INVARIANT IDEALS AND MATSUSHIMA'S CRITERION

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Abstract. Let G be a reductive algebraic group and H a closed subgroup of G. Explicit constructions of G-invariant ideals in the algebra $\mathbb{K}[G/H]$ are given. This allows to obtain an elementary proof of Matsushima's criterion: a homogeneous space G/H is an affine variety if and only if H is reductive.

1. Algebraic homogeneous spaces

Let G be an affine algebraic group over an algebraically closed field $\mathbb{K}$. A G-module V is said to be rational if any vector in V is contained in a finite-dimensional rational G-submodule. Below all modules are supposed to be rational. By V^G denote the subspace of G-fixed vectors in V.

The group $G \times G$ acts on G by translations, $(g_1, g_2)g := g_1 g g_2^{-1}$. This action induces the action on the algebra of regular functions on G:

$$(G \times G) : \mathbb{K}[G], \quad ((g_1, g_2)f)(g) := f(g_1^{-1} g g_2).$$

For any closed subgroup H of G, H_l, H_r denote the groups of all left and right translations of $\mathbb{K}[G]$ by elements of H. Under these actions, the algebra $\mathbb{K}[G]$ becomes a rational H_l- (and H_r-) module.

By Chevalley's Theorem, the set G/H of left H-cosets in G admits a structure of a quasi-projective algebraic variety such that the projection $p : G \to G/H$ is a surjective G-equivariant morphism. Moreover, a structure of an algebraic variety on G/H satisfying these conditions is unique. It is easy to check that the morphism p is open and the algebra of regular functions on G/H may be identified with the subalgebra $\mathbb{K}[G]^{H_r}$ in $\mathbb{K}[G]$. We refer to [6, Ch. IV] for details.

2. Matsushima's criterion

Let G be a reductive algebraic group and H a closed subgroup of G. It is known that the homogeneous space G/H is affine if and only if H if reductive. The first proof was given over the field of complex numbers and used some results from algebraic topology, see [8] and [9, Th. 4]. An algebraic proof in characteristic zero was obtained in [2]. A characteristic-free proof that uses the Mumford conjecture proved by W.J. Haboush is given in [11]. Another proof based on the Morozov-Jacobson Theorem may be found in [7].

Below we give an elementary proof of Matsushima's criterion in terms of representation theory. The ground field $\mathbb{K}$ is assumed to be algebraically closed and of characteristic zero.

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Theorem 2.1. *Let G be a reductive algebraic group and H its closed subgroup. Then the homogeneous space G/H is affine if and only if H is reductive.*

Proof. We begin with the "easy half".

Proposition 2.2. *Let G be an affine algebraic group and H its reductive subgroup. Then G/H is affine.*

Proof. If a reductive group H acts on an affine variety X, then the algebra of invariants $\mathbb{K}[X]^H$ is finitely generated, the quotient morphism $\pi : X \to \operatorname{Spec} \mathbb{K}[X]^H$ is surjective and any fiber of π contains a unique closed H-orbit [10, Sec .4.4]. In the case $X = G$ this shows that G/H is isomorphic to $\operatorname{Spec} \mathbb{K}[G]^{H_r}$.

□

Now assume that G is reductive and consider a decomposition

$$\mathbb{K}[G] = \mathbb{K} \oplus \mathbb{K}[G]_G,$$

where the first component corresponds to constant functions on G, and the second one is the sum of all simple non-trivial G_l- (or G_r-) submodules in $\mathbb{K}[G]$. Let $\mathrm{pr} : \mathbb{K}[G] \to \mathbb{K}$ be the projection on the first component. Clearly, pr is a $(G_l \times G_r)$-invariant linear map.

Let H be a closed subgroup of G. Consider

$$I(G,H) = \{f \in \mathbb{K}[G]^{H_r} \mid \mathrm{pr}(fl) = 0 \ \forall l \in \mathbb{K}[G]^{H_r}\}.$$

This is a G_l-invariant ideal in $\mathbb{K}[G]^{H_r}$ with $1 \notin I(G,H)$. Assume that G/H is affine. Then $G/H \cong \operatorname{Spec} \mathbb{K}[G]^{H_r}$ and $\mathbb{K}[G]^{H_r}$ does not contain proper G_l-invariant ideals. Thus $I(G,H) = 0$. Our aim is to deduce from this that any H-module is completely reducible.

Lemma 2.3. *If W is an H_r-submodule in $\mathbb{K}[G]$ and $f \in W$ is a non-zero H_r-fixed vector, then $W = \langle f \rangle \oplus W'$, where W' is an H_r-submodule.*

Proof. Since $I(G,H) = 0$, there exists $l \in \mathbb{K}[G]^{H_r}$ such that $\mathrm{pr}(fl) \neq 0$. The submodule W' is defined as $W' = \{w \in W \mid \mathrm{pr}(wl) = 0\}$.

□

Lemma 2.4. *If $f \in \mathbb{K}[G]$ is an H_r-semi-invariant of the weight ξ, then there exists an H_r-semi-invariant in $\mathbb{K}[G]$ of the weight $-\xi$.*

Proof. Let Z be the zero set of f in G. Since Z is H_r-invariant, one has $Z = p^{-1}(p(Z))$. This implies that $p(Z)$ is a proper closed subset of G/H. There exists a non-zero $\alpha \in \mathbb{K}[G/H]$ with $\alpha|_{p(Z)} = 0$. Then $p^*\alpha \in \mathbb{K}[G]^{H_r}$ and $p^*\alpha|_Z = 0$. By Hilbert's Nullstellensatz, there are $n \in \mathbb{N}$, $s \in \mathbb{K}[G]$ such that $(p^*\alpha)^n = fs$. This shows that s is an H_r-semi-invariant of the weight $-\xi$.

□

Lemma 2.5. *(1) Any cyclic G-module V may be embedded (as a G_r-submodule) into $\mathbb{K}[G]$.*

(2) Any n-dimensional H-module W may be embedded (as an H_r-submodule) into $(\mathbb{K}[H])^n$.

(3) Any finite-dimensional H-module may be embedded (as an H-submodule) into a finite-dimensional G-module.

Proof. (1) Suppose that $V = \langle Gv \rangle$. The map $\phi : G \to V$, $\phi(g) = g^{-1}v$, induces the embedding of the dual module $\phi^* : V^* \to \mathbb{K}[G]$. Consider the G_r-submodule $U = \{f \in \mathbb{K}[G] \mid \mathrm{pr}(fl) = 0 \ \forall l \in \phi^*(V^*)\}$. By the complete reducibility, $\mathbb{K}[G] = U \oplus U'$ for some G_r-submodule U'. Obviously, $I(G,G) = 0$ and U' is G_r-isomorphic to V.

(2) Let $\lambda_1, \dots, \lambda_n$ be a basis of W^*. The embedding may be given as

$$w \to (f_1^w, \dots, f_n^w), \ f_i^w(h) := \lambda_i(hw).$$

(3) Note that the restriction homomorphism $\mathbb{K}[G] \to \mathbb{K}[H]$ is surjective. By (2), any finite-dimensional H-module W has the form W_1/W_2, where W_1 is a finite-dimensional H-submodule in a G-module V and W_2 is an H-submodule of W_1. Consider $W_1 \bigwedge (\bigwedge^m W_2)$ as an H-submodule in $\bigwedge^{m+1} W_1$, where $m = \dim W_2$. Note that $W \cong (W_1 \bigwedge (\bigwedge^m W_2)) \otimes (\bigwedge^m W_2)^*$. By (1), the cyclic G-submodule of $\bigwedge^m V$ generated by $\bigwedge^m W_2$ may be embedded into $\mathbb{K}[G]$. By Lemma 2.4, $(\bigwedge^m W_2)^*$ also may be embedded into a G-module.

□

Lemma 2.6. *For any H-module W and any non-zero $w \in W^H$ there is an H-submodule W' such that $W = \langle w \rangle \oplus W'$.*

Proof. Embed W into a G-module V. Let $V_1 = \langle Gw \rangle$. Then $V = V_1 \oplus V_2$ for some G-submodule V_2. Embed V_1 into $\mathbb{K}[G]$ as a G_r-submodule. By Lemma 2.3, $V_1 = \langle w \rangle \oplus W_1$ for some H-submodule W_1. Finally, $W' = W \cap (W_1 \oplus V_2)$.

□

Lemma 2.7. *Any H-module is completely reducible.*

Proof. Assume that W_1 is a simple submodule in an H-module W. Consider two submodules in the H-module $\mathrm{End}(W, W_1)$:

$$L_2 = \{p \in \mathrm{End}(W, W_1) \mid p|_{W_1} = 0\} \ \subset \ L_1 = \{p \in \mathrm{End}(W, W_1) \mid p|_{W_1} \text{ is scalar}\}.$$

Clearly, L_2 is a hyperplane in L_1. Consider an H-eigenvector $l \in (L_1)^*$ corresponding to L_2. Taking the tensor product with a one-dimensional H-module, one may assume that l is H-fixed. By Lemma 2.6, $(L_1)^* = \langle l \rangle \oplus M$, implying $L_1 = L_2 \oplus \langle P \rangle$, where M and $\langle P \rangle$ are H-submodules. Then $\mathrm{Ker}\, P$ is a complementary submodule to W_1.

□

Theorem 2.1 is proved.

□

Remark 2.8. In [13], for any action $G : X$ of a reductive group G on an affine variety X with the decomposition $\mathbb{K}[X] = \mathbb{K}[X]^G \oplus \mathbb{K}[X]_G$ and the projection $\mathrm{pr} : \mathbb{K}[X] \to \mathbb{K}[X]^G$, the $\mathbb{K}[X]^G$-bilinear scalar product $(f, g) = \mathrm{pr}(fg)$ on $\mathbb{K}[X]$ was introduced and the kernel of this product was considered. Our ideal $I(G,H)$ is such kernel in the case $X = \mathrm{Spec}\, \mathbb{K}[G]^{H_r}$ provided $\mathbb{K}[G]^{H_r}$ is finitely generated.

Remark 2.9. For convenience of the reader we include all details in the proof of Theorem 2.1. Lemma 2.4 and Lemma 2.5 are taken from [3]. They show that for a quasi-affine G/H any H-module may be realized as an H-submodule of a G-module. The converse is also true [3], [4]. Proposition 2.2 is a standart fact. The proof of Lemma 2.7 is a part of the proof of the Weyl Theorem on complete reducibility [5], see also [12, Prop. 2.2.4].

3. Some additional remarks

The following lemma may be found in [2].

Lemma 3.1. *Let G be an affine algebraic group and H its reductive subgroup. Then $\mathbb{K}[G]^{H_r}$ does not contain proper G_l-invariant ideals.*

Proof. Consider a decomposition

$$\mathbb{K}[G] = \mathbb{K}[G]^{H_r} \oplus \mathbb{K}[G]_{H_r},$$

where $\mathbb{K}[G]_{H_r}$ is the sum of all non-trivial simple H_r-submodules in $\mathbb{K}[G]$. Clearly, $\mathbb{K}[G]^{H_r}\mathbb{K}[G]_{H_r} \subseteq \mathbb{K}[G]_{H_r}$. Hence any proper G_l-invariant ideal in $\mathbb{K}[G]^{H_r}$ generates a proper G_l-invariant ideal in $\mathbb{K}[G]$, a contradiction.

□

By Hilbert's Theorem on invariants, the algebra $\mathbb{K}[G]^{H_r}$ is finitely generated. It is easy to see that functions from $\mathbb{K}[G]^{H_r}$ separate (closed) right H-cosets in G. These observations and Lemma 3.1 give another proof of Proposition 2.2. Moreover, it is proved in [2, Prop. 1] that for a quasi-affine G/H the algebra $\mathbb{K}[G]^{H_r}$ does not contain proper G_l-invariant ideals if and only if G/H is affine.

Now assume that G is reductive.

Proposition 3.2. [13, Prop. 1] *The ideal $I(G,H)$ is the biggest G_l-invariant ideal in $\mathbb{K}[G]^{H_r}$ different from $\mathbb{K}[G]^{H_r}$.*

Proof. Any proper G_l-invariant ideal I of $\mathbb{K}[G]^{H_r}$ is contained in $\mathbb{K}[G]^{H_r} \cap \mathbb{K}[G]_G$. Thus $\mathrm{pr}(il) = 0$ for any $l \in \mathbb{K}[G]^{H_r}$, $i \in I$. This implies $I \subseteq I(G,H)$.

□

Remark 3.3. For non-reductive G the biggest invariant ideal in $\mathbb{K}[G]^{H_r}$ may not exist. For example, one may take

$$G = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}, \quad H = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{pmatrix} \right\}.$$

Here $G/H \cong \mathbb{K}^3 \setminus \{x_2 = x_3 = 0\}$, $\mathbb{K}[G]^{H_r} \cong \mathbb{K}[x_1, x_2, x_3]$, and the maximal ideals $(x_1 - a, x_2, x_3)$ are G_l-invariant for any $a \in \mathbb{K}$.

4. The boundary ideal

In this section we assume that H is an observable subgroup of G, i.e., G/H is quasi-affine.

If the algebra $\mathbb{K}[G]^{H_r}$ is finitely generated, then the affine G-variety $X = \operatorname{Spec} \mathbb{K}[G]^{H_r}$ has an open G-orbit isomorphic to G/H and may be considered as the canonical embedding $G/H \hookrightarrow X$. Moreover, this embedding is uniquely characterized by two properties: X is normal and $\operatorname{codim}_X(X \setminus G/H) \geq 2$, see [4]. There are two remarkable G_l-invariant ideals in $\mathbb{K}[G]^{H_r}$, namely

$$I^b(G,H) = I(X \setminus (G/H)) = \{f \in \mathbb{K}[G]^{H_r} \mid f|_{X \setminus (G/H)} = 0\},$$

and, if G is reductive, the ideal $I^m(G,H)$ of the unique closed G-orbit in X. If G/H is affine, then $I^b(G,H) = \mathbb{K}[G]^{H_r}$, $I^m(G,H) = 0$. In other cases $I^b(G,H)$ is the smallest proper radical G_l-invariant ideal, and $I^m(G.H)$ is the biggest proper G_l-invariant ideal of $\mathbb{K}[G]^{H_r}$. By Proposition 3.2, $I^m(G,H) = I(G,H)$. Moreover, $\mathbb{K}[G]^{H_r}/I^m(G,H) \cong \mathbb{K}[G]^{S_r}$, where S is a minimal reductive subgroup of

G containing H. (Such a subgroup may be not unique, but all of them are G-conjugate, see [1, Sec. 7].) It follows from the Slice Theorem [7] and [1, Prop. 4] that $I^b(G,H) = I^m(G,H)$ if and only if H is a quasi-parabolic subgroup of a reductive subgroup of G.

Now assume that $\mathbb{K}[G]^{H_r}$ is not finitely generated. If G is reductive, then $I(G,H)$ may be consider as an analog of $I^m(G,H)$ in this situation (Proposition 3.2). We claim that $I^b(G,H)$ also has an analog, even for non-reductive G.

Proposition 4.1. *Let $\hat{X}$ be a quasi-affine variety, $\hat{X} \hookrightarrow X$ be an (open) embedding into an affine variety X, $I(X \setminus \hat{X}) \lhd \mathbb{K}[X]$, and $\mathcal{I} = \mathcal{I}(\hat{X})$ be the radical of the ideal of $\mathbb{K}[\hat{X}]$ generated by $I(X \setminus \hat{X})$. Then*

(1) the ideal $\mathcal{I} \lhd \mathbb{K}[\hat{X}]$ does not depend on X;

(2) $I(X \setminus \hat{X})$ is the smallest radical ideal of $\mathbb{K}[X]$ generating an ideal in $\mathbb{K}[\hat{X}]$ with the radical $\mathcal{I}$.

Proof. (1) Consider two affine embeddings: $\phi_i : \hat{X} \hookrightarrow X_i$, $i = 1, 2$. Let X_{12} be the closure of $(\phi_1 \times \phi_2)(\hat{X})$ in $X_1 \times X_2$ with the projections $r_i : X_{12} \to X_i$. Let us identify the images of $\hat{X}$ in X_1, X_2, and X_{12}. We claim that $r_i(X_{12} \setminus \hat{X}) \subseteq X_i \setminus \hat{X}$. Indeed, the diagonal image of $\hat{X}$ is closed in $\hat{X} \times X_j$, $j \neq i$, as the graph of a morphism.

It follows from what was proved above that the ideal of $\mathbb{K}[X_{12}]$ generated by $r_i^*(I(X_i \setminus \hat{X}))$ has the radical $I(X_{12} \setminus \hat{X})$. This shows that the radical of the ideal generated by $I(X_i \setminus \hat{X})$ in $\mathbb{K}[\hat{X}]$ does not depend on i.

(2) Assume that there is a radical ideal $I_1 \lhd \mathbb{K}[X]$ not containing $I = I(X \setminus \hat{X})$ and generating an ideal in $\mathbb{K}[\hat{X}]$ with the radical $\mathcal{I}$. There is $x_0 \in \hat{X}$ such that $h(x_0) = 0$ for any $h \in I_1$. Take $f \in I$ such that $f(x_0) \neq 0$. One has $f^k = \alpha_1 h_1 + \cdots + \alpha_k h_k$ for some $\alpha_i \in \mathbb{K}[\hat{X}]$, $h_i \in I_1$, $k \in \mathbb{N}$, and this implies $f(x_0) = 0$, a contradiction. □

So $\mathcal{I}(G/H)$ is a radical G_l-invariant ideal of $\mathbb{K}[G]^{H_r}$, and $\mathcal{I}(G/H) = I^b(G,H)$ provided $\mathbb{K}[G]^{H_r}$ is finitely generated.

Proposition 4.2. *$\mathcal{I}(G/H)$ is the smallest non-zero radical G_l-invariant ideal of $\mathbb{K}[G]^{H_r}$.*

Proof. Let $f \in \mathbb{K}[G]^{H_r}$ and $I(f)$ be the ideal of $\mathbb{K}[G]^{H_r}$ generated by the orbit $G_l f$. It is sufficient to prove that $\mathcal{I}(G/H) \subseteq \operatorname{rad} I(f)$. Take any G-equivariant affine embedding $G/H \hookrightarrow X$ with $f \in \mathbb{K}[X]$. For the ideal $I'(f)$ generated by $G_l f$ in $\mathbb{K}[X]$ one has $I(X \setminus (G/H)) \subseteq \operatorname{rad} I'(f)$, hence $\mathcal{I}(G/H) \subseteq \operatorname{rad} I(f)$. □

Corollary 4.3. *Let G be an affine algebraic group and H its observable subgroup. Then G/H is affine if and only if $\mathcal{I}(G/H) = \mathbb{K}[G]^{H_r}$.*

It should be interesting to give a description of the ideal $\mathcal{I}(G/H)$ similar to the definition of $I(G,H)$, and to find a geometric meaning of the G_l-algebras $\mathbb{K}[G]^{H_r}/I(G,H)$ and $\mathbb{K}[G]^{H_r}/\mathcal{I}(G/H)$ for non-finitely generated $\mathbb{K}[G]^{H_r}$.

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