad p_{ij+1} < p_{ij}, \quad \lim_{j\to\infty} p_{ij} = -\infty, \quad i \in \{0,1,2\}, \tag{4}$$

uniformly absolutely convergent in a neighborhood of $x = +\infty$. We assume that Eq. (1) is nonhomogeneous, since the homogeneous equation is the Bernoulli

National Research University Higher School of Economics, Moscow, 101000 Russia
e-mail: 555svs@mail.ru

equation, which is directly integrable. Thus, we assume in what follows that $c_{01}c_{21} \neq 0$ in (4).

Hereinafter, we use the terminology adopted in power geometry [5]. The Newton polygon N of Eq. (1) under condition (4) is the closed convex hull of the points $Q = (-1, 1)$, $Q_{ij} = (p_{ij}, i)$, $i \in \{0, 1, 2\}$, $j \in \{1, 2, ...\}$. In this study, we consider the case when the point Q belongs to the right-hand boundary of N, that is, we have the inequalities

$$\frac{p_{01} + p_{21}}{2} \leq -1, \quad p_{11} \leq -1. \tag{5}$$

We will show that continuable solutions to Eq. (1) in this case can be represented as functional series convergent in a neighborhood of $x = +\infty$. If condition (5) is violated, we can calculate formal (generally speaking, divergent in the neighborhood of $x = +\infty$) series for existing continuable solutions to Eq. (1). An analysis of this situation is beyond the scope of this study.

In what follows, when mentioning solutions to Eq. (1), we mean continuable solutions to this equation.

Consider two cases. The first is when the condition

$$\frac{p_{01} + p_{21}}{2} < -1, \quad p_{11} \leq -1 \tag{6}$$

is satisfied. If condition (6) is satisfied, an important role in calculating expansions of solutions to Eq. (1) is played by the two edges, $[QQ_{01}]$ and $[QQ_{21}]$, and the vertex Q of the polygon N. These edges and the vertex are associated with truncated equations, whose solutions are the first approximations of solutions to Eq. (1) as $x \to +\infty$.

In the second case when

$$\frac{p_{01} + p_{21}}{2} = -1, \quad p_{11} \leq -1, \tag{7}$$

the edge $[Q_{01}Q_{21}]$ is the right-hand boundary of the polygon N. This edge is also associated with a truncated equation, whose solutions are the first approximations of solutions to Eq. (1).

Under condition (6), we will prove the existence of solutions to Eq. (1) in the form of series in powers of x with coefficients being functions that depend on $\ln x$. These series uniformly absolutely converge in a neighborhood of the point $x = +\infty$. In case (7), continuable solutions do not always exist. Using a truncated equation, we will determine conditions for their existence and also calculate convergent series that are expansions of these solutions in a neighborhood of the point $x = +\infty$.

Note that, in this study, when we mention power series (expansions), the exponents are not necessarily integers, unlike in traditional power series.

Theorem 1 describes, in the case (6), a series expansion of a continuable solution to Eq. (1) whose first approximation is determined by the edge $[QQ_{01}]$ of the Newton polygon of Eq. (1). The solution has the form of a uniformly absolutely convergent series in powers of x in a neighborhood of the point $x = +\infty$ with coefficients being polynomials in $\ln x$. According to the terminology adopted in power geometry (see [5]), the expansion of the solution can be of two types: a power expansion (the solution is expandable into a power series with constant coefficients) and a power-logarithmic expansion (the solution is expandable into a power series and the coefficients of the terms of the series are polynomials in $\ln x$). A significant role in determining the type of the expansion is played by the presence or absence of critical values of the first approximation of the solution (see [5]).

Theorem 2 describes a family of solutions whose first approximations are determined by the vertex Q. These solutions have the form of uniformly absolutely convergent power series in a neighborhood of the point $x = +\infty$ with constant coefficients, that is, the expansions of solutions of the family are power expansions in this case.

Theorems 3 and 4 consider the case when condition (7) holds. Theorem 3 presents conditions for the existence of continuable solutions to Eq. (1) in this case, and Theorem 4 describes expansions of these solutions into convergent series. The first approximations of the solutions are solutions to the truncated equation corresponding to the edge $[Q_{01}Q_{21}]$ of the polygon N and have a power form. The expansion can be of power or power-logarithmic type.

A fractional-rational transformation of the variable y reduces Riccati equation (1) to an equation of the same type. After applying the power transformation $y = z^{-1}$, the edges $[QQ_{01}]$ and $[QQ_{21}]$ are interchanged. Based on Theorem 1, this makes it possible to obtain series expansions of the solutions to Eq. (1) whose first approximations are determined by the edge $[QQ_{21}]$ of the polygon N. A detailed description of these expansions is left to the reader. We only note that the solutions are expandable into series in powers of x that are uniformly absolutely convergent in a neighborhood of the point $x = +\infty$ with coefficients being functions of $\ln x$.

Remark 1. Theorem 2 describes families of solutions whose first approximations are solutions to the truncated equation corresponding to the vertex Q. Similar families can arise in some other situations. It is straightforward to obtain corresponding propositions based on Theorems 1, 2, and 4. To avoid overloading the presentation, only one of these families is described in Theorem 5.

STATEMENT OF THE MAIN RESULTS

Theorem 1. *It conditions* (4) *and* (6) *are satisfied, then Eq.* (1) *has the following continuable solution*:

(i) *If* $p_{11} < -1$ *and* $p_{01} \neq -1$, *then*

$$y = a_1x^{s_1} + \sum_{i=2}^{\infty} a_i(x)x^{s_i},$$
$$s_{i+1} < s_i, \quad \lim_{i\to\infty} s_i = -\infty, \tag{8}$$

$$s_1 = p_{01} + 1, \quad a_1 = \frac{-c_{01}}{p_{01}+1}.$$

(ii) *If* $p_{11} < -1$ *and* $p_{01} \neq -1$, *then*

$$y = -c_{01}\ln x + \sum_{i=2}^{\infty} a_i(x)x^{s_i},$$
$$s_{i+1} < s_i < 0, \quad \lim_{i\to\infty} s_i = -\infty, \tag{9}$$

(iii) *If* $p_{11} = -1$ *and* $p_{01} + c_{11} \neq -1$, *then the expansion of the solution has form* (8), *where*

$$s_1 = p_{01} + 1, \quad a_1 = \frac{-c_{01}}{p_{01} + c_{11} + 1}.$$

(iv) *If* $p_{11} = -1$ *and* $p_{01} + c_{11} \neq -1$, *then the solution has the form*

$$y = -c_{01}x^{p_{01}+1}\ln x + \sum_{i=2}^{\infty} a_i(x)x^{s_i},$$
$$s_{i+1} < s_i < p_{01} + 1, \quad \lim_{i\to\infty} s_i = -\infty. \tag{10}$$

Here, $a_i(x)$ *are functions being polynomials in* $\ln x$. *All the series uniformly absolutely converge in a neighborhood of the point* $x = +\infty$.

Remark 2. As can be seen from Theorem 1, the expansions of the solution are power-logarithmic. If Theorem 1 additionally assumes that $p_{01} < -1$ and $p_{11} < -1$, then we have a power expansion

$$y = a_1x^{s_1} + \sum_{i=2}^{\infty} a_ix^{s_i}, \quad a_i = \text{const},$$
$$s_{i+1} < s_i, \quad \lim_{i\to\infty} s_i = -\infty, \tag{11}$$

$$s_1 = p_{01} + 1, \quad a_1 = \frac{-c_{01}}{p_{01}+1}.$$

If Theorem 1 additionally assumes that $p_{11} = -1$ and $p_{01} + c_{11} < -1$, then the expansion of the solution is also a power series of form (11), where $s_1 = p_{01} + 1$, $a_1 = \dfrac{-c_{01}}{p_{01} + c_{11} + 1}$.

Theorem 2. *If condition* (4) *is satisfied and* $p_{i1} < -1$, $i \in \{0, 1, 2\}$, *then Eq.* (1) *has a one-parameter family of solutions of the form*

$$y = ax^{-c_{11}} + \sum_{i=1}^{\infty} a_ix^{s_i}, \quad s_{i+1} < s_i < -c_{11},$$
$$\lim_{i\to\infty} s_i = -\infty, \quad a_i = \text{const} \tag{12}$$

if condition (4) *is satisfied and* $p_{11} = -1$, $p_{01} + 1 < -c_{11} < -p_{21} - 1$, *then Eq.* (1) *has a one-parameter family of solutions of the form*

$$y = ax^{-c_{11}} + \sum_{i=1}^{\infty} a_ix^{s_i}, \quad s_{i+1} < s_i < -c_{11},$$
$$\lim_{i\to\infty} s_i = -\infty, \quad a_i = \text{const}. \tag{13}$$

Here, $a \neq 0$ *is an arbitrary constant. All the series uniformly absolutely converge in a neighborhood of the point* $x = +\infty$.

We now consider the case (7). In this case, Eq. (1) does not always have continuable solutions. We present the necessary conditions for the existence of solutions of this type.

Theorem 3. *If condition* (4) *is satisfied and* $\dfrac{p_{01} + p_{21}}{2} = -1$, $p_{11} < -1$, *then Eq.* (1) *does not have continuable solutions if* $4c_{01}c_{21} > (p_{01} + 1)^2$.

If condition (4) *is satisfied and* $\dfrac{p_{01} + p_{21}}{2} = p_{11} = -1$, *then Eq.* (1) *does not have continuable solutions if* $4c_{01}c_{21} > (p_{01} + c_{11} + 1)^2$.

Remark 3. Condition (4) in Theorem 3 is excessive. For the theorem to hold, it suffices that the functions $f_i(x)$, $i \in \{0, 1, 2\}$, have power asymptotics as $x \to +\infty$:

$$f_i(x) = c_{i1}x^{p_{i1}}(1 + o(x^{-\varepsilon})),$$
$$\varepsilon = \text{const} > 0, \quad i \in \{0,1,2\}.$$

Theorem 4. *If condition* (4) *is satisfied,*

$$\frac{p_{01} + p_{21}}{2} = -1, \quad p_{11} < -1, \quad \text{and} \quad 4c_{01}c_{21} \le (p_{01} + 1)^2,$$

then Eq. (1) *has continuable solutions that can be expanded into series of form* (8) *that are uniformly absolutely convergent in a neighborhood of the point* $x = +\infty$; *here*, $s_1 = p_{01} + 1$ *and* a_1 *is any of the roots of the quadratic equation*

$$c_{21}v^2 + (p_{01} + 1)v + c_{01} = 0. \tag{14}$$

If condition (4) *is satisfied*, $\dfrac{p_{01} + p_{21}}{2} = p_{11} = -1$, *and* $4c_{01}c_{21} \le (c_{11} + p_{01} + 1)^2$, *then Eq.* (1) *has continuable solutions that can be expanded into series of form* (8) *that are uniformly absolutely convergent in a neighborhood of the point* $x = +\infty$; *here*, $s_1 = p_{01} + 1$ *and* a_1 *is any of the roots of the quadratic equation*

$$c_{21}v^2 + (c_{11} + p_{01} + 1)v + c_{01} = 0. \tag{15}$$

It follows from this theorem that the expansions of solutions in this case are either power expansions or power-logarithmic expansions.

Theorem 5. *Let conditions* (4) *and* (6) *be satisfied. If* $p_{01} = -1$, $p_{11} < -1$, *then Eq.* (1) *has a one-parameter family of solutions of the form*

$$y = a - c_{01}\ln x + \sum_{i=2}^{\infty} a_i(x)x^{s_i},$$
$$s_{i+1} < s_i < 0, \quad \lim_{i\to\infty} s_i = -\infty, \tag{16}$$

while, if $p_{01} > -1$, $p_{11} < -1$, *then Eq.* (1) *has a one-parameter family of solutions of the form*

$$y = a + a_1x^{s_1} + \sum_{i=2}^{\infty} a_i(x)x^{s_i},$$
$$s_1 = p_{01} + 1, \quad a_1 = \frac{-c_{01}}{p_{01}+1}, \tag{17}$$
$$s_{i+1} < s_i, \quad \lim_{i\to\infty} s_i = -\infty.$$

Here, $a \neq 0$ *is an arbitrary constant and* $a_i(x)$ *are functions being polynomials in* $\ln x$. *All the series uniformly absolutely converge in a neighborhood of the point* $x = +\infty$.

We can describe similar families of solutions using Theorem 1 (cases (iii) and (iv)) and Theorem 2. Families of solutions of this type can also be obtained based on Theorems 4 and 2.

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