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Homogeneous Toric Varieties

Ivan Arzhantsev* and Sergey Gaifullin†

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Abstract. A description of transitive actions of a semisimple algebraic group G on toric varieties is obtained. Every toric variety admitting such an action lies between a product of punctured affine spaces and a product of projective spaces. The result is based on the Cox realization of a toric variety as a quotient space of an open subset of a vector space V by a quasitorus action and on investigation of the G-module structure of V.
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1. Introduction

We study toric varieties X equipped with a transitive action of a connected semisimple algebraic group G. In this case X is called a *homogeneous toric variety.* The ground field $\mathbb{K}$ is algebraically closed and of characteristic zero.

Consider a quasiaffine variety

$$\mathcal{X} = \mathcal{X}(n_1, \dots, n_m) := (\mathbb{K}^{n_1} \setminus \{0\}) \times \cdots \times (\mathbb{K}^{n_m} \setminus \{0\})$$

with $n_i \geq 2$. The group $G = G_1 \times \ldots \times G_m$, where every component G_i is either $\mathrm{SL}(n_i)$ or $\mathrm{Sp}(n_i)$, and n_i is even in the second case, acts on $\mathcal{X}$ transitively and effectively. Let $\mathbb{S} = (\mathbb{K}^\times)^m$ be an algebraic torus acting on $\mathcal{X}$ by component-wise scalar multiplication, and

$$p : \mathcal{X} \to \mathcal{Y} := \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_m-1}$$

be the quotient morphism. Fix a closed subgroup $S \subseteq \mathbb{S}$. The action of the group S on $\mathcal{X}$ admits a geometric quotient $p_X \colon \mathcal{X} \to X := \mathcal{X}/S$. The variety X is toric, it carries the induced action of the quotient group $\mathbb{S}/S$, and there is a quotient morphism $p^X \colon X \to \mathcal{Y}$ for this action closing the commutative diagram

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$$\begin{tikzcd} \mathcal{X} \arrow[rr, "p_X"] \arrow[dr, "p"'] & & X \arrow[dl, "p^X"] \\ & \mathcal{Y} & \end{tikzcd}$$

The induced action of the group G on X is transitive and locally effective. We say that the G-variety X is obtained from $\mathcal{X}$ by *central factorization*. The following theorem gives a classification of transitive actions of semisimple groups on toric varieties up to a twist by a diagram automorphism of the acting group.

Theorem 1.1. *Let X be a toric variety with a transitive locally effective action of a connected simply connected semisimple algebraic group G. Then $G = G_1 \times \ldots \times G_m$, where every simple component G_i is either $\mathrm{SL}(n_i)$ or $\mathrm{Sp}(n_i)$, and the variety X is obtained from $\mathcal{X} = \mathcal{X}(n_1, \ldots, n_m)$ by central factorization. Conversely, any variety obtained from $\mathcal{X}$ by central factorization is a homogeneous toric variety.*

Theorem 1.1 also describes homogeneous spaces of a semisimple group that have a toric structure. It is natural to apply the Cox realization of a variety in order to search for toric varieties in a given class of varieties. This idea is already used in [8], where toric affine SL(2)-embeddings are characterized.

In Section 2 we recall basic facts on the Cox realization and its generalization. Criterions of existence of an open G-orbit on X in terms of G- and $(G \times S)$-actions on the total coordinate space Z are also given there. In Section 3 we prove Theorem 1.1. The next section is devoted to special classes of toric homogeneous varieties and to a characterization of their fans. In the last section we consider transitive actions of reductive groups on toric varieties.

Our results are closely connected with the results of E.B. Vinberg [17], where algebraic transformation groups of maximal rank were classified. Recall that an *algebraic transformation group of maximal rank* is an effective generically transitive (i.e., with an open orbit) action of an algebraic group $\mathcal{G}$ on an algebraic variety X such that $\dim X = \mathrm{rk}\,\mathcal{G}$, where $\mathrm{rk}\,\mathcal{G}$ is the rank of a maximal torus T of the group $\mathcal{G}$. In this situation the induced action of the torus T on X is effective and generically transitive, see [6]. If the group $\mathcal{G}$ is semisimple, then an open $\mathcal{G}$-orbit on X is a homogeneous toric variety. It turns out that in this case X is a product of projective spaces and $\mathcal{G}$ acts on X transitively. Theorem 1.1 implies that every homogeneous toric variety determines a reductive transformation group of maximal rank; here $\mathcal{G}$ is the quotient group $(\mathrm{GL}(n_1) \times \ldots \times \mathrm{GL}(n_m))/S$.

Finally, let us mention a related result from toric topology. A *torus manifold* is a smooth real even-dimensional manifold M^{2n} with an effective action of a compact torus $(S^1)^n$ such that the set of $(S^1)^n$-fixed points is nonempty. In [15], homogeneous torus manifolds are studied. The latter are torus manifolds M^{2n} with a transitive action of a compact Lie group K such that the induced action of a maximal torus of K coincides with the given $(S^1)^n$-action. It is proved that every homogeneous torus manifold may be realized as

$$M = \mathbb{CP}^{n_1} \times \ldots \times \mathbb{CP}^{n_k} \times (S^{2m_1} \times \ldots \times S^{2m_l})/F,$$

where S^{2m} is a sphere of dimension $2m$, F is a subgroup of $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ (l copies), and each copy of $\mathbb{Z}_2$ acts on the corresponding sphere by central symmetry. A compact Lie group

$$K = \mathrm{PSU}(n_1+1) \times \ldots \times \mathrm{PSU}(n_k+1) \times \mathrm{SO}(2m_1+1) \times \ldots \times \mathrm{SO}(2m_l+1)$$

acts on M transitively. Moreover, the manifold M is orientable if and only if $F \subset \mathrm{SO}(2m_1 + 2m_2 + \ldots + 2m_l + l)$.

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2. The Cox construction

A *toric variety* is a normal algebraic variety with an effective generically transitive action of an algebraic torus T. A toric variety X is *non-degenerate* if any invertible regular function on X is constant.

Let $\mathrm{Cl}(X)$ be the divisor class group of the variety X. It is well-known that the group $\mathrm{Cl}(X)$ of a toric variety X is finitely generated, see [7, Section 3.4]. Recall that a *quasitorus* is an affine algebraic group S isomorphic to a direct product of an algebraic torus S^0 and a finite abelian group Γ. Every closed subgroup of a torus is a quasitorus. The group of characters of a quasitorus S is a finitely generated abelian group. The *Neron-Severi quasitorus* of a toric variety X is a quasitorus S whose group of characters is identified with $\mathrm{Cl}(X)$.

We come to a canonical quotient realization of a non-degenerate toric variety X obtained in [5]. Let d be the number of prime T-invariant Weil divisors on X. Consider the vector space $\mathbb{K}^d$ and the torus $\mathbb{T} = (\mathbb{K}^\times)^d$ of all invertible diagonal matrices acting on $\mathbb{K}^d$. Then there are a closed embedding of the Neron-Severi quasitorus S into $\mathbb{T}$ and an open subset $U \subseteq \mathbb{K}^d$ such that

- the complement $\mathbb{K}^d \setminus U$ is a union of some coordinate subspaces of dimension $\leq d-2$;
- there exist a categorical quotient $p_X \colon U \to U/\!/S$ and an isomorphism $\varphi \colon X \to U/\!/S$;
- via isomorphism φ, the T-action on X corresponds to the action of the quotient group $\mathbb{T}/S$ on $U/\!/S$.

Later this realization was generalized to a wider class of normal algebraic varieties, see [11], [4], [10]. One of the conditions that determines this class is finite generation of the divisor class group $\mathrm{Cl}(X)$. This allows to define the Neron-Severi quasitorus S of the variety X. The space $\mathbb{K}^d$ is replaced by an affine factorial (or, more generally, factorially graded, see [1]) S-variety Z. It is called the *total coordinate space* of the variety X. Further, X appears as the quotient space of the categorical quotient $p_X \colon U \to U/\!/S$, where U is an open S-invariant subset of Z such that the complement $Z \setminus U$ is of codimension at least two in Z. The morphism $p_X \colon U \to X \cong U/\!/S$ is called the *universal torsor* over X.

Let a connected affine algebraic group G act on a normal variety X. Passing to a finite covering we may assume that $\mathrm{Cl}(G) = 0$ [12, Proposition 4.6]. Then the action of G on X can be lifted to an action of G on the total coordinate space Z that commutes with the S-action, see [3, Section 4]. It turns out that the set U is $(G \times S)$-invariant and $p_X : U \to X$ is a G-equivariant morphism.

Lemma 2.1. *The following conditions are equivalent.*

(i) *The action of the group G on X is generically transitive.*

(ii) *The action of the group $G \times S$ on Z is generically transitive.*

Proof. Let $X_0 \subseteq X$ be an open G-orbit. Each point $x \in X_0$ is smooth on X, and thus the fiber $p_X^{-1}(x)$ is isomorphic to the quasitorus S [10, Proposition 2.2, (iii)]. It shows that the group $G \times S$ acts on $p_X^{-1}(X_0)$ transitively.

Conversely, if $Z_0 \subseteq Z$ is an open $(G \times S)$-orbit, then $Z_0 \subseteq U$ and the action of G on the quotient space $U /\!\!/ S$ is generically transitive. ■

Assume that the group G has trivial group of characters. Then the lifting of the action of the group G to Z is unique, compare [3, Remark 4.1] and [9, Proposition 1.8]. Let H be a closed subgroup of G. Every invertible regular function on the homogeneous space G/H is constant, see [13, Proposition 1.2].

Proposition 2.2. *The following conditions are equivalent.*

(i) *The action of the group G on X is generically transitive and the complement of an open G-orbit has codimension at least two in X.*

(ii) *The action of the group G on the total coordinate space Z is generically transitive.*

(iii) *The action of the group G on the total coordinate space Z is generically transitive and the complement of an open G-orbit has codimension at least two in Z.*

Proof. We check "(i) $\Rightarrow$ (iii)". Let $X_0 \subseteq X$ be an open G-orbit. The condition $\operatorname{codim}_X(X \setminus X_0) \geq 2$ implies that $p_X : p_X^{-1}(X_0) \to X_0$ is the universal torsor over X_0 and that the complement to $p_X^{-1}(X_0)$ in Z does not contain divisors, see [2, Section 2]. By [2, Lemma 3.14] (see also [1, Theorem 4.1]), the universal torsor over a homogeneous space G/H is the projection $G/H_1 \to G/H$, where H_1 is the intersection of kernels of all characters of the subgroup H. This shows that the group G acts on $p_X^{-1}(X_0)$ transitively.

In order to obtain "(iii) $\Rightarrow$ (i)" note that $p_X(Z_0)$, where Z_0 is the open G-orbit in Z, is an open G-orbit in X whose complement does not contain divisors. The implication "(iii) $\Rightarrow$ (ii)" is obvious.

To verify "(ii) $\Rightarrow$ (iii)" let $Z_0 \subseteq Z$ be an open G-orbit. Since the subset Z_0 is S-invariant, for every prime divisor $D \subset Z$ in the complement to Z_0 the set $S \cdot D$ is an S-invariant Weil divisor. Each S-invariant Weil divisor on Z is a principal divisor $\operatorname{div}(f)$ of a regular function $f \in \mathbb{K}[Z]$, see [10, Proposition 2.2, (iv)]. Then the non-constant function f is invertible on Z_0, a contradiction. ■

The same arguments lead to the following result.

Proposition 2.3. *The action of the group G on X is transitive if and only if the open subset $U \subseteq Z$ is a G-orbit.*

3. Classification of homogeneous toric varieties

In this section we prove Theorem 1.1. Since the variety X is toric, its total coordinate space Z is an affine space.

Lemma 3.1. *Let a semisimple group G act on a toric variety X with an open orbit. Then X is non-degenerate and the action of the group $G \times S$ on the affine space Z is equivalent to a linear one.*

Proof. Since any invertible function of the open G-orbit is constant, the variety X is non-degenerate. By Lemma 2.1, the action of the group $G \times S$ on the space Z is generically transitive, and the second statement follows from [14, Proposition 5.1]. ∎

Later on we assume that $G = G_1 \times \ldots \times G_m$ acts on X transitively. Denote by V the total coordinate space Z of the variety X regarded as the $(G \times S)$-module. We proceed with a description of the G-module structure on V.

Proposition 3.2. *Let $V = V_1 \oplus \ldots \oplus V_s$ be a decomposition into irreducible summands. Then every simple component G_i acts not identically only on one summand V_i (up to renumbering), and thus $m = s$. Moreover, every G_i acts on the set of nonzero vectors in V_i transitively.*

Proof. By Proposition 2.3, the complement of the open G-orbit U in V is a union of coordinate subspaces (in some, possibly nonlinear, coordinate system). Thus each irreducible component of the complement is a smooth variety. The linear action of the group G on V commutes with the group $\mathbb{K}^\times$ of scalar operators, and the open orbit U as well as any component of the complement $V \setminus U$ is $(G \times \mathbb{K}^\times)$-invariant. But a cone is a smooth variety if and only if it is a subspace. This shows that each component of $V \setminus U$ is a maximal proper submodule of V. In particular, the number of maximal proper submodules is finite and thus the G-modules $V_1, \ldots, V_s$ are pairwise non-isomorphic. The orbit U is the set of vectors $v \in V$ whose projection on each V_i is nonzero. This implies that the group G acts on the set of nonzero vectors of each submodule V_i transitively.

If several components of G act on some V_i not identically, then V_i is isomorphic to the tensor product of simple modules of these components. Then the cone of decomposable tensors in V_i is G-invariant, a contradiction.

Suppose that a simple component G_l acts on both V_i and V_j not identically. Then G_l acts transitively on the set of pairs (v_i, v_j) with nonzero v_i and v_j. In particular, any such pair is an eigenvector of a Borel subgroup of G_l. Fix a Borel subgroup $B \subset G_l$ and a highest vector for B in V_i as v_i and a lowest vector for B in V_j as v_j. Since the intersection of two opposite parabolic subgroups of G_l

does not contain a Borel subgroup, we get a contradiction. ∎

The following lemma is well known. We give a short self-contained proof suggested by the referee.

Lemma 3.3. *Finite-dimensional rational modules of a simple group G such that G acts on the set of nonzero vectors transitively are*

1. *the tautological* $\mathrm{SL}(n)$*-module* $\mathbb{K}^n$ *and* $\mathrm{Sp}(2n)$*-module* $\mathbb{K}^{2n}$*;*

2. *the dual* $\mathrm{SL}(n)$*-module* $(\mathbb{K}^n)^*$*.*

Proof. Since G acts on $V\setminus\{0\}$ transitively, V is a simple G-module of highest weight λ and $V = \mathfrak{g}v_\lambda$, where $\mathfrak{g}$ is the tangent algebra of the group G and v_λ is a highest weight vector. In particular, a lowest weight vector is $v_{-\lambda^*} = e_{-\alpha}v_\lambda$, where α is a positive root, whence $\alpha = \lambda + \lambda^*$ is the highest root. This occurs only for $G = \mathrm{SL}(n)$ with fundamental weights $\lambda = \omega_1, \omega_{n-1}$, and $G = \mathrm{Sp}(2n)$ with $\lambda = \omega_1$. ∎

Applying an outer automorphism of G, we may assume that

$$G = G_1 \times \ldots \times G_m \text{ and } V = V_1 \oplus \ldots \oplus V_m,$$

where every component G_i is either $\mathrm{SL}(n_i)$ or $\mathrm{Sp}(n_i)$, and V_i is the tautological G_i-module with identical action of other components. The open G-orbit U in V coincides with the subvariety $\mathcal{X} = \mathcal{X}(n_1, \ldots, n_m)$. Therefore the variety X is obtained from $\mathcal{X}$ by central factorization.

Let $\mathbb{S} = (\mathbb{K}^\times)^m$ be an algebraic torus acting on $V = V_1 \oplus \ldots \oplus V_m$ by component-wise scalar multiplication. It remains to explain why for any subgroup $S \subseteq \mathbb{S}$ there exists a geometric quotient $\mathcal{X} \to \mathcal{X}/S$. This follows from the fact that $\mathcal{X}$ is a homogeneous space of the group $\overline{G} := \mathrm{GL}(n_1) \times \ldots \times \mathrm{GL}(n_m)$, and S is a central subgroup of $\overline{G}$. The proof of Theorem 1.1 is completed.

Remark 3.4. The collection $(n_1, \ldots, n_m)$ is determined by a homogeneous toric variety X uniquely. Indeed, if $\mathbb{K}^d \supset U \to X$ is the Cox realization of X and $C_1, \ldots, C_m$ are irreducible components of the complement $\mathbb{K}^d \setminus U$, then $n_i = d - \dim C_i$.

4. Properties of homogeneous toric varieties

In this section we use standard notation of toric geometry, see [7]. Let $\mathcal{N}$ be the lattice of one-parameter subgroups of a d-dimensional torus $\mathbb{T}$ and $\mathcal{M}$ be the lattice of characters of $\mathbb{T}$. The torus $\mathbb{T}$ acts diagonally on the space $\mathbb{K}^d = V = V_1 \oplus \ldots \oplus V_m$, and $\mathbb{S} \subset \mathbb{T}$ is the m-dimensional subtorus acting on every V_i by scalar multiplication. Identification of $\mathbb{T}$ with $(\mathbb{K}^\times)^d$ defines standard bases in $\mathcal{N}$ and $\mathcal{M}$. Moreover, the decomposition $V = V_1 \oplus \ldots \oplus V_m$ divides the standard basis of $\mathcal{N}$ into m groups $I_1, \ldots, I_m$, where each group I_j contains n_j basis vectors and $n_j := \dim V_j$. The open subvariety $\mathcal{X}(n_1, \ldots, n_m) = U \subset V$ is a toric $\mathbb{T}$-variety. Its fan $\mathcal{C} = \mathcal{C}(n_1, \ldots, n_m)$ in the lattice $\mathcal{N}$ consists of the cones generated by all collections of standard basis vectors that do not contain any subset I_j.

Let $S \subseteq \mathbb{S}$ be a closed subgroup. There is a sequence of lattices of one-parameter subgroups $\mathcal{N}_S \subseteq \mathcal{N}_{\mathbb{S}} \subset \mathcal{N}$, where the lattice $\mathcal{N}_S$ is determined by the connected component S^0 of the quasitorus S. The fan $\mathcal{C}_{S^0}$ of the quotient space $\mathcal{X}/S^0$ is the image of the fan $\mathcal{C}$ under the projection

$$\mathcal{N}_{\mathbb{Q}} \to (\mathcal{N}/\mathcal{N}_S)_{\mathbb{Q}}.$$

The fan $\mathcal{C}_S$ of the variety $\mathcal{X}/S$ coincides with the fan $\mathcal{C}_{S^0}$ considered with regard to an overlattice of $\mathcal{N}/\mathcal{N}_S$ of finite index, see [7, Section 2.2]. In particular, the fan $\mathcal{C}_{\mathbb{S}}$ coincides with the fan $\mathcal{P}$ of the product of projective spaces $\mathbb{P}^{n_1-1}\times\ldots\times\mathbb{P}^{n_m-1}$, and $\mathcal{C}_S$ may be considered as an intermediate step of the projection:

$$\mathcal{C} \to \mathcal{C}_S \to \mathcal{P}.$$

Let us define a sublattice $\mathcal{M}_S \subseteq \mathcal{M}$ as the set of characters of the torus $\mathbb{T}$ containing S in the kernel. Elements of $\mathcal{M}_S$ are linear functions on the space $(\mathcal{N}/\mathcal{N}_S)_{\mathbb{Q}}$.

Proposition 4.1. *Let $X = \mathcal{X}/S$ be a homogeneous toric variety. Then*

1. *the variety X is quasiprojective;*
2. *the variety X is not affine;*
3. *the variety X is projective if and only if it coincides with $\mathbb{P}^{n_1-1}\times\ldots\times\mathbb{P}^{n_m-1}$;*
4. *the variety X is quasiaffine if and only if the lattice $\mathcal{M}_S$ contains a vector with positive coordinates;*
5. *the variety X has a nonconstant regular function if and only if the lattice $\mathcal{M}_S$ contains a nonzero vector with nonnegative coordinates.*

Proof. (1) By Chevalley's Theorem, any homogeneous space of an affine algebraic group is a quasiprojective variety.

(2) A toric variety obtained via Cox construction is affine if and only if $U = V$. In our situation this is not the case.

(3) Maximal dimension of a cone in the fan $\mathcal{C}$ equals $n_1+\ldots+n_m-m$. Therefore the fan $\mathcal{C}_S$ is complete if and only if it is obtained from $\mathcal{C}$ by projection to $(\mathcal{N}/\mathcal{N}_{\mathbb{S}})_{\mathbb{Q}}$, and thus $\mathcal{C}_S$ coincides with $\mathcal{P}$.

(4) A toric variety is quasiaffine if and only if its fan is a collection of faces of a strongly convex polyhedral cone. In our case, this condition implies that the projection K of the support of the fan $\mathcal{C}$ to $(\mathcal{N}/\mathcal{N}_S)_{\mathbb{Q}}$ is a strongly convex cone. The latter is equivalent to existence of a linear function on the space $(\mathcal{N}/\mathcal{N}_S)_{\mathbb{Q}}$ that is positive on $K\setminus\{0\}$. This gives the desired element of the lattice $\mathcal{M}_S$.

Conversely, assume that the lattice $\mathcal{M}_S$ contains a vector v with positive coordinates. We have to show that the projection of each cone of the fan $\mathcal{C}$ is a face of K. Fix proper subsets $J_1 \subset I_1, \ldots, J_m \subset I_m$ of the sets of standard basis vectors of the lattice $\mathcal{N}$. We claim that there is an element of the lattice $\mathcal{M}_S$, which vanishes on the vectors of $J_1\cup\ldots\cup J_m$ and is positive on other standard basis vectors. Indeed, the sublattice $\mathcal{M}_S$ is defined in terms of the sums of coordinates of a character over all m groups of its coordinates. The desired vector should have the same sums of coordinates over the groups as the vector v.

(5) Since regular functions on X form a rational $\mathbb{T}$-module, one may consider only $\mathbb{T}$-semiinvariant regular functions. Further, regular $\mathbb{T}$-semiinvariants on X correspond to characters from $\mathcal{M}_S$ that are nonnegative on the rays of the fan $\mathcal{C}$, see [7, Section 3.3]. ■

Remark 4.2. Let X be a homogeneous toric variety. Then X is projective if and only if X contains a $\mathbb{T}$-fixed point. Indeed, the latter condition means that the fan $\mathcal{C}_S$ contains a cone of full dimension, thus $\mathcal{N}_S = \mathcal{N}_{\mathbb{S}}$ and $S = \mathbb{S}$.

Example 4.3. Let $m = 2$ and $n_1 = n_2 = 2$. Then $\mathcal{X} = (\mathbb{K}^2 \setminus \{0\}) \times (\mathbb{K}^2 \setminus \{0\})$. Set $S = \{(s, s, s, s) : s \in \mathbb{K}^\times\}$. Then

$$\mathcal{M}_S = \{(x_1, x_2, x_3, x_4)\,;\, x_i \in \mathbb{Z},\ x_1 + x_2 + x_3 + x_4 = 0\},$$

and the variety X is $\mathbb{P}^3 \setminus (D_1 \cup D_2)$, where $D_i \cong \mathbb{P}^1$. If we set

$$S = \{(s, s, s^{-1}, s^{-1}) : s \in \mathbb{K}^\times\},$$

then

$$\mathcal{M}_S = \{(x_1, x_2, x_3, x_4)\,;\, x_i \in \mathbb{Z},\ x_1 + x_2 = x_3 + x_4\},$$

and X is a three-dimensional quadratic cone with the apex removed.

Let us characterize the fans of homogeneous toric varieties. Let N be a lattice, Δ be a fan in $N_{\mathbb{Q}}$ and P be the set of primitive vectors on the rays of Δ. Denote by N_0 a sublattice of N generated by P. Fix a positive integer m.

Definition 4.4. A fan Δ is called *m-partite* if

- the set P spans the vector space $N_{\mathbb{Q}}$;
- the set P can be decomposed into m subsets $P = I_1 \sqcup \ldots \sqcup I_m$, where each I_j contains at least two elements, and the cones of Δ are exactly the cones generated by subsets $J \subset P$ that do not contain any I_j.

Set $I_j = \{e_1^j, \ldots, e_{n_j}^j\}$ and $q_j = e_1^j + \ldots + e_{n_j}^j$. Let Q be a sublattice of N generated by $q_1, \ldots, q_m$, and $Q_{\mathbb{Q}} = Q \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proposition 4.5. *A fan Δ is the fan of a homogeneous toric variety if and only if*

1. *Δ is m-partite for some $m \geq 1$;*
2. *every linear relation among elements of P has the form $\lambda_1 q_1 + \ldots + \lambda_m q_m = 0$ for some rational λ_i;*
3. *$N \subset N_0 + Q_{\mathbb{Q}}$.*

Proof. A fan is m-partite if and only if it is a projection of the fan $\mathcal{C}(n_1,\ldots,n_m)$ with some $n_i \geq 2$. Condition 2 means that the kernel of the projection is of the form $(\mathcal{N}_{S^0})_{\mathbb{Q}}$, where $S \subseteq \mathbb{S}$. Finally, condition 3 means that N is generated by P and some elements

$$\frac{r_{1i}}{R_i}q_1 + \ldots + \frac{r_{mi}}{R_i}q_m, \quad \text{where } r_{ji} \in \mathbb{Z}_{\geq 0}, \ \ R_i \in \mathbb{Z}_{>0}, \ \ r_{ji} < R_i, \ \text{ and } \ i = 1,\ldots,l.$$

Equivalently, the corresponding toric variety is obtained as the quotient of the variety $\mathcal{X}(n_1,\ldots,n_m)/S^0$ by an action of the group $\Gamma = \Gamma_1 \times \ldots \times \Gamma_l$, where Γ_i is the cyclic group of R_i-th roots of unity and an element $\epsilon \in \Gamma_i$ multiplies the j-th factor of $\mathcal{X}(n_1,\ldots,n_m)$ by $\epsilon^{r_{ji}}$. ∎

5. Some generalizations

Let a connected reductive group G act on a toric variety X transitively. One may assume that $G = G^s \times L$, where G^s is a simply connected semisimple group, L is a central torus, and the G-action on X is locally effective. It is well known that any toric variety X is isomorphic to a direct product $X_0 \times X_1$, where X_0 is a non-degenerate toric variety and X_1 is an algebraic torus.

Let us give a construction of a transitive G-action on a toric variety X. Take a G^s-homogeneous toric variety X_0 with a locally effective and G^s-equivariant action of a quasitorus L'. Fix an inclusion $L' \subseteq L$ into an algebraic torus L as a closed subgroup. The group $G = G^s \times L$ acts on $X_0 \times L$, where G^s acts on the first factor and L acts on the second one by multiplication. Consider the G-equivariant action of L' on $X_0 \times L$ given by $(x_0, l) \mapsto (sx_0, s^{-1}l)$ for every $s \in L'$. Then

$$X(X_0, G^s, L', L) := (X_0 \times L)/L'$$

is a G-homogeneous toric variety.

Proposition 5.1. *Let X be a toric variety endowed with a transitive and locally effective action of a connected reductive group $G = G^s \times L$. Then the non-degenerate factor X_0 of X is a G^s-homogeneous toric variety. Moreover, if L' is the stabilizer of a G^s-orbit on X in the torus L, then X is G-equivariantly isomorphic to $X(X_0, G^s, L', L)$.*

Proof. Since the G^s- and L-actions on X commute, all G^s-orbits are of the same dimension. Let Y be one of these orbits. Any invertible function on Y is constant. Consider the above decomposition $X = X_0 \times X_1$. Since points on X_1 are separated by invertible functions, Y is contained in a subvariety $X_0 \times \{x_1\}$, where $x_1 \in X_1$. Let L' be the stabilizer of the subvariety Y in the torus L. Then the stabilizer H of a point $x \in Y$ is contained in the subgroup $G^s \times L'$ and the homogeneous space G/H projects onto $G/(G^s \times L') \cong L/L'$. Points on L/L' are separated by invertible functions, hence $X_0 \times \{x_1\}$ is contained in a fiber of the projection. But the fibers coincide with G^s-orbits on X. This implies $Y = X_0 \times \{x_1\}$.

Let us identify the variety X_0 with the subvariety $Y \subseteq X$. Consider the morphism

$$\varphi\colon X_0 \times L \to X, \quad (x_0, l) \mapsto lx_0.$$

Two pairs (x_0, l) and $(\widetilde{x_0}, \widetilde{l})$ are in the same fiber of φ if and only if $(\widetilde{x_0}, \widetilde{l}) = (sx_0, s^{-1}l)$ with $s = \widetilde{l}^{-1}l$. This shows that φ induces a bijective morphism $X(X_0, G^s, L', L) \to X$. Clearly, this is an isomorphism of G-homogeneous spaces. ∎

If the subgroup L' is connected, then $L \cong L' \times L''$ with some complementary subtorus L'', and $X \cong X_0 \times L''$. But unlike the case of algebraic transformation groups of maximal rank [17, Theorem 2], this situation does not always occur. Indeed, one may consider a toric variety $(\mathbb{K}^2 \setminus \{0\}) \times \mathbb{K}^\times$ with a transitive locally effective action of the group $\mathrm{SL}(2) \times \mathbb{K}^\times$ given as $(g, t) \cdot (v, a) = (g(tv), t^2 a)$.

Remark 5.2. It would be interesting to generalize [17, Theorem 3] and to describe toric varieties with transitive actions of non-reductive affine algebraic groups.

Besides homogeneous toric varieties, our method allows to describe toric varieties with a generically transitive action of a semisimple group G. By Lemma 2.1, they are quasitorus quotients of open subsets of generically transitive $(G \times \mathbb{S})$-modules. Such modules are known as $(G \times \mathbb{S})$-prehomogeneous vector spaces. For an explicit description, one needs a list of prehomogeneous vector spaces. The classification results here are known only under some restrictions on the group and on the module. For example, if G is simple and the number of irreducible summands of the module does not exceed three, the classification is given in a series of papers of M.Sato, T.Kimura, K.Ueda, T.Yoshigiaki and others.

If the complement of an open G-orbit on a toric variety X has codimension at least two in X, then X comes from a G-prehomogeneous vector space (Proposition 2.2). When the group G is simple, the list of G-prehomogeneous vector spaces is obtained in [16, Theorems 7-8], and the corresponding toric varieties are described in [2, Proposition 4.7]. In constrast to the homogeneous case, here appear singular [2, Example 5.8] and non-quasiprojective [2, Example 5.9] varieties.

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Ivan V. Arzhantsev
Department of Higher Algebra
Faculty of Mechanics and Mathematics
Moscow State University
Leninskie Gory 1
Moscow, 119991, Russia
arjantse@mccme.ru

Sergey A. Gaifullin
Department of Higher Algebra
Faculty of Mechanics and Mathematics
Moscow State University
Leninskie Gory 1
Moscow, 119991, Russia
sgayf@yandex.ru

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