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Geometric invariant theory via Cox rings

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ABSTRACT

We consider actions of reductive groups on a variety with finitely generated Cox ring, e.g., the classical case of a diagonal action on a product of projective spaces. Given such an action, we construct via combinatorial data in the Cox ring all maximal open subsets such that the quotient is quasiprojective or embeddable into a toric variety. As applications, we obtain an explicit description of the chamber structure of the linearized ample cone and several Gelfand–MacPherson type correspondences relating quotients by reductive groups to quotients by torus actions. Moreover, our approach provides a general access to the geometry of many of the resulting quotient spaces.

1. Introduction

The passage to a quotient by an algebraic group action is often an essential step in classical moduli space constructions of Algebraic Geometry, and it is the task of Geometric Invariant Theory (GIT) to provide such quotients. Starting with Mumford's approach of constructing quotients for actions of reductive groups on projective varieties via linearized line bundles and their sets of semistable points [17], the notion of a "good quotient" became a central concept in GIT, compare [24,5]. Recall that a good quotient for an action of a reductive group G on a variety X is an affine morphism $\pi: X \to Y$ of varieties such that Y carries the sheaf of invariants $(\pi_* \mathcal{O}_X)^G$ as its structure sheaf. In general, a G-variety X need not admit a good quotient, but there may be many different invariant open subsets $U \subseteq X$ with a good quotient; we will call them the good G-sets.

In this paper, we consider G-varieties X with a finitely generated Cox ring, e.g. X being a product of projective spaces, and ask for good G-sets $U \subseteq X$, which are maximal with respect to the properties either that the quotient space $U /\!\!/ G$ is quasiprojective or, more generally, that it comes with the A2-property; the latter means that any two points of $U /\!\!/ G$ admit a common affine neighbourhood, or, equivalently, that $U /\!\!/ G$ admits a closed embedding into some toric variety, see [28]. Our aim is to provide a constructive approach to such good G-sets, thus splitting the explicit computation into two parts: firstly computations of invariant rings in the spirit of classical Invariant Theory and, secondly, combinatorial computations with convex polyhedral cones. Another feature is that our approach opens an access to the geometry of quotient spaces via the methods developed in [3].

Let us present our results in more detail. A first step is to consider actions of G on factorial affine varieties X. The basic data for the construction of good G-sets of X are *orbit cones*. They live in the rational character space $\mathbb{X}_{\mathbb{Q}}(G)$, and for any $x \in X$ its orbit cone $\omega(x)$ is the convex cone generated by all $\chi \in \mathbb{X}(G)$ admitting a semiinvariant f with weight χ such that $f(x) \neq 0$ holds. It turns out that there are only finitely many orbit cones and all of them are polyhedral.

Based on the concept of orbit cones, we introduce the data describing the good G-sets of the factorial affine variety X. First, we associate to any character $\chi \in \mathbb{X}(G)$ its *GIT-cone*, namely

$$\lambda(\chi) := \bigcap_{\chi \in \omega(x)} \omega(x) \subseteq \mathbb{X}_{\mathbb{Q}}(G).$$

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Second, we say that a collection Φ of orbit cones is *2-maximal*, if for any two members their relative interiors overlap and Φ is maximal with respect to this property. Here comes the first result, see Theorems 3.2 and 3.5.

Theorem. *Let a connected reductive group G act on a factorial affine variety X.*

(i) *The GIT-cones form a fan in $\mathbb{X}_{\mathbb{Q}}(G)$, and this fan is in a canonical order reversing bijection with the collection of sets of semistable points of X.*
(ii) *There is a canonical bijection from the set of 2-maximal collections of orbit cones onto the collection of A2-maximal good G-sets of X.*

For the case of a torus G this result was already known. The first statement is given in [4]. Moreover, a result similar to the second statement was obtained in [8] for linear torus actions on vector spaces, and for torus actions on any affine factorial X, statement (ii) is given in [2].

To obtain the general statement, we reduce to the case of a torus action as follows. Consider the quotient $Y := X /\!\!/ G^s$ by the semisimple part $G^s \subseteq G$. It comes with an induced action of the torus $T := G/G^s$, and the key observation is that the good T-sets in Y are in a canonical bijection with the good G-sets in X, see Proposition 3.6. Note that this is the place, where in explicit computations, Classical Invariant Theory comes in, as it provides often the necessary information on the algebra $\mathbb{K}[X]^{G^s}$ of invariants, see the examples discussed in Sections 6 and 8.

The second step is passing to the case of a normal variety X with a finitely generated Cox ring $\mathcal{R}(X)$; recall that, for its definition, one assumes that the divisor class group $\mathrm{Cl}(X)$ is free and finitely generated, and then sets

$$\mathcal{R}(X) := \bigoplus_{D \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)).$$

The "total coordinate space" $\overline{X}$ of X is the spectrum of the Cox ring $\mathcal{R}(X)$. This $\overline{X}$ is a factorial affine variety, see [4], acted on by the Neron-Severi torus H having the divisor class group $\mathrm{Cl}(X)$ as its character lattice. Moreover, X can be reconstructed from $\overline{X}$ as a good quotient $q: \widehat{X} \to X$ by H for an open subset $\widehat{X} \subseteq \overline{X}$, see Section 4 for details.

After replacing G with a simply connected covering group, its action on X can be lifted to the total coordinate space $\overline{X}$. The actions of H and G on $\overline{X}$ commute, and thus define an action of the direct product $\overline{G} := H \times G$. Given a good $\overline{G}$-set $W \subseteq \overline{X}$, we introduce in 4.3 a "saturated intersection" $W \sqcap_G \widehat{X}$. The main feature of this construction is the following, see Theorem 4.5.

Theorem. *The canonical assignment $W \mapsto q(W \sqcap_G \widehat{X})$ defines a surjection from the collection of good $\overline{G}$-sets in $\overline{X}$ to the collection of good G-sets in X.*

So this result reduces the construction of good G-sets on X to the construction of good $\overline{G}$-sets in $\overline{X}$, and the latter problem, as noted before, is reduced to the case of a torus action. Again, this allows explicit computations. Note that our way to reduce the construction of quotients to the case of a torus action has nothing in common with the various approaches based on the Hilbert–Mumford Criterion, see [5,10,17,22], but is rather in the spirit of [26, Sec. 3].

As a first application of this result, we give an explicit description of the ample GIT-fan, i.e. the chamber structure of the linearized ample cone, for a given normal projective G-variety X with finitely generated Cox ring, see Proposition 6.1; recall that existence of the ample GIT-fan for any normal projective G-variety was proven in [10,26], and, finally,[22]. As an example, we compute the ample GIT-fan for the diagonal action of $\mathrm{Sp}(2n)$ on a product of projective spaces $\mathbb{P}^{2n-1}$, see Theorem 6.2.

As a second application, we obtain Gelfand–MacPherson type correspondences. Classically [11], this correspondence relates orbits of the diagonal action of the special linear group G on a product of projective spaces to the orbits of an action of a torus T on a Grassmannian. Kapranov [20] extended this correspondence to isomorphisms of certain GIT-quotients and used it in his study of the moduli space of point configurations on the projective line. Similarly, Thaddeus [27] proceeded with complete collineations. In Section 7, we put these correspondences into a general framework, relating GIT-quotients and also their inverse limits. As examples, we retrieve a result of [27] and also an isomorphism of GIT-limits in the setting of [20].

Finally, we use our approach to study the geometry of quotient spaces of a connected reductive group G on a normal variety X with finitely generated Cox ring. The basic observation is that in many cases our quotient construction provides the Cox ring of the quotient spaces. This allows to apply the language of bunched rings developed in [4], which encodes information on the geometry of a variety in terms of combinatorial data living in the divisor class group.

2. Some background on good quotients

In this section, we recall the concept of a good quotient and state basic properties, which will be used freely in the subsequent text.

Throughout this paper, we work in the category of algebraic varieties over an algebraically closed field $\mathbb{K}$ of characteristic zero. By a point we always mean a closed point. If we say that an algebraic group G acts on a variety X, then we tacitly assume that this action is given by a morphism $G \times X \to X$, and we refer to X as a G-variety. As usual, we say that a morphism

$\varphi: X \to Y$ of G-varieties is equivariant if it is compatible with the actions in the sense that always $\varphi(g \cdot x) = g \cdot \varphi(x)$ holds. Moreover, a morphism is called invariant, if it is constant along the orbits.

The classical finiteness theorem in Invariant Theory says that for an action of a reductive linear algebraic group on an affine variety $X = \mathrm{Spec}(A)$, the algebra A^G of invariant functions is finitely generated. This allows to define the classical invariant theory quotient $Y := \mathrm{Spec}(A^G)$, which comes with a morphism $p: X \to Y$. The notion of a good quotient is locally modeled on this concept:

Definition 2.1. Let G be a reductive linear algebraic group. A *good quotient* for a G-variety X is an affine morphism $p: X \to Y$ onto a variety Y such that the pullback $p^*: \mathcal{O}_Y \to (p_*\mathcal{O}_X)^G$ to the sheaf of invariants is an isomorphism. A good quotient is called *geometric*, if its fibers are precisely the orbits.

The basic properties of a good quotient $p: X \to Y$ of a G-variety are that it sends closed G-invariant subsets $A \subseteq X$ to closed sets $p(A) \subseteq Y$, and that for any two disjoint closed G-invariant subsets $A, A' \subseteq X$ their images $p(A), p(A') \subseteq Y$ are again disjoint. An immediate consequence is that each fiber $p^{-1}(y)$ of a good quotient $p: X \to Y$ contains precisely one closed G-orbit, and this orbit lies in the closure of any further orbit in $p^{-1}(y)$.

These basic properties imply that a good quotient $X \to Y$ for a G-variety X is categorical, i.e. any G-invariant morphism $X \to Z$ factors uniquely through $X \to Y$. In particular, good quotient spaces are unique up to isomorphism. This justifies the notation $X \to X /\!\!/ G$ for good and $X \to X/G$ for geometric quotients, which we will use frequently later on.

Proposition 2.2. *Let G be a connected reductive group, $H \subseteq G$ a normal, reductive subgroup, and X be a G-variety.*

(i) *If the good quotient $X \to X /\!\!/ H$ exists, then there is a unique G-action on $X /\!\!/ H$ making $X \to X /\!\!/ H$ equivariant, and this action uniquely induces an action of G/H on $X /\!\!/ H$.*
(ii) *The good quotient $X \to X /\!\!/ G$ exists if and only if the good quotients $X \to X /\!\!/ H$ and $X /\!\!/ H \to (X /\!\!/ H) /\!\!/ (G/H)$ exist. In this case, one has a commmutative diagram*

$$\begin{tikzcd} X \arrow[r, "/\!\!/ H"] \arrow[d, "/\!\!/ G"'] & X /\!\!/ H \arrow[d, "/\!\!/ (G/H)"] \\ X /\!\!/ G \arrow[r, leftrightarrow, "\cong"'] & (X /\!\!/ H) /\!\!/ (G/H). \end{tikzcd}$$

Proof. In the setting of (i), universality of the good quotient allows to push down the G-action to $X /\!\!/ H$, see [5, Thm. 7.1.4]. In the setting of (ii), if $X \to X /\!\!/ G$ exists, then also $X \to X /\!\!/ H$ exists, see [7, Cor. 10], and one directly verifies that the induced morphism $X /\!\!/ H \to X /\!\!/ G$ is a good quotient for the action of G/H. Conversely, if the stepwise good quotients exist, then one directly verifies that their composition is a good quotient for the G-variety X. $\square$

In general, quite a few non-affine G-varieties X admit a good quotient $X \to X /\!\!/ G$. However, there may be many open invariant subsets $U \subset X$ with a good quotient $U \to U /\!\!/ G$, and it is one of the main tasks of the theory of good quotients to describe all these sets. Here we fix the basic terminology.

Definition 2.3. Let a connected reductive group G act on a variety X.

(i) By a *good G-set* we mean an open, G-invariant subset $U \subseteq X$ admitting a good quotient $U \to U /\!\!/ G$.
(ii) If $U \subseteq X$ is a good G-set, then the *G-limit of* $x \in U$ is the unique closed G-orbit in the closure of $G \cdot x$ with respect to U; we denote it by $\lim_G(x, U)$.
(iii) For a good G-set $U \subseteq X$, we say that $U' \subseteq U$ is a *G-saturated inclusion* if U' is open, G-invariant, and for any $x \in U'$ one has $\lim_G(x, U) \subseteq U'$.

Given a good G-set $U \subseteq X$, we have mutually inverse bijections between the collection of G-saturated subsets $U' \subseteq U$ and the collection of open subsets $V \subseteq U /\!\!/ G$, sending $U' \subseteq U$ to $p(U') \subseteq U /\!\!/ G$ and $V \subseteq U /\!\!/ G$ to $p^{-1}(V)$. Moreover, for any G-saturated $U' \subseteq U$, the restriction $p: U' \to p(U')$ is a good quotient for the G-variety U'.

The preceding observation allows us to concentrate on the study of good G-sets to certain maximal ones, where one also may impose properties on the quotient spaces, like quasiprojectivity or the A2-property, i.e. any two points admit a common affine neighbourhood. Here are the precise notions, compare [5].

Definition 2.4. Let a connected reductive group G act on a variety X. We say that a good G-set $U \subseteq X$ is *maximal (qp-maximal, A2-maximal)*, if it is maximal with respect to saturated inclusion among all good G-sets $W \subseteq X$ (among those with $W /\!\!/ G$ quasiprojective, having the A2-property).

We conclude this section by recalling the construction of good G-sets with quasiprojective quotient spaces as sets of semistable points presented by Mumford in [17]. In fact here we use a slightly more general version, based on Weil divisors instead of line bundles, see [13]; compared to Mumford's original approach this has the advantage of producing all qp-maximal subsets, see Proposition 2.7.

Let X be a normal G-variety, where G is a reductive linear algebraic group. To any Weil divisor D on X, we associate a sheaf of $\mathcal{O}_X$-algebras, and consider the corresponding relative spectrum with its canonical morphism, compare [12, I.9.4.9] and [14, pp. 128/129]:

$$\mathcal{A} := \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathcal{O}_X(nD), \qquad X(D) := \mathrm{Spec}_X(\mathcal{A}), \qquad q_D\colon X(D) \to X.$$

The $\mathbb{Z}_{\geq 0}$-grading of the sheaf of algebras $\mathcal{A}$ defines a $\mathbb{K}^*$-action on $X(D)$ having the canonical morphism $q_D\colon X(D) \to X$ as a good quotient. For these constructions, we tacitly assumed that $\mathcal{A}$ is locally of finite type over $\mathcal{O}_X$, and thus $X(D)$ is in fact a variety; this holds for all Cartier divisors D, and will later be guaranteed by a global finiteness condition on X.

Definition 2.5. A *G-linearization* of the divisor D is a (morphical) G-action on $X(D)$ that commutes with the $\mathbb{K}^*$-action on $X(D)$ and makes $q_D\colon X(D) \to X$ into a G-equivariant morphism.

Note that for a Cartier divisor D, the sheaf $\mathcal{A}$ is locally free of rank one, and hence $X(D) \to X$ is a line bundle. So, in this context a G-linearization of D is a fiberwise linear action on the total space $X(D)$ making the projection $X(D) \to X$ equivariant; this is precisely Mumford's original notion of a linearization of a line bundle.

On the (invariant) set $X_{\mathrm{reg}} \subseteq X$ of smooth points, any G-linearized Weil divisor D is Cartier, and hence $X(D) \to X$ is a G-linearized line bundle over X_{reg} in the usual sense. This allows to define the (linearized) sum $D + D'$ of linearized divisors D, D' by extending the canonical action on $X(D) \otimes X(D')$ from X_{reg} to all of X, compare [4, Sec. 1].

There is also the concept of linearized divisor class group $\mathrm{Cl}_G(X)$. Call two linearized divisors D, D' equivalent, if there is an $(\mathbb{K}^* \times G)$-equivariant isomorphism $X(D) \to X(D')$. Then the addition defined just before induces a well defined addition on the set $\mathrm{Cl}_G(X)$ of classes of linearized divisors turning it into a group. Note that we have a canonical restriction isomorphism $\mathrm{Cl}_G(X) \to \mathrm{Pic}_G(X_{\mathrm{reg}})$ to the linearized Picard group of X_{reg}.

Finally, we come to the definition of semistability. Given a G-linearized Weil divisor D, one has a natural rational G-representation on the vector space of its global sections $\Gamma(X, \mathcal{O}(D)) = \Gamma(X(D), \mathcal{O})$, namely

$$G \times \Gamma(X(D), \mathcal{O}) \to \Gamma(X(D), \mathcal{O}), \qquad (g \cdot f)(x) := f(g^{-1} \cdot x).$$

In particular, this allows to speak about the space $\Gamma(X(D), \mathcal{O})^G$ of invariant sections of D. Moreover, for any section $f \in \Gamma(X, \mathcal{O}(D))$, one defines its set of zeroes $Z(f) \subseteq X$ by setting

$$Z(f) := \mathrm{Supp}(\mathrm{div}(f) + D).$$

Definition 2.6. Let G be a linear algebraic group, X a normal G-variety and D a G-linearized Weil divisor on X.

(i) We call $x \in X$ *semistable* with respect to D if there are $n \in \mathbb{Z}_{>0}$ and $f \in \Gamma(X, \mathcal{O}(nD))^G$ such that $X \setminus Z(f)$ is an affine neighbourhood of x.
(ii) The set of semistable points of a G-linearized Weil divisor D on X will be denoted by $X^{ss}(D)$, or $X^{ss}(D, G)$, if the group G needs to be specified.
(iii) If D' is another G-linearized Weil divisor on X, then we say that D and D' are *GIT-equivalent* if their associated sets of semistable points coincide.

Note that two linearized divisors defining the same class in $\mathrm{Cl}_G(X)$ have the same set of semistable points. From [13, Theorem 3.3] we infer the following features of the sets of semistable points:

Proposition 2.7. *Let G be a reductive linear algebraic group and X a normal G-variety.*

(i) *If D is a G-linearized Weil divisor on X, then there exists a good quotient $X^{ss}(D) \to X^{ss}(D) /\!\!/ G$ with a quasiprojective quotient space.*
(ii) *If $U \subset X$ is a G-invariant open subset having a good quotient $U \to U /\!\!/ G$ with $U /\!\!/ G$ quasiprojective, then U is G-saturated in some set $X^{ss}(D)$.*

3. Good quotients of factorial affine varieties

In this section, we consider an action of a connected reductive group G on an irreducible affine variety Z. In the first result, we describe the collection of sets of semistable points arising from the possible linearizations of the trivial line bundle over Z, and in the second one, we describe the collection of A2-maximal subsets of Z provided that Z is factorial. Both descriptions are of combinatorial nature and are given in terms of certain convex polyhedral cones. The first setting was also studied in [21]; there a numerical criterion for semistability was given.

Let us briefly fix the necessary notation. Given a polyhedral cone σ in some rational vector space, we denote by $\sigma^\circ \subseteq \sigma$ its relative interior, and for $\tau \subseteq \sigma$, we write $\tau \preceq \sigma$ if τ is a face of σ. By a fan in a rational vector space, we mean a finite collection Σ of convex, polyhedral cones such that for $\sigma \in \Sigma$ also every face $\tau \preceq \sigma$ belongs to Σ and for any two $\sigma_1, \sigma_2 \in \Sigma$ one has $\sigma_1 \cap \sigma_2 \preceq \sigma_i$; note that we don't require the cones of Σ to be pointed.

Now we turn to the G-linearizations of the trivial bundle $Z \times \mathbb{K} \to Z$; they arise from the elements $\chi \in M$ of the character group $M := \mathbb{X}(G)$ as follows

$$G \times (Z \times \mathbb{K}) \to Z \times \mathbb{K}, \qquad g \cdot (z, z') = (g \cdot z, \chi(g)z'). \tag{1}$$

Every such G-linearization defines a set $Z^{ss}(\chi) \subseteq Z$ of semistable points, and this set is explicitly given by

$$Z^{ss}(\chi) = \{z \in Z; f(z) \neq 0 \text{ for some } f \in \Gamma(Z, \mathcal{O})_{n\chi}, n > 0\}.$$

As outlined before, the set $Z^{ss}(\chi)$ admits a good quotient for the action of G; the quotient space is given by

$$Z^{ss}(\chi) /\!\!/ G = \operatorname{Proj}(A(\chi)), \quad \text{where } A(\chi) := \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Gamma(Z, \mathcal{O})_{n\chi}.$$

In particular, $Z^{ss}(\chi) /\!\!/ G$ is projective over $Z /\!\!/ G$. Our description of the collection of sets $Z^{ss}(\chi) \subseteq Z$ is formulated in terms of the following combinatorial data.

Definition 3.1. Let G be a connected reductive group, denote by $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ its rational character space, and let Z be an irreducible affine G-variety.

(i) The *weight cone* of the G-variety Z is the convex cone $\omega(Z) \subseteq M_{\mathbb{Q}}$ generated by all $\chi \in M$ with $\Gamma(Z, \mathcal{O})_\chi \neq 0$.
(ii) The *orbit cone* of a point $z \in Z$ is the convex cone $\omega(z) \subseteq M_{\mathbb{Q}}$ generated by all $\chi \in M$ that admit an $f \in \Gamma(Z, \mathcal{O})_\chi$ with $f(z) \neq 0$.
(iii) The *GIT-cone* $\lambda(\chi) \subseteq M_{\mathbb{Q}}$ of a character $\chi \in M$ is the intersection of all orbit cones containing χ:

$$\lambda(\chi) := \bigcap_{\chi \in \omega(x)} \omega(x).$$

(iv) The *GIT-fan* of the G-variety Z is the collection $\Sigma(Z)$ of all GIT-cones $\lambda(\chi)$, where $\chi \in M$.

Theorem 3.2. *Let G be a connected reductive group and Z be an irreducible affine G-variety.*

(i) *The weight cone, the orbit cones and the GIT-cones of the G-action on Z are all polyhedral, and there are only finitely many of them.*
(ii) *For every character $\chi \in M$, the associated set of semistable points $Z^{ss}(\chi) \subseteq Z$ is given by*

$$Z^{ss}(\chi) = \{z \in Z; \chi \in \omega(z)\}.$$

(iii) *The GIT-fan $\Sigma(Z)$ is in fact a fan in the rational character space $M_{\mathbb{Q}}$, and the union of all $\lambda(\chi) \in \Sigma(Z)$ is precisely the weight cone $\omega(Z)$.*
(iv) *For any $\chi \in M$, the set $Z^{ss}(\chi) \subseteq Z$ is nonempty if and only if $\chi \in \omega(Z)$. Moreover, for any two $\chi, \chi' \in \omega(Z) \cap M$, one has*

$$Z^{ss}(\chi) \subseteq Z^{ss}(\chi') \iff \lambda(\chi) \succeq \lambda(\chi').$$

(v) *If Z is factorial, then $\Sigma(Z)$ is in bijection to the qp-maximal good G-sets of Z via $\lambda \mapsto Z^{ss}(\chi)$, with any χ taken from the relative interior of λ.*

We prove these assertions by reducing them to the known case of a torus action. For this, we consider the semisimple part of G, i.e. the maximal connected semisimple subgroup $G^s \subseteq G$. Recall that $G^s \subseteq G$ is a normal subgroup, the factor group $T := G/G^s$ is a torus, and $G \to T$ induces an isomorphism of the character groups. The latter allows us to identify the character groups of G and T; we denote them both by M.

Lemma 3.3. *Let G be a connected reductive group and Z an irreducible affine G-variety. Consider the quotient $\pi: Z \to Z /\!\!/ G^s$ and the induced action $T := G/G^s$ on $Y := Z /\!\!/ G^s$.*

(i) *For every point $z \in Z$, we have $\omega(\pi(z)) = \omega(z)$.*
(ii) *For every character $\chi \in M$, we have $Z^{ss}(\chi) = \pi^{-1}(Y^{ss}(\chi))$.*
(iii) *If Z is factorial, then also $Z /\!\!/ G^s$ is factorial.*

Proof. The first two assertions follow directly from the fact that the algebra $\Gamma(Z, \mathcal{O})^{G^s}$ of G^s-invariants in $\Gamma(Z, \mathcal{O})$ equals the algebra of G-semiinvariants of $\Gamma(Z, \mathcal{O})$. The third one is well known, see [19, Thm. 3.17]. $\square$

Proof of Theorem 3.2. We first turn to statements (i)–(iv). By Lemma 3.3 it suffices to have the corresponding statements for the action of the torus $T := G/G^s$ on the affine variety $Y := Z /\!\!/ G^s$. In that case, the statements were proven in [4, 2.5, 2.7, 2.9 and 2.11].

To see (v), recall first from Proposition 2.7 that every good G-set $W \subseteq Z$ with $W /\!\!/ G$ quasiprojective is G-saturated in the set of semistable points $Z^{ss}(D)$ of some linearized divisor. Since Z is factorial, we have $Z^{ss}(D) = Z^{ss}(\chi)$ for some $\chi \in M$. This consideration shows in particular that every qp-maximal good G-set $W \subseteq Z$ is of the form $W = Z^{ss}(\chi)$ for some $\chi \in M$.

So, in order to establish the bijection as claimed, we only have to show that any $Z^{ss}(\chi) \subseteq Z$ is qp-maximal. Suppose that some $Z^{ss}(\chi) \subseteq Z$ is not. Then we have a G-saturated inclusion $Z^{ss}(\chi) \subseteq Z^{ss}(\chi')$ with a qp-maximal $Z^{ss}(\chi') \subseteq Z$ and a commutative diagram

$$\begin{tikzcd}
Z^{ss}(\chi) \arrow[rr] \arrow[d] & & Z^{ss}(\chi') \arrow[d] \\
Z^{ss}(\chi)/\!\!/G \arrow[rr] \arrow[dr] & & Z^{ss}(\chi')/\!\!/G. \arrow[dl] \\
& Z/\!\!/G &
\end{tikzcd}$$

Since the quotient spaces in the middle line are projective over $Z/\!\!/G$, the induced morphism $Z^{ss}(\chi)/\!\!/G \to Z^{ss}(\chi')/\!\!/G$ is projective. On the other hand, by G-saturatedness, it is an open embedding. This implies $Z^{ss}(\chi)/\!\!/G = Z^{ss}(\chi')/\!\!/G$, and thus, again by G-saturatedness, $Z^{ss}(\chi) = Z^{ss}(\chi')$. $\square$

Our next aim is a description of all A2-maximal good G-sets of a factorial affine G-variety Z. The necessary data are again given in terms of orbit cones.

Definition 3.4. Let G be a connected reductive group and Z an irreducible affine G-variety. Let $\Omega(Z)$ denote the collection of all orbit cones $\omega(z)$, where $z \in Z$.

(i) By a *2-connected collection* we mean a subcollection $\Psi \subseteq \Omega(Z)$ such that $\tau_1^\circ \cap \tau_2^\circ \neq \emptyset$ holds for any two $\tau_1, \tau_2 \in \Psi$.
(ii) By a *2-maximal collection*, we mean a 2-connected collection, which is not a proper subcollection of any other 2-connected collection.
(iii) We say that a 2-connected collection Ψ is a *face* of a 2-connected collection Ψ' (written $\Psi \preceq \Psi'$), if for any $\omega' \in \Psi'$ there is an $\omega \in \Psi$ with $\omega \preceq \omega'$.
(iv) To any collection $\Psi \subseteq \Omega(Z)$, we associate a G-invariant subset $U(\Psi) \subseteq Z$ as follows:

$$U(\Psi) := \{z \in Z;\ \omega_0 \preceq \omega(z) \text{ for some } \omega_0 \in \Psi\}.$$

(v) To any G-invariant open subset $U \subseteq Z$, we associate a set of orbit cones, namely

$$\Psi(U) := \{\omega(z);\ z \in U \text{ with } G \cdot z \text{ closed in } U\}.$$

Theorem 3.5. *Let G be a connected reductive group, and let Z be a factorial affine G-variety. Then we have mutually inverse bijections of finite sets:*

$$\begin{array}{rcl}
\{\text{2-maximal collections in } \Omega(Z)\} & \longleftrightarrow & \{\text{A2-maximal good } G\text{-sets of } Z\} \\
\Psi & \mapsto & U(\Psi) \\
\Psi(U) & \leftarrow\!\shortmid & U.
\end{array}$$

These bijections are order-reversing maps of partially ordered sets in the sense that we always have

$$\Psi \preceq \Psi' \iff U(\Psi) \supseteq U(\Psi').$$

As before, the idea of proof is to reduce the problem via passing to the quotient $Z/\!\!/G^s$ to the case of a torus action. This time the reduction step is a statement of independent interest.

Proposition 3.6. *Let G be a connected reductive group and Z a factorial affine G-variety. Consider the semisimple part $G^s \subseteq G$, the torus $T := G/G^s$, the quotient $\pi: Z \to Z/\!\!/G^s$, and the induced T-action on $Z/\!\!/G^s$. Then we have mutually inverse bijections*

$$\begin{array}{rcl}
\{\text{good } G\text{-sets of } Z\} & \longleftrightarrow & \left\{\text{good } T\text{-sets of } Z/\!\!/G^s\right\} \\
U & \mapsto & \pi(U), \\
\pi^{-1}(V) & \leftarrow\!\shortmid & V.
\end{array}$$

Both assignments preserve saturated inclusions, and they send maximal (A2-maximal, qp-maximal) subsets into maximal (A2-maximal, qp-maximal) ones.

Lemma 3.7. *Let a semisimple group G^s act on a factorial affine variety Z. Then every good G^s-set $U \subseteq Z$ is G^s-saturated in Z.*

Proof. Consider the quotient morphism $\pi: U \to U /\!\!/ G^s$, and cover $U /\!\!/ G^s$ by affine open subsets $V_i \subseteq U /\!\!/ G^s$. Then each $U_i := \pi^{-1}(V_i)$ is affine, and hence, the complement $A_i := Z \setminus U_i$ is of pure codimension one in Z. Since Z is factorial affine, A_i is the set of zeroes of a function $f_i \in \Gamma(Z, \mathcal{O})$.

We claim that f_i is G^s-invariant. In fact, for any $z \in U_i$, the map $g \mapsto f_i(g \cdot z)$ is an invertible function on G^s, and thus, by semisimplicity of G^s, is constant. So, U_i is the complement of the zero set of a G^s-invariant function, and thus it is G^s-saturated in Z. Thus, U as the union of the U_i, is G^s-saturated as well. $\square$

Proof of Proposition 3.6. First, we check that the assignments are well defined. Let $U \subseteq Z$ be a good G-set. Then U is as well a good G^s-set. Lemma 3.7 ensures that U is G^s-saturated in Z. Thus, $\pi(U)$ is open in $Y := Z /\!\!/ G^s$. Moreover, the induced morphism $\pi(U) \to U /\!\!/ G$ is a good quotient for the T-action, see Proposition 2.2.

If $V \subseteq Y$ is a good T-set, then $\pi^{-1}(V) \to V$ is a good quotient for the G^s-action, and thus $\pi^{-1}(V) \to V /\!\!/ T$ is a good quotient for the G-action, see again Proposition 2.2. Thus, the assignments are well defined. Since every good G^s-set $U \subseteq Z$ is saturated with respect to $\pi: Z \to Y$, they are moreover inverse to each other.

The fact that the assignments $U \mapsto \pi(U)$ and $V \mapsto \pi^{-1}(V)$ preserve saturated inclusion relies on the fact that we have the induced isomorphisms $U /\!\!/ G \cong \pi(U) /\!\!/ T$ and $V /\!\!/ T \cong \pi^{-1}(V) /\!\!/ G$. Moreover, this implies that maximality (A2-maximality, qp-maximality) is preserved. $\square$

Proof of Theorem 3.5. In [2, Sec. 1], the assertions were proven for torus actions on factorial affine varieties. In particular, they hold for the action of $T := G/G^s$ on $Y := Z /\!\!/ G^s$. Now, we have the canonical bijection between the respective sets of orbit cones

$$\Omega(Z) \to \Omega(Y), \qquad \omega(z) \mapsto \omega(\pi(z)).$$

In particular, this gives a one-to-one correspondence between the sets of 2-maximal collections $\Psi \subseteq \Omega(Z)$ and 2-maximal collections $\Psi \subseteq \Omega(Y)$. Thus, denoting by $V(\Psi) \subseteq Y$ the A2-maximal good T-set corresponding to a 2-maximal collection $\Psi \subseteq \Omega(Y)$, Proposition 3.6 reduces the problem to showing

$$U(\Psi) = \pi^{-1}(V(\Psi)), \qquad \Psi(U) = \Psi(\pi(U)).$$

The first equality is obvious. For "$\subseteq$" in the second one, note that $U \mapsto \pi(U)$ is a good quotient for the G^s-action. Thus, if $G \cdot z \subseteq U$ is closed, then $T \cdot \pi(z) = \pi(G \cdot z)$ is closed in $\pi(U)$. For "$\supseteq$", let $T \cdot z \subseteq \pi(U)$ be closed. Then $\pi^{-1}(T \cdot z)$ is a closed G-invariant subset in $\pi^{-1}(\pi(U)) = U$ and thus contains a closed G-orbit, which is mapped onto $T \cdot z$. $\square$

4. Lifting to the total coordinate space

Here we reduce the problem of finding good G-sets for a given G-variety X to the problem of finding good $(H \times G)$-sets, H a torus, in a certain affine factorial variety $\overline{X}$, called the "total coordinate space" of X. We begin with fixing the setup and recalling basic constructions from [3].

Let X be a normal algebraic variety with finitely generated free divisor class group $\mathrm{Cl}(X)$, and suppose that $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ holds. To define the *Cox ring* (also *total coordinate ring*) $\mathcal{R}(X)$ of X, choose a subgroup $K \subseteq \mathrm{WDiv}(X)$ of the group of Weil divisors mapping isomorphically onto $\mathrm{Cl}(X)$, and set

$$\mathcal{R}(X) := \Gamma(X, \mathcal{R}), \quad \text{where } \mathcal{R} := \bigoplus_{D \in K} \mathcal{O}(D).$$

Then $\mathcal{R}(X)$ is a ring, where multiplication takes place in the field $\mathbb{K}(X)$ of rational functions. The definition of $\mathcal{R}(X)$ is (up to isomorphism) independent from the choice of $K \subset \mathrm{WDiv}(X)$. An important property of $\mathcal{R}(X)$ is that it admits unique factorization, compare [3].

Throughout this section, we assume that $\mathcal{R}(X)$ is finitely generated as a $\mathbb{K}$-algebra; this holds for spherical varieties X, and, more generally for unirational varieties X with a complexity one group action, i.e. some Borel subgroup has an orbit of codimension one, see [15]. We consider the following geometric objects associated to the K-graded sheaf $\mathcal{R}$ of $\mathcal{O}_X$-algebras:

$$H := \mathrm{Spec}(\mathbb{K}[K]), \qquad \overline{X} := \mathrm{Spec}(\mathcal{R}(X)), \qquad \widehat{X} := \mathrm{Spec}_X(\mathcal{R}).$$

The relative spectrum $\widehat{X}$ as well as $\overline{X}$ come with actions of the Neron-Severi torus H, both defined by the K-gradings of $\mathcal{R}$ and $\mathcal{R}(X)$ respectively. The canonical morphism $q: \widehat{X} \to X$ is a good quotient for the action of H, and there is a canonical open H-equivariant embedding $\widehat{X} \subseteq \overline{X}$ with $\overline{X} \setminus \widehat{X}$ of codimension at least two in $\overline{X}$, compare also [3]. We will call $q: \widehat{X} \to X$ a *Cox construction* for X, as it naturally generalizes the often studied case of toric varieties [9]. Moreover, we refer to $\overline{X}$ as the *total coordinate space*.

In this section, we consider G-equivariant Cox constructions in the sense that a linear algebraic group G acts on $\overline{X}$ and X such that the actions of G and H on $\overline{X}$ commute, $\widehat{X} \subseteq \overline{X}$ is G-invariant and $q: \widehat{X} \to X$ is G-equivariant. The following two remarks can be helpful for finding equivariant Cox constructions.

Remark 4.1. If a connected linear algebraic group G acts on X, then the simply connected covering group $\widetilde{G}$ does as well. After fixing a basis $E_1, \dots, E_k$ of K one may choose a $\widetilde{G}$-linearization of each E_i. This induces a $\widetilde{G}$-linearization of any $D \in K$ and thus defines a $\widetilde{G}$-action on $\widehat{X}$ making $\widehat{X} \to X$ equivariant. The lifted $\widetilde{G}$-action extends to $\overline{X}$. Note that the actions of G and $\widetilde{G}$ on X have the same quotients.

Remark 4.2. Suppose that a linear algebraic group G acts on a factorial affine variety $\overline{X}$, and that, moreover, there is an action of an algebraic torus H on $\overline{X}$ commuting with the action of G. Let $\widehat{X} \subseteq \overline{X}$ be invariant under the actions of H and G, and suppose that there is a good quotient $q: \widehat{X} \to X$. If there is an H-saturated subset $W \subseteq \widehat{X} \subseteq \overline{X}$ with $\overline{X} \setminus W$ of codimension at least two in $\overline{X}$ such that H acts freely on W, then $q: \widehat{X} \to X$ is an equivariant Cox construction for X.

Given a G-equivariant Cox construction $q: \widehat{X} \to X$ with some reductive group G, the (commuting) actions of H and G on $\overline{X}$ define an action of the direct product $H \times G$ on the factorial affine variety $\overline{X}$. Our aim is to relate the good $(H \times G)$-sets of $\overline{X}$ to the good G-sets of X. The key construction for this is the following.

Definition 4.3. Let G be a connected reductive group, X a G-variety with equivariant Cox construction $q: \widehat{X} \to X = \widehat{X} /\!\!/ H$ and total coordinate space $\overline{X}$. For every good $(H \times G)$-set $W \subseteq \overline{X}$, we set

$$W \sqcap_G \widehat{X} := \left\{ x \in W \cap \widehat{X};\ \lim_{H \times G}(x, W) \subseteq \widehat{X},\ H \cdot x_0 \text{ is closed in } \widehat{X} \text{ for every } x_0 \in \lim_{H \times G}(x, W) \right\}.$$

Remark 4.4. Let G be a connected reductive group, X a G-variety with equivariant Cox construction $q: \widehat{X} \to X = \widehat{X} /\!\!/ H$ and total coordinate space $\overline{X}$.

(i) If X is $\mathbb{Q}$-factorial, then $q: \widehat{X} \to X$ is even a geometric quotient for the action of H and, for every good $(H \times G)$-set $W \subseteq \overline{X}$, one has

$$W \sqcap_G \widehat{X} := \left\{ x \in W \cap \widehat{X};\ \lim_{H \times G}(x, W) \subseteq \widehat{X} \right\}.$$

(ii) If X is affine, then $\widehat{X} = \overline{X}$ holds and, for every good $(H \times G)$-set $W \subseteq \overline{X}$, one has

$$W \sqcap_G \widehat{X} := \left\{ x \in W;\ x_0 \in \lim_{H \times G}(x, W) \Rightarrow H \cdot x_0 \subseteq \overline{X} \text{ is closed} \right\}.$$

The following result shows how to relate the good $(H \times G)$-sets of the total coordinate space $\overline{X}$ to the good G-sets of X using the assignment $W \mapsto W \sqcap_G \widehat{X}$.

Theorem 4.5. *Let G be a connected reductive group, X a G-variety with equivariant Cox construction $q: \widehat{X} \to X = \widehat{X} /\!\!/ H$ and total coordinate space $\overline{X}$. Then, for every good $(H \times G)$-set $W \subseteq \overline{X}$, the set $W \sqcap_G \widehat{X}$ is $(H \times G)$-saturated in W and H-saturated in $\widehat{X}$. This gives a surjection*

$$\begin{array}{ccc} \{\text{good } (H \times G)\text{-sets of } \overline{X}\} & \longrightarrow & \{\text{good } G\text{-sets of } X\} \\ W & \mapsto & q(W \sqcap_G \widehat{X}). \end{array}$$

This map has $U \mapsto q^{-1}(U)$ as a right inverse. Moreover, any maximal (A2-maximal, qp-maximal) good G-set $U \subseteq X$ is of the form $U = q(W \sqcap_G \widehat{X})$ with a maximal (A2-maximal, qp-maximal) good $(H \times G)$-set $W \subseteq \overline{X}$.

Proof. The first thing we have to show is that, for any good $(H \times G)$-set $W \subseteq \overline{X}$, the set $W \sqcap_G \widehat{X} \subseteq X$ is open and $(H \times G)$-saturated in W and H-saturated in $\widehat{X}$. We do this by constructing $W \sqcap_G \widehat{X} \subseteq W$ via stepwise removing suitable closed subsets from W.

Let $p: W \to W /\!\!/ (H \times G)$ be the quotient, and consider the closed $(H \times G)$-invariant subset $A := W \setminus \widehat{X}$ of W. By the general properties of good quotients, we obtain an open, $(H \times G)$-saturated subset $V \subseteq W$ by setting

$$V := W \setminus p^{-1}(p(A)) = \left\{ x \in W \cap \widehat{X};\ \lim_{H \times G}(x, W) \subseteq \widehat{X} \right\} \subseteq W \cap \widehat{X}.$$

Now, we consider the quotient $q: \widehat{X} \to X$ and the $(H \times G)$-invariant, closed complement $B := \widehat{X} \setminus V$. Using G-equivariance and again the properties of good quotients, we obtain an H-saturated, $(H \times G)$-invariant open subset $V' \subseteq \widehat{X}$ by setting

$$V' := \widehat{X} \setminus q^{-1}(q(B)) = \left\{ x \in V;\ \lim_{H}(x, \widehat{X}) \subseteq V \right\} \subseteq V.$$

In the third shrinking step, we consider the quotient $p: V \to V /\!\!/ (H \times G)$ and the closed $(H \times G)$-invariant subset $C := V \setminus V'$ of V. Then we obtain an open $(H \times G)$-saturated subset $V'' \subseteq V$ by setting

$$V'' := V \setminus p^{-1}(p(C)) = \left\{ x \in V;\ \lim_{H \times G}(x, V) \subseteq V' \right\} \subseteq V'.$$

Thus, $V'' \subseteq W$ is $(H \times G)$-saturated with good quotient $p: V'' \to V'' /\!\!/ (H \times G)$. Moreover, $V'' \subseteq V'$ and hence $V'' \subseteq \widehat{X}$ are H-saturated inclusions, because for $x \in V''$ the limit $\lim_H(x, V')$ is contained in the closure of $(H \times G) \cdot x$ taken in V, which in turn is contained in V''. In particular, we have a good quotient $q: V'' \to q(V'')$, where $q(V'') \subseteq X$ is open. Hence, in order to finish the proof, we have to verify

$$V'' = W \sqcap_G \widehat{X}.$$

Given $x \in V''$, we have $\lim_{H \times G}(x, W) = \lim_{H \times G}(x, V) \subseteq V'$. In particular, $\lim_{H \times G}(x, W)$ is contained in $\widehat{X}$. Moreover, for $x_0 \in \lim_{H \times G}(x, W)$ one obtains $\lim_H(x_0, \widehat{X}) \subseteq V$, which gives $\lim_H(x_0, \widehat{X}) \subseteq \lim_{H \times G}(x, V)$. Since all H-orbits in $\lim_{H \times G}(x, V)$ have the same dimension in $\widehat{X}$, we see that $H \cdot x_0$ is closed in $\widehat{X}$. Thus, $x \in W \sqcap_G \widehat{X}$ holds.

Conversely, for any $x \in W \sqcap_G \widehat{X}$, one obviously has $x \in V$. Moreover, for $x_0 \in \lim_{H \times G}(x, W)$, the orbit $H \cdot x_0$ is closed in $\widehat{X}$, which gives $x_0 \in V'$. This in turn means $x \in V''$.

Having seen that $W \sqcap_G \widehat{X}$ is $(H \times G)$-saturated in W and H-saturated in $\widehat{X}$ for every good $(H \times G)$-set $W \subseteq \overline{X}$, it is clear that we have the surjection $W \mapsto q(W \sqcap_G \widehat{X})$ as in the assertion. Moreover, $U \mapsto q^{-1}(U)$ is obviously a right inverse.

A few words of explanation are needed concerning the claim that any maximal good G-set $U \subseteq X$ arises as $U = q(W \sqcap_G \widehat{X})$ with a maximal good $(H \times G)$-set $W \subseteq \overline{X}$. In fact, $q^{-1}(U)$ is an $(H \times G)$-saturated subset of some maximal good $(H \times G)$-set $W \subseteq \overline{X}$. Since $W \sqcap_G \widehat{X} \subseteq W$ is $(H \times G)$-saturated as well, we can conclude that $q^{-1}(U) \subseteq W \sqcap_G \widehat{X}$ is $(H \times G)$-saturated. It follows that $U \subseteq q(W \sqcap_G \widehat{X})$ is G-saturated, and thus, using maximality, we obtain $U = q(W \sqcap_G \widehat{X})$. □

We now consider the setting of sets of semistable points. This needs to recall a pullback construction for G-linearized divisors, which was performed for the case of a torus G in [4, Sec. 3], but generalizes without changes to any linear algebraic group G.

Let D be any G-linearized Weil divisor on X. Then the restriction D_{reg} to the set X_{reg} of regular points on X is a linearized Cartier divisor, and thus has a canonically $(H \times G)$-linearized pullback divisor q^*D_{reg}, where the $(H \times G)$-action on

$$q^{-1}(X_{\mathrm{reg}})(q^*D_{\mathrm{reg}}) \cong q^{-1}(X_{\mathrm{reg}}) \times_{X_{\mathrm{reg}}} X_{\mathrm{reg}}(D_{\mathrm{reg}})$$

is given by the diagonal G-action and the H-action on the first factor. Since the complement $\overline{X} \setminus q^{-1}(X_{\mathrm{reg}})$ has codimension at least two, we may close the components of q^*D_{reg}, and obtain in this way a $(H \times G)$-linearized Weil divisor $\overline{D}$ on $\overline{X}$. As shown in [4, Lemma 3.3], this construction sets up an isomorphism

$$\mathrm{Cl}_G(X) \to \mathrm{Cl}_{H \times G}(\overline{X}), \qquad [D] \mapsto [\overline{D}]. \tag{2}$$

The following statement shows that all sets of semistable points of G-linearized divisors on X arise from those of $(H \times G)$-linearized divisors on $\overline{X}$; the proof is identical to that in the case of a torus G, see [4, Theorem 3.5], and therefore will be omitted.

Theorem 4.6. *Let G be a connected reductive group, X a G-variety with equivariant Cox construction $q: \widehat{X} \to X = \widehat{X} /\!\!/ H$ and total coordinate space $\overline{X}$. Then, for any G-linearized Weil divisor D on X, we have a $(H \times G)$-saturated inclusion*

$$q^{-1}(X^{ss}(D, G)) = \overline{X}^{ss}(\overline{D}, H \times G) \sqcap_G \widehat{X} \subseteq \overline{X}^{ss}(\overline{D}, H \times G).$$

An important finiteness result by Dolgachev and Hu [10] and, independently, Thaddeus [26] says that on any projective G-variety, where G is a reductive group, there are only finitely many GIT-equivalence classes arising from *ample* bundles. In our setting, Theorem 4.6 gives more:

Corollary 4.7. *Let G be a connected reductive group, and X a G-variety with finitely generated total coordinate ring. Then the G-action on X has only finitely many GIT-equivalence classes.*

Proof. According to Theorem 4.6, the number of GIT-equivalence classes of the G-action on X is bounded by the number of GIT-classes of the $(H \times G)$-action on $\overline{X}$. But the latter number is finite by Theorem 3.2. □

5. Computing a first example

The previous sections suggest the following strategy for constructing good G-sets of a given G-variety X. First, take an equivariant Cox construction $q: \widehat{X} \to X = \widehat{X} /\!\!/ H$ and consider the associated total coordinate space $\overline{X}$. Then Theorem 4.5 reduces the problem of finding good G-sets $U \subseteq X$ to finding good $(H \times G)$-sets of $\overline{X}$. By Proposition 3.6, the latter problem is equivalent to finding the quotients of a torus action on $\overline{Y} = \overline{X} /\!\!/ G^s$.

In fact, in many concrete cases, the equivariant Cox construction is given from the beginning, and Classical Invariant Theory often provides enough information on the quotient $\overline{X} /\!\!/ G$, see the examples treated later. So the general difficulties remain in understanding the step $W \mapsto W \sqcap_G \widehat{X}$ and the computation of GIT-fan and A2-maximal collections for torus actions.

The first problem disappears, for example, when we restrict to GIT-quotients arising from ample bundles, see Section 6. For the second one, we begin with a general observation showing that one may work in terms of *walls*, i.e. orbit cones of codimension one. Let us first say a few words on the combinatorial framework.

Remark 5.1. Let $\omega_1, \ldots, \omega_r$ be polyhedral cones in a rational vector space $K_{\mathbb{Q}}$ such that their union is a convex cone $\omega \subseteq K_{\mathbb{Q}}$. Suppose that

$$\Sigma := \Sigma(\omega_1, \ldots, \omega_r) := \{\lambda(u); u \in \omega\}, \quad \text{where } \lambda(u) := \bigcap_{u \in \omega_i} \omega_i,$$

is a fan, any ω_i is a face of some full dimensional $\omega_j \subseteq K_{\mathbb{Q}}$, and the facets $\eta_1, \ldots, \eta_s$ of the full dimensional ω_j occur among the ω_i. Then

- the maximal cones $\lambda(u) \in \Sigma$ are precisely the closures of the connected components of $\omega \setminus (\eta_1 \cup \cdots \cup \eta_s)$,
- every nonmaximal cone $\lambda(u) \in \Sigma$ is the intersection over the facets η_i with $\lambda(u) \subseteq \eta_i$.

If we are in the setting 5.1, then we call the facets $\eta_1, \ldots, \eta_s$ of the full-dimensional ω_j the *walls* and we say that Σ is *determined by the walls*.

Proposition 5.2. *Let a reductive group G act on a factorial affine variety Z. Then the associated GIT-fan is determined by its walls.*

Proof. According to Lemma 3.3, we may assume that G is a torus, acting effectively. To obtain the setting 5.1, two things have to be verified. Firstly, given an orbit cone of full dimension, then also its facets are orbit cones; this is obvious. Secondly, every orbit cone is a face of some orbit cone of full dimension; this will be done below.

We have to show that any G-orbit is contained in the closure of a G-orbit of maximal dimension. Otherwise, we find some orbit $G \cdot z$ such that $\dim(G \cdot z)$ is not maximal and $G \cdot z$ is not contained in the closure of any other G-orbit. Then $G' := (G_z)^0$ is a proper subtorus of G, and G' acts nontrivially on Z. Semicontinuity of fiber dimension tells us that the fiber $\pi^{-1}(\pi(z))$ of the quotient map $\pi : Z \to Z /\!\!/ G'$ must contain a G'-orbit $G' \cdot z'$ of positive dimension. As a G'-fixed point, z lies in the closure of $G' \cdot z'$. It follows that $G \cdot z$ is contained in the closure of the orbit $G \cdot z'$, which is different from $G \cdot z$; a contradiction. $\square$

Example 5.3. Consider the homogeneous space $X := \mathrm{SL}(3)/H$, where $H \subset \mathrm{SL}(3)$ is a maximal torus. Then X is a smooth affine variety of dimension 6, and the special orthogonal group $G := \mathrm{SO}(3) \subset \mathrm{SL}(3)$ acts on X from the left. The generic G-orbit on X is of dimension 3 and it is closed in X, see [16].

Consider $\widehat{X} := \mathrm{SL}(3)$ with the left G-action. Then the projection $q : \widehat{X} \to X$ is a G-equivariant Cox construction, and the total coordinate space is given as $\overline{X} = \widehat{X}$. Moreover, $\overline{Y} := \overline{X} /\!\!/ G$ is the homogeneous space $\mathrm{SO}(3) \setminus \mathrm{SL}(3)$ with respect to the left $\mathrm{SO}(3)$-action. The situation is summarized in the following commutative diagram

$$\begin{tikzcd}
 & \overline{X} = \hat{X} = \mathrm{SL}(3) \arrow[dl, "q"'] \arrow[dr, "\pi"] & \\
X = \mathrm{SL}(3)/H \arrow[dr, "\varphi"'] & & \overline{Y} = \mathrm{SO}(3)\backslash\mathrm{SL}(3) \arrow[dl, "\psi"] \\
 & Y := \mathrm{SO}(3)\backslash\mathrm{SL}(3)/H. &
\end{tikzcd}$$

Combining Proposition 3.6 and Theorem 4.5, we see that, in the present setting, the good G-sets of X are in bijection with the good H-sets of $\overline{Y}$ via $U \mapsto \pi(q^{-1}(U))$. Moreover, the quotient of U by G is geometric if and only if the quotient of $\pi(q^{-1}(U))$ by H is so. So, our task is reduced to describing the H-quotients of $\overline{Y}$.

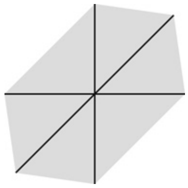
First, recall that $\overline{Y}$ can be identified as the variety of symmetric (3×3)-matrices with determinant one via

$$G \cdot A \mapsto A^t \cdot A.$$

The H-action is given as $(t_1, t_2, t_3)(a_{ij}) = (t_i t_j a_{ij})$, where $t_3 = t_1^{-1} t_2^{-1}$. The following matrices have one-dimensional H-orbits:

$$A_1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad A_3 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

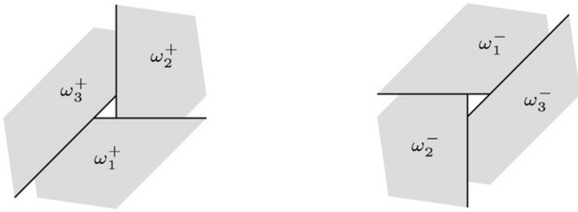
The associated orbit cones in $\mathbb{X}(H) = \mathbb{Q}^2$ are the lines $\omega(A_1) = \mathbb{Q} \cdot e_1$, $\omega(A_2) = \mathbb{Q} \cdot e_2$ and $\omega(A_3) = \mathbb{Q} \cdot (e_1 + e_2)$. According to Proposition 5.2 the GIT-fan looks as follows.



The six full dimensional cones of this fan define geometric GIT-quotients $U_i \to U_i/G$, where $1 \le i \le 6$, of the G-variety X. Moreover, there are two 2-maximal collections defining A2-maximal quotients $U_i \to U_i/G$, where $7 \le i \le 8$, with a non-quasiprojective quotient space, namely

$$(\mathbb{Q}^2, \omega_1^+, \omega_2^+, \omega_3^+), \qquad (\mathbb{Q}^2, \omega_1^-, \omega_2^-, \omega_3^-),$$

where ω_i^+ are half spaces bounded by $\omega(A_i)$, and $\omega_i^- = -\omega_i^+$ such that the intersection over the interiors of the ω_i^+ as well as that over the interiors of the ω_i^- are empty.



It is not hard to check that the quotient $X \to X/\!\!/G$ has precisely three exceptional fibers, each of which consists of one closed orbit and two one-parameter families of three-dimensional orbits. From this one may guess that there are eight maximal open subsets with geometric quotient. In any case, the quotient space U_i/G is a small modification of the affine threefold $X/\!\!/G$, having three exceptional fibers, each isomorphic to a projective line. A priori it is not clear why six of the U_i/G should be quasiprojective and two not.

6. Chambers of the linearized ample cone

In this section, we consider projective G-varieties. In [10,26] it was first proved that the cone of linearized ample divisors has a "chamber structure" describing the GIT-equivalence, see also [22]. In this section, we describe this chamber structure in our setting and then treat a concrete example.

Let G be a connected reductive group, and let X be a projective G-variety with equivariant Cox construction $q: \widehat{X} \to X = \widehat{X}/\!\!/H$ and total coordinate space $\overline{X}$. Let K denote the character group of the torus H and M that of G. Then we have canonical isomorphisms

$$\mathrm{Cl}(X) \cong K, \qquad \mathrm{Cl}_G(X) \cong \mathrm{Cl}_{H\times G}(\overline{X}) \cong K \times M,$$

see formulae (1) and (2) for the latter two. This allows us to denote the G-linearized divisors on X by pairs $(D, \chi) \in K \times M$. Moreover, we denote by $\kappa_X \subseteq K_{\mathbb{Q}}$ the cone of semiample divisor classes on X; recall from [4] that κ_X is a GIT-cone for the H-action on $\overline{X}$. Finally, we denote by $\omega(\overline{X}) \subseteq K_{\mathbb{Q}} \times M_{\mathbb{Q}}$ the weight cone and by $\Sigma(\overline{X})$ the GIT-fan of the $(H \times G)$-action on $\overline{X}$.

Proposition 6.1. *Let G be a connected reductive group, and X a projective G-variety with equivariant Cox construction $q: \widehat{X} \to X = \widehat{X}/\!\!/H$ and total coordinate space $\overline{X}$. Then, in the above notation, the following holds.*

(i) *The cone $\alpha(X) \subseteq K \times M$ of ample G-linearized divisor classes with nonempty set of semistable points is given by*

$$\alpha(X) = (\kappa_X^\circ \times M_{\mathbb{Q}}) \cap \omega(\overline{X}).$$

(ii) *The partial fan $\Sigma(X) := \{\lambda \cap \alpha(X);\ \lambda \in \Sigma(\overline{X})\}$ describes the GIT-equivalence on X in the sense that for any two (D, χ) and (D', χ') in $\alpha(X)$, one has*

$$X^{ss}(D, \chi) \subseteq X^{ss}(D', \chi') \iff \lambda(D, \chi) \succeq \lambda(D', \chi').$$

Proof. We prove (i). Given any G-linearized divisor on X, represented by some $(D, \chi) \in K \times M$, its invariant sections are exactly the semiinvariants with respect to the weight (D, χ) in $\Gamma(\overline{X}, \mathcal{O})$. So, the weight cone $\omega(\overline{X})$ contains precisely the G-linearized divisors of X admitting invariant sections.

In order to verify the description of $\alpha(X)$, it suffices to show that for any $(D, \chi) \in \omega(\overline{X})$ with D ample, some positive multiple nD admits a section f with $X \setminus Z(f)$ affine. But this follows from the general observation that for any section of an ample divisor the complement of its zero set is affine.

To see assertion (ii), note that for any ample G-linearized divisor D on the projective variety X, the quotient space $X^{ss}(D)/\!\!/G$ is again projective. Thus Theorem 4.6 gives $q^{-1}(X^{ss}(D)) = \overline{X}^{ss}(\overline{D}, H \times G)$. Consequently, the assertion follows from Theorem 3.2. $\square$

We now treat a concrete example. For $n \geq 2$, let $G = \mathrm{Sp}(2n)$ be the symplectic group, i.e. the group of invertible matrices preserving a non-degenerate skew-symmetric bilinear form $\langle , \rangle$ on $\mathbb{K}^{2n}$. Then G acts diagonally on the m-fold product

$$X := (\mathbb{P}^{2n-1})^m.$$

Fix a hyperplane E_i on each factor $\mathbb{P}^{2n-1}$ and consider its pullback D_i on X. Then the lattice $K \subseteq \mathrm{WDiv}(X)$ generated by $D_1, \ldots, D_m$ maps isomorphically to the divisor class group $\mathrm{Cl}(X)$. We have the identification $K \cong \mathbb{Z}^m$ via $D_i \mapsto e_i$ and, moreover,

$$\overline{X} = \mathrm{Spec}\left(\bigoplus_{D \in K} \Gamma(X, \mathcal{O}(D))\right) \cong (\mathbb{K}^{2n})^m =: V.$$

The torus acting on $\overline{X}$ is $H \cong (\mathbb{K}^*)^m$, and its action is componentwise scalar multiplication. As G has trivial character group, the GIT-fan of the $(H \times G)$-action lives in $K_{\mathbb{Q}} = \mathbb{X}_{\mathbb{Q}}(H) \cong \mathbb{Q}^m$.

For our description of the GIT-fan, we need one more notation. Given a set of vectors $w_1, \ldots, w_k$, consider the collection $\omega_1, \ldots, \omega_r$ of all convex cones generated by some of the w_i, and set

$$\Sigma(w_1, \ldots, w_k) := \Sigma(\omega_1, \ldots, \omega_r).$$

Note that $\Sigma(w_1, \ldots, w_k)$ is the coarsest common refinement of all fans having precisely $\mathbb{Q}_{\geq 0} \cdot w_1, \ldots, \mathbb{Q}_{\geq 0} \cdot w_k$ as their set of rays.

Theorem 6.2. *For $1 \leq i < j \leq m$, let $u_{ij} = (u_{ij}^1, \ldots, u_{ij}^m) \in \mathbb{Z}^m$ be the vector with entries 1 at the i-th and j-th place and 0 elsewhere. Then the weight cone $\omega(\overline{X})$ of the $(H \times G)$-action on $\overline{X}$ and the G-ample cone $\alpha(X)$ of the G-action on X are*

$$\begin{aligned} \omega(\overline{X}) &= \mathrm{cone}(u_{ij};\ 1 \leq i < j \leq m) \\ &= \left\{(s_1, \ldots, s_m) \in \mathbb{Q}^m_{\geq 0};\ 2s_i \leq s_1 + \cdots + s_m,\ 1 \leq i \leq m\right\} \end{aligned}$$
$$\alpha(X) = \mathbb{Q}^m_{>0} \cap \omega(\overline{X}).$$

The GIT-fan of the action of $H \times G$ on $\overline{X}$ is $\Sigma(u_{ij};\ 1 \leq i < j \leq m)$; it is determined by the walls, and these are precisely the orbit cones

$$\omega(\overline{X}) \cap \left\{(s_1, \ldots, s_m);\ \sum_{j \in J} s_j = \sum_{l \notin J} s_l\right\}, \quad J \subset \{1, \ldots, m\},\ 1 < |J| \leq \frac{m}{2},$$

$$\mathrm{cone}\left(u_{ij};\ \sum_{k \in J_1} u_{ij}^k = \sum_{l \in J_2} u_{ij}^l\right), \quad J_1, J_2 \subset \{1, \ldots, m\}, J_1 \cap J_2 = \emptyset,$$
$$|J_1| + |J_2| \leq m - 3.$$

Proof. Lemma 3.3 tells us that the GIT-fan of the action of $H \times G$ on $\overline{X}$ is the same as that of the H-action on $\overline{Y} = \mathrm{Spec}(\mathbb{K}[V]^G)$. The algebra of invariants $\mathbb{K}[V]^G$ is generated by the functions $f_{ij} \in \mathbb{K}[V]$ given by

$$f_{ij}(v_1, \ldots, v_m) := \langle v_i, v_j \rangle.$$

Each f_{ij} is H-semiinvariant with weight u_{ij}. Moreover, $\overline{Y} = \mathrm{Spec}(\mathbb{K}[V]^G)$ is the variety of skew-symmetric $m \times m$-matrices of rank $\leq 2n$, see [19, Sec. 9]. The quotient morphism $\pi : V \to \overline{Y}$ sends $(v_1, \ldots, v_m)$ to the matrix $(\langle v_i, v_j \rangle)$, and an element $(t_1, \ldots, t_m) \in H$ moves (c_{ij}) to $(t_i t_j c_{ij})$.

According to Proposition 5.2, the problem of describing the GIT-fan is reduced to computing the walls of the H-action on $\overline{Y}$. To any subset $J \subset \{1, \ldots, m\}$ with $1 < |J| \leq m/2$, we associate a hyperplane

$$\mathcal{H}_J := \left\{(s_1, \ldots, s_m);\ \sum_{j \in J} s_j = \sum_{l \notin J} s_l\right\}.$$

Moreover, for any pair $J_1, J_2 \subset \{1, \ldots, m\}$ of disjoint subsets satisfying $|J_1| + |J_2| \leq m - 3$ and if J_1 is empty then $J_2 = \{i\}$, we set

$$\mathcal{H}_{J_1, J_2} := \left\{(s_1, \ldots, s_m);\ \sum_{j_1 \in J_1} s_{j_1} = \sum_{j_2 \in J_2} s_{j_2}\right\}.$$

With these definitions, the description of the walls, i.e., the orbit cones of codimension one, is an immediate consequence of the following two claims.

Claim 1. *If C is a skew-symmetric matrix with a one-dimensional stabilizer H_C, then $\omega(C)$ lies in either some $\mathcal{H}_J$ or some $\mathcal{H}_{J_1,J_2}$.*

Let $C = (c_{ij})$. First observe, that if $(t_1, \dots, t_m)$ stabilizes C, then for any two i, j with $c_{ij} \neq 0$ we have $t_i = t_j^{-1}$. Next we associate a graph Γ_C to C: the set of vertices is $\{1, \dots, m\}$, and the edges are the (ij) with $c_{ij} \neq 0$. Let $\Gamma_C^1, \dots, \Gamma_C^k$ be the connected components of Γ_C.

If Γ_C^s contains a cycle of odd length (type I), then $t_i^2 = 1$ holds for all vertices i in Γ_C^s. If Γ_C^s contains no cycle of odd length (type II), then one may divide the set of vertices of Γ_C^s into subsets J_1, J_2 with $J_1 \cap J_2 = \emptyset$ such that for any edge (ij) of Γ_C^s we have $i \in J_1, j \in J_2$.

Any connected component of type II gives a free parameter in H_C. Thus, if the stabilizer H_C is one-dimensional, there is exactly one connected component, say Γ_C^1, of type II and all others are of type I. If $\Gamma_C^1 = \Gamma_C$, then we have $\omega(C) \subset \mathcal{H}_{J_1} = \mathcal{H}_{J_2}$, and otherwise we have $\omega(C) \subset \mathcal{H}_{J_1,J_2}$ (any component of type I contains ≥ 3 vertices). This proves Claim 1.

Claim 2. *For any $\mathcal{H}_J$ (resp. $\mathcal{H}_{J_1,J_2}$), there exists a skew-symmetric matrix C of rank ≤ 4 such that $\omega(C)$ is generated by all $u_{ij} \in \mathcal{H}_J$ (resp. $u_{ij} \in \mathcal{H}_{J_1,J_2}$), and $\omega(C)$ generates $\mathcal{H}_J$ (resp. $\mathcal{H}_{J_1,J_2}$).*

First, consider $\mathcal{H}_J$. By renumbering, we may assume that $J = \{1, 2, \dots, k\}$. Then the hyperplane $\mathcal{H}_J$ is generated by $\omega(C(k, m-k))$, where we set

$$C(k,l) := \begin{pmatrix} 0 & \mathbb{1}_{k\times l} \\ -\mathbb{1}_{l\times k} & 0 \end{pmatrix}$$

and $\mathbb{1}_{k\times l}$ denotes the $(k \times l)$-matrix with all entries equal one. One easily sees that all weights u_{ij} lying in the hyperplane $\mathcal{H}_J$ already belong to $\omega(C(k, m-k))$.

Now we turn to $\mathcal{H}_{J_1,J_2}$. Again, by renumbering, we may assume that $J_1 = \{1, \dots, k_1\}$ and $J_2 = \{k_1+1, \dots, k_1+k_2\}$ holds. Set $s := m - k_1 - k_2$, and take pairwise non-proportional vectors $w_1, \dots, w_s$ in some two dimensional symplectic vector space W; then these vectors define a skew symmetric matrix $C(s) = (\langle w_i, w_j\rangle)$ of rank two having only non-zero non-diagonal elements. The hyperplane $\mathcal{H}_{J_1,J_2}$ is generated by $\omega(C(J_1, J_2))$, where

$$C(J_1, J_2) = \begin{pmatrix} C(k_1, k_2) & 0 \\ 0 & C(s) \end{pmatrix}.$$

Again one directly checks that all weights u_{ij} in $\mathcal{H}_{J_1,J_2}$, are already contained in the orbit cone $\omega(C(J_1, J_2))$. This proves Claim 2.

Finally, observe that for $n \geq 2$ the GIT-fan of the action of $G = \mathrm{Sp}(2n)$ on $(\mathbb{P}^{2n-1})^m$ does not depend on n. But for $2n \geq m$, the variety $\overline{Y}$ is a vector space, and hence, the corresponding GIT-fan coincides with $\Sigma(u_{ij};\ 1 \leq i, j \leq m)$. $\square$

Remark 6.3. It is proved in [22, Prop. 17] that for an SL(2)-action on a projective variety any wall is the intersection of the weight cone with a hyperplane. Theorem 6.2 shows that for $G = \mathrm{Sp}(2n)$, $n \geq 2$, this is not the case. Indeed, the intersection $\mathcal{H}_{J_1,J_2} \cap \omega(\overline{X})$ has extremal rays different from any of the u_{ij}; e.g. the ones generated by

$$(\underbrace{0, \dots, 0, 1, 0, \dots, 0}_{|J_1|}, \underbrace{0, \dots, 0, 1, 0, \dots, 0}_{|J_2|}, 0, \dots, 0, 2, 0, \dots, 0).$$

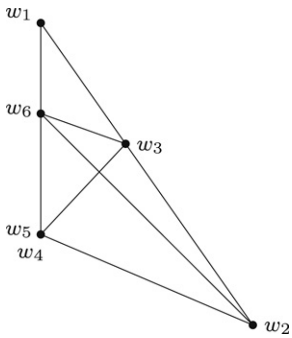
Remark 6.4. In the setting of Theorem 6.2, none of the quotients $X^{ss}(D) \to X^{ss}(D) /\!\!/ G$ is geometric.

A further class of examples arises from (reducible) representations of simple groups having a free algebra of invariants. They are all known and can be found in [1,23]; the multidegrees of basic invariants are also indicated in tables there.

Example 6.5. Consider the action of the special linear group $G := \mathrm{SL}(6)$ on the product $X = \mathbb{P}(\mathbb{K}^6) \times \mathbb{P}(\Lambda^2\mathbb{K}^6) \times \mathbb{P}(\Lambda^3\mathbb{K}^6)$. Then the total coordinate space is $\overline{X} = \mathbb{K}^6 \times \Lambda^2\mathbb{K}^6 \times \Lambda^3\mathbb{K}^6$, and $H = (\mathbb{K}^*)^3$ acts by scalar multiplication on the factors. The weights of the canonical generators of the algebra $\mathbb{K}[\overline{X}]^G$ are listed as well in [23]; they are

$$\begin{array}{lll} w_1 = (0, 0, 4), & w_2 = (0, 3, 0), & w_3 = (0, 3, 4), \\ w_4 = (1, 1, 1), & w_5 = (2, 2, 2), & w_6 = (1, 1, 3). \end{array}$$

As the algebra $\mathbb{K}[\overline{X}]^G$ of invariants is a polynomial ring, the GIT-fan of the H-action on $\overline{Y} = \overline{X} /\!\!/ G$ is $\Sigma(w_1, \dots, w_6)$. Here comes a figure, showing the intersection of the weight cone $\omega(\overline{Y})$ with a transversal hyperplane.



All resulting quotients are toric varieties; the computation of the respective fans is a standard calculation.

7. Gelfand–MacPherson type correspondences

The classical Gelfand–MacPherson correspondence [11] relates generic orbits of the diagonal action of $\mathrm{SL}(n)$ on $(\mathbb{P}^{n-1})^m$ to generic orbits of a torus action on the Grassmannian $G(n, m)$. This may even be extended to isomorphisms between certain quotient spaces on both sides, see [20, Theorem 2.4.7]. Combining Proposition 3.6 and Theorem 4.5, we obtain the following general way to relate quotients for a reductive group action to quotients of a torus action.

Construction 7.1. Let G_X be a connected reductive group, X a G_X-variety with equivariant Cox construction $q_X\colon \widehat{X} \to X = \widehat{X} /\!\!/ H_X$ and total coordinate space $\overline{X}$. Consider the induced T-action on $\overline{Y}$, where

$$T := (H_X \times G_X)/(H_X \times G_X)^s = H_X \times (G_X/G_X^s),$$
$$\overline{Y} := \overline{X} /\!\!/ (H_X \times G_X)^s = \overline{X} /\!\!/ G_X^s.$$

Suppose that for some T-invariant open set $\widehat{Y} \subseteq \overline{Y}$ and some subtorus $H_Y \subseteq T$ we obtain a Cox construction $q_Y\colon \widehat{Y} \to Y = \widehat{Y} /\!\!/ H_Y$, and consider the induced action of $T_Y := T/H_Y$ on Y. Then the good G_X-sets $U \subseteq X$ and the good T_Y-sets $V \subseteq Y$ fit into the diagram

$$\begin{tikzcd}
\widehat{U} \arrow[d, "/\!\!/ H_X"'] & \subseteq & \widehat{X} \arrow[d, "/\!\!/ H_X"] & \subseteq & \overline{X} \arrow[r, "\pi", "/\!\!/ G_X^s"'] & \overline{Y} & \supseteq & \widehat{Y} \arrow[d, "/\!\!/ H_Y"'] & \supseteq & \widehat{V}, \arrow[d, "/\!\!/ H_Y"] \\
U \arrow[d] & \subseteq & X & & & & & Y & \supseteq & V \arrow[d] \\
U /\!\!/ G_X & & & & & & & & & V /\!\!/ T_Y
\end{tikzcd}$$

where we set $\widehat{U} := q_X^{-1}(U)$ and $\widehat{V} := q_Y^{-1}(V)$. Moreover, combining Proposition 3.6 and Theorem 4.5 gives canonical assignments from good G_X-sets $U \subseteq X$ to good T_Y-sets $V \subseteq Y$ and vice versa:

(i) If $U \subseteq X$ is a good G_X-set, then $V := q_Y(\pi(\widehat{U}) \sqcap_{T_Y} \widehat{Y})$ is a good T_Y-set in Y, and there is a canonical open embedding $V /\!\!/ T_Y \to U /\!\!/ G_X$. This embedding is an isomorphism if and only if one has an H_Y-saturated inclusion $\pi(\widehat{U}) \subseteq \widehat{Y}$.

(ii) If $V \subseteq Y$ is a good T_Y-set, then $U := q_X(\pi^{-1}(\widehat{V}) \sqcap_{G_X} \widehat{X})$ is a good G_X-set in X, and there is a canonical open embedding $U /\!\!/ G_X \to V /\!\!/ T_Y$. This embedding is an isomorphism if and only if one has an H_X-saturated inclusion $\pi^{-1}(\widehat{V}) \subseteq \widehat{X}$.

To consider these assignments for sets of semistable points recall first that (1) and (2) provide canonical isomorphisms relating the respective groups of linearized divisor classes

$$\mathrm{Cl}_{G_X}(X) \cong \mathrm{Cl}_{H \times G_X}(\overline{X}) \cong \mathrm{Cl}_T(\overline{Y}) \cong \mathrm{Cl}_{T_Y}(Y).$$

If $U \subseteq X$ is a set of semistable points of a G_X-linearized divisor, then Lemma 3.3 and Theorem 4.6 ensure that the associated set $V \subseteq Y$ is a saturated (possibly proper) subset of the set of semistable points of the corresponding T_Y-linearized divisor and vice versa.

In certain cases, the above Gelfand–MacPherson type correspondence can even be extended to the respective inverse limits of the GIT-quotients, which in turn gives interesting descriptions of moduli spaces, see [27]. For giving a general statement in this context, we now recall the basic facts on inverse limits of GIT-quotients.

Consider a projective variety X with an action of a connected reductive group G. If, for two ample G-linearized divisors D, D', we have an inclusion $X^{ss}(D) \subseteq X^{ss}(D')$, then there is an induced morphism of the associated quotient spaces. These induced morphisms of GIT-quotients form an inverse system, the *ample GIT-system* of the G-variety X. The *GIT-limit* of X is the inverse limit of this system. As in the case of fiber products, the GIT-limit can be realized as a subvariety of the product over all GIT-quotients arising from ample bundles.

In order to compare GIT-limits in our setting, recall that for a projective G-variety X with equivariant Cox construction $\widehat{X} \to X = \widehat{X} /\!\!/ H_X$ its ample cone is the relative interior of a GIT-cone κ_X of the H_X-action on the total coordinate space $\overline{X}$. By the *open G-ample cone*, we mean the relative interior of the cone $(\kappa_X \times M_{\mathbb{Q}}) \cap \omega(\overline{X})$ in $(K \times M)_{\mathbb{Q}}$, where K and M stand for the character lattices of H_X and G, respectively.

Theorem 7.2. *Consider a G_X-variety X and a T_Y-variety Y as in 7.1, and suppose that both are projective. If the canonical isomorphism $\mathrm{Cl}_{G_X}(X) \to \mathrm{Cl}_{T_Y}(Y)$ sends the open G_X-ample cone onto the open T_Y-ample cone, then the GIT-limits of X and Y are isomorphic.*

Proof. First, note that for determining the GIT-limit, it suffices to consider GIT-quotients given by the classes inside the open linearized ample cone. Any class inside the open G_X-ample cone defines an ample class on Y and vice versa. Moreover, since X and Y are projective, all quotients arising from ample classes are projective again. This implies

$$q_X^{-1}(X^{ss}(D, G_X)) = \overline{X}^{ss}(\overline{D}, H_X \times G_X), \qquad q_Y^{-1}(Y^{ss}(D, T_Y)) = \overline{Y}^{ss}(\overline{D}, T),$$

for any ample D. Consequently, the morphism of 7.1(ii) comparing the G_X-quotient with the T_Y-quotient is an isomorphism. Obviously, the family of these comparing morphisms is compatible with the respective GIT-systems, and thus defines an isomorphism of their inverse limits. □

As an immediate consequence we obtain the following rather special looking statements, which however give back the known Gelfand–MacPherson type isomorphisms of GIT-limits.

Corollary 7.3. *Consider a G_X-variety X and a T_Y-variety Y as in 7.1 and both projective. If on X and Y every effective divisor is semiample, then X and Y have isomorphic GIT-limits.*

This applies to the case where X as well as Y are products of projective varieties having free divisor class group of rank one. In particular, it applies to the following setting.

Corollary 7.4. *Suppose that in the setting of 7.1, we have $X = \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_r)$ and $Y = \mathrm{Proj}((\mathbb{K}[V_1] \otimes \cdots \otimes \mathbb{K}[V_r])^{G_X^s})$ with some G_X-modules $V_1, \ldots, V_r$. Then X and Y have isomorphic GIT-limits.*

We conclude the section with a couple of examples. The first one shows that the classical Gelfand–MacPherson correspondence gives rise to an isomorphism of GIT-limits; this was observed by Kapranov [20]. The second one is an analogous statement in the setting of complete collineations; this result is due to Thaddeus [27].

Example 7.5. Consider the product $X = (\mathbb{P}^{n-1})^m$, where $m \geq n$, with the diagonal action of $G_X = \mathrm{SL}(n)$. The total coordinate space and a Cox construction of X are given by

$$\overline{X} = (\mathbb{K}^n)^m, \qquad \widehat{X} = \{(v_1, \ldots, v_m) \in \overline{X};\ v_i \neq 0 \text{ for } 1 \leq i \leq m\}, \qquad H_X = (\mathbb{K}^*)^m.$$

The G_X-action canonically lifts to the total coordinate space, we have $(H_X \times G_X)^s = G_X$, and the algebra of G_X-invariants is generated by the $(n \times n)$-minors of the matrices $(v_1, \ldots, v_m) \in \overline{X}$.

Thus, $\overline{Y} = \overline{X} /\!\!/ G_X$ is the cone over the Grassmannian $Y := G(n, m)$, and the basic G_X-invariants give Plücker coordinates on $\overline{Y}$. Moreover, we obtain a Cox construction $\widehat{Y} \to Y = \widehat{Y} /\!\!/ H_Y$ with a one-dimensional torus $H_Y \subseteq T = H_X$ and $\widehat{Y} := \overline{Y} \setminus \{\pi(0)\}$, where $\pi : \overline{X} \to \overline{Y}$ is the quotient morphism.

The situation fits into Corollary 7.4 and thus we obtain that the actions of G_X on X and T/H_Y on Y have isomorphic GIT-limits.

Example 7.6. For finite dimensional vector spaces U, V, W, consider the action of $G_X := \mathrm{SL}(U)$ on the product

$$X := \mathbb{P}(\mathrm{Hom}(U, V)) \times \mathbb{P}(\mathrm{Hom}(U, W)).$$

This action lifts canonically to the total coordinate space $\overline{X} = \mathrm{Hom}(U, V) \times \mathrm{Hom}(U, W)$, and $\overline{Y}$ is the cone over the Grassmannian $G(n, V \oplus W)$, where $n = \dim(U)$, acted on by the two dimensional torus $T = H_X$.

Take $H_Y \subseteq H_X$ such that $Y = \widehat{Y} /\!\!/ H_Y$ is the Grassmannian $G(n, V \oplus W)$. Then the action of $T_Y \cong \mathbb{K}^*$ on Y comes from letting act $\mathbb{K}^*$ with weight 1 on V and weight -1 on W. By Corollary 7.4, the action of G_X on X and T_Y on Y have isomorphic GIT-limits.

8. Geometry of (many) quotient spaces

Let a connected reductive group G act on a normal variety X with finitely generated Cox ring. In this section, we show that the description of good G-sets $U \subseteq X$ in terms of orbit cones also opens an approach to study the geometry of the quotient spaces $U /\!\!/ G$; it turns out that in many cases the language of bunched rings developed in [3] can be applied.

First, we recall the basic concepts of [3]. Consider a factorial, finitely generated $\mathbb{K}$-algebra R, graded by some lattice $K \cong \mathbb{Z}^k$ such that $R^* = \mathbb{K}^*$ holds. The latter condition enables us to fix a system $\mathfrak{F} = \{f_1, \dots, f_r\} \subset R$ of homogeneous pairwise non associated nonzero prime generators for R.

The *projected cone* $(E \xrightarrow{Q} K, \gamma)$ *associated* to the system of generators $\mathfrak{F} \subset R$ consists of the surjection Q of the lattices $E := \mathbb{Z}^r$ and K sending the i-th canonical base vector $e_i \in \mathbb{Z}$ to the degree $\deg(f_i) \in K$, and the cone $\gamma \subset E_{\mathbb{Q}}$ generated by $e_1, \dots, e_r$.

(i) We call $\mathfrak{F} \subset R$ is *admissible*, if, for each facet $\gamma_0 \preceq \gamma$, the image $Q(\gamma_0 \cap E)$ generates the lattice K.
(ii) A face $\gamma_0 \preceq \gamma$ is called an $\mathfrak{F}$*-face* if the product over all f_i with $e_i \in \gamma_0$ does not belong to the ideal $\sqrt{\langle f_j;\, e_j \not\in \gamma_0 \rangle} \subset R$.
(iii) An $\mathfrak{F}$*-bunch* is a nonempty collection Φ of projected $\mathfrak{F}$-faces with the following properties:
- a projected $\mathfrak{F}$-face τ belongs to Φ if and only if for each $\tau \neq \sigma \in \Phi$ we have $\emptyset \neq \tau^\circ \cap \sigma^\circ \neq \sigma^\circ$,
- for each facet $\gamma_0 \prec \gamma$, there is a cone $\tau \in \Phi$ such that $\tau^\circ \subseteq Q(\gamma_0)^\circ$ holds.

Given an $\mathfrak{F}$-bunch Φ in the projected cone $(E \xrightarrow{Q} K, \gamma)$ associated to an admissible system of generators $\mathfrak{F} \subset R$ as above, we call the triple $(R, \mathfrak{F}, \Phi)$ a *bunched ring*.

Given a bunched ring $(R, \mathfrak{F}, \Phi)$ with corresponding projected cone $(E \xrightarrow{Q} K, \gamma)$, consider the affine variety $Z := \mathrm{Spec}(R)$, the torus $H := \mathrm{Spec}(\mathbb{K}[K])$, and the action $H \times Z \to Z$ given by the K-grading of R. Then the projected $\mathfrak{F}$-faces are precisely the orbit cones of the H-action on Z, and there is a canonical injection

$$\{\mathfrak{F}\text{-bunches}\} \to \{2\text{-maximal collections in } \Omega(Z)\}$$
$$\Phi \mapsto \Psi(\Phi) := \{\omega(z);\, z \in Z,\, \tau^\circ \subset \omega(z)^\circ \text{ for some } \tau \in \Phi\}.$$

Using this observation, we may associate to the bunched ring $(R, \mathfrak{F}, \Phi)$ a variety by setting

$$X(R, \mathfrak{F}, \Phi) = U(\Psi(\Phi)) /\!\!/ H.$$

The main object of [3] is to read off geometric properties of this variety from its defining data. Let us briefly provide the necessary notions. Call an $\mathfrak{F}$-face $\gamma_0 \preceq \gamma$ *relevant* if $Q(\gamma_0)^\circ \supset \tau^\circ$ holds for some $\tau \in \Phi$, and denote by $\mathrm{rlv}(\Phi)$ the collection of relevant $\mathfrak{F}$-faces. The *covering collection* of Φ is the collection $\mathrm{cov}(\Phi) \subset \mathrm{rlv}(\Phi)$ of set-theoretically minimal members of $\mathrm{rlv}(\Phi)$.

Theorem 8.1. *Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring with corresponding projected cone $(E \xrightarrow{Q} K, \gamma)$, and let $X := X(R, \mathfrak{F}, \Phi)$ be the associated variety.*

(i) *The variety X is locally factorial if and only if $Q(\gamma_0 \cap E)$ generates the lattice K for every $\gamma_0 \in \mathrm{rlv}(\Phi)$.*
(ii) *The variety X is $\mathbb{Q}$-factorial if and only if any cone of Φ is of full dimension in $K_{\mathbb{Q}}$.*
(iii) *The dimension of X is $\dim(R) - \dim(K_{\mathbb{Q}})$, its divisor class group is $\mathrm{Cl}(X) \cong K$, and the Picard group of X sits in $\mathrm{Cl}(X)$ as*

$$\mathrm{Pic}(X) = \bigcap_{\gamma_0 \in \mathrm{cov}(\Phi)} Q(\mathrm{lin}(\gamma_0) \cap E).$$

(iv) *The effective cone, the moving cone, and the cones of semiample and ample divisor classes of X are given by*

$$\mathrm{Eff}(X) = Q(\gamma), \qquad \mathrm{Mov}(X) = \bigcap_{\gamma_0 \text{ facet of } \gamma} Q(\gamma_0),$$
$$\mathrm{SAmple}(X) = \bigcap_{\tau \in \Phi} \tau, \qquad \mathrm{Ample}(X) = \bigcap_{\tau \in \Phi} \tau^\circ.$$

(v) *Suppose that for $d := r - \dim(R)$, we have K-homogeneous generators $g_1, \dots, g_d$ for the relations between the members $f_1, \dots, f_r$ of $\mathfrak{F}$. Then the canonical divisor class of X is*

$$\sum \deg(g_j) - \sum \deg(f_i).$$

Now consider a factorial affine algebra R graded by a lattice K, let $\mathfrak{F} \subset R$ be an admissible system of generators, and let $(E \xrightarrow{Q} K, \gamma)$ be the associated projected cone. Consider the affine variety $Z := \mathrm{Spec}(R)$ and the action of the torus $H := \mathrm{Spec}(\mathbb{K}[K])$ on it given by the K-grading of R. Define a subset of the weight cone $\omega(Z)$ by

$$\omega(R)^\odot := \omega(Z)^\odot := \bigcap_{\gamma_0 \preceq \gamma} Q(\gamma_0)^0,$$

where γ_0 runs through all facets of γ. With any $\chi \in \omega(Z)$ associate the collection $\Psi(\chi)$ of projected cones τ with $\chi \in \tau^0$ and the collection $\Phi(\chi)$ of set-theoretically minimal elements of $\Psi(\chi)$.

Proposition 8.2. *Suppose that for each facet $\gamma_0 \preceq \gamma$, the image $Q(\gamma_0 \cap E)$ generates the lattice K and that $R_0 = \mathbb{K}$ holds. Then for any $\chi \in \omega(Z)^{\odot}$ the triple $(R, \mathfrak{F}, \Phi(\chi))$ is the bunched ring representing the quotient space $Z^{ss}(\chi)/\!\!/H$.*

Proof. The condition $R_0 = \mathbb{K}$ implies that $Z^{ss}(\chi)/\!\!/H$ is projective and thus the collection $\Psi(\chi)$ is 2-maximal. This shows that $\Phi(\chi)$ satisfies the first condition in the definition of an $\mathfrak{F}$-bunch. Note also that for any facet $\gamma_0 \preceq \gamma$ the cone $Q(\gamma_0)$ is an orbit cone. Indeed, since $\mathfrak{F}$ is a system of pairwise non associated nonzero prime generators of R, the condition $f_i(z) = 0$ does not imply $f_j(z) = 0$ for any $j \neq i$. Since $Q(\gamma_0) \in \Psi(\chi)$, there is an element $\tau \in \Phi(\chi)$ with $\tau^0 \subseteq Q(\gamma_0)^0$. We have checked that $\Phi(\chi)$ is an $\mathfrak{F}$-bunch. The other statements follow from the definition of variety associated with a bunched ring. $\square$

Corollary 8.3. *For any $\chi \in \omega(Z)^{\odot}$ the cone $\omega(Z)$ is the cone of effective divisors and the GIT-cone $\lambda(\chi)$ is the cone of semiample divisors of the variety $Z^{ss}(\chi)/\!\!/H$.*

Now, let a connected reductive group G act on a normal projective variety X. Suppose that there is a G-equivariant Cox construction $\widehat{X} \to X = \widehat{X}/\!\!/H_X$ with total coordinate space $\overline{X}$. Then the invariant Cox ring $R := \mathcal{R}(X)^{G^s}$ comes with a grading by the character group K of $H := H_X \times G/G^s$ corresponding to the induced H-action on $\overline{X}/\!\!/G^s$, and Proposition 8.2 gives the following.

Corollary 8.4. *Let $\mathfrak{F}$ be an admissible system of generators for $R = \mathcal{R}(X)^{G^s}$. Then for any G-linearized ample divisor D on X defining an element $\chi \in \omega(R)^{\odot}$, the associated quotient space $X^{ss}/\!\!/G$ arises from the bunched ring $(R, \mathfrak{F}, \Phi(\chi))$.*

In a first example, we consider once more the diagonal action of the special linear group $\mathrm{SL}(n)$ on a product of projective spaces $\mathbb{P}^{n-1}$. It has quite a big variation of GIT-quotients, but there is one "canonical" candidate, namely the (unique) set of semistable points, which is invariant under permuting the factors $\mathbb{P}^{n-1}$. The case $n = 2$, is studied by several authors, see [6] for a uniqueness result and [18] for an approach to the geometry. We will see, also for higher n, that the quotient fits into the setting of bunched rings.

Example 8.5. Consider the diagonal action of $G = \mathrm{SL}(n)$ on $X = (\mathbb{P}^{n-1})^m$, where $m \geq n+2$. We identify $\mathbb{Z}^m \to \mathrm{Cl}(X)$ by sending e_i to the class of D_i of the pullback of a hyperplane in the i-th factor $\mathbb{P}^{n-1}$. Then the action of $H = (\mathbb{K}^*)^m$ on the total coordinate space $\overline{X} = (\mathbb{K}^n)^m$ is componentwise scalar multiplication.

The quotient $\overline{Y} = \overline{X}/\!\!/G$ is the cone over the Grassmannian $G(n, m)$, and $R := \mathbb{K}[\overline{Y}]$ is generated by the $(n \times n)$-minors of the matrices $(v_1, \ldots, v_m) \in \overline{X}$. The weights of these generators with respect to the H-action are the $w \in \mathbb{Z}^m$ having exactly n entries 1 and the others 0. Consequently, the weight cone is

$$\omega(\overline{X}) = \omega(\overline{Y}) = \left\{(s_1, \ldots, s_m) \in \mathbb{Q}^m_{\geq 0};\ s_1 + \cdots + s_m \geq n s_j,\ 1 \leq j \leq m\right\}.$$

The associated GIT-fan is known, see [10, 3.3.24]. It is most conveniently described by giving the walls; these are the intersections of $\omega(\overline{X})$ with the hyperplanes

$$\mathcal{H}_{k,J} := \left\{(s_1, \ldots, s_m);\ (n-k)\sum_{j \in J} s_j = k \sum_{l \notin J} s_l\right\}, \quad 1 \leq k \leq \frac{n}{2}, J \subset \{1, \ldots, m\}, k < |J| < m + k - n.$$

The $(n \times n)$-minors mentioned before form a system of pairwise nonassociated prime generators for R, and one easily checks that we are in the situation of Proposition 8.2. Thus, in order to figure out the GIT-quotients, for which we get a describing bunched ring for free, we need the cone $\omega(\overline{X})^{\odot}$; it is given as

$$\left\{(s_1, \ldots, s_m) \in \mathbb{Q}^m_{\geq 0};\ \sum_{j \in J} s_j < (n-1)\sum_{l \notin J} s_l, J \subset \{1, \ldots, m\}, |J| = n\right\}.$$

There is a unique set $U \subseteq X$ of semistable points, which is invariant under permutation of the factors $\mathbb{P}^{n-1}$ of X; it is defined by the divisor $D := D_1 + \cdots + D_m$. In our picture, the class of D is the point $\chi := (1, \ldots, 1)$. The corresponding GIT-cone $\lambda(\chi)$ is defined by inequalities

$$(n-k)\sum_{j \in J} s_j \leq k \sum_{l \notin J} s_l, \quad 1 \leq k \leq \frac{n}{2}, J \subset \{1, \ldots, m\}, k \leq |J| \leq \frac{km}{n},$$

$$k \sum_{l \notin J} s_l \leq (n-k)\sum_{j \in J} s_j, \quad 1 \leq k \leq \frac{n}{2}, J \subset \{1, \ldots, m\}, \frac{km}{n} \leq |J| \leq m + k - n.$$

Note that here only the cases $km/n - 1 < |J| \leq km/n + 1$ are essential. In particular, we see that χ belongs to $\omega(\overline{X})^{\odot}$ provided $m \geq 5$ and $n = 2$, or $m \geq n+2$ and $n \geq 3$. From Theorem 8.1 we infer that $\lambda(\chi)$ is the semiample cone of the quotient space $U/\!\!/G$. Let us have a closer look at it.

Case 1. We have $\mathrm{GCD}(m, n) = d > 1$. Then χ lies in all walls corresponding to $k = n/d$ and $|J| = m/d$. It follows that the GIT-cone $\lambda(\chi)$ is a ray. In particular, the $U /\!\!/ G$ comes with non-$\mathbb{Q}$-factorial singularities, and its Picard group is of rank one.

Case 2. The numbers m and n are coprime. Then χ is not contained in any wall, so $\lambda(\chi)$ has full dimension. In the cases $n = 2, 3$ we obtain the following describing inequalities (S_m stands for the symmetric group).

- For $n = 2$ and $m = 2r + 1$, the cone $\lambda(\chi)$ is given by the inequalities
$$s_{\pi(1)} + \cdots + s_{\pi(r)} \le s_{\pi(r+1)} + \cdots + s_{\pi(m)}, \quad \pi \in S_m.$$
- For $n = 3$ and $m = 3r + s$, $s = 1, 2$, the cone $\lambda(\chi)$ is given by
$$2(s_{\pi(1)} + \cdots + s_{\pi(r)}) \le s_{\pi(r+1)} + \cdots + s_{\pi(m)}, \qquad \pi \in S_m,$$
$$2(s_{\pi(1)} + \cdots + s_{\pi(r+1)}) \ge s_{\pi(r+2)} + \cdots + s_{\pi(m)}, \quad \pi \in S_m.$$

The number of extremal rays of the semiample cone is certainly an invariant for any variety; here we obtain, for example, that for $n = 3$ and $m = 5$ the quotient $U /\!\!/ G$ is a smooth projective surface having a semiample cone with 10 extremal rays.

Also for the symplectic group action on a product of projective spaces studied in Theorem 6.2, we have a unique set of semistable points being invariant under permuting the factors. Here comes more information.

Example 8.6. Consider the diagonal action of the symplectic group $\mathrm{Sp}(2n)$ on $(\mathbb{P}^{2n-1})^m$ as in Theorem 6.2. Then the cone $\omega(\overline{Y})^{\odot} \subseteq \mathbb{Q}^m$ is given by the additional inequalities
$$s_{\pi(1)} + s_{\pi(2)} \le s_{\pi(3)} + \cdots + s_{\pi(m)}, \qquad \pi \in S_m.$$
In particular, the point $\chi = (1, \dots, 1)$ is contained in $\omega(Z)^{\odot}$ for $m \ge 5$. Moreover, this point belongs to the walls K_{J_1,J_2} with $|J_1| = |J_2|$. Taking $|J_1| = |J_2| = 1$, one gets that the GIT-cone $\lambda(\chi)$ is one-dimensional. So the resulting quotient space has non-$\mathbb{Q}$-factorial singularities and its Picard group is of rank one.

The next example belongs to a large class, arising from reducible G-representations whose algebra of invariants has a single relation. A complete classification for $G = \mathrm{SL}(n)$ is given in [25]. There also the weights of generators and of the relation are listed. However, the relation itself is sometimes not easy to write down explicitly. This may cause difficulties in determining orbit cones. The following observation helps.

Lemma 8.7. *Let a torus T act diagonally on $\mathbb{K}^r$ with weights $(w_1, \dots, w_r)$, and consider a T-invariant hypersurface $Z := V(\mathbb{K}^r; f)$ with a polynomial f of the form*
$$f = T_r^k + g, \ \text{where } g \in \mathbb{K}[T_1, \dots, T_{r-1}].$$
Then the orbit cones of the T-action on Z are precisely the cones $\mathrm{cone}(w_j; j \in J)$ *with $J \subseteq \{1, \dots, r-1\}$, and the GIT-fan of the T-action on Z is $\Sigma(w_1, \dots, w_{r-1})$.*

Proof. First, we show that any orbit cone $\omega(z)$, $z \in Z$, is of the form $\mathrm{cone}(w_j; j \in J)$ with a subset $J \subseteq \{1, \dots, r-1\}$. If $z_r = 0$ holds, then $\omega(z)$ is necessarily generated by weights from $\{w_1, \dots, w_{r-1}\}$. For $z_r \neq 0$, we have $g(z_1, \dots, z_{r-1}) \neq 0$. Thus, some monomial $g_0 = T_{i_1}^{\nu_{i_1}} \cdots T_{i_l}^{\nu_{i_l}}$ occurring in g satisfies $g_0(z) \neq 0$. This implies
$$kw_r = \deg(g) \in \mathrm{cone}(w_{i_1}, \dots, w_{i_l}) \subseteq \omega(z).$$
Consequently, we see that w_r is not needed as a generator of the orbit cone $\omega(z)$, and we are done.

Conversely, let $\omega = \mathrm{cone}(w_j; j \in J)$ with a subset $J \subseteq \{1, \dots, r-1\}$ be given. Then we need a point $z \in Z$ with $\omega(z) = \omega$. For $1 \le j \le r-1$, we set
$$z_j := \begin{cases} 1 & \text{for } j \in J, \\ 0 & \text{for } j \notin J. \end{cases}$$
Next take $z_r \in \mathbb{K}$ with $z_r^k = -g(z_1, \dots, z_{r-1})$. Then $z := (z_1, \dots, z_r)$ belongs to Z because of $f(z) = 0$, and we directly see $\omega \subseteq \omega(z)$.

For $z_r = 0$, also the inclusion $\omega(z) \subseteq \omega$ is obvious. If $z_r \neq 0$ holds, then we have $g(z_1, \dots, z_{r-1}) \neq 0$, and a similar reasoning as before gives $w_r \in \omega$. Moreover, if h is any semiinvariant with $h(z) \neq 0$, then some monomial $h_0 = T_{i_1}^{\nu_{i_1}} \cdots T_{i_l}^{\nu_{i_l}}$ satisfies $h_0(z) \neq 0$. The latter condition implies $i_1, \dots, i_l \in J \cup \{r\}$, which in turn gives $\deg(h) \in \omega$. $\square$

Example 8.8. Let $G := \mathrm{SL}(4)$ and consider the irreducible representations $G \to \mathrm{GL}(V)$ and $G \to \mathrm{GL}(W)$ with the respective highest weights ω_2 and ω_2^2, where ω_2 denotes the second fundamental weight, i.e., we have $V := \bigwedge^2 \mathbb{K}^4$ and W is a 20-dimensional subspace of S^2V. Then we have an induced G-action on
$$X := \mathbb{P}(V) \times \mathbb{P}(W).$$

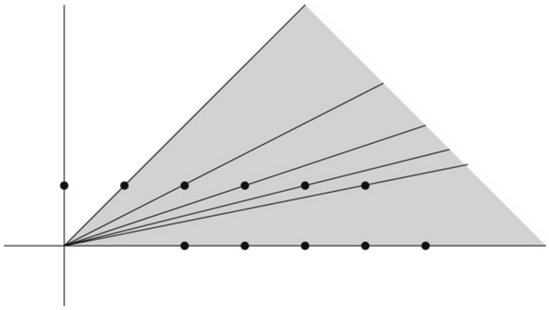
By construction, $\overline{X} = V \times W$ is the equivariant total coordinate space. The Neron-Severi torus $H = (\mathbb{K}^*)^2$ acts by componentwise scalar multiplication on $\overline{X}$. According to [25, Table 4], the algebra of invariants is of the form

$$R := \mathbb{K}[\overline{X}]^G = \mathbb{K}[T_1, \ldots, T_{12}]/\langle f \rangle, \qquad f = T_{12}^2 + g(T_1, \ldots, T_{11}).$$

The classes $f_i \in R$ of the T_i form an admissible system $\mathfrak{F} = (f_1, \ldots, f_{12})$ of pairwise nonassociated prime homogeneous generators. Their degrees $w_1 \ldots, w_{12}$ were also calculated in [25]; in $\mathbb{Z}^2$ they can be given as

$$w_i = (2+i, 0), \quad i = 0, \ldots, 4, \qquad w_{6+j} = (j, 1), \quad j = 0, \ldots, 5, \qquad w_{12} = (15, 3).$$

According to Lemma 8.7, the GIT-fan of the $(H \times G)$-action on $\overline{X}$ is $\Sigma(w_1, \ldots, w_{11})$. The situation is sketched in the following figure, where the bullets stand for the weights, and the shadowed area indicates the cone $\omega(\overline{X})^{\odot}$.



Thus, we have five $\mathbb{Q}$-factorial and four non-$\mathbb{Q}$-factorial quotient spaces, to which Theorem 8.1 directly can be applied. Note that one of them is $\mathbb{Q}$-Fano (but not Fano), and each of them comes with singularities.

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