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Robust performance analysis of linear discrete-time systems in presence of colored noise☆

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Abstract

In this paper, linear discrete-time time-invariant (LDTI) normal and descriptor systems with norm-bounded parametric uncertainties are under consideration. The input signal is supposed to be a "colored" noise with bounded known mean anisotropy level (spectral color). The conditions of anisotropic norm boundedness for such class of systems are derived. The algorithm is based on convex optimization technique. A numerical example is given.

Keywords: descriptor systems, uncertain linear systems, norm-bounded uncertainties, colored noise, stochastic systems.
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1. Introduction

In recent years problems of robust control and performance analysis of uncertain systems affected by external disturbances have become one of the most popular research areas in modern control theory. Considerable attention is paid to problems of robust stabilization and robust performance analysis

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of uncertain normal dynamical systems in both continuous and discrete-time cases [1, 2, 3, 4].

$\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms have become the most popular criteria in robust performance analysis of linear systems. The squared $\mathcal{H}_2$-norm of a linear time-invariant system can be interpreted as a trace of its steady-state output covariance under the assumption that the system is driven by Gaussian white noise with identity covariance matrix. So, $\mathcal{H}_2$-norm is a useful measure of performance when the system is affected by the Gaussian white noise. In order to find solutions of robust $\mathcal{H}_2$-control and robust $\mathcal{H}_2$-performance analysis see [5, 6, 7] and references therein.

In discrete-time $\mathcal{H}_\infty$-approach the input signal is assumed to be square summable. The so-called bounded real lemma plays an important role in solving this problem. In the context of both normal and descriptor systems, results on $\mathcal{H}_\infty$-control are proposed, for example, in [4, 8].

A descriptor system framework for mathematical modelling and control design has been extensively developed [8, 9, 10, 11]. Linear discrete-time descriptor systems found their application in biological complex systems [16], in hydraulic systems [17], and in electrical systems [18]. In the last few years, a number of papers dedicated to descriptor systems appear in different research topics, see [12, 13, 14, 15]. Firstly, this approach simplifies the design of mathematical models, secondly, simulation of the system's dynamics is more illustrative, and, thirdly, some state variables are redundant, this fact allows to have more freedom while designing control laws [10, 12]. Finally, descriptor systems are a general case of standard state-space systems: a descriptor system without algebraic equations can be easily transformed into the standard state-space one. Despite obvious advantages, in discrete-time case, a specific behavior such as noncausality may occur while solving the system's equations, due to the presence of algebraic constraints in the model. Motivated by this fact, many efforts have been made towards developing methods for solving a number of control problems.

A number of fundamental results on $\mathcal{H}_2$- and $\mathcal{H}_\infty$-performance analysis based on the theory of normal state-space systems are successfully extended to descriptor systems [8, 19, 20, 21]. All of the results mentioned above deal with uncertainty-free discrete-time descriptor systems. Interest in stability analysis and control of descriptor systems with parametric uncertainties has grown recently due to their frequent presence in dynamical systems, which are often causes of instability and poor performance of control systems. It is known that analysis and control problems for uncertain descriptor systems are much more complicated comparing to the normal ones. Discrete-time descriptor systems with norm-bounded parametric uncertainties are investigated in [8, 22, 23]. These papers provide different versions of bounded real lemma for uncertain descriptor systems. $\mathcal{H}_2$-performance analysis for uncertain descriptor systems is still an open problem.

However, in practical applications, the system is affected by a correlated random disturbance. In this case, the above mentioned $\mathcal{H}_2$- and $\mathcal{H}_\infty$-criteria have an important drawback, namely mathematical models of input disturbances do not capture main properties of real signals. To overcome this drawback, the anisotropy-based concept was suggested by I.Vladimirov and described in details in the papers [24, 25]. Anisotropy-based theory studies disturbance attenuation capabilities of the system in presence of colored random input disturbances. Mean anisotropy quantitatively characterizes spectral color of Gaussian sequences, and it is strongly connected with theoretical information approaches of signals' description. Anisotropic norm is a special case of the stochastic norm. In case of bounded mean anisotropy anisotropic norm of the system defines its performance index. Other results on control of stochastic systems based on theoretical information approach are presented in [26, 27].

Some results on anisotropy-based robust control design and performance analysis for normal systems with norm-bounded uncertainties are obtained in [29, 30]. These results are based on solving of algebraic Riccati equations and have large computational complexity. To this end, the problem of developing numerically effective methods of robust performance analysis of uncertain linear systems is an open problem.

The key feature of the anisotropy-based approach is that anisotropic norm of the system lies between the scaled $\mathcal{H}_2$-norm and $\mathcal{H}_\infty$-norm [24, 28]. Anisotropy-based control theory allows developing a unified theoretical framework for performance analysis and control synthesis, which covers popular $\mathcal{H}_2$- and $\mathcal{H}_\infty$-approaches as limiting cases. From the practical point of view, additional information about the input disturbance allows to expend less energy for control, and, at the same time, remove strong assumption that the input disturbance is a white noise sequence.

This paper addresses the robust anisotropy-based analysis based on convex optimization technique for both normal and descriptor systems with norm-bounded parametric uncertainties. We state a novel condition for the robust anisotropy-based performance analysis of discrete-time systems. It is shown that this condition is expressed as strict LMIs. An algorithm of

a-anisotropic norm computation, based on this condition, is also considered. We explain how this condition can be applied for $\mathcal{H}_2-$ and $\mathcal{H}_\infty-$ robust performance analysis of discrete-time uncertain systems. The numerical effectiveness of proposed methods is confirmed by numerical examples.

The paper is organized as follows. In section 2 problem statement is discussed. Section 3 provides necessary background on linear discrete-time descriptor systems and anisotropy-based control theory. Section 4 deals with preliminary results on the anisotropy-based analysis of certain linear systems. Section 5 deals with main results of the paper. In section 6 an illustrative numerical example is given. Finally, in section 7 some conclusive remarks and future work are discussed.

Throughout the paper, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{R}^{m\times n}$ denote the set of real numbers, the set of complex numbers, and the set of all real matrices of dimension $m\times n$, respectively; I_n is $n\times n$ identity matrix; Z^* is the Hermitian conjugate of the matrix $Z=[z_{ij}]\in\mathbb{C}^{m\times n}$: $Z^*=[z^*_{ji}]\in\mathbb{C}^{n\times m}$; $\rho(A)$ is the spectral radius of a square matrix A: $\rho(A)=\max\limits_j|\lambda_j(A)|$; $\overline{\sigma}(A)$ stands for the maximal singular value of the matrix A: $\overline{\sigma}(A)=\sqrt{\rho(A^*A)}$; $\mathrm{sym}\,(A)$ stands for symmetrization of matrix A: $\mathrm{sym}\,(A)=A+A^{\mathrm{T}}$.

2. Problem Statement

In this paper, we deal with uncertain linear discrete-time time-invariant systems. A general state-space representation of such systems is given by

$$Ex(k+1) = (A+M_A\Delta N_A)x(k)+(B+M_B\Delta N_B)w(k), \quad (1)$$
$$y(k) = (C+M_C\Delta N_C)x(k)+(D+M_D\Delta N_D)w(k) \quad (2)$$

where $x(k)\in\mathbb{R}^n$ is a state vector, $w(k)\in\mathbb{R}^m$ is an input signal, $y(k)\in\mathbb{R}^p$ is a measurable output, $\Delta\in\mathbb{R}^{q\times q}$ is an unknown spectral norm-bounded matrix:

$$\|\Delta\|_2 := \overline{\sigma}(\Delta)\leqslant 1$$

iff

$$\Delta^{\mathrm{T}}\Delta\leqslant I_q$$

(or Frobenius norm-bounded matrix as $\|\Delta\|_2\leqslant\|\Delta\|_F$). The matrices A, B, C, D, M_A, N_A, M_B, N_B, M_C, N_C, M_D and N_D are constant real of appropriate dimensions. We will use the following notations $A_\Delta=A+M_A\Delta N_A$, $B_\Delta=B+M_B\Delta N_B$, $C_\Delta=C+M_C\Delta N_C$, $D_\Delta=D+M_D\Delta N_D$.

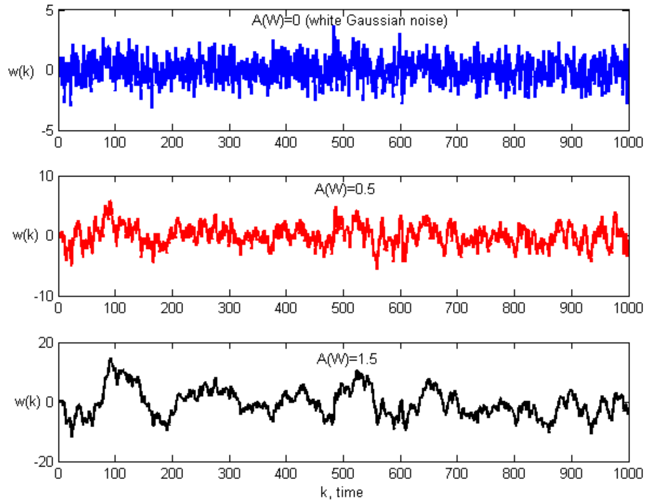

Figure 1: Random signals with different mean anisotropy level $\overline{\mathbf{A}}(W)$.

Then, the transfer function of system (1)–(2) is defined as $P_\Delta(z) = C_\Delta(zE - A_\Delta)^{-1}B_\Delta + D_\Delta$.

The input signal $w(k)$ is supposed to be a random stationary sequence with bounded mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a$. The parameter a is supposed to be known and always nonnegative. Mean anisotropy of the signal characterizes its "spectral color". The realizations of random signals with different mean anisotropy levels are presented in fig. 1. The detailed necessary background on the anisotropy-based concept is introduced below.

According to [31, 32] a-anisotropic norm of a system is a particular case of the stochastic norm and is defined as the supremum of the ratio of the root mean square value of the system output to that of the input over all stationary Gaussian inputs with mean anisotropy upper-bounded by a nonnegative parameter a. Thus, a-anisotropic norm characterizes a disturbance attenuation level from random input $w(k)$ to measurable output $y(k)$ of the system. It is denoted as

$$\|P_\Delta(z)\|_a .$$

In equation (1) we introduce matrix E. If matrix E is equal to I_n or has a full rank, then system (1) is a normal one. If matrix E is singular, i.e. $\operatorname{rank} E = r < n$, then system (1) is called descriptor or singular system. In this paper both cases are concerned.

Hereinafter, we suppose that system (1)–(2) satisfies the following assumptions:

$$\operatorname{rank} E^{\mathrm{T}} = \operatorname{rank}\left[E^{\mathrm{T}}, C^{\mathrm{T}}, N_C^{\mathrm{T}}\right], \quad (3)$$
$$\operatorname{rank} E = \operatorname{rank}\left[E, B, M_B\right]. \quad (4)$$

It is easy to see that in case of a normal system ($\operatorname{rank}(E) = n$) assumptions (3)–(4) are automatically satisfied.

Remark 1. Assumption (3) means that there are no disturbances in algebraic equations. In this case, algebraic constraints don't contain random disturbance as an input signal. Assumption (4) means that measurable output contains no state variables connected with the algebraic subsystem. On the one hand, these assumptions restrict a set of descriptor systems to be considered in this paper, on the other hand, most of the practical systems satisfy these assumptions [9, 16, 17]. It should be noted that these assumptions are also required for correct computation of a-anisotropic norm of a descriptor system in time domain.

The problem of robust anisotropy-based performance analysis is formulated as follows.

Consider system (1)–(2) with known matrices of appropriate dimensions and unknown but norm-bounded uncertainty Δ. For the given scalar value $\gamma > 0$, and known nonnegative mean anisotropy level of random input disturbance $\overline{\mathbf{A}}(W) \leqslant a$ the problem is to check if

1. the system is robustly stable (admissible);
2. its a-anisotropic norm is less than $\gamma > 0$, i.e. $|||P_\Delta(z)|||_a < \gamma$.

The term "admissible" relates to the descriptor systems theory. This term will be discussed in detail in the next section.

3. Basics of LDTI descriptor systems and anisotropy-based theory

This section introduces main definitions and concepts from descriptor system theory [8, 9] and anisotropy-based theory [25, 31, 32].

3.1. LDTI descriptor systems

A state-space representation of linear discrete-time descriptor systems is

$$Ex(k+1) = Ax(k) + Bf(k), \tag{5}$$
$$y(k) = Cx(k) + Df(k) \tag{6}$$

where $x(k) \in \mathbb{R}^n$ is the state, $f(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ are the input and output signals, respectively, A, B, C and D are constant real matrices of appropriate dimensions. The matrix $E \in \mathbb{R}^{n\times n}$ is singular, i.e. $\operatorname{rank}(E) = r < n$.

The transfer function of system (5)–(6) is defined by the expression

$$P(z) = C(zE - A)^{-1}B + D, \; z \in \mathbb{C} \tag{7}$$

where z represents z-Transform.

Definition 1. System (5) is called regular if $\exists\lambda \neq 0 : \det(\lambda E - A) \neq 0$, $\lambda \in \mathbb{C}$.

Regularity stands for existence and uniqueness of the solution for consistent initial conditions [9]. Hereinafter, we suppose that the considered systems are regular. Now we give some definitions, necessary for further discussion.

$\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of the transfer function $P(z)$ are defined as follows

$$\|P\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Trace}\left(P^*(e^{i\omega})P(e^{i\omega})\right) d\omega\right)^{\frac{1}{2}},$$
$$\|P\|_\infty = \sup_{\omega\in[0,2\pi]} \overline{\sigma}\left(P(e^{i\omega})\right).$$

Definition 2. System (5) is called admissible if it is regular, causal ($\deg\det(zE-A) = \operatorname{rank} E$), and stable ($\rho(E, A) = \max|\lambda|_{\lambda\in\{z|\det(zE-A)=0\}} < 1$).

For more information, see [8, 9].

For regular system (5)–(6) there exist two nonsingular matrices $\overline{W}$ and $\overline{V}$ such that $\overline{W}E\overline{V} = \operatorname{diag}(I_r, 0)$ see [9].

Consider the following change of variables:

$$\overline{V}^{-1}x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{8}$$

where $x_1(k) \in \mathbb{R}^r$ and $x_2(k) \in \mathbb{R}^{n-r}$.

By left multiplying equation (5) on the matrix $\overline{W}$ and using the change of variables (8), one can rewrite the system (5)–(6) in the form

$$\begin{aligned} x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) + B_1 f(k), && (9) \\ 0 &= A_{21}x_1(k) + A_{22}x_2(k) + B_2 f(k), && (10) \\ y(k) &= C_1 x_1(k) + C_2 x_2(k) + D f(k) && (11) \end{aligned}$$

where

$$A_d = \overline{W}A\overline{V} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B_d = \overline{W}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C_d = C\overline{V} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \tag{12}$$

Matrices $\overline{W}$ and $\overline{V}$ are found from singular value decomposition (SVD) [33]

$$E = U \operatorname{diag}(S, 0) H^{\mathrm{T}}.$$

Here U and H are unitary matrices, S is a diagonal $r \times r$-matrix, it is formed by nonzero singular values of the matrix E

$$\overline{W} = \operatorname{diag}(S^{-1}, I_{n-r})U^{\mathrm{T}}, \qquad \overline{V} = H.$$

Representation (9)–(11) is called SVD equivalent form of system (5)–(6) see [9].

Causality is an important property of discrete-time descriptor systems. Noncausal behavior means that the system's state depends not only on the current values of the input signal but also on the future ones. In SVD equivalent form (9)–(11) system (5)–(6) is causal if A_{22} is nonsingular [8]. Hence, noncausal behavior does not allow to implement equivalent transformation from a descriptor system to a normal state-space one. Consider the following simple illustrative example.

Let the system have the following state-space representation:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(k+1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} f(k). \tag{13}$$

The solution is defined as

$$\begin{aligned} x_2(k) &= -f(k), \\ x_1(k) &= -f(k) - f(k+1). \end{aligned}$$

The transformation matrices $\overline{W}$ and $\overline{V}$ for system (13) can be selected as

$$\overline{W} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \overline{V} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then $\overline{W}E\overline{V} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\overline{W}A\overline{V} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Here $A_{22} = 0$.

3.2. Basics of anisotropy-based theory

Now, we provide some background material on anisotropy-based analysis of linear discrete-time systems. The concepts of mean anisotropy of Gaussian random sequences and anisotropic norm of linear systems are introduced in [24, 31]. More detailed discussion on mean anisotropy and anisotropic norm can be found in [28].

Let $W = \{w_k\}_{-\infty<k<\infty}$ be a stationary sequence of square summable random vectors $w_k \in \mathbb{R}^m$. Assembling the elements of W, associated with a time interval $[0, N]$, into a random vector

$$W_{0:N} = \begin{bmatrix} w_0 \\ \vdots \\ w_N \end{bmatrix}, \tag{14}$$

we assume that $W_{0:N}$ is absolutely continuously distributed for any $N \geq 0$. Anisotropy $\mathbf{A}(W_{0:N})$ is defined as the minimal value of the relative entropy (Kullback-Leibler information divergence) with respect to Gaussian distributions in $\mathbb{R}^{m(N+1)}$ with zero mean and scalar covariance matrix given by

$$\mathbf{A}(W_{0:N}) = \frac{m}{2} \ln \left(\frac{2\pi e}{m} \mathbf{E}(|W_{0:N}|^2) \right) - h(W_{0:N}),$$

where

$$h(W_{0:N}) = -\mathbf{E} \ln f(W_{0:N}) = -\int_{\mathbb{R}^{m(N+1)}} f(x) \ln f(x) dx,$$

and $f(x)$ is a probability density function of the vector $W_{0:N}$. Note that Kullback-Leibler information divergence is a nonnegative functional. Hence,

$\mathbf{A}(W_{0:N}) \geqslant 0$. Clearly, $\mathbf{A}(W_{0:N}) = 0$ if $W_{0:N}$ is a Gaussian vector with zero mean and scalar covariance matrix.

The mean anisotropy of the sequence W is defined by

$$\overline{\mathbf{A}}(W) = \lim_{N\to+\infty} \frac{\mathbf{A}(W_{0:N})}{N}. \tag{15}$$

It is shown in [28] that

$$\overline{\mathbf{A}}(W) = \mathbf{A}(w_0) + \mathbf{I}\left(w_0; \{w_k\}_{k<0}\right) \tag{16}$$

where $\mathbf{I}\left(w_0; \{w_k\}_{k<0}\right) = \lim_{s\to-\infty} \mathbf{I}\left(w_0; W_{s:-1}\right)$ is the Shannon mutual information [34] between w_0 and all previous values $\{w_k\}_{k<0}$ of the sequence W.

Remark 2. Shannon mutual information between random vectors ξ and η is defined as

$$\mathbf{I}(\xi, \eta) = D(f_{\xi\eta} \| f_\xi f_\eta),$$

where f_* is a probability density function, $D(\cdot\|\cdot)$ is Kullback-Leibler information divergence. So, mutual information is always nonnegative.

For the Gaussian stationary random sequence W Shannon mutual information between w_0 and the past history $\{w_k\}_{k<0}$ is calculated as follows

$$\mathbf{I}\left(w_0; \{w_k\}_{k<0}\right) = \frac{1}{2} \ln \det(\mathbf{cov}(w_0)\mathbf{cov}(\tilde{w}_0)^{-1}), \tag{17}$$

where

$$\tilde{w}_0 = w_0 - \mathbf{E}(w_0 | \{w_k\}_{k<0}) \tag{18}$$

is the error of the mean-square optimal prediction of w_0 by the past history $(w_k)_{k<0}$, provided by the conditional expectation.

The random sequence W can be generated from white Gaussian noise V by a shaping filter $G(z) \in \mathcal{H}_2{}^{m\times m}$. Then

$$\|G\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \text{Trace}\,\left(\widehat{G}(\omega)\widehat{G}^*(\omega)\right) d\omega\right)^{1/2} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \text{Trace}\, S(\omega) d\omega\right)^{1/2}$$

where $S(\omega) = \widehat{G}(\omega)\widehat{G}^*(\omega)$, $(-\pi \leqslant \omega \leqslant \pi)$, $\widehat{G}(\omega) = \lim_{l\to 1} G(l e^{i\omega})$ is a boundary value of the transfer function $G(z)$.

Covariance matrix of prediction error (18) and spectral density $S(\omega)$ are interrelated by Szegö-Kolmogorov formula (19):

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det S(\omega)d\omega = \ln\det \mathbf{cov}(\tilde{w}_0). \tag{19}$$

Mean anisotropy of the sequence may be defined by the filter's parameters, using the expression

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det \frac{mS(\omega)}{\|G\|_2^2}d\omega = -\frac{1}{4\pi}\ln\det\frac{m\mathbf{cov}(\tilde{w}_0)}{\|G\|_2^2}. \tag{20}$$

Note that [32] matrix

$$H = \frac{m}{2\pi\|G\|_2^2}\int_{-\pi}^{\pi} S(\omega)d\omega$$

has trace equal to m. By the geometric-arithmetic mean inequality,

$$\det H \leqslant \left(\frac{\operatorname{Trace} H}{m}\right) = 1.$$

So, taking into account the concavity of function $\ln\det(\cdot)$ on the cone of positive definite Hermitian matrices and using the Jensen inequality, it follows from (20) that $\overline{\mathbf{A}}(W) \geqslant -\frac{1}{2}\ln\det H \geqslant 0$.

Remark 3. Mean anisotropy of the random sequence W, generated by shaping filter $G(z)$, is fully defined by its parameters, so the notation $\overline{\mathbf{A}}(G)$ will be used further instead of $\overline{\mathbf{A}}(W)$.

Let $Y = PW$ be an output of the linear discrete-time (normal or descriptor) system $P \in \mathcal{H}_\infty{}^{p\times m}$ with a transfer function $P(z)$, which is analytic outside unit disk $|z| > 1$, and has a finite $\mathcal{H}_\infty$-norm.

Definition 3. For a given constant value $a \geqslant 0$ a-anisotropic norm of the system P is defined as

$$|\!|\!| P |\!|\!|_a = \sup\left\{\frac{\|PG\|_2}{\|G\|_2} :\ G \in \mathbf{G}_a\right\}, \tag{21}$$

i.e. the maximum value of the system's gain with respect to the class of shaping filters

$$\mathbf{G}_a = \left\{G \in \mathcal{H}_2{}^{m\times m} :\ \overline{\mathbf{A}}(G) \leqslant a\right\}.$$

So, a-anisotropic norm $\|\!|P|\!\|_a$ describes the stochastic gain of system P with respect to the input sequence W with known mean anisotropy level. a-Anisotropic norm for both normal and descriptor systems is similarly defined in the frequency domain. Formulas of a-anisotropic norm in frequency domain were obtained and discussed in [32]. All of the existing frequency-domain approaches to computation of a-anisotropic norm require the knowledge of the system's parameters, being thus inapplicable to systems with uncertainties. Also, these methods are computationally hard since they are related to inverting a function of a scalar parameter, depending on integrals of the spectral density logarithms, and involves a factorization problem which cannot be expressed explicitly in general case. The time-domain approach is more attractive as it deals with well-known convex optimization methods of polynomial complexity [35] and can be generalized on uncertain systems.

If $\overline{\mathbf{A}}(W) = 0$, then the input signal is the Gaussian white noise sequence. In this case, the shaping filter $G(z)$ can be represented, for example, by an identity $m \times m$-matrix, i.e. $G(z) = I_m$. Therefore, $P(z)G(z) = P(z)$ and $\|G\|_2 = \sqrt{m}$. In this case $\|\!|P|\!\|_a = \frac{\|P\|_2}{\sqrt{m}}$.

If $\overline{\mathbf{A}}(W) \to \infty$, then $\lim_{a\to\infty} \|\!|P|\!\|_a = \|P\|_\infty$. For more information, see [24, 28].

4. Preliminary results for solving robust anisotropy-based analysis problem

This section represents preliminary results on anisotropy-based performance analysis for normal and descriptor systems with known parameters. Some important results related to transformations of linear matrix inequalities are also recalled here.

4.1. Modified anisotropy-based bounded real lemma for normal systems

Consider a LDTI normal system in the following form:

$$x(k+1) = Ax(k) + Bw(k), \tag{22}$$

$$y(k) = Cx(k) + Dw(k) \tag{23}$$

where $x(k) \in \mathbb{R}^n$ is a state vector, $w(k) \in \mathbb{R}^m$ is a random stationary sequence with bounded mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a$ $(a \geqslant 0)$, $y(k) \in \mathbb{R}^p$

is a measurable output, A, B, C, D are constant real matrices of appropriate dimensions. The transfer function of system (22)–(23) is given by the expression

$$T(z) = C(zI - A)^{-1}B + D.$$

System (22)–(23) is supposed to be stable, scalar values $a \geqslant 0$ and $\gamma > 0$ are known. The problem is to satisfy the inequality

$$|\!|\!| T |\!|\!|_a < \gamma.$$

In [35] the condition of norm boundedness, using convex optimization, is formulated as follows.

Theorem 1. *For given scalar values $a \geqslant 0$ and $\gamma > 0$ system* (22)–(23) *is stable and its a-anisotropic norm is bounded by γ, i.e.*

$$|\!|\!| T |\!|\!|_a < \gamma$$

if there exist a scalar value $\eta > \gamma^2$, $n \times n$-matrix $\Phi = \Phi^{\mathrm{T}} > 0$, and $n \times n$-matrix Y, such that the following inequalities hold true

$$\eta - (\mathrm{e}^{-2a} \det(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D))^{1/m} < \gamma^2, \tag{24}$$

$$\begin{bmatrix} -\frac{1}{2}Y - \frac{1}{2}Y^{\mathrm{T}} & YA & YB & \Phi^{\mathrm{T}} - Y^{\mathrm{T}} - \frac{1}{2}Y & 0 \\ A^{\mathrm{T}}Y^{\mathrm{T}} & -\Phi & 0 & A^{\mathrm{T}}Y^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}}Y^{\mathrm{T}} & 0 & -\eta I_m & B^{\mathrm{T}}Y^{\mathrm{T}} & D^{\mathrm{T}} \\ \Phi - Y - \frac{1}{2}Y^{\mathrm{T}} & YA & YB & -Y - Y^{\mathrm{T}} & 0 \\ 0 & C & D & 0 & -I_p \end{bmatrix} < 0. \tag{25}$$

4.2. Modified bounded real lemma for descriptor systems in SVD equivalent form

LDTI descriptor systems in a state-space representation are given as

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bw(k), && (26) \\ y(k) &= Cx(k) + Dw(k) && (27) \end{aligned}$$

where $x(k) \in \mathbb{R}^n$, $w(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^p$, $\operatorname{rank} E = r < n$. Hereinafter we suppose that the following assumption holds true for system (26)

$$\operatorname{rank} \begin{bmatrix} E & B \end{bmatrix} = \operatorname{rank} E. \tag{28}$$

We identify system (26)–(27) with the system $P \in \mathcal{H}_\infty{}^{p\times m}$, given by its transfer function

$$P(z) = C(zE - A)^{-1}B + D.$$

For an admissible system P, given by (26)–(27), for a known mean anisotropy level of the input disturbance $a \geqslant 0$ and a scalar $\gamma > 0$ we have to find the conditions of anisotropic norm boundedness $|||P|||_a$ by a scalar value γ.

In [35] the condition of anisotropic norm boundedness for descriptor systems is formulated in the following form.

Theorem 2. *For given scalar values $a \geqslant 0$ and $\gamma > 0$ the system* (26)–(27) *with a transfer function $P(z) \in \mathcal{H}_\infty{}^{p\times m}$ is admissible and its a-anisotropic norm is bounded by γ, i.e.*

$$|||P|||_a < \gamma$$

if there exist matrices $L \in \mathbb{R}^{r\times r}$, $L > 0$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, scalar values $\eta > \gamma^2$ and $\alpha > 0$, for which the following inequalities hold true

$$\eta - (\mathrm{e}^{-2a}\det(\eta I_m - B_d^{\mathrm{T}}\Theta B_d - D_d^{\mathrm{T}}D_d))^{1/m} < \gamma^2, \tag{29}$$

$$\begin{bmatrix} -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}} & \Gamma A_d & \Gamma B_d & L^{\mathrm{T}} - Q^{\mathrm{T}} - \frac{1}{2}Q & 0 \\ A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Pi A_d + A_d^{\mathrm{T}}\Pi^{\mathrm{T}} - \Theta & \Pi B_d & A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & C_d^{\mathrm{T}} + \alpha A_d^{\mathrm{T}}\Pi^{\mathrm{T}}C_d^{\mathrm{T}} \\ B_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & B_d^{\mathrm{T}}\Pi^{\mathrm{T}} & -\eta I_m & B_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & D_d^{\mathrm{T}} + \alpha B_d^{\mathrm{T}}\Pi^{\mathrm{T}}C_d^{\mathrm{T}} \\ L - Q - \frac{1}{2}Q^{\mathrm{T}} & \Gamma A_d & \Gamma B_d & -Q - Q^{\mathrm{T}} & 0 \\ 0 & C_d + \alpha C_d\Pi A_d & D_d + \alpha C_d \Pi B_d & 0 & -I_p \end{bmatrix} < 0 \tag{30}$$

where

$$\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}, \Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}, \Gamma = \begin{bmatrix} Q & R \end{bmatrix}.$$

The matrices A_d, B_d, C_d are defined by (12), *and $D_d = D$.*

Consider now limiting cases of anisotropic norm boundedness conditions for $a = 0$ and $a \to \infty$.

1. If $a = 0$, then inequality (29) is equal to

$$\eta - (\det(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D))^{1/m} < \gamma^2. \tag{31}$$

Taking into account the relation between arithmetic and geometric means, we get

$$(\det(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D))^{1/m} \leqslant \frac{1}{m}\mathrm{Trace}\,(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D)).$$

Inequality (31) leads to

$$\mathrm{Trace}\,(B^{\mathrm{T}}\Phi B + D^{\mathrm{T}}D) < m\gamma^2. \tag{32}$$

It can be verified (see [35]) that the inequality (30) is valid if

$$A^{\mathrm{T}}\Phi A - E^{\mathrm{T}}\Phi E + C^{\mathrm{T}}C < 0. \tag{33}$$

Conditions (32) and (33) are equivalent to the inequality

$$\frac{1}{\sqrt{m}}\|P\|_2 < \gamma.$$

Note that (33) holds true when $E = I_n$.

2. Clearly, there always exists sufficiently large value a such that (29) is fulfilled. However, if $a \to \infty$, then $\eta \to \gamma^2$, and condition (29) is violated. So, for $a \to \infty$ the condition $\|P\|_a < \gamma$ coincides with the condition $\|P\|_\infty < \gamma$ (see [21] for details).

Now recall some important results required for further discussion.

Lemma 1. *(Petersen's lemma [36])*
Let matrices $M \in \mathbb{R}^{n\times p}$ and $N \in \mathbb{R}^{q\times n}$ be nonzero, and $G = G^{\mathrm{T}} \in \mathbb{R}^{n\times n}$. The inequality

$$G + M\Delta N + N^{\mathrm{T}}\Delta^{\mathrm{T}}M^{\mathrm{T}} \leqslant 0 \tag{34}$$

is true for all $\Delta \in \mathbb{R}^{p\times q}$: $\overline{\sigma}(\Delta) \leqslant 1$ if there exists a scalar value $\varepsilon > 0$ such that

$$G + \varepsilon MM^{\mathrm{T}} + \frac{1}{\varepsilon}N^{\mathrm{T}}N \leqslant 0. \tag{35}$$

Lemma 2. *(Schur lemma [37])*
For any symmetric matrix

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \end{bmatrix}$$

where X_{11} and X_{22} are symmetric matrices.

If $X_{11} > 0$, then $X > 0$ if and only if

$$X_{22} - X_{12}^{\mathrm{T}} X_{11}^{-1} X_{12} > 0. \tag{36}$$

If $X_{22} > 0$, then $X > 0$ if and only if

$$X_{11} - X_{12} X_{22}^{-1} X_{12}^{\mathrm{T}} > 0. \tag{37}$$

5. Main Results

In this section problem of anisotropy-based analysis for uncertain normal and descriptor systems is discussed. Anisotropy-based bounded real lemmas for both uncertain normal and descriptor systems are proposed. The derived conditions are based on the convex optimization techniques and guarantee a-anisotropic norm boundedness for the considered systems.

5.1. Robust anisotropy-based analysis for normal systems with norm-bounded uncertainties

Briefly, recall problem statement of anisotropy-based robust analysis in the case of normal systems.

$$\begin{aligned} x(k+1) &= (A + M_A \Delta N_A)x(k) + (B + M_B \Delta N_B)w(k), & (38)\\ y(k) &= (C + M_C \Delta N_C)x(k) + (D + M_D \Delta N_D)w(k) & (39) \end{aligned}$$

where $x(k) \in \mathbb{R}^n$ is a state vector, $w(k) \in \mathbb{R}^m$ is a random stationary sequence with bounded mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a$, $y(k) \in \mathbb{R}^p$ is a measurable output, $\Delta \in \mathbb{R}^{q \times q}$ is an unknown spectral norm-bounded matrix. The transfer function of system (38)–(39) is given by $T_\Delta(z) = C_\Delta(zI - A_\Delta)^{-1}B_\Delta + D_\Delta$.

The problem is to check if system (38)–(39) is stable and its a-anisotropic norm is less than given positive value, i.e.

$$|\!|\!| T_\Delta |\!|\!|_a < \gamma$$

is satisfied for a known mean anisotropy level $a \geqslant 0$ and a given scalar value $\gamma > 0$.

To solve this problem, we now formulate the following theorem.

Theorem 3. *(Anisotropy-based bounded real lemma for uncertain normal systems.) For the given scalars $a \geqslant 0$ and $\gamma > 0$ system* (38)–(39) *is stable and $|||T_\Delta|||_a < \gamma$ if there exist such scalars $\eta > \gamma^2$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $n \times n$-matrix $\Phi = \Phi^{\mathrm{T}} > 0$, $n \times n$-matrix $\Pi = \Pi^{\mathrm{T}} > 0 : \Phi\Pi = I_n$, $m \times m$-matrix $\Psi = \Psi^{\mathrm{T}} > 0$ and $n \times n$-matrix Y, that the following inequalities hold true:*

$$\eta - (\mathrm{e}^{-2a}\det(\Psi))^{1/m} < \gamma^2, \tag{40}$$

$$\begin{bmatrix} \Xi + \varepsilon_1 M_1 M_1^{\mathrm{T}} & N_1^{\mathrm{T}} \\ N_1 & -\varepsilon_1 I_{2q} \end{bmatrix} < 0, \tag{41}$$

$$\begin{bmatrix} \Omega + \varepsilon_2 N_2^{\mathrm{T}} N_2 & M_2 \\ M_2^{\mathrm{T}} & -\varepsilon_2 I_{4q} \end{bmatrix} < 0. \tag{42}$$

Here

$$\Xi = \begin{bmatrix} \Psi - \eta I_m & B^{\mathrm{T}} & D^{\mathrm{T}} \\ B & -\Pi & 0 \\ D & 0 & -I_p \end{bmatrix}, \; M_1 = \begin{bmatrix} 0 & 0 \\ M_B & 0 \\ 0 & M_D \end{bmatrix}, N_1 = \begin{bmatrix} N_B & 0 & 0 \\ N_D & 0 & 0 \end{bmatrix},$$

$$\Omega = \begin{bmatrix} -\frac{1}{2}Y - \frac{1}{2}Y^{\mathrm{T}} & YA & YB & \Phi^{\mathrm{T}} - Y^{\mathrm{T}} - \frac{1}{2}Y & 0 \\ A^{\mathrm{T}}Y^{\mathrm{T}} & -\Phi & 0 & A^{\mathrm{T}}Y^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}}Y^{\mathrm{T}} & 0 & -\eta I_m & B^{\mathrm{T}}Y^{\mathrm{T}} & D^{\mathrm{T}} \\ \Phi - Y - \frac{1}{2}Y^{\mathrm{T}} & YA & YB & -Y - Y^{\mathrm{T}} & 0 \\ 0 & C & D & 0 & -I_p \end{bmatrix},$$

$$M_2 = \begin{bmatrix} YM_A & YM_B & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ YM_A & YM_B & 0 & 0 \\ 0 & 0 & M_C & M_D \end{bmatrix}, \; N_2 = \begin{bmatrix} 0 & N_A & 0 & 0 & 0 \\ 0 & 0 & N_B & 0 & 0 \\ 0 & N_C & 0 & 0 & 0 \\ 0 & 0 & N_D & 0 & 0 \end{bmatrix}.$$

Proof. Inequality (24) does not allow to separate out certain matrices and uncertainties. For this reason consider the new variable $\Psi = \Psi^{\mathrm{T}} > 0$ such that

$$\Psi < \eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}}D. \tag{43}$$

Under (43), inequality (24) equivalent to (40).

Using Schur complement inequality (43) is equivalent to

$$\Psi - \eta I_m - \begin{bmatrix} B^{\mathrm{T}} & D^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} -\Phi^{-1} & 0 \\ 0 & -I_p \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} < 0$$

or

$$\begin{bmatrix} \Psi - \eta I_m & B^{\mathrm{T}} & D^{\mathrm{T}} \\ B & -\Phi^{-1} & 0 \\ D & 0 & -I_p \end{bmatrix} < 0. \tag{44}$$

Rewrite the conditions of norm boundedness from Theorem 1 for system (38)–(39). Inequality (44) is transformed to

$$\begin{bmatrix} \Psi - \eta I_m & B^{\mathrm{T}} & D^{\mathrm{T}} \\ B & -\Phi^{-1} & 0 \\ D & 0 & -I_p \end{bmatrix} + \mathrm{sym}\left(\begin{bmatrix} 0 & 0 \\ M_B & 0 \\ 0 & M_D \end{bmatrix} \Delta \begin{bmatrix} N_B & 0 & 0 \\ N_D & 0 & 0 \end{bmatrix} \right) < 0. \tag{45}$$

Applying sequentially the conditions of Lemmas 1 and 2 to inequality (45) and taking into account that $\Phi^{-1} = \Pi$, we get (41).
Inequality (25) for system (38)–(39) takes the form

$$\begin{bmatrix} -\frac{1}{2}Y - \frac{1}{2}Y^{\mathrm{T}} & YA & YB & \Phi^{\mathrm{T}} - Y^{\mathrm{T}} - \frac{1}{2}Y & 0 \\ A^{\mathrm{T}}Y^{\mathrm{T}} & -\Phi & 0 & A^{\mathrm{T}}Y^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}}Y^{\mathrm{T}} & 0 & -\eta I_m & B^{\mathrm{T}}Y^{\mathrm{T}} & D^{\mathrm{T}} \\ \Phi - Y - \frac{1}{2}Y^{\mathrm{T}} & YA & YB & -Y - Y^{\mathrm{T}} & 0 \\ 0 & C & D & 0 & -I_p \end{bmatrix} +$$
$$+ \mathrm{sym}\left(\begin{bmatrix} YM_A & YM_B & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ YM_A & YM_B & 0 & 0 \\ 0 & 0 & M_C & M_D \end{bmatrix} \Delta \begin{bmatrix} 0 & N_A & 0 & 0 & 0 \\ 0 & 0 & N_B & 0 & 0 \\ 0 & N_C & 0 & 0 & 0 \\ 0 & 0 & N_D & 0 & 0 \end{bmatrix} \right) < 0. \tag{46}$$

Using the conditions of Lemma 1, we can easily transform inequality (46) to (42). Expression (40) coincides with (24) from Theorem 1, so, $|||T_\Delta|||_a < \gamma$.

The proposed theorem provides a procedure of checking stability and a-anisotropic norm boundedness of system (38)–(39). It should be noted that condition (40) also defines a convex set. Function $(\det \Psi)^p$ of the $(m \times m)$-matrix $\Psi = \Psi^{\mathrm{T}} \geq 0$ is concave upward over its argument for any $0 \leq p \leq \frac{1}{m}$. The function $(\det \Psi)^{\frac{1}{m}}$ of the $(m \times m)$-matrix $\Psi = \Psi^{\mathrm{T}} \geq 0$ is the geometric mean of its eigenvalues $\sqrt[m]{\lambda_1(\Psi) \ldots \lambda_m(\Psi)}$. See [39] for details.

The derived constraints allows to calculate a-anisotropic norm of system (38)–(39). Denoting $\xi = \gamma^2$ the problem of a-anisotropic norm calculation is to find

$$\xi_* = \inf \xi$$

on the set

$$\{\eta,\ \xi,\ \Phi,\ \Psi,\ \Pi,\ Y,\ \varepsilon_1,\ \varepsilon_2\}$$

that satisfies (40)–(42), and $\Phi\Pi = I_n$. If the minimum value ξ_* is found, then a-anisotropic norm of the system $T_\Delta(z)$ can be approximately calculated as

$$|\!|\!| T_\Delta |\!|\!|_a \approx \sqrt{\xi_*}. \tag{47}$$

The derived above conditions require mutual inverse matrices searching procedure, which can be found, for example, in [38].

Algorithm 1. An algorithm of a-anisotropic norm computation for an uncertain system can be represented as follows.

1. Set $j = 0$, select some matrices $G_1 = G_1^{\mathrm{T}}$ and $G_2 = G_2^{\mathrm{T}}$.
2. Solve optimization problem

$$\{\lambda_*, \xi_*\} = \inf\{\lambda, \xi\}$$

on the set

$$\{\eta,\ \xi,\ \lambda,\ \Phi,\ \Psi,\ \Pi,\ Y,\ \varepsilon_1,\ \varepsilon_2\}$$

that satisfies (40)–(42), and

$$\begin{bmatrix} I_n & G_1 \end{bmatrix} \begin{bmatrix} \Phi & I_n \\ I & \Pi \end{bmatrix} \begin{bmatrix} I_n \\ G_1 \end{bmatrix} + \\ + \begin{bmatrix} G_2 & I_n \end{bmatrix} \begin{bmatrix} \Phi & I_n \\ I & \Pi \end{bmatrix} \begin{bmatrix} G_2 \\ I_n \end{bmatrix} - \lambda I_{2n} < 0, \tag{48}$$

and

$$\begin{bmatrix} -\Phi & I_n \\ I_n & -\Pi \end{bmatrix} - \lambda I_{2n} < 0. \tag{49}$$

3. If $\lambda_* < \delta$ where δ is a given accuracy, then

$$|||T_\Delta|||_a \approx \sqrt{\xi_*},$$

and algorithm stops, else go to the next step.

4. Set $G_1 = -\Pi_j^{-1}$, $G_2 = -\Phi_j^{-1}$, $j = j + 1$. Go to step 2.

As it was shown in [38], for any initial condition G_1^0 and G_2^0, the sequence λ_j generated by the algorithm is a nondecreasing one and there exist

$$\lim_{j\to\infty} \lambda_j = \lambda^\star \geqslant 0, \quad \lim_{j\to\infty} \Phi_j = \Phi^\star, \quad \lim_{j\to\infty} \Pi_j = \Pi^\star.$$

Clearly, if $\lambda^\star = 0$, then we get exact equality $\Phi\Pi = I_n$, otherwise an accuracy of Algorithm 1 is defined by preassigned value $\delta > 0$.

5.2. Robust anisotropy-based analysis for descriptor systems with norm-bounded uncertainties

Finally, in order to introduce anisotropy-based bounded real, we recall problem statement for uncertain discrete-time systems. A discrete-time descriptor system is given by

$$\begin{aligned} Ex(k+1) &= (A + M_A\Delta N_A)x(k) + (B + M_B\Delta N_B)w(k), &(50)\\ y(k) &= (C + M_C\Delta N_C)x(k) + (D + M_D\Delta N_D)w(k) &(51)\end{aligned}$$

System (50)–(51) satisfies assumptions

$$\begin{aligned} \operatorname{rank} E^{\mathrm{T}} &= \operatorname{rank}\left[E^{\mathrm{T}}, C^{\mathrm{T}}, N_C^{\mathrm{T}}\right], &(52)\\ \operatorname{rank} E &= \operatorname{rank}\left[E, B, M_B\right]. &(53)\end{aligned}$$

Its transfer function is given by $P_\Delta(z) = C_\Delta(zE - A_\Delta)^{-1}B_\Delta + D_\Delta$. We will use the notations

$$M_B^d = \overline{W}M_B = \begin{bmatrix} M_{B1}^d \\ M_{B2}^d \end{bmatrix},\ N_B^d = N_B,\ M_A^d = \overline{W}M_A,\ N_A^d = N_A\overline{V},\ M_C^d = M_C,$$
$$N_C^d = N_C\overline{V}$$

where matrices $\overline{W}$ and $\overline{V}$ transform system (50)–(51) to SVD equivalent form (9)–(11).

Other notations are taken from subsection 4.2.

The anisotropy-based bounded real lemma for uncertain descriptor systems can be formulated as follows.

Theorem 4. *(Anisotropy-based bounded real lemma for uncertain descriptor systems.) For the given scalars $a \geqslant 0$ and $\gamma > 0$ system (50)–(51) is admissible and $\|P_\Delta\|_a < \gamma$ if there exist such scalars $\eta > \gamma^2$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and matrices $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $\Psi \in \mathbb{R}^{m\times m}$, $L \in \mathbb{R}^{r\times r}$, $L > 0$, $\Upsilon \in \mathbb{R}^{r\times r}$, $\Upsilon > 0 : \Upsilon L = I_r$, such that:*

$$\eta - (\mathrm{e}^{-2a}\det(\Psi))^{1/m} < \gamma^2, \tag{54}$$

$$\begin{bmatrix} \$ + \varepsilon_1 M_1 M_1^{\mathrm{T}} & N_1^{\mathrm{T}} \\ N_1 & -\varepsilon_1 I_{2q} \end{bmatrix} < 0, \tag{55}$$

$$\begin{bmatrix} \Sigma + \varepsilon_2 N_2^{\mathrm{T}} N_2 & M_2 \\ M_2^{\mathrm{T}} & -\varepsilon_2 I_{4q} \end{bmatrix} < 0, \tag{56}$$

here

$$\$ = \begin{bmatrix} \Psi - \eta I_m & D_d^{\mathrm{T}} & B_1^{\mathrm{T}} \\ D_d & -I_p & 0 \\ B_1 & 0 & -\Upsilon \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 0 & 0 \\ M_D & 0 \\ 0 & M_{B1}^d \end{bmatrix}, N_1 = \begin{bmatrix} N_D & 0 & 0 \\ N_B^d & 0 & 0 \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}} & \Gamma A_d & \Gamma B_d & L^{\mathrm{T}} - Q^{\mathrm{T}} - \frac{1}{2}Q & 0 \\ A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Pi A_d + A_d^{\mathrm{T}}\Pi^{\mathrm{T}} - \Theta & \Pi B_d & A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & C_d^{\mathrm{T}} \\ B_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & B_d^{\mathrm{T}}\Pi^{\mathrm{T}} & -\eta I_m & B_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & D_d^{\mathrm{T}} \\ L - Q - \frac{1}{2}Q^{\mathrm{T}} & \Gamma A_d & \Gamma B_d & -Q - Q^{\mathrm{T}} & 0 \\ 0 & C_d & D_d & 0 & -I_p \end{bmatrix},$$

$$M_2 = \begin{bmatrix} \Gamma M_A^d & \Gamma M_B^d & 0 & 0 \\ \Pi M_A^d & \Pi M_B^d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Gamma M_A^d & \Gamma M_B^d & 0 & 0 \\ 0 & 0 & M_C^d & M_D \end{bmatrix}, N_2 = \begin{bmatrix} 0 & N_A^d & 0 & 0 & 0 \\ 0 & 0 & N_B^d & 0 & 0 \\ 0 & N_C^d & 0 & 0 & 0 \\ 0 & 0 & N_D & 0 & 0 \end{bmatrix},$$

$$\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}, \Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}, \Gamma = \begin{bmatrix} Q & R \end{bmatrix}.$$

Proof. Consider inequality (29) from Theorem 2. It can be rewritten similar to (24) into the form

$$\eta - (\mathrm{e}^{-2a}\det(\Psi))^{1/m} < \gamma^2,$$

$$\begin{bmatrix} \Psi - \eta I_m + B_d^{\mathrm{T}} \Theta B_d & D_d^{\mathrm{T}} \\ D_d & -I_p \end{bmatrix} < 0. \tag{57}$$

Taking into account condition (53), we get $B_2 = 0$. Transform the expression $B_d^{\mathrm{T}} \Theta B_d = \begin{bmatrix} B_1^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = B_1^{\mathrm{T}} L B_1 > 0$. So, inequality (57) is equivalent to

$$\begin{bmatrix} \Psi - \eta I_m + B_1^{\mathrm{T}} L B_1 & D_d^{\mathrm{T}} \\ D_d & -I_p \end{bmatrix} < 0,$$

using the conditions of Lemma 2, we have

$$\begin{bmatrix} \Psi - \eta I_m & D_d^{\mathrm{T}} & B_1^{\mathrm{T}} \\ D_d & -I_p & 0 \\ B_1 & 0 & -\Upsilon \end{bmatrix} < 0. \tag{58}$$

Here $\Upsilon = L^{-1}$. Now we write the inequality of the form (58) for system (50)–(51) with norm-bounded uncertainties.

$$\begin{bmatrix} \Psi - \eta I_m & (D_d + M_D \Delta N_D)^{\mathrm{T}} & (B_1 + M_{B1}^d \Delta N_B^d)^{\mathrm{T}} \\ D_d + M_D \Delta N_D & -I_p & 0 \\ B_1 + M_{B1}^d \Delta N_B^d & 0 & -\Upsilon \end{bmatrix} < 0 \tag{59}$$

or

$$\begin{bmatrix} \Psi - \eta I_m & D_d^{\mathrm{T}} & B_1^{\mathrm{T}} \\ D_d & -I_p & 0 \\ B_1 & 0 & -\Upsilon \end{bmatrix} + \mathrm{sym} \left(\begin{bmatrix} 0 & 0 \\ M_D & 0 \\ 0 & M_{B1}^d \end{bmatrix} \Delta \begin{bmatrix} N_D & 0 & 0 \\ N_B^d & 0 & 0 \end{bmatrix} \right) < 0. \tag{60}$$

Using the conditions of Lemmas 1 and 2, we can rewrite inequality (60) as (55).

Under assumptions (52) and (53) we get that $B_2 = 0$ and $C_2 = 0$. It's easy to check that in this case $\alpha C_d \Pi A_d = 0$ and $\alpha C_d \Pi B_{wd} = 0$ in (30). Now we transform expression (30) for system (50)–(51)

$$\begin{bmatrix} -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}} & \Gamma A_d & \Gamma B_d & L^{\mathrm{T}} - Q^{\mathrm{T}} - \frac{1}{2}Q & 0 \\ A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Pi A_d + A_d^{\mathrm{T}}\Pi^{\mathrm{T}} - \Theta & \Pi B_d & A_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & C_d^{\mathrm{T}} \\ B_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & B_d^{\mathrm{T}}\Pi^{\mathrm{T}} & -\eta I_m & B_d^{\mathrm{T}}\Gamma^{\mathrm{T}} & D_d^{\mathrm{T}} \\ L - Q - \frac{1}{2}Q^{\mathrm{T}} & \Gamma A_d & \Gamma B_d & -Q - Q^{\mathrm{T}} & 0 \\ 0 & C_d & D_d & 0 & -I_p \end{bmatrix} +$$
$$+ \operatorname{sym}\left(\begin{bmatrix} \Gamma M_A^d & \Gamma M_B^d & 0 & 0 \\ \Pi M_A^d & \Pi M_B^d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Gamma M_A^d & \Gamma M_B^d & 0 & 0 \\ 0 & 0 & M_C^d & M_D \end{bmatrix} \Delta \begin{bmatrix} 0 & N_A^d & 0 & 0 & 0 \\ 0 & 0 & N_B^d & 0 & 0 \\ 0 & N_C^d & 0 & 0 & 0 \\ 0 & 0 & N_D & 0 & 0 \end{bmatrix} \right) < 0. \tag{61}$$

Applying Lemmas 1 and 2 to inequality (61), we get

$$\Sigma + \frac{1}{\varepsilon_2} M_2 M_2^{\mathrm{T}} + \varepsilon_2 N_2^{\mathrm{T}} N_2 < 0,$$
$$\Sigma + \varepsilon_2 N_2^{\mathrm{T}} N_2 - M_2(-\varepsilon_2 I_{4q})^{-1} M_2^{\mathrm{T}} < 0,$$
$$\begin{bmatrix} \Sigma + \varepsilon_2 N_2^{\mathrm{T}} N_2 & M_2 \\ M_2^{\mathrm{T}} & -\varepsilon_2 I_{4q} \end{bmatrix} < 0.$$

The last inequality coincides with (56). Expression (54) is equivalent to (29). Consequently, the conditions of Theorem 2 hold true for system (50)–(51), it means that its anisotropic norm is bounded by a positive scalar value, i.e. $|||P_\Delta|||_a < \gamma$.

The procedure of a-anisotropic norm calculation of the uncertain descriptor system (50)–(51) is based on conditions of Theorem 4 and can be formulated as follows. Similar to normal systems case introduce the notation $\xi = \gamma^2$. So, a-anisotropic norm calculation problem is to find

$$\xi_* = \inf \xi$$

on the set

$$\{\eta,\, \xi,\, L,\, \Psi,\, \Upsilon,\, Y,\, Q,\, R,\, S,\, \varepsilon_1,\, \varepsilon_2\}$$

that satisfies (54)–(56), and $\Upsilon L = I_r$. If the minimum value ξ_* is found, then a-anisotropic norm of the system $P_\Delta(z)$ can be approximately calculated as

$$|||P_\Delta|||_a \approx \sqrt{\xi_*}. \tag{62}$$

When the problem of robust anisotropy-based analysis involves mutually inverse matrices search procedure, an algorithm of a-anisotropic norm estimation is similar to Algorithm 1.

Remark 4. If mean anisotropy level goes to infinity, i.e. $a \to +\infty$, then inequality (54) holds true for any Ψ, and (55) is violated. Expression (56) defines the condition of $\mathcal{H}_\infty$-norm boundedness for system (50)–(51). Moreover, the rank restrictions (52)–(53) are no longer required.

Remark 5. Let the system be given by (50)–(51). If $M_B = 0$ and $N_B = 0$, then the conditions of Theorem 4 become simpler:

$$\eta - (\mathrm{e}^{-2a} \det(\Psi))^{1/m} < \gamma^2, \tag{63}$$

$$\begin{bmatrix} \$ + \varepsilon_1 M_1 M_1^{\mathrm{T}} & N_1^{\mathrm{T}} \\ N_1 & -\varepsilon_1 I_q \end{bmatrix} < 0, \tag{64}$$

$$\begin{bmatrix} \Sigma + \varepsilon_2 N_2^{\mathrm{T}} N_2 & M_2 \\ M_2^{\mathrm{T}} & -\varepsilon_2 I_{3q} \end{bmatrix} < 0. \tag{65}$$

Here

$$\$ = \begin{bmatrix} \Psi - \eta I_m + B_d^{\mathrm{T}} \Theta B_d & D_d^{\mathrm{T}} \\ D_d & -I_p \end{bmatrix}, \; M_1 = \begin{bmatrix} 0 \\ M_D \end{bmatrix}, \; N_1 = \begin{bmatrix} N_D & 0 \end{bmatrix}.$$

$$M_2 = \begin{bmatrix} \Gamma M_A^d & 0 & 0 \\ \Pi M_A^d & 0 & 0 \\ 0 & 0 & 0 \\ \Gamma M_A^d & 0 & 0 \\ 0 & M_C^d & M_D \end{bmatrix}, \; N_2 = \begin{bmatrix} 0 & N_A^d & 0 & 0 & 0 \\ 0 & N_C^d & 0 & 0 & 0 \\ 0 & 0 & N_D & 0 & 0 \end{bmatrix},$$

In this case, we do not use the algorithm of mutually inverse matrices computation in order to find Υ.

6. Numerical example

Example 1. Consider a fully uncertain normal system with parameters:

$$A = \begin{bmatrix} -0.25 & 0 & 0 \\ -0.5 & 0.5 & 2 \\ 0.13 & -0.18 & -0.66 \end{bmatrix},$$

Table 1: Estimation of a-anisotropic norm

Mean anisotropy level $\overline{\mathbf{A}}(W)$	0	0.1	0.5	1.5	10
Estimation of $\|\|\|T_\Delta\|\|\|_a$	0.8233	1.0173	1.2115	1.3305	1.3782
Exact upper bound $\|\|\|T_\Delta\|\|\|_a$	0.7601	0.9396	1.1212	1.2695	1.3398
Norm of nominal system $\|\|\|T\|\|\|_a$	0.6464	0.7859	0.9258	1.0423	1.0988
Exact lower bound $\|\|\|T_\Delta\|\|\|_a$	0.5500	0.6650	0.7757	0.8694	0.9159

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \ D = \begin{bmatrix} 0.1 & -0.05 \end{bmatrix},$$

Uncertainties are presented by

$$M_A = \begin{bmatrix} 0.25 & -0.7 & 0.15 \end{bmatrix}^{\mathrm{T}}, \ N_A = \begin{bmatrix} 0 & 0.15 & 0.21 \end{bmatrix},$$
$$M_B = \begin{bmatrix} 0 & 0 & 0.2 \end{bmatrix}^{\mathrm{T}}, \ N_B = \begin{bmatrix} 0.1 & 0.3 \end{bmatrix},$$
$$M_C = M_D = 0.2, \ N_C = \begin{bmatrix} 0.05 & 0.2 & 0 \end{bmatrix}, \ N_D = \begin{bmatrix} 0.02 & 0.08 \end{bmatrix}.$$

The results of a-anisotropic norm computation for different values of mean anisotropy are presented in Table 1 with tolerance $\delta = 10^{-7}$. In considered example, there are large deviations between lower bound and upper bound of a-anisotropic norm of an uncertain system. The results show that proposed algorithm provides a good estimation of a-anisotropic norm of an uncertain system.

Example 2. We consider a model of hydraulic tank system with three tanks represented in Fig. 2. A linearized and discretized state-space model with discretization step $h = 3$ s in descriptor form is given by [17]

$$Eq(k+1) = Aq(k) + B_u u(k) + B_\xi \xi(k), \quad (66)$$
$$y(k) = Cq(k) + 0.3\eta(k), \quad (67)$$

where $q(k)$ is a vector consisting of the volumes in the tanks, $u(k)$ is a pump flow, $\xi(k)$ is a plant noise, $\eta(k)$ is a measurement noise. Matrices in state-space representation (66)–(67) are given by

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 0.9692 & 0 & 0 \\ 0.0095 & 0.9867 & 0 \\ 1 & 2.3328 & 1 \end{bmatrix},$$

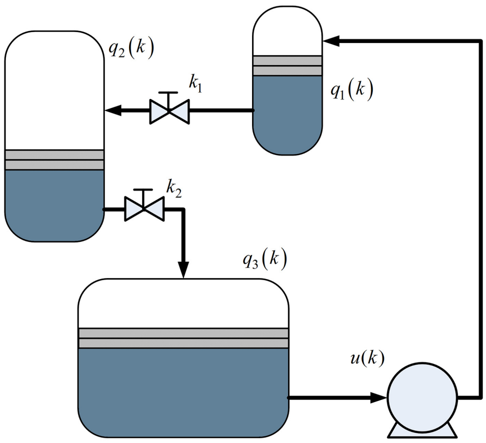

Figure 2: Hydraulic tank system

$$B_u = \begin{bmatrix} 0.056 \\ 0.003 \\ 0 \end{bmatrix}, \; B_\xi = \begin{bmatrix} 0.02 \\ 0.01 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},$$

Define $w = [\xi \;\; \eta]^{\mathrm{T}}$. Then $B_w = \begin{bmatrix} 0.02 & 0 \\ 0.01 & 0 \\ 0 & 0 \end{bmatrix}$, $D_w = \begin{bmatrix} 0 & 0.3 \end{bmatrix}$. In addition,

$$M_A = \begin{bmatrix} 0.1 & -0.1 & 0.3 \end{bmatrix}^{\mathrm{T}} \text{ and } N_A = \begin{bmatrix} 0.2 & 0.1 & 0.1 \end{bmatrix}.$$

One can check, that the system is admissible for all $\Delta \in [-1; \; 1]$, its worst case generalized spectral radius is $\rho(E, A) < 1$.

The exact upper bound of a-anisotropic norm of uncertain systems for different mean anisotropy levels a is shown in Fig. 3. The dashed line in Fig. 3 displays the result of γ-minimization using conditions of Theorem 4. The estimation error is shown in Fig. 4.

7. Conclusions and Future Work

This paper is addressed to the problem of anisotropy-based performance analysis for a class of uncertain discrete-time linear (normal and descriptor) systems with Frobenius norm-bounded parametric uncertainties. A new class of systems is taken up. Novelty and importance of proposed result consist

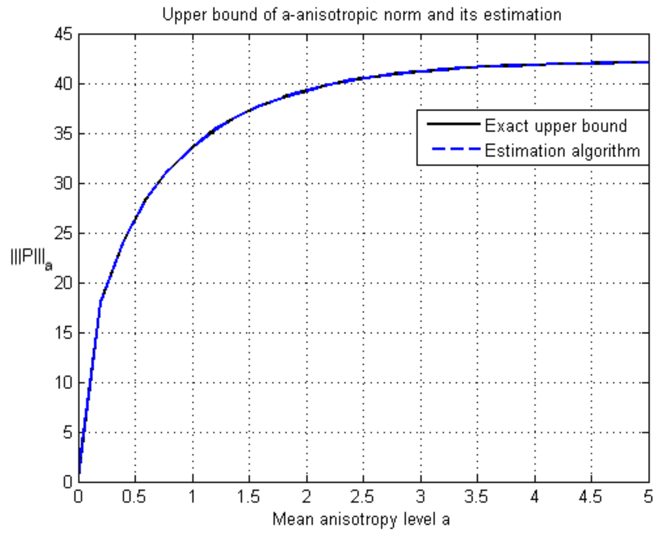

Figure 3: a-Anisotropic norm of uncertain system and its estimation.

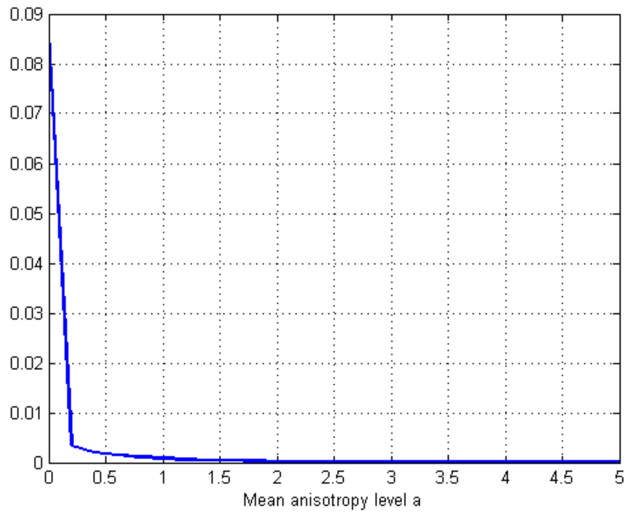

Figure 4: Error of a-anisotropic norm estimation.

in a general approach to robust performance analysis as it covers $\mathcal{H}_2$- and $\mathcal{H}_\infty$- approaches for normal and descriptor systems. The problem of robust performance analysis is solved via matrix inequality approach involving no parameter uncertainties.

Since proposed methods are based on LMIs, the computational complexity of both algorithms is defined by number of decision variables. The number of decision variables in algorithm, proposed by Theorem 3, is $2n^2 + n + \frac{m(m+1)}{2} + 4$ with all uncertainties in the system's model, and $\frac{3n^2+n}{2} + \frac{m(m+1)}{2} + 3$ when no uncertainties are presented in matrix B. Similar to normal systems, algorithm of robust anisotropy-based analysis for descriptor systems involves $n^2 + 2r^2 + r - nr + \frac{m(m+1)}{2} + 4$ and $n^2 + \frac{3r(r+1)}{2} - nr + \frac{m(m+1)}{2} + 3$ for fully uncertain system and for system without uncertainties in matrix B respectively. Clearly, if initial system is robustly admissible, then mutually inverse matrices always exist. Hence, Algorithm 1 provides estimation of the upper bound of a-anisotropic norm.

Obtained conditions open a prospective field for further theoretical research of discrete-time systems affected by random disturbances. First of all, the results can be applied to the design of robust anisotropy-based output-feedback and state-feedback controllers. Further, an anisotropy-based performance analysis of certain and uncertain systems with time-varying delays are also a point of concern.

A number of interesting research papers on robust fault-tolerant controllers with respect to sensor and actuator faults appeared in last few years (see. [40, 41]). Another possible extension of this work is developing of anisotropy-based fault tolerant controller design for uncertain systems in presence of colored noise.

Finally, proposed methods can be extended to robust filter design problem for sensor networks with guaranteed random disturbance attenuation. Some existing results on filtering problem in sensor networks are presented in [42, 43].

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