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Cox rings, semigroups and automorphisms of affine algebraic varieties

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Abstract. We study the Cox realization of an affine variety, that is, a canonical representation of a normal affine variety with finitely generated divisor class group as a quotient of a factorially graded affine variety by an action of the Neron-Severi quasitorus. The realization is described explicitly for the quotient space of a linear action of a finite group. A universal property of this realization is proved, and some results in the divisor theory of an abstract semigroup emerging in this context are given. We show that each automorphism of an affine variety can be lifted to an automorphism of the Cox ring normalizing the grading. It follows that the automorphism group of an affine toric variety of dimension $\geqslant 2$ without nonconstant invertible regular functions has infinite dimension. We obtain a wild automorphism of the three-dimensional quadratic cone that rises to the Anick automorphism of the polynomial algebra in four variables.

Bibliography: 22 titles.

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§ 1. Introduction

Let X be a normal algebraic variety without nonconstant invertible functions over an algebraically closed field $\mathbb{K}$ of characteristic zero. Suppose that the divisor class group $\mathrm{Cl}(X)$ is free and finitely generated. Denote by $\mathrm{WDiv}(X)$ the group of Weil divisors on X. Let us fix a subgroup $K \subset \mathrm{WDiv}(X)$ that projects onto $\mathrm{Cl}(X)$ isomorphically. Following Cox's famous construction from toric geometry [1] (see also [2], Definition 2.6), we define the Cox ring (or the total coordinate ring) of the variety X as

$$R(X) = \bigoplus_{D\in K} \mathscr{O}(X, D), \quad \text{where } \mathscr{O}(X, D) = \{f \in \mathbb{K}(X) \mid \mathrm{div}(f) + D \geqslant 0\}.$$

Multiplication on graded components of $R(X)$ is given by multiplication of rational functions, and extends to other elements by distributivity. It is not difficult to check that the ring $R(X)$ depends on the choice of the lattice K only up to isomorphism. The ring R(X) is proved to be factorial, see [3] and [4]. There is a proof of this fact based on the concept of graded factoriality [5], that is, uniqueness of the

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prime decomposition in the semigroup $\mathrm{Ass}(R(X))$ of association classes of nonzero homogeneous elements of $R(X)$. This semigroup is naturally isomorphic to the semigroup $\mathrm{WDiv}(X)_+$ of effective Weil divisors on X, hence it is free.

If the group $\mathrm{Cl}(X)$ is finitely generated but not free, the definition of the Cox ring $R(X)$ requires more effort (see § 2). Here $R(X)$ is also factorially graded, but may be not factorial; see [5], Example 4.2 and Example 3.4 below.

The *Neron-Severi quasitorus* of a variety X is a quasitorus (that is, a closed subgroup of a linear algebraic torus) N with a fixed isomorphism between the character group and the group $\mathrm{Cl}(X)$. Suppose that $R(X)$ is finitely generated. The quasitorus N acts naturally on the spectrum $\overline{X}$ of $R(X)$, and X is the categorical quotient for this action. The quotient morphism $q\colon \overline{X} \to X$ is said to be the *Cox realization* of the variety X. This construction has already been used in affine algebraic geometry. For example, the calculation of the Cox ring has enabled one to characterize affine SL(2)-embeddings that admit the structure of a toric variety [6]. In [7] it is shown that the Cox realization leads to a remarkable unified description of all affine SL(2)-embeddings, which was not known before. The Cox realization can be defined for a wide class of normal (not necessarily affine) varieties. If the Cox ring is finitely generated, the realization results in a combinatorial description of varieties via the theory of bunched rings. This approach enables one to describe many geometric properties; see [8] and [9].

We construct explicitly the Cox realization of the quotient space of a finite linear group. As a corollary, quotient spaces that are toric varieties are described. In § 4 we prove the following universal property: the quotient morphism of a factorially graded affine variety by the quasitorus action that does not contract divisors can be factored through the Cox realization of the quotient space. For an affine variety X, the embedding of the algebra of regular functions $\mathbb{K}[X]$ into $R(X)$ as the zero homogeneous component defines an embedding of the semigroup Γ of principal effective divisors on X into $\mathrm{WDiv}(X)_+$. This is the divisor theory for the semigroup Γ in the sense of [10], Ch. III and [11]. Some problems on Cox rings and varieties corresponding to them can be reformulated and solved in terms of semigroup theory. Conversely, geometric intuition suggests assertions about semigroups that are of independent interest. One of the aims of this paper is to show the effectiveness of this correspondence. For example, the universal property of the Cox realization mentioned above leads to the following result: an embedding of a semigroup Γ into a free semigroup S under certain conditions can be extended to an embedding of the divisor theory of Γ into S (Theorem 7.3). All definitions and statements concerning semigroups are gathered in the last section.

One of the main results of [1] is a description of the automorphism group of a complete simplicial toric variety using homogeneous automorphisms of its Cox ring. Later Bühler [12] generalized this result to arbitrary complete toric varieties. We prove that each automorphism of a normal affine variety lifts to an automorphism of the Cox ring normalizing the grading; see Theorem 5.1. It is known that the Cox ring of a toric variety is a polynomial ring [1]. This allows us to prove that the automorphism group of an affine toric variety of dimension $\geqslant 2$ without nonconstant invertible regular functions has infinite dimension (Theorem 5.4).

We say that an automorphism of an affine toric variety X is *tame* if it can be represented as a composition of automorphisms inducing elementary automorphisms of the polynomial algebra $R(X)$ in the sense of the theory of automorphisms of the polynomial ring; see Definition 5.3. Other automorphisms are called wild. In § 6 we give an example of a wild automorphism of a three-dimensional quadratic cone. Here we do not use recent results of Shestakov and Umirbaev [13]. Our automorphism lifts to the famous Anick automorphism of the polynomial algebra; see [14], p. 49, p. 96 and [15]. The Anick automorphism is one of the candidates to be a wild automorphism of the algebra of polynomials in four variables. We show that this automorphism cannot be decomposed into a composition of elementary automorphisms preserving the grading.

In the text standard notations of invariant theory are used. If G is a reductive group acting on an affine variety X, then $X//G$ is an affine variety isomorphic to the spectrum of the algebra of invariants $\mathbb{K}[X]^G$, and $q\colon X \to X//G$ is the quotient morphism corresponding to the inclusion $\mathbb{K}[X]^G \subseteq \mathbb{K}[X]$. If the quotient morphism is geometric, we use the notation X/G. Note that the quotient morphism is automatically geometric if the group G is finite.

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§ 2. The Cox ring of an affine variety

Consider an irreducible normal variety X with finitely generated divisor class group Cl(X) and without nonconstant invertible regular functions. Let $K \subset \mathrm{WDiv}(X)$ be a finitely generated subgroup projecting surjectively onto $\mathrm{Cl}(X)$. Consider the ring $T_K(X) = \bigoplus_{D\in K} \mathscr{O}(X, D)$. Let $K^0 \subset K$ be the kernel of the projection $K \to \mathrm{Cl}(X)$. Fix some bases $D_1, \dots, D_s$ in K and $D_1^0 = d_1D_1, \dots, D_r^0 = d_rD_r$ in K^0, $r \leqslant s$. We call a set of rational functions

$$\mathscr{F} = \{F_D \in \mathbb{K}(X)^\times : D \in K^0\}$$

coherent if the principal divisor $\mathrm{div}(F_D)$ coincides with D and $F_{D+D'} = F_D F_{D'}$. It is clear that the set $\{F_D\}$ is determined by the functions $F_{D_i^0}$, $i = 1, \dots, r$: if $D = a_1D_1^0 + \dots + a_rD_r^0$, then $F_D = F_{D_1^0}^{a_1} \cdots F_{D_r^0}^{a_r}$. Fix a coherent set $\mathscr{F}$.

Suppose that $D_1, D_2 \in K$ and $D_1 - D_2 \in K^0$. The map $f \mapsto F_{D_1-D_2}f$ is an isomorphism of vector spaces $\mathscr{O}(X, D_2)$ and $\mathscr{O}(X, D_1)$. It is easy to see that the vector space spanned by the vectors $f - F_{D_1-D_2}f$ for all $D_1, D_2 \in K$, $D_1 - D_2 \in K^0$, $f \in \mathscr{O}(X, D_2)$, is an ideal $I(K, \mathscr{F}) \lhd T_K(X)$. We define the Cox ring of the variety X as

$$R_{K,\mathscr{F}}(X) = T_K(X)/I(K, \mathscr{F}).$$

The ring $R_{K,\mathscr{F}}(X)$ does not depend on the choice of K and $\mathscr{F}$ up to isomorphism (see, for example, [5], Proposition 3.2). Further we denote it by $R(X)$. Since $K/K^0 \cong \mathrm{Cl}(X)$, the ring $R(X)$ has a natural $\mathrm{Cl}(X)$-grading.

Definition 2.1. Let M be a finitely generated Abelian group and $R = \bigoplus_{u\in M} R_u$ an M-graded associative commutative algebra with unit. A nonzero homogeneous element $a \in R \setminus R^\times$ is called *M-irreducible* if the condition $a = bc$, where b and c

are homogeneous, implies that either b or c is invertible. The graded algebra R is called *factorially graded* if R does not contain nonconstant homogeneous invertible elements, each nonconstant homogeneous element can be represented as a product of M-irreducible elements and this representation is unique up to association and order of factors.

Recall that two homogeneous elements $a, b \in R$ are *associated* if $a = bc$, where c is invertible. The set $\mathrm{Ass}(R)$ of association classes of nonzero homogeneous elements of R is a semigroup with multiplication induced by multiplication in R. It is clear that graded factoriality of an algebra R means that the semigroup $\mathrm{Ass}(R)$ is free. If $R = R(X)$ and $M = \mathrm{Cl}(X)$, then the semigroup $\mathrm{Ass}(R)$ may be identified with $\mathrm{WDiv}(X)_+$. Hence R(X) contains neither homogeneous zero divisors nor nonconstant homogeneous invertible functions, and is factorially graded.

Recall that a *quasitorus* is an affine algebraic group isomorphic to the direct product of a torus and a finite Abelian group. The group of characters of a quasitorus is a finitely generated Abelian group whose rank is equal to the dimension of the quasitorus. It is easy to check that each finitely generated Abelian group can be realized as the group of characters of a quasitorus, and a quasitorus is determined by its group of characters up to isomorphism. We define the *Neron-Severi quasitorus* of a variety X as a quasitorus N whose character group is identified with $\mathrm{Cl}(X)$. The quasitorus N acts by automorphisms of the algebra $R(X)$: a homogeneous component $R(X)_u$ is the weight subspace corresponding to the weight u. The subalgebra $R(X)^N$ of invariants of this action coincides with $R(X)_0 = \mathbb{K}[X]$.

Further we assume that the variety X is affine and $R(X)$ is finitely generated. Consider the spectrum $\overline{X} = \mathrm{Spec}(R(X))$ and the induced regular N-action on the affine variety $\overline{X}$. Since $X \cong \mathrm{Spec}(R(X)^N)$, the variety X is realized as the N-quotient of $\overline{X}$. The quotient morphism $q\colon \overline{X} \to X \cong \overline{X}/\!/N$ is said to be the *Cox realization* of the variety X.

Proposition 2.2. *The variety $\overline{X}$ is irreducible and normal.*

Proof. Let X^{reg} be the set of smooth points of X. It is an open subset with complement of codimension $\geqslant 2$. Fix a finitely generated subgroup $K \subset \mathrm{WDiv}(X)$ that projects onto $\mathrm{Cl}(X)$ surjectively, and consider the ring $T_K(X)$. This ring has no homogeneous zero divisors and hence has no zero divisors at all ([5], Lemma 2.1). Every point $x \in X^{\mathrm{reg}}$ has an affine neighbourhood V, where any divisor is principal. Thus the spectrum of the algebra $T_K(V)$ is isomorphic to the direct product $V \times T$, where T is a torus with a character lattice identified with the lattice K. It is known that such a spectrum is isomorphic to the preimage of V in the spectrum of the algebra $T_K(X)$; see the proof of Theorem 4.3 in [8]. This shows that the projection of the preimage W of the variety X^{reg} onto X^{reg} is a locally trivial bundle $p\colon W \to X^{\mathrm{reg}}$ with fibre T; see also [9], Proposition 2.2. Factorization by $I(K, \mathscr{F})$ corresponds to a closed subvariety $U = q^{-1}(X^{\mathrm{reg}}) \subseteq W$. Since $\mathrm{Cl}(X) \cong K/K_0$, the intersection of this subvariety with $p^{-1}(V) \cong V \times T$ is identified with $V \times N$.

It suffices to show that the variety U is irreducible. Indeed, for dimension reasons U is contained in an irreducible component of $\overline{X}$ that is the closure of U. Consequently, this component is N-invariant. The union of other components is also N-invariant, and if it is nonempty, then there are homogeneous zero divisors in $R(X)$.

The subvariety $q^{-1}(V)$ is defined in $p^{-1}(V) = V \times T$ by the equations $t_i^{d_i} = F_{D_i^0}$, where the functions $F_{D_i^0}$ are irreducible in $\mathbb{K}[X]$. Hence $q^{-1}(V)$ is irreducible and is contained in one component of U. Such components are N-invariant and cover U. Since X^{reg} is irreducible and the image of a closed N-invariant subset under the morphism q is closed, the variety U is irreducible.

The proof of the fact that $R(X)$ is integrally closed can be found in [5], Proposition 2.7, Remark 2.8 and [9], Proposition 2.2(i). The proof is complete.

It is easy to check that the algebra $R(X)$ is factorially graded if and only if each N-invariant Weil divisor on $\overline{X}$ is principal. With any N-linearized line bundle over $\overline{X}$ one may associate an N-invariant Cartier divisor corresponding to an N-semi-invariant regular section. This shows that any N-linearized line bundle over $\overline{X}$ is nothing but an N-linearization of the trivial line bundle.

The following proposition contains a necessary and sufficient condition for the quotient morphism of an affine variety by an action of a quasitorus to be the Cox realization of the quotient space. This condition emerged in several former papers; see for example [16], Remark 4.2 and [7], Theorem 2.2.

Proposition 2.3. *Let $q\colon \overline{X} \to X$ be the Cox realization of an affine variety X. Suppose Q is a quasitorus acting on an irreducible affine variety Z, which is factorially graded with regard to this action. Let $\pi\colon Z \to Z/\!/Q$ be the quotient morphism. Suppose that there exists an isomorphism $\varphi\colon X \to Z/\!/Q$. Then the following conditions are equivalent*:

(i) *there exists an open Q-invariant subset $U \subseteq Z$ such that* $\operatorname{codim}_Z(Z \setminus U) \geqslant 2$, *the Q-action on U is free and each fibre of π having nonempty intersection with U is a Q-orbit*;

(ii) *there exist an isomorphism $\mu\colon Q \to N$ and an isomorphism $\xi\colon Z \to \overline{X}$ such that $\xi(gz) = \mu(g)\xi(z)$ for all $g \in Q$, $z \in Z$, and the following diagram is commutative*:

$$\begin{array}{ccc} Z & \xrightarrow{\ \xi\ } & \overline{X} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle q} \\ Z/\!/Q & \xrightarrow{\ \varphi\ } & X. \end{array}$$

Proof. For the Cox realization $q\colon \overline{X} \to X$ one can take $q^{-1}(X^{\mathrm{reg}})$ for U (see the proof of Proposition 2.2).

Conversely, consider the restriction of π to U:

$$\pi\colon U \to U_0 \subset Z/\!/Q.$$

Reducing U we may assume that U_0 consists of smooth points of the variety $Z/\!/Q$. For each divisor D its intersection with U_0 is a locally principal divisor, thus it determines a line bundle L_D. The lift of this bundle to U gives a Q-linearization of the trivial bundle on U, and hence on Z. Conversely, every Q-linearization of the trivial line bundle on Z comes from a divisor on U_0. This defines an isomorphism between the group of characters of the quasitorus Q and the Picard group of U_0 coinciding with $\mathrm{Cl}(X)$ ([17], Proposition 4.2). The space of invariant sections of the Q-linearized trivial line bundle on Z with respect to a character ξ of Q is identified with the subspace of semi-invariants of weight $-\xi$ in $\mathbb{K}[Z]$. The algebra $\mathbb{K}[Z]$ is a direct sum of subspaces of Q-semi-invariants, so we get a homogeneous isomorphism between $\mathbb{K}[Z]$ and $R(X)$, which completes the proof.

Finally we give an example of an affine variety with nonfinitely generated Cox ring. If $Y \subseteq \mathbb{P}^n$ is an irreducible projectively normal projective subvariety and $X \subseteq \mathbb{K}^{n+1}$ is the affine cone over Y, then $R(Y)$ and $R(X)$ are isomorphic as (nongraded) algebras. Indeed, for every divisor D on Y the quotient morphism $\pi\colon X \setminus \{0\} \xrightarrow{//\mathbb{K}^\times} Y$ determines an isomorphism between the space $\mathscr{O}(Y, D)$ and a weight component with respect to the action of the torus $\mathbb{K}^\times$ in the space $\mathscr{O}(X, D')$, where D' is the image of the class of D in the exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Cl}(Y) \to \mathrm{Cl}(X) \to 0.$$

But $\mathscr{O}(X, D')$ is a direct sum of weight components corresponding to divisors D mapping to D' ([17], Proposition 4.2).

Let us take as Y the blow-up of the projective plane $\mathbb{P}^2$ at nine generic points. By [18] (see also [4], Remark 3.2) the ring $R(Y)$ is not finitely generated.

§ 3. The Cox realization of a quotient space

Let V be a finite-dimensional vector space over the field $\mathbb{K}$ and $G \subset \mathrm{GL}(V)$ a finite subgroup. We shall describe the Cox realization of the quotient space $V/G := \operatorname{Spec} \mathbb{K}[V]^G$. Recall that a linear operator of finite order $A \in \mathrm{GL}(V)$ is called a *pseudoreflection* if the subspace V^A of A-fixed points is a hyperplane. By the classical Chevalley-Shephard-Todd theorem the algebra of invariants $\mathbb{K}[V]^H$ of a finite subgroup $H \subset \mathrm{GL}(V)$ is free if and only if the subgroup H is generated by pseudoreflections. We denote by H the subgroup generated by all pseudoreflections in G. It is clear that H is a normal subgroup of G. Let $\varphi\colon G \to F := G/H$ be the projection, $[F, F]$ the commutant of the group F, $N := F/[F, F]$ and $\widetilde{H} := \varphi^{-1}([F, F])$. Put $Z := \operatorname{Spec} \mathbb{K}[V]^{\widetilde{H}}$. The finite Abelian group N acts naturally on the variety Z.

Theorem 3.1. *The quotient morphism $Z \xrightarrow{/N} V/G$ is the Cox realization of the variety V/G.*

Proof. We denote the variety $\operatorname{Spec} \mathbb{K}[V]^H$ by W. It is easy to prove that the group F acts linearly on W. Let $\widetilde{F} \subseteq F$ be a subgroup generated by pseudoreflections. Then the algebra of invariants $\mathbb{K}[W]^{\widetilde{F}} \cong \mathbb{K}[V]^{\varphi^{-1}(\widetilde{F})}$ is free. This shows that F does not include any element acting on W as a pseudoreflection. Hence there exists an open F-invariant subset $U \subseteq W$ such that $\operatorname{codim}_W(W \setminus U) \geqslant 2$ and the F-action on U is free. Let

$$\zeta\colon W \to W/[F, F] \cong Z$$

be the quotient morphism. Then $\operatorname{codim}_Z(Z \setminus \zeta(U)) \geqslant 2$ and the N-action on $\zeta(U)$ is free. According to Proposition 2.3, it is sufficient to prove that the N-variety Z is factorially graded. Let D be an N-invariant Weil divisor on Z and $\zeta^{-1}(D)$ its preimage in W. The divisor $\zeta^{-1}(D)$ coincides with $\operatorname{div}(f)$, where f is an F-semi-invariant function on W. This implies that $f \in \mathbb{K}[W]^{[F,F]}$ and $D = \operatorname{div}(f)$ on Z. The proof is complete.

Corollary 3.2. *The variety V/G is toric if and only if the group G/H is commutative.*

Proof. The Cox ring of a toric variety is a polynomial ring [1]. Therefore the group $[F,F]$ is generated by pseudoreflections, and hence it is trivial. The converse assertion follows from the fact that a linear action of the finite Abelian group G/H on the space W is diagonalizable.

Remark 3.3. Assume that $R(X)$ is a polynomial ring. It is natural to ask whether the variety X is toric. In general, the answer is negative: one may take a toric variety of dimension $\geqslant 2$ and remove a finite set of points. Nevertheless, this is true for a complete variety with a free finitely generated divisor class group ([8], Corollary 4.4). The question whether this is true for affine varieties reduces to the linearization problem for actions of quasitori (see, for example, [19]), which is open.

Example 3.4. Let $V = \mathbb{K}^2$ and

$$G = Q_8 = \left\{ \pm E, \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\},$$

where $i^2 = -1$. Here $H = \{e\}$, $\widetilde{H} = \{\pm E\}$ and the variety Z is a two-dimensional quadratic cone. Thus the Cox ring of the variety V/G is not factorial.

Theorem 3.1 can be partially generalized to the case of infinite groups. With any affine algebraic group G we associate the group of characters $\Xi(G)$ and the intersection of kernels of all characters $G_1 := \bigcap_{\xi \in \Xi(G)} \operatorname{Ker}(\xi)$. It is clear that G_1 is a normal subgroup of G and $N := G/G_1$ is a quasitorus. If G is reductive, then G_1 is reductive too.

Proposition 3.5. *Let G be a reductive algebraic group and Y an irreducible affine G-variety such that*

(i) $\mathbb{K}[Y]^\times = \mathbb{K}^\times$;

(ii) *each G-stable Weil divisor on Y is principal*;

(iii) *for the quotient morphism $\pi\colon Y \to Y/\!/G := \operatorname{Spec} \mathbb{K}[Y]^G$ there exists an open subset $U \subseteq Y/\!/G$ such that $\operatorname{codim}_Y(Y \setminus \pi^{-1}(U)) \geqslant 2$ and the G-action on $\pi^{-1}(U)$ is free.*

Let $Z = \operatorname{Spec} \mathbb{K}[Y]^{G_1}$. Then the quotient morphism $p\colon Z \xrightarrow{/\!/N} Y/\!/G$ is the Cox realization of $Y/\!/G$.

Proof. It is easy to see that $\operatorname{codim}_Z(Z \setminus p^{-1}(U)) \geqslant 2$ and the N-action on $p^{-1}(U)$ is free. Hence it is sufficient to prove that each N-stable divisor D on Z is principal. The preimage $\zeta^{-1}(D)$ of the divisor D under the quotient morphism $\zeta\colon Y \xrightarrow{/\!/G_1} Z$ is a G-stable divisor on Y. Hence $\zeta^{-1}(D) = \operatorname{div}(f)$ for some G-semi-invariant $f \in \mathbb{K}[Y]$. Then $f \in \mathbb{K}[Y]^{G_1}$ and $D = \operatorname{div}(f)$ on Z.

Example 3.6 (see [20], Ch. 3 and [5], Theorem 4.1). Let $\widehat{G}$ be a semisimple connected simply connected affine algebraic group and $G \subseteq \widehat{G}$ a reductive subgroup. It is known that $\widehat{G}$ is a factorial variety, $\mathbb{K}[\widehat{G}]^\times = \mathbb{K}^\times$ and the homogeneous space $\widehat{G}/G$ is affine. Hence, the map $\widehat{G}/G_1 \xrightarrow{/N} \widehat{G}/G$ is the Cox realization.

More generally, consider the Cox realization of the variety of double cosets. Let G and G' be reductive subgroups of a simply connected semisimple group $\widehat{G}$. Then $G \times G'$ acts on $\widehat{G}$ as $(g, g')\hat{g} = g\hat{g}g'^{-1}$. We denote by $G\backslash\widehat{G}/G'$ the categorical quotient for this action. To prove that $G_1\backslash\widehat{G}/G'_1 \to G\backslash\widehat{G}/G'$ is the Cox realization one needs to check condition (iii) of Proposition 3.5.

Example 3.7. Let $\widehat{G} = \mathrm{SL}(3)$, $G' = \mathrm{SO}(3)$ and

$$G = T^2 = \mathrm{diag}(t_1, t_2, t_3 : t_1t_2t_3 = 1).$$

Here $Z = G_1\backslash\widehat{G}/G'_1 = \mathrm{SL}(3)/\mathrm{SO}(3)$ is the variety of symmetric matrices (a_{ij}), $i, j = 1, 2, 3$, with determinant 1, $N = T^2$ acts on Z as $a_{ij} \to t_it_ja_{ij}$, and $X = G\backslash\widehat{G}/G' = Z/\!/T^2$ is a three-dimensional hypersurface. As the subset $\pi^{-1}(U)$, one may take the preimage in $\mathrm{SL}(3)$ of the set of symmetric matrices with at most one zero element a_{ij}, $i \geqslant j$. Thus the spectrum of the Cox ring of the hypersurface X:

$$2x_1^3 + x_2x_3x_4 - x_1^2 - x_1^2x_2 - x_1^2x_3 - x_1^2x_4 = 0$$

is the hypersurface Z:

$$y_1y_2y_3 + 2y_4y_5y_6 - y_1y_4^2 - y_2y_5^2 - y_3y_6^2 = 1.$$

Example 3.8. Let $\widehat{G} = \mathrm{SL}(4)$, $G' = \mathrm{Sp}(4)$ and

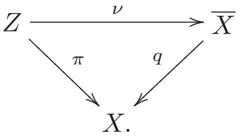
$$G = T^3 = \mathrm{diag}(t_1, t_2, t_3, t_4 : t_1t_2t_3t_4 = 1).$$

Here $Z = \mathrm{SL}(4)/\mathrm{Sp}(4)$ is the variety of skew-symmetric matrices (b_{ij}), $i, j = 1, 2, 3, 4$, with Pfaffian $\mathrm{Pf} = b_{12}b_{34} - b_{13}b_{24} + b_{14}b_{23}$ equal to 1. In this case condition (iii) of Proposition 3.5 is not fulfilled: the closure of the T^3-orbit of any point on the divisor $\{b_{12} = 0\}$ contains a point with $b_{34} = 0$. It is easy to prove that $X = T^3\backslash\mathrm{SL}(4)/\mathrm{Sp}(4) \cong \mathbb{K}^2$. Therefore the spectrum of the Cox ring cannot be realized as a homogeneous space of the group $\mathrm{SL}(4)$.

§ 4. A universal property of the Cox realization

Let Z be an irreducible affine variety with an action of a quasitorus Q. We call the action *uncontracting* if for any prime divisor $D \subset Z$ the closure of its image $\overline{\pi(D)}$ under the quotient morphism $\pi\colon Z \to Z/\!/Q$ has codimension $\leqslant 1$ in $Z/\!/Q$. Note that if Q is a finite Abelian group, then every Q-action is uncontracting.

Theorem 4.1. *Let Z be an irreducible normal affine variety with an uncontracting action of a quasitorus Q and let $\pi\colon Z \to Z/\!/Q$ be the quotient morphism. Suppose that the Q-variety Z is factorially graded and the quotient space $X = Z/\!/Q$ has* *the Cox realization* $q\colon \overline{X} \to X$. *Then there exists a surjective homomorphism* $\mu\colon Q \to N$ *and a dominant morphism* $\nu\colon Z \to \overline{X}$ *such that* $\nu(gz) = \mu(g)\nu(z)$ *for all* $g \in Q$, $z \in Z$, *and the following diagram is commutative*:



Moreover, the homomorphism μ *is unique, and the morphism* ν *is determined up to composition with the automorphism of* Z *defined by an element of* Q.

This result can be obtained using restriction of divisors to the subset X^{reg} and lifting the corresponding line bundles to Z; see the proof of Proposition 2.3. We leave the details of this proof to the reader.

Now we prove Theorem 4.1 in terms of semigroups of association classes of homogeneous elements. In § 7 we present an independent proof of Theorem 7.3, which may be considered as a semigroup version of Theorem 4.1. Here our aim is to show that Theorem 7.3 implies Theorem 4.1.

Proof of Theorem 4.1. *Existence*. Let $A = \mathbb{K}[Z]$ and let $A = \bigoplus_{u\in M} A_u$ be the grading defined by the Q-action. By assumption, there is an isomorphism $\delta\colon \mathbb{K}[X] \to A_0$. We have to prove that there exists a homogeneous embedding $\psi\colon R(X) \to A$ coinciding with δ on the zero component and mapping different homogeneous components of $R(X)$ to different components of A. Let $S = \mathrm{Ass}(A)$ be the semigroup of association classes of nonzero homogeneous elements of A, $\Gamma = \mathrm{Ass}(\mathbb{K}[X])$ and $D = \mathrm{WDiv}(X)_+$. The embedding τ of Γ in D as the semigroup of principal effective divisors defines the divisor theory of the semigroup Γ. The isomorphism δ defines an embedding $\alpha\colon \Gamma \hookrightarrow S$ satisfying conditions (i) and (ii) of Theorem 7.3. Indeed, the first condition follows from the existence of gradings and the second is satisfied because the Q-action is uncontracting. Thus there is an embedding β of the semigroup D, which is identified with $\mathrm{Ass}(R(X))$, into the semigroup S extending the embedding α. We shall 'lift' this embedding to a homogeneous embedding of algebras.

Suppose the group $\mathrm{Cl}(X)$ has rank r. Let $C_1,\dots,C_r,B_1,\dots,B_s$ be Weil divisors on X such that their classes generate $\mathrm{Cl}(X)$ with relations $d_1[B_1]=\dots=d_s[B_s]=0$, $d_i \in \mathbb{N}$. Fix nonzero elements of the Cox ring

$$f_1,\dots,f_r,h_1,\dots,h_s, \qquad f_i \in R(X)_{[C_i]}, \quad h_j \in R(X)_{[B_j]},$$

and nonzero homogeneous elements $a_1,\dots,a_r,a_1',\dots,a_s'$ of A such that the classes of the f_i (respectively, of the h_j) in D are mapped to the classes of the a_i (respectively, a_j') in S under the embedding β. Note that $h_j^{d_j} = F_j \in \mathbb{K}[X]$. Therefore, we need the isomorphism δ to map F_j to $(a_j')^{d_j}$. This can be achieved by changing the elements a_j' with proportional ones.

The isomorphism δ extends to an isomorphism of the field of fractions $\delta\colon \mathbb{K}(X) \to QA_0$. Consider an element $u \in \mathrm{Cl}(X)$,

$$u = u_1[C_1]+\dots+u_r[C_r]+v_1[B_1]+\dots+v_s[B_s], \qquad u_i \in \mathbb{Z}, \quad v_j \in \{0,\dots,d_j-1\},$$

and the element $f_u := f_1^{u_1} \cdots f_r^{u_r} h_1^{v_1} \cdots h_s^{v_s} \in (QR(X))_u$. For any $f \in R(X)_u$ we have $f = \frac{f}{f_u} f_u$ and $\frac{f}{f_u} \in (QR(X))_0 = \mathbb{K}(X)$. Put

$$\psi(f) = \delta\left(\frac{f}{f_u}\right)\psi(f_u) = \delta\left(\frac{f}{f_u}\right) a_1^{u_1} \cdots a_r^{u_r} (a_1')^{v_1} \cdots (a_s')^{v_s}.$$

It is easy to check that the map ψ is linear on any homogeneous component $R(X)_u$ and corresponds to the embedding β of semigroups. In particular, the image of ψ lies in A. The grading defines a homomorphism of the semigroup D (resp. S) to the group $\mathrm{Cl}(X)$ (resp. M), and the kernel of this homomorphism is the semigroup Γ (resp. $\alpha(\Gamma)$). The semigroup D projects onto the group $\mathrm{Cl}(X)$ surjectively. Extending the above homomorphisms to homomorphisms of groups generated by the semigroups, we obtain an injective homomorphism $\mathrm{Cl}(X) \to M$. This proves the assertion of the theorem concerning homogeneous components.

We have to check that for $f \in R(X)_u$ and $f' \in R(X)_{u'}$ the condition

$$\psi(ff') = \psi(f)\psi(f')$$

holds. It is sufficient to prove this for $f = f_u$, $f' = f_{u'}$, where it follows from $\psi(F_j) = \delta(F_j)$, $j = 1, \ldots, s$.

Thus, the restriction of ψ to any component $R(X)_u$ is linear, ψ is multiplicative on homogeneous elements, and one can extend ψ to a homogeneous embedding of algebras $\psi\colon R(X) \to A$ via distributivity.

Uniqueness. Theorem 7.3 implies that the embedding of semigroups $\beta\colon D \to S$ is unique. Hence two embeddings ψ and ψ' of algebras can differ on $a_1, \ldots, a_r$, $a_1', \ldots, a_s'$ only by scalar multiples; moreover, the multiple for a_j' has to be a root of unity of degree d_j. Since the group $\mathrm{Cl}(X)$ is embedded into M, this corresponds to the action of an element of Q on A. This proves Theorem 4.1.

The condition of graded factoriality of Z in Theorem 4.1 is essential: one may take Z to be the quotient of $\overline{X}$ by a proper subgroup $Q \subset N$. The following examples show that the condition of uncontracting is also essential.

Example 4.2. Let $Z = \mathbb{K}^3$, $Q = \mathbb{K}^\times$ and $t(z_1, z_2, z_3) = (tz_1, tz_2, t^{-2}z_3)$. Here $Z/\!/Q$ is isomorphic to a two-dimensional quadratic cone X, and the torus Q does not admit a surjective homomorphism onto the Neron-Severi quasitorus N consisting of two elements. An analogous example of a variety X with a free divisor class group gives the cone $X = \{(x_1, x_2, x_3, x_4) : x_1^2 x_2 - x_3 x_4 = 0\}$ realized as the quotient $Z/\!/Q$, where $Z = \mathbb{K}^5$, $Q = (\mathbb{K}^\times)^2$ and

$$(t_1, t_2)(z_1, z_2, z_3, z_4, z_5) = (t_1 z_1, t_2 z_2, t_1^{-1} t_2 z_3, t_1 t_2^{-1} z_4, t_1^{-1} t_2^{-1} z_5).$$

Example 4.3. Let $Z = \mathbb{K}^5$, $Q = (\mathbb{K}^\times)^2$ and

$$(t_1, t_2)(z_1, z_2, z_3, z_4, z_5) = (t_1 z_1, t_1 z_2, t_2 z_3, t_2 z_4, t_1^{-1} t_2^{-1} z_5).$$

The quotient space $Z/\!/Q$ is a three-dimensional quadratic cone X. Here $\overline{X} = \mathbb{K}^4$, $N = \mathbb{K}^\times$ and the action is given by

$$s(y_1, y_2, y_3, y_4) = (sy_1, sy_2, s^{-1}y_3, s^{-1}y_4).$$

In this case, there is a coherent embedding of the Cox ring $\mathbb{K}[y_1, y_2, y_3, y_4]$ into $\mathbb{K}[Z]$ extending the embedding of semigroups, but it is not unique: one may take either

$$y_1 \to z_1, \qquad y_2 \to z_2, \qquad y_3 \to z_3z_5, \qquad y_4 \to z_4z_5,$$

or

$$y_1 \to z_1z_5, \qquad y_2 \to z_2z_5, \qquad y_3 \to z_3, \qquad y_4 \to z_4.$$

§ 5. Lifting automorphisms

Let $R = \bigoplus_{u\in M} R_u$ be an associative commutative algebra with unit graded by a finitely generated Abelian group M. We define a subgroup $\widetilde{\mathrm{Aut}}(R)$ of the automorphism group of R as

$$\widetilde{\mathrm{Aut}}(R) = \{\varphi \in \mathrm{Aut}(R) \mid \exists\, \varphi_0 \in \mathrm{Aut}(M) : \varphi(R_u) = R_{\varphi_0(u)}\ \forall\, u \in M\}.$$

We say that elements of $\widetilde{\mathrm{Aut}}(R)$ *normalize the grading* of the algebra R.

Theorem 5.1. *Let X be an irreducible normal affine variety with a finitely generated divisor class group* $\mathrm{Cl}(X)$ *and the Neron-Severi quasitorus N. Assume that* $\mathbb{K}[X]^\times = \mathbb{K}^\times$. *Then there is the following exact sequence*:

$$1 \to N \xrightarrow{\alpha} \widetilde{\mathrm{Aut}}(R(X)) \xrightarrow{\beta} \mathrm{Aut}(X) \to 1.$$

Proof. Each automorphism $\phi \in \widetilde{\mathrm{Aut}}(R(X))$ induces an automorphism of the algebra $R(X)_0 = \mathbb{K}[X]$, and this defines the map β. The quasitorus N acts by homogeneous automorphisms of $R(X)$, which are identical on $R(X)_0$. This defines α and shows that the composition $\beta\circ\alpha$ maps elements of N to the identity automorphism of X.

Lemma 5.2. *Let $u \in \mathrm{Cl}(X)$ be a nonzero element. Then the $R(X)_0$-module $R(X)_u$ is not cyclic.*

Proof. Suppose that there is $f \in R(X)_u$ such that any $g \in R(X)_u$ has the form $g = fh$ for some $h \in R(X)_0$. Let D be a Weil divisor from the fixed set of representatives corresponding to the class u. Then the divisor

$$(D + \mathrm{div}(g)) - (D + \mathrm{div}(f)) = \mathrm{div}(h)$$

is effective. Put

$$D + \mathrm{div}(f) = \sum_i a_iD_i, \qquad a_i \in \mathbb{N}.$$

There exists a rational function $F \in \mathbb{K}(X)$ with pole of order one on D_1. Multiplying it by a suitable regular function we get $\mathrm{div}(F) = -D_1 + \sum_j b_jD_j$, $b_j \in \mathbb{N}$. Then $fF \in R(X)_u$, but $\mathrm{div}(fF) - \mathrm{div}(f)$ is not effective. The proof is complete.

Let us show that $\mathrm{Ker}\,\beta = \mathrm{Im}\,\alpha$. Suppose $\varphi \in \mathrm{Ker}\,\beta$ and f is a prime homogeneous element of $R(X)$. If $f \in R(X)_0$, then $\varphi(f) = f$. Let $f \in R(X)_u$, $u \neq 0$, and let $g \in R(X)_u$ be an element not divisible by f. The automorphism φ induces an automorphism of the quotient field $QR(X)$ which is the identity on $QR(X)_0$. Since

$f/g \in QR(X)_0$, one gets $f\varphi(g) = g\varphi(f)$. Since $R(X)$ is factorially graded, f divides $\varphi(f)$. The element $\varphi(f)$ is also prime, therefore $\varphi(f) = \lambda f$, $\lambda \in \mathbb{K}^\times$. Thus ϕ acts by scalar multiplication on prime homogeneous (and on all homogeneous) elements. Moreover, $\varphi\,|_{R(X)_u}$ is a scalar operator, so φ defines a homomorphism from $\mathrm{Cl}(X)$ to $\mathbb{K}^\times$, that is, it belongs to $\alpha(N)$.

Now we have to prove that the map β is surjective. Let $\psi \in \mathrm{Aut}(X)$. Then ψ induces automorphisms of the groups WDiv(X) and $\mathrm{Cl}(X)$. Suppose K is a subgroup of WDiv(X) from the definition of $R(X)$ (see § 2), and $\psi(K)$ is its image. Since the ring $R(X)$ does not depend (up to isomorphism) on the choices of K and a coherent set $\mathscr{F}$, we can fix an isomorphism τ between the Cox rings $R_{\psi(K),\psi^*(\mathscr{F})}(X)$ and $R_{K,\mathscr{F}}(X)$ with the identical restriction to $R(X)_0 = \mathbb{K}[X]$. Then the composition of $\psi^*\colon R_{K,\mathscr{F}}(X) \to R_{\psi(K),\psi^*(\mathscr{F})}(X)$ and τ is an element of the subgroup $\widetilde{\mathrm{Aut}}(R(X))$, which induces the automorphism ψ^* on $R(X)_0$. The proof is complete.

Let X be an affine toric variety. The Cox ring $R(X)$ is a (graded) polynomial algebra with homogeneous generators [1]. Hence the description of the group $\mathrm{Aut}(X)$ may be reduced to the description of the group of automorphisms of the polynomial algebra normalizing the grading. In the study of automorphisms of the polynomial algebra the concepts of tame and wild automorphisms play an important role.

Definition 5.3. (i) An automorphism of the algebra $\mathbb{K}[y_1, \ldots, y_m]$ is called *elementary* if it is either a linear map or a map of the following form:

$$(y_1, \ldots, y_m) \to (y_1, \ldots, y_{i-1}, y_i + f, y_{i+1}, \ldots, y_m),$$

where $f \in \mathbb{K}[y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m]$.

(ii) An automorphism is called *tame* if it can be decomposed into a composition of elementary automorphisms.

(iii) An automorphism, which is not tame, is called *wild*.

Define an *elementary* automorphism of the algebra $\mathbb{K}[X]$ as the image under the map β of an elementary automorphism of the algebra $R(X)$ belonging to the subgroup $\widetilde{\mathrm{Aut}}(R(X))$. Accordingly, we call an automorphism *tame* if it can be represented as a composition of elementary automorphisms and *wild* otherwise. In the next section we illustrate these concepts with the example of quadratic cone.

Recall that for any affine variety X the group $\mathrm{Aut}(X)$ has the structure of an infinite-dimensional affine algebraic group in the sense of [21]. This group is said to be *finite-dimensional* if $\mathrm{Aut}(X)$ has the structure of an affine algebraic group such that the action $\mathrm{Aut}(X) \times X \to X$ is algebraic (an equivalent definition of a finite-dimensional group of automorphisms is given in [22]).

Theorem 5.4. *Let X be an affine toric variety of dimension $\geqslant 2$ and $\mathbb{K}[X]^\times = \mathbb{K}^\times$. Then the group $\mathrm{Aut}(X)$ is not finite-dimensional.*

Proof. Let $R(X) = \mathbb{K}[y_1, \ldots, y_m]$ be the Cox ring with the $\mathrm{Cl}(X)$-grading. The component $R(X)_0$ contains a monomial h, which does not depend on y_1. Indeed, if this is not the case, the Cox realization $q\colon \mathbb{K}^m \to X$ contracts the divisor $y_1 = 0$, in contradiction with Proposition 2.3. Lemma 5.2 implies that the homogeneous component containing y_1 contains a monomial f which does not depend on y_1. Consider homogeneous automorphisms of $R(X)$ mapping y_1 to $y_1 + fF(h)$ and preserving the variables $y_2, \dots, y_m$, where $F(t) \in \mathbb{K}[t]$. They induce homomorphisms of the algebra $R(X)_0$. The images of any monomial depending on y_1 are not contained in a finite-dimensional subspace. Thus the group $\mathrm{Aut}(X)$ is not an affine algebraic group. The proof is complete.

§ 6. A wild automorphism of the quadratic cone

Consider the quadratic cone

$$X = \{(x_1, x_2, x_3, x_4) : x_1x_4 - x_2x_3 = 0\}.$$

The variety X can be realized as the cone of singular 2×2 matrices. Here $R(X)$ is the polynomial algebra $\mathbb{K}[y_1, y_2, y_3, y_4]$ with $\mathbb{Z}$-grading $\deg(y_1) = \deg(y_2) = 1$, $\deg(y_3) = \deg(y_4) = -1$ and $x_1 = y_1y_3$, $x_2 = y_1y_4$, $x_3 = y_2y_3$, $x_4 = y_2y_4$.

Consider the automorphism τ of the cone X defined by the formula

$$\tau\colon \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_1 & x_2 + x_1(x_3 - x_2) \\ x_3 + x_1(x_3 - x_2) & x_4 + (x_3 + x_2)(x_3 - x_2) + x_1(x_3 - x_2)^2 \end{pmatrix}.$$

Its inverse is

$$\tau^{-1}\colon \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_1 & x_2 - x_1(x_3 - x_2) \\ x_3 - x_1(x_3 - x_2) & x_4 - (x_3 + x_2)(x_3 - x_2) + x_1(x_3 - x_2)^2 \end{pmatrix}.$$

The lift of this automorphism to the Cox ring is defined as

$$\zeta\colon (y_1, y_2, y_3, y_4) \mapsto (y_1, y_2 + y_1(y_1y_4 - y_2y_3), y_3, y_4 + y_3(y_1y_4 - y_2y_3)).$$

This is known as the *Anick automorphism* of the polynomial algebra in four variables.

Theorem 6.1. *The automorphism τ is wild.*

Proof. An elementary automorphism ϕ of $R(X)$ that preserves the grading either multiplies the variables by nonzero scalars or maps (y_1, y_2, y_3, y_4) to one of the sets

$$(y_1 + y_2H_1(y_2y_3, y_2y_4), y_2, y_3, y_4), \qquad (y_1, y_2 + y_1H_2(y_1y_3, y_1y_4), y_3, y_4),$$
$$(y_1, y_2, y_3 + y_4H_3(y_1y_4, y_2y_4), y_4), \qquad (y_1, y_2, y_3, y_4 + y_3H_4(y_1y_3, y_2y_3)).$$

Therefore, the group of tame automorphisms is generated by these automorphisms and the transpose automorphism $(y_1, y_2, y_3, y_4) \to (y_3, y_4, y_1, y_2)$, which reverses the grading. Suppose that the Anick automorphism is equal to a composition of elementary automorphisms normalizing the grading:

$$\zeta = \varphi_n \circ \dots \circ \varphi_2 \circ \varphi_1.$$

Since ζ preserves the grading and the result of conjugation of an elementary automorphism preserving the grading with the transpose automorphism is an elementary automorphism preserving the grading, one may assume that each φ_i preserves the grading.

Let us replace all nonlinear automorphisms changing y_3 or y_4 in the above composition by their linear parts. Denote the new composition by ρ. Since the linear parts of ζ and the identity automorphism are the same, we obtain $\rho(y_3) = y_3$, $\rho(y_4) = y_4$. Let $f = \zeta(y_1) - \rho(y_1)$ and $g = \zeta(y_2) - \rho(y_2)$. Then

$$\rho\colon (y_1, y_2, y_3, y_4) \to (y_1 - f, y_2 + y_1(y_1y_4 - y_2y_3) - g, y_3, y_4).$$

Consider the ideal $I = (y_1, y_2) \lhd \mathbb{K}[y_1, y_2, y_3, y_4]$.

Lemma 6.2. *The polynomials f and g are in I^3.*

Proof. Let $\zeta_j = \varphi_j \circ \cdots \circ \varphi_1$ and ρ_j be the composition obtained from ζ_j by replacing automorphisms changing y_3 or y_4 with their linear parts. We set $f_j = \zeta_j(y_1) - \rho_j(y_1)$, $g_j = \zeta_j(y_2) - \rho_j(y_2)$, $h_j = \zeta_j(y_3) - \rho_j(y_3)$ and $s_j = \zeta_j(y_4) - \rho_j(y_4)$.

Let us prove by induction on j that $f_j, g_j \in I^3$ and $h_j, s_j \in I$. The case $j = 1$ is easy. At the step from $j = k$ to $j = k + 1$, only one of the polynomials f_j, g_j, h_j, and s_j is changed. We consider the cases when f_j and h_j are changed, other cases are similar.

1) Assume that f_j is changed. Then φ_{k+1} has the form

$$(y_1, y_2, y_3, y_4) \mapsto (y_1 + y_2H_1(y_2y_3, y_2y_4), y_2, y_3, y_4).$$

Thus,

$$\begin{aligned} f_{k+1} &= \zeta_{k+1}(y_1) - \rho_{k+1}(y_1) \\ &= \zeta_k(y_1) + \zeta_k(y_2)H_1\bigl(\zeta_k(y_2)\zeta_k(y_3), \zeta_k(y_2)\zeta_k(y_4)\bigr) \\ &\quad - \rho_k(y_1) - \rho_k(y_2)H_1\bigl(\rho_k(y_2)\rho_k(y_3), \rho_k(y_2)\rho_k(y_4)\bigr) \\ &= f_k + g_kH_1\bigl(\zeta_k(y_2)\zeta_k(y_3), \zeta_k(y_2)\zeta_k(y_4)\bigr) \\ &\quad + \rho_k(y_2)\bigl(H_1(\zeta_k(y_2)\zeta_k(y_3), \zeta_k(y_2)\zeta_k(y_4)) - H_1(\rho_k(y_2)\rho_k(y_3), \rho_k(y_2)\rho_k(y_4))\bigr). \end{aligned}$$

By the inductive assumption, f_k and g_k belong to I^3. We have to prove that the last term belongs to I^3. It is sufficient to do it when H_1 is a monomial. Let $H_1(u, v) = u^lv^r$. The automorphisms ζ_k and ρ_k preserve the grading, hence $\zeta_k(y_2) \in I$ and $\rho_k(y_2) \in I$. Therefore, if $l + r \geqslant 2$, then

$$\rho_k(y_2)\bigl(H_1(\zeta_k(y_2)\zeta_k(y_3), \zeta_k(y_2)\zeta_k(y_4)) - H_1(\rho_k(y_2)\rho_k(y_3), \rho_k(y_2)\rho_k(y_4))\bigr) \in I^3.$$

If $l + r < 2$, then either $l + r = 0$, so that $H_1 = \mathrm{const}$ and

$$H_1\bigl(\zeta_k(y_2)\zeta_k(y_3), \zeta_k(y_2)\zeta_k(y_4)\bigr) - H_1\bigl(\rho_k(y_2)\rho_k(y_3), \rho_k(y_2)\rho_k(y_4)\bigr) = 0,$$

or $l + r = 1$ and without loss of generality one may assume that $l = 1$ and $r = 0$. We get

$$\begin{aligned} &\rho_k(y_2)\bigl(H_1(\zeta_k(y_2)\zeta_k(y_3), \zeta_k(y_2)\zeta_k(y_4)) - H_1(\rho_k(y_2)\rho_k(y_3), \rho_k(y_2)\rho_k(y_4))\bigr) \\ &\qquad = \rho_k(y_2)\bigl(\zeta_k(y_2)\zeta_k(y_3) - \rho_k(y_2)\rho_k(y_3)\bigr) \\ &\qquad = \rho_k(y_2)\bigl((\rho_k(y_2) + g_k)(\rho_k(y_3) + h_k) - \rho_k(y_2)\rho_k(y_3)\bigr) \\ &\qquad = \rho_k(y_2)\bigl(\rho_k(y_2)h_k + g_k\rho_k(y_3) + g_kh_k\bigr). \end{aligned}$$

Since $\rho_k(y_2), h_k \in I$ and $g_k \in I^3$, the sum belongs to I^3. Thus, $f_{k+1} \in I^3$.

2) Suppose h_j is changed. Then φ_{k+1} has the form

$$(y_1, y_2, y_3, y_4) \mapsto (y_1, y_2, y_3 + \mu y_4 + y_4 F(y_1 y_4, y_2 y_4), y_4),$$

where F is a polynomial without constant term. Hence

$$\begin{aligned} h_{k+1} &= \zeta_{k+1}(y_3) - \rho_{k+1}(y_3) \\ &= \zeta_k(y_3) + \mu\zeta_k(y_4) + \zeta_k(y_4) F\big(\zeta_k(y_1)\zeta_k(y_4), \zeta_k(y_2)\zeta_k(y_4)\big) - \rho_k(y_3) - \mu\rho_k(y_4) \\ &= h_k + \mu s_k + \zeta_k(y_4) F\big(\zeta_k(y_1)\zeta_k(y_4), \zeta_k(y_2)\zeta_k(y_4)\big). \end{aligned}$$

Since $h_k \in I$, $s_k \in I$, $\zeta_k(y_1) \in I$ and $\zeta_k(y_2) \in I$, we obtain $h_{k+1} \in I$. The proof of the lemma is complete.

Let us calculate the Jacobian matrix J of the automorphism ρ. Since a partial derivative of a polynomial from I^3 is in I^2, we obtain

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2y_1y_4 - y_2y_3 & 1 - y_1y_3 & -y_1y_2 & y_1^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + Q,$$

where Q is a matrix with elements in I^2. Therefore,

$$\det(J) = 1 - y_1y_3 + c, \qquad c \in I^2.$$

Since the determinant is not a constant, ρ is not an automorphism. This contradiction completes the proof of the theorem.

It is interesting to note that if one adds the fifth ζ-stable variable of degree zero, the automorphism ζ becomes tame in the class of automorphisms preserving the grading [15]. Let us write down a decomposition of this automorphism into elementary ones:

$$\begin{aligned} (y_1, y_2, y_3, y_4, y_5) &\mapsto \big(y_1, y_2, y_3, y_4, y_5 + (y_1y_4 - y_2y_3)\big) \\ &\mapsto \big(y_1, y_2 + y_1y_5 + y_1(y_1y_4 - y_2y_3), y_3, y_4, y_5 + (y_1y_4 - y_2y_3)\big) \\ &\mapsto \big(y_1, y_2 + y_1y_5 + y_1(y_1y_4 - y_2y_3), y_3, \\ &\qquad y_4 + y_3y_5 + y_3(y_1y_4 - y_2y_3), y_5 + (y_1y_4 - y_2y_3)\big) \\ &\mapsto \big(y_1, y_2 + y_1y_5 + y_1(y_1y_4 - y_2y_3), y_3, y_4 + y_3y_5 + y_3(y_1y_4 - y_2y_3), y_5\big) \\ &\mapsto \big(y_1, y_2 + y_1(y_1y_4 - y_2y_3), y_3, y_4 + y_3y_5 + y_3(y_1y_4 - y_2y_3), y_5\big) \\ &\mapsto \big(y_1, y_2 + y_1(y_1y_4 - y_2y_3), y_3, y_4 + y_3(y_1y_4 - y_2y_3), y_5\big). \end{aligned}$$

Remark 6.3. The famous Nagata automorphism

$$(y_1, y_2, y_3) \to (y_1 - 2y_2(y_1y_3 + y_2^2) - y_3(y_1y_3 + y_2^2)^2, y_2 + y_3(y_1y_3 + y_2^2), y_3)$$

is a wild automorphism of the polynomial algebra in three variables [13]. This automorphism is homogeneous with respect to a grading if and only if

$$\deg(y_1) = 3\deg(y_2), \qquad \deg(y_3) = -\deg(y_2).$$

For a $\mathbb{Z}$-grading, one gets the action $t \cdot (y_1, y_2, y_3) = (t^3 y_1, t y_2, t^{-1} y_3)$ of a one-dimensional torus with a quotient space isomorphic to an affine plane. Here the quotient morphism is not the Cox realization of the quotient space. Nevertheless, in the case of a $\mathbb{Z}_n$-grading, the quotient morphism for the action $(\varepsilon^3 y_1, \varepsilon y_2, \varepsilon^{-1} y_3)$, $\varepsilon^n = 1$, is the Cox realization of $X_n := \mathbb{K}^3/\mathbb{Z}_n$, and the Nagata automorphism induces a wild automorphism of the variety X_n. In particular, the Nagata automorphism defines a wild automorphism of the cone $X_2 \subset \mathbb{K}^6$ over the image of the Veronese map $\mathbb{P}^2 \subset \mathbb{P}^5$.

§ 7. Appendix. The divisor theory of a semigroup

In this section a semigroup is a commutative semigroup with unit e such that all nonunit elements are not invertible. We use multiplicative notation for the operation. A semigroup S is called *free* if there is a subset $P \subset S \setminus \{e\}$ such that each $s \in S \setminus \{e\}$ can be expressed as $s = p_1^{k_1} \cdots p_m^{k_m}$, $p_i \in P$, $k_i \in \mathbb{N}$, and this representation is unique up to the order of factors. Elements of P are prime (or indecomposable) elements of S, so P is uniquely determined by S. Sometimes it is convenient to say 'factorial semigroup' instead of 'free semigroup'.

Let Γ be an arbitrary semigroup. The following definition can be found in [10], Ch. III, § 3 (see also [11]).

Definition 7.1. The *divisor theory* of a semigroup Γ is an embedding $\tau\colon \Gamma \to D$ into a free semigroup D such that

(i) if $a, b \in \Gamma$ and $\tau(a) = \tau(b)c_1$ for some $c_1 \in D$, then $c_1 = \tau(c)$ for some $c \in \Gamma$;

(ii) if for $d_1, d_2 \in D$ one has

$$\{a \in \Gamma \mid \exists\, d \in D : \tau(a) = dd_1\} = \{b \in \Gamma \mid \exists\, d \in D : \tau(b) = dd_2\},$$

then $d_1 = d_2$.

In [10], Ch. III, § 3, Theorem 1 it is proved that if a semigroup Γ has a divisor theory, then this theory is unique up to isomorphism. So we denote D as $D(\Gamma)$. Also we do not distinguish between elements of Γ and their images in $D(\Gamma)$.

Suppose Γ is a finitely generated semigroup. Let us explain when Γ admits a divisor theory and recall a known realization of this theory. Since there are only finitely many prime elements of $D(\Gamma)$ in the decomposition of any generating element of Γ, the semigroup $D(\Gamma)$ is finitely generated and Γ can be embedded into the lattice $\mathbb{Z}D(\Gamma)$. The saturation condition emerging in the following lemma is necessary for the existence of the divisor theory of Γ (cf. [10], Ch. III, § 3, Theorem 3).

Lemma 7.2. *Let $\Gamma \subseteq D(\Gamma)$ be the divisor theory of a semigroup Γ and L the subgroup of the free Abelian group $\mathbb{Z}D(\Gamma)$ generated by Γ. If for an element $l \in L$ there is $m \in \mathbb{N}$ such that $l^m \in \Gamma$, then $l \in \Gamma$.*

Proof. The condition $l^m \in \Gamma$ implies $l \in D(\Gamma)$. Since $l = ab^{-1}$, $a, b \in \Gamma$, we have $a = bl$, and condition (i) implies $l \in \Gamma$. The proof is complete.

It is known that for a finitely generated Γ the saturation condition is equivalent to the condition $\Gamma = \sigma \cap L$, where σ is the cone in the space $L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by (generators of) Γ. Let $\sigma^\vee$ be the dual cone in the dual space $L^*_{\mathbb{Q}}$, that is,

$$\sigma^\vee = \{f \in L^*_{\mathbb{Q}} : f(x) \geqslant 0 \ \forall\, x \in \sigma\},$$

and let $f_1, \dots, f_r$ be primitive vectors on the edges of the cone $\sigma^\vee$. Then the embedding

$$\Gamma \hookrightarrow \mathbb{Z}^r_{\geqslant 0}, \qquad a \to (f_1(a), \dots, f_r(a)),$$

is the divisor theory of Γ. Note that this embedding defines a minimal realization of σ as the intersection of the positive octant and a subspace $V \subseteq \mathbb{Q}^r$.

Conversely, let $V \subseteq \mathbb{Q}^r$ be a subspace such that for the cone $\sigma := V \cap \mathbb{Q}^r_{\geqslant 0}$ the subsets $\sigma_i := \sigma \cap \{x_i = 0\}, i = 1, \dots, r$, are pairwise distinct and form the set of facets of the cone σ. Suppose also that $L \subseteq V \cap \mathbb{Z}^r$ is a sublattice whose images under the coordinate projections to each axis Ox_i coincide with the set of integer points on the Ox_i. Then the embedding $\Gamma \hookrightarrow \mathbb{Z}^r_{\geqslant 0}$ is the divisor theory of the semigroup $\Gamma := L \cap \sigma$. Indeed, the set of coordinate functions x_i coincides with the set of primitive vectors on the edges of $\sigma^\vee$.

Now we focus on non-finitely generated semigroups. The theory of Weil divisors provides an example of the divisor theory in algebraic geometry. Let X be a normal irreducible affine algebraic variety and Γ the semigroup of association classes of the algebra $\mathbb{K}[X]$. The semigroup Γ can be identified with the semigroup of principal effective divisors on X, and the semigroup $D(\Gamma)$ can be realized as the semigroup of effective Weil divisors on X. Examples of divisor theories appearing in number theory can be found in [10], Ch. III.

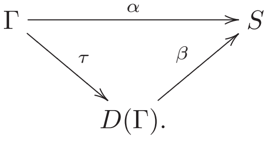
The aim of this appendix is to prove the following result generalizing the uniqueness theorem for the divisor theory.

Theorem 7.3. *Let $\tau\colon \Gamma \to D(\Gamma)$ be the divisor theory of a semigroup Γ and $\alpha\colon \Gamma \to S$ an embedding of Γ into a free semigroup S satisfying the following conditions*:

(i) *if for some $a, b \in \Gamma$ there exists $s \in S$ such that $\alpha(a) = \alpha(b)s$, then $s = \alpha(c)$ for some $c \in \Gamma$;*

(ii) *if all elements of some subset $A \subseteq \Gamma$ are coprime in $D(\Gamma)$, that is, they are not divisible by a nonunit element of $D(\Gamma)$, then all elements of $\alpha(A) \subseteq S$ are coprime in S.*

Then there exists a unique embedding $\beta\colon D(\Gamma) \hookrightarrow S$ such that the following diagram is commutative:



Proof. Existence. Let P be the set of all prime elements of S and $P_1 \subseteq P$ the subset of elements that divide at least one of the elements $\alpha(a)$, $a \in \Gamma \setminus \{0\}$. Without loss of generality we may assume that $P = P_1$. For any $d \in D := D(\Gamma)$ and $s \in S$ we define

$$L(d) = \{a \in \Gamma \mid \exists\, d' \in D : a = dd'\}, \qquad N(s) = \{b \in \Gamma \mid \exists\, s' \in S : \alpha(b) = ss'\}.$$

Lemma 7.4. (i) *Let $p \in P$. Then there is a prime element $q \in D$ with $L(q) \subseteq N(p)$.*

(ii) *Let $q \in S$ be a prime element. Then there is $p_1 \in P$ such that $N(p_1) \subseteq L(q)$.*

Proof. Since $N(p) \neq \varnothing$, there exists $a \in \Gamma$ such that $\alpha(a) = ps$, $s \in S$. Let $a = q_1^{k_1} \dots q_l^{k_l}$ be the prime decomposition in D. If $L(q_i) \not\subseteq N(p)$, then for every $i = 1, \dots, l$ there is $a_i \in \Gamma$ divisible by q_i such that $\alpha(a_i)$ is not divisible by p. Consider $b = a_1^{k_1} \dots a_l^{k_l}$. Then a divides b. But $\alpha(a)$ is divisible by p. This implies that p divides $\alpha(a_i)$, a contradiction. Assertion (ii) can be proved in the same way. The proof of Lemma 7.4 is complete.

Lemma 7.5. *Let $q, q_1 \in D$ be prime elements with $L(q) \subseteq L(q_1)$. Then $q = q_1$.*

For the proof it suffices to observe that there is $a \in \Gamma$, which is divisible by q and is not divisible by qq_1, hence it is in $L(q) \setminus L(q_1)$.

For a fixed $q \in D$ there exist $p \in P$ and prime $q_1 \in D$ such that

$$L(q_1) \subseteq N(p) \subseteq L(q).$$

Hence $q = q_1$ and $N(p) = L(q)$. Let $\{p_1, \dots, p_t\}$ be all the prime elements of S satisfying $N(p_i) = L(q)$. Since $\bigcap_{j \in \mathbb{N}} N(p_i^j) = \varnothing$, for any i there exists $r_i \in \mathbb{N}$ such that

$$N(p_i) = N(p_i^2) = \dots = N(p_i^{r_i}) \neq N(p_i^{r_i+1}).$$

Thus, with any prime $q \in D$ one can associate the set of prime elements $\{p_1, \dots, p_t\}$ and the set of exponents $\{r_1, \dots, r_t\}$.

Let us define $\beta(q) = p_1^{r_1} \dots p_t^{r_t}$ and extend this map to a homomorphism $\beta\colon D \to S$. Since the sets $\{p_1, \dots, p_t\}$ corresponding to different q have empty intersection, the map β is injective. It remains to prove that $\beta(a) = \alpha(a)$ for any $a \in \Gamma$.

Assume that a prime divisor $q \in D$ appears in the decomposition of an element a with multiplicity k, $\beta(q) = p_1^{r_1} \cdots p_t^{r_t}$, and any p_i appears in the decomposition of $\alpha(a)$ with multiplicity n_i. For every p_i there exists $q' \in D$ with $L(q') \subseteq N(p_i)$. If this inclusion is strict, then there is an element $b \in N(p_i)$ which is not divisible by q'. But then the elements of $L(q')$ and the element b are coprime in D. This contradicts condition (ii) of Theorem 7.3. Therefore, $L(q') = N(p_i)$. Hence each prime divisor p_i of the element $\alpha(a)$ corresponds to a (unique) prime divisor q of the element a, and it is sufficient to prove that $n_i = kr_i$ for every $i = 1, \dots, t$.

Lemma 7.6. *For $s \in S$ suppose that $p_1, \dots, p_t$ do not divide s. Then there exists $b \in \Gamma$ such that $\alpha(b)$ is divisible by s and is not divisible by $p_1, \dots, p_t$.*

Proof. Let $s = s_1 \dots s_n$ be the prime decomposition. If $N(s_j) \subseteq N(p_i) = L(q)$, then, as we have just seen, $N(s_j) = L(q)$ and s_j coincides with one of the elements $p_1, \dots, p_t$, a contradiction. Hence there are $b_j \in \Gamma$ such that $\alpha(b_j)$ is divisible by s_j and not divisible by $p_1, \dots, p_t$. We put $b = b_1 \dots b_t$ and the proof is complete.

We have $L(q) = N(p_1^{r_1} \cdots p_t^{r_t}) \supsetneqq N(p_j^{r_j+1})$. Therefore, for every j there exists $a_j \in \Gamma$ such that $\alpha(a_j)$ is divisible by $p_1^{r_1} \cdots p_t^{r_t}$, but not by $p_j^{r_j+1}$. Then

$$\alpha(a_j) = p_1^{r_{1j}} \cdots p_t^{r_{tj}} h_j, \qquad r_{ij} \geqslant r_i, \quad r_{jj} = r_j,$$

and the element h_j is coprime with $p_1, \dots, p_t$.

Lemma 7.7. *$r_{ij} = r_i$ for all $i, j = 1, \dots, t$.*

Proof. Assume that there is a pair (u, v) with $r_{uv} > r_u$. We subtract the vector $(r_{1v}, \dots, r_{tv})$ from the vector $(r_{1u}, \dots, r_{tu})$. At the uth position we obtain a negative number. Let k_1 be the first position in the difference corresponding to a negative coordinate. We add the vector $(r_{1k_1}, \dots, r_{tk_1})$ to this difference. We repeat this operation till a vector $(z_1, \dots, z_t)$ with nonnegative coordinates is obtained. Then we have

$$(z_1, \dots, z_t) + (r_{1v}, \dots, r_{tv}) = (r_{1u}, \dots, r_{tu}) + \sum_{i=1}^{m} (r_{1k_i}, \dots, r_{tk_i}).$$

Note that $m \geqslant 1$ and $z_{k_m} < r_{k_m}$ because z_{k_m} is obtained from a negative number by summation with $r_{k_m k_m} = r_{k_m}$. Put $c = a_u a_{k_1} \dots a_{k_m}$. Recall that $\alpha(a_v) = p_1^{r_{1v}} \dots p^{r_{tv}} h_v$, and Lemma 7.6 implies the existence of $b \in \Gamma$ such that $\alpha(b)$ is divisible by h_v and not divisible by $p_1, \dots, p_t$. Then $bc = a_v b'$ for some $b' \in \Gamma$. Therefore $\alpha(b') = p_1^{z_1} \dots p_t^{z_t} f$, where f is not divisible by $p_1, \dots, p_t$. From the conditions $z_{k_m} < r_{k_m}$ and $N(p_{k_m}) = N(p_{k_m}^{r_m})$ it follows that $z_{k_m} = 0$. But $N(p_i) = N(p_{k_m})$, so all the z_i are equal to zero. On the other hand,

$$z_v = r_{vu} + \sum_{i=1}^{m} r_{vk_i} - r_{vv} \geqslant \sum_{i=1}^{m} r_{vk_i} > 0,$$

a contradiction. The proof is complete.

Lemma 7.8. *There is an element $c \in \Gamma$ divisible by q, but not divisible by q^2 such that $\alpha(c)$ is divisible by $p_i^{r_i}$, but not divisible by $p_i^{r_i+1}$, $i = 1, \dots, t$.*

Proof. The previous lemma implies that the element a_1 satisfies the last two properties and is divisible by q. Let us prove that any element c with these properties is not divisible by q^2. Suppose that $c = q^r h$, where the element $h \in D$ is coprime with q, and $r > 1$. Fix an element $b \in \Gamma$ that is divisible by qh and not divisible by q^2. Then $b^r = cg$, $g \in \Gamma$ and g is coprime with q. Hence $\alpha(b)^r = \alpha(c)\alpha(g)$ and $\alpha(g)$ is not divisible by $p_1, \dots, p_t$. Therefore $\alpha(b)$ is not divisible by $p_i^{r_i}$ since $r > 1$. On the other hand, b is divisible by q, and hence $p_1^{r_1} \dots p_t^{r_t}$ divides $\alpha(b)$, a contradiction.

Let us return to the proof of the equality $n_i = kr_i$. One can find m such that $mr_i \leqslant n_i$ for all $i = 1, \dots, t$ and $(m+1)r_j > n_j$ for at least one j. Then there exist $f_1, f_2 \in \Gamma$ such that $f_1 a = f_2 c^m$ and $\alpha(f_1)$ is coprime with $p_1, \dots, p_t$. The multiplicity of p_j in the factorization of $\alpha(f_2)$ is less than r_j, hence f_2 is coprime with $p_1, \dots, p_t$ and f_2 is coprime with q. Thus, $n_i = mr_i$. On the other hand, comparing the multiplicities of q in the equation $f_1 a = f_2 c^m$ we obtain $k = m$. This completes the proof of the equality $n_i = kr_i$ and demonstrates the existence part of Theorem 7.3.

Uniqueness. Let $\gamma\colon D(\Gamma) \to S$ be another embedding satisfying the conditions of Theorem 7.3. Let $a \in \Gamma$, $a = q_1^{k_1} \cdots q_l^{k_l}$. Then

$$\alpha(a) = \beta(q_1)^{k_1} \cdots \beta(q_l)^{k_l} = \gamma(q_1)^{k_1} \cdots \gamma(q_l)^{k_l}. \qquad (*)$$

If p is a prime factor in the factorization of $\gamma(q_i)$, then $L(q_i) \subseteq N(p)$, hence $L(q_i) = N(p)$. Therefore $\gamma(q_i) = p_1^{m_1} \dots p_t^{m_t}$, where $p_1, \dots, p_t$ are the prime elements corresponding to q_i. If some m_j is greater than r_j, then one gets a contradiction with $N(p_j^{r_j+1}) \subsetneq L(q_i)$. Hence $m_j \leqslant r_j$ for all j, and $(*)$ implies $m_j = r_j$. This proves that $\beta = \gamma$ and completes the proof of the theorem.

Let us show that conditions (i) and (ii) of Theorem 7.3 are essential.

Example 7.9. Let Γ be the subsemigroup of the multiplicative semigroup $\mathbb{N}$ of positive integers generated by 10, 14, 15 and 21. The divisor theory of Γ is the embedding of Γ into the subsemigroup D generated by 2, 3, 5, and 7. On the other hand, the embedding $\alpha\colon \Gamma \to \mathbb{N}$ defined on the generators by $\alpha(10) = 10$, $\alpha(14) = 2$, $\alpha(15) = 15$, and $\alpha(21) = 3$ cannot be extended to an embedding of D into $\mathbb{N}$. Here condition (i) of Theorem 7.3 is not satisfied.

Example 7.10. Let Γ be the subsemigroup of the multiplicative semigroup $\mathbb{N}$ of positive integers generated by 4, 6, and 9. The divisor theory of Γ is the embedding of Γ into the subsemigroup D generated by 2 and 3. On the other hand, the embedding $\alpha\colon \Gamma \to \mathbb{N}$ defined on generators by $\alpha(4) = 20$, $\alpha(6) = 30$ and $\alpha(9) = 45$ cannot be extended to an embedding of D into $\mathbb{N}$. Here condition (ii) of Theorem 7.3 is not satisfied.

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