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Quantum groups, Verma modules and q -oscillators: general linear case

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Abstract

The Verma modules over the quantum groups $U_q(\mathfrak{gl}_{l+1})$ for arbitrary values of l are analysed. The explicit expressions for the action of the generators on the elements of the natural basis are obtained. The corresponding representations of the quantum loop algebras $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ are constructed via Jimbo’s homomorphism. This allows us to find certain representations of the positive Borel subalgebras of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ as degenerations of the shifted representations. The latter are the representations used in the construction of the so-called Q -operators in the theory of quantum integrable systems. The interpretation of the corresponding simple quotient modules in terms of representations of the q -deformed oscillator algebra is given.

Keywords: quantum groups, Verma modules, submodules, q -oscillators

1. Introduction

One of the most advanced methods of investigation of quantum integrable systems was developed on the basis of quantum groups [1–3]. The latter are, in a sense, the objects which have replaced the classical Lie groups and Lie algebras within the framework of the group-theoretic, or algebraic, approach to physical models. The primary problem in the study of quantum integrable systems is to describe the spectrum of the corresponding transfer matrices. This task reduces to the examination of functional relations in the system of transfer and Q -operators, being a substitution of the Bethe ansatz equations [4]⁴.

⁴ Actually, there is a vast list of interesting papers on this subject, see here some part of it [5–14].

In the quantum-group formalism the derivation of the functional relations is based on a thorough analysis of the appropriate representations of quantum groups and their Borel subalgebras. For the case of the quantum groups related to the Lie algebras \mathfrak{sl}_2 and \mathfrak{sl}_3 the corresponding work has been carried out in the papers [15–22].

In this paper we consider the case of the quantum groups $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$, $l = 1, 2, \dots$, see section 5 for the definition. These quantum groups are deformations of enveloping algebras of the loop algebras of Lie algebras $\mathcal{L}(\mathfrak{sl}_{l+1})$. Presently, in this case it is common instead of the term a quantum group to use the term a quantum loop algebra. From the point of view of quantum integrable systems the most interesting representations of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ are those which are obtained from the Verma modules over the quantum groups $U_q(\mathfrak{gl}_{l+1})$ via the Jimbo's homomorphism [23]. Thus, it is very useful and interesting to study the Verma modules over $U_q(\mathfrak{gl}_{l+1})$. In sections 2–4 we find the explicit form of the corresponding defining relations. By this we mean the explicit expressions for the action of the generators of $U_q(\mathfrak{gl}_{l+1})$ on the vectors of the natural basis of the Verma module. The corresponding representations of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ are considered in section 5.

In fact, to investigate a quantum integrable system one does not need to know representations of the whole quantum loop algebra but only of its Borel subalgebras. Furthermore, the representations of the Borel subalgebras which cannot be extended to representations of the whole quantum loop algebra are of special interest. Such representations are used to construct Q -operators. They can be constructed as certain degeneration of the shifted Verma modules, see, for example, [18] and [21, 22] for the case of $U_q(\mathcal{L}(\mathfrak{sl}_2))$ and $U_q(\mathcal{L}(\mathfrak{sl}_3))$. In the present paper we consider the general case of the quantum loop algebra $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ (section 6). The obtained representations appear to be reducible. We find the corresponding submodules and construct the irreducible quotient modules (section 7). Finally, we give an interpretation of the corresponding irreducible modules in terms of representations of the q -oscillator algebra (section 8). Almost the same expression for q -oscillator representations was suggested by Kojima [7]. The advantage of our approach is that we get it as the result of degeneration of shifted Verma modules. This allows one to present Q -operators as a limit of transfer operators, see the papers [19–22] for the case $l = 1, 2$.

We assume that the deformation parameter $q \in \mathbb{C}^\times$ is not a root of unity. The notation $\kappa_q = q - q^{-1}$ is often used, so that the definition of the q -number can be written as

$$[\nu]_q = \frac{q^\nu - q^{-\nu}}{q - q^{-1}} = \kappa_q^{-1}(q^\nu - q^{-\nu}), \quad \nu \in \mathbb{C}.$$

For a nonnegative integer n and the corresponding q -number, we also use the notation

$$[n]_q! = [1]_q [2]_q \dots [n]_q.$$

It is assumed here that $[0]_q! = 1$.

2. Quantum group $U_q(\mathfrak{gl}_{l+1})$

We start with a short reminder of some basics facts on the Cartan subalgebras and root systems of the general linear and special linear Lie algebras \mathfrak{gl}_{l+1} and \mathfrak{sl}_{l+1} . The standard basis of the standard Cartan subalgebra \mathfrak{h}_{l+1} of \mathfrak{gl}_{l+1} is formed by the matrices K_i , $i = 1, \dots, l+1$, with the matrix entries

$$(K_i)_{jk} = \delta_{ij} \delta_{ik}.$$

There are l simple roots $\alpha_i \in \mathfrak{k}_{l+1}^*$, which are usually defined by the equation

$$\langle \alpha_i, K_j \rangle = c_{ji}, \quad (2.1)$$

where

$$c_{ij} = \delta_{ij} - \delta_{i,j+1}. \quad (2.2)$$

Then, the full system of positive roots is formed by the roots

$$\alpha_{ij} = \sum_{k=i}^{j-1} \alpha_k, \quad 1 \leq i < j \leq l+1.$$

It is clear that $\alpha_i = \alpha_{i,i+1}$. Certainly, the negative roots are $-\alpha_{ij}$.

The standard basis of the standard Cartan subalgebra \mathfrak{h}_{l+1} of \mathfrak{sl}_{l+1} is formed by the matrices

$$H_i = K_i - K_{i+1}, \quad i = 1, \dots, l.$$

As the positive and negative roots we take the restriction of α_{ij} and $-\alpha_{ij}$ to \mathfrak{h}_{l+1} . For the simple roots we have

$$\langle \alpha_i, H_j \rangle = a_{ji}, \quad (2.3)$$

where

$$a_{ij} = c_{ij} - c_{i+1,j}$$

are the matrix entries of the Cartan matrix of \mathfrak{sl}_{l+1} .

Let q be the exponential of a complex number \hbar , such that q is not a root of unity. We define the quantum group $U_q(\mathfrak{gl}_{l+1})$ as a unital associative \mathbb{C} -algebra generated by the elements

$$E_i, \quad F_i, \quad i = 1, \dots, l, \quad q^X, \quad X \in \mathfrak{k}_{l+1}, \quad (2.4)$$

satisfying the following defining relations

$$q^0 = 1, \quad q^{X_1} q^{X_2} = q^{X_1+X_2}, \quad (2.5)$$

$$q^X E_i q^{-X} = q^{\langle \alpha_i, X \rangle} E_i, \quad q^X F_i q^{-X} = q^{-\langle \alpha_i, X \rangle} F_i, \quad (2.6)$$

$$[E_i, F_j] = \delta_{ij} \frac{q^{K_i - K_{i+1}} - q^{-K_i + K_{i+1}}}{q - q^{-1}}. \quad (2.7)$$

Relations (2.6) can equivalently be written as

$$q^{\nu K_i} E_j q^{-\nu K_i} = q^{\nu c_{ij}} E_j, \quad q^{\nu K_i} F_j q^{-\nu K_i} = q^{-\nu c_{ij}} F_j, \quad \nu \in \mathbb{C}. \quad (2.8)$$

Besides, we have the Serre relations

$$E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i, \quad |i - j| \geq 2, \\ E_i^2 E_{i\pm 1} - [2]_q E_i E_{i\pm 1} E_i + E_{i\pm 1} E_i^2 = 0, \quad F_i^2 F_{i\pm 1} - [2]_q F_i F_{i\pm 1} F_i + F_{i\pm 1} F_i^2 = 0.$$

Note that the notation q^X , $X \in \mathfrak{k}_{l+1}$, is used to emphasize that the Cartan subalgebra \mathfrak{k}_{l+1} parametrizes the corresponding set of elements of $U_q(\mathfrak{gl}_{l+1})$.

The quantum group $U_q(\mathfrak{sl}_{l+1})$ is generated by the same generators (2.4) as $U_q(\mathfrak{gl}_{l+1})$, however, in this case $X \in \mathfrak{h}_{l+1}$. The defining relations are also the same, except that (2.7) should be written in the form

$$[E_i, F_j] = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}.$$

From the point of view of quantum integrable systems, it is important that $U_q(\mathfrak{gl}_{l+1})$ and $U_q(\mathfrak{sl}_{l+1})$ are Hopf algebras with respect to appropriately defined co-multiplication, antipode and counit. However, we do not use the Hopf algebra structure in the present paper.

Below we assume that

$$q^{X+\nu} = q^\nu q^X, \\ [X + \nu]_q = \frac{q^{X+\nu} - q^{-X-\nu}}{q - q^{-1}} = \frac{q^\nu q^X - q^{-\nu} q^{-X}}{q - q^{-1}}$$

for any complex number ν and element X of \mathfrak{k}_{l+1} .

3. Higher root vectors and q -commutation relations

The abelian group

$$Q = \bigoplus_{i=1}^l \mathbb{Z} \alpha_i$$

is called the root lattice of \mathfrak{gl}_{l+1} . The algebra $U_q(\mathfrak{gl}_{l+1})$ can be considered as Q -graded if we assume that

$$E_i \in U_q(\mathfrak{gl}_{l+1})_{\alpha_i}, \quad F_i \in U_q(\mathfrak{gl}_{l+1})_{-\alpha_i}, \quad q^X \in U_q(\mathfrak{gl}_{l+1})_0$$

for any $i = 1, \dots, l$ and $X \in \mathfrak{k}_{l+1}$. An element a of $U_q(\mathfrak{gl}_{l+1})$ is called a root vector corresponding to a root γ of \mathfrak{gl}_{l+1} if $a \in U_q(\mathfrak{gl}_{l+1})_\gamma$. In particular, E_i and F_i are root vectors corresponding to the roots α_i and $-\alpha_i$. It is possible to find linearly independent root vectors corresponding to all roots of \mathfrak{gl}_{l+1} . To this end, we denote

$$\Lambda_l = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i < j \leq l+1\}$$

and, following Jimbo [23], introduce elements E_{ij} and F_{ij} , $(i, j) \in \Lambda_l$, with the help of the relations

$$E_{i,i+1} = E_i, \quad i = 1, \dots, l,$$

$$E_{ij} = E_{i,j-1} E_{j-1,j} - q E_{j-1,j} E_{i,j-1}, \quad j - i > 1,$$

and

$$F_{i,i+1} = F_i, \quad i = 1, \dots, l,$$

$$F_{ij} = F_{j-1,j} F_{i,j-1} - q^{-1} F_{i,j-1} F_{j-1,j}, \quad j - i > 1.$$

It is clear that the vectors E_{ij} and F_{ij} correspond to the roots α_{ij} and $-\alpha_{ij}$ respectively. These vectors are linearly independent, and together with the elements q^X , $X \in \mathfrak{k}_{l+1}$, are called Cartan–Weyl generators of $U_q(\mathfrak{gl}_{l+1})$. It appears that the ordered monomials constructed from the Cartan–Weyl generators form a Poincaré–Birkhoff–Witt basis of $U_q(\mathfrak{gl}_{l+1})$. In this paper

we choose the following ordering. First endow Λ_l with the colexicographical order. It means that $(i, j) < (m, n)$ if $j < n$, or if $j = n$ and $i < m$ ⁵. Now we say that a monomial is ordered if it has the form

$$F_{i_1 j_1} \dots F_{i_r j_r} q^X E_{m_1 n_1} \dots E_{m_s n_s}, \quad (3.1)$$

where $(i_1, j_1) \leq \dots \leq (i_r, j_r)$, $(m_1, n_1) \leq \dots \leq (m_s, n_s)$ and X is an arbitrary element of \mathfrak{h}_{l+1} . The monomials of the same form with $X \in \mathfrak{h}_{l+1}$, form a Poincaré–Birkhoff–Witt basis of $U_q(\mathfrak{sl}_{l+1})$.

Let us demonstrate that any monomial can be written as a sum of ordered monomials of the form (3.1). We will write the necessary equations in the form of commutation or q -commutation relations. First of all, using (2.1), (2.3) and (2.8), we obtain

$$q^{\nu K_i} E_{mn} q^{-\nu K_i} = q^{\nu \sum_{j=m}^{n-1} c_{ij}} E_{mn}, \quad q^{\nu K_i} F_{mn} q^{-\nu K_i} = q^{-\nu \sum_{j=m}^{n-1} c_{ij}} F_{mn}, \quad (3.2)$$

and, similarly,

$$q^{\nu H_i} E_{mn} q^{-\nu H_i} = q^{\nu \sum_{j=m}^{n-1} a_{ij}} E_{mn}, \quad q^{\nu H_i} F_{mn} q^{-\nu H_i} = q^{-\nu \sum_{j=m}^{n-1} a_{ij}} F_{mn}.$$

To describe the relations allowing one to order the elements E_{ij} and F_{ij} , we follow H. Yamane [27]. Note that for $(i, j), (m, n) \in \Lambda_l$ such that $(i, j) < (m, n)$, there are six cases⁶

$$\mathcal{C}_I : i = m < j < n, \quad \mathcal{C}_{II} : m < i < j < n, \quad \mathcal{C}_{III} : i < m < j = n, \quad (3.3)$$

$$\mathcal{C}_{IV} : i < m < j < n, \quad \mathcal{C}_V : i < j = m < n, \quad \mathcal{C}_{VI} : i < j < m < n. \quad (3.4)$$

Here, the symbol \mathcal{C}_a , $a = I, \dots, VI$, means a branch in $\Lambda_l \times \Lambda_l$, where (i, j) and (m, n) are subject to the corresponding conditions (3.3)–(3.4). In any of these branches, or in a certain union of them, the relations in question are form-invariant.

According to our definitions, which are slightly different from those of the paper [27], we obtain

$$E_{ij} E_{mn} = q^{-1} E_{mn} E_{ij}, \quad ((i, j), (m, n)) \in \mathcal{C}_I \cup \mathcal{C}_{III}, \quad (3.5)$$

$$E_{ij} E_{mn} = E_{mn} E_{ij}, \quad ((i, j), (m, n)) \in \mathcal{C}_{II} \cup \mathcal{C}_{VI}, \quad (3.6)$$

$$E_{ij} E_{mn} - q E_{mn} E_{ij} = E_{in}, \quad ((i, j), (m, n)) \in \mathcal{C}_V, \quad (3.7)$$

$$E_{ij} E_{mn} - E_{mn} E_{ij} = -\kappa_q E_{in} E_{mj}, \quad ((i, j), (m, n)) \in \mathcal{C}_{IV}. \quad (3.8)$$

Similarly, we have

$$F_{ij} F_{mn} = q^{-1} F_{mn} F_{ij}, \quad ((i, j), (m, n)) \in \mathcal{C}_I \cup \mathcal{C}_{III}, \quad (3.9)$$

$$F_{ij} F_{mn} = F_{mn} F_{ij}, \quad ((i, j), (m, n)) \in \mathcal{C}_{II} \cup \mathcal{C}_{VI}, \quad (3.10)$$

$$F_{ij} F_{mn} - q F_{mn} F_{ij} = -q F_{in}, \quad ((i, j), (m, n)) \in \mathcal{C}_V, \quad (3.11)$$

$$F_{ij} F_{mn} - F_{mn} F_{ij} = -\kappa_q F_{in} F_{mj}, \quad ((i, j), (m, n)) \in \mathcal{C}_{IV}. \quad (3.12)$$

⁵ Note that if we define an ordering of the positive roots so that $\alpha_{ij} < \alpha_{mn}$ if $(i, j) < (m, n)$ we will have a normal ordering in the sense of [24, 25], see also [26].

⁶ Since we chose for the elements of Λ_l the colexicographical order, but not the lexicographical one as in [27], we define \mathcal{C}_{II} in a different way in comparison with [27].

It is convenient to write (3.11) also as

$$F_{mn} F_{ij} - q^{-1} F_{ij} F_{mn} = F_{in}.$$

Further, we obtain

$$[E_{ij}, F_{ij}] = \kappa_q^{-1} (q^{H_{ij}} - q^{-H_{ij}}),$$

where, and in what follows, $[,]$ means the usual commutator, and we denote

$$q^{\nu H_{ij}} = q^{\nu \sum_{k=i}^{j-1} H_k}, \quad \nu \in \mathbb{C}, \quad (i, j) \in \Lambda_l.$$

We also obtain the following commutation relations:

$$[E_{ij}, F_{mn}] = -q^{-1} F_{jn} q^{-H_{ij}}, \quad ((i, j), (m, n)) \in \mathcal{C}_I, \quad (3.13)$$

$$[E_{ij}, F_{mn}] = q E_{im} q^{-H_{mn}} = q^{-H_{mn}} E_{im}, \quad ((i, j), (m, n)) \in \mathcal{C}_{III}, \quad (3.14)$$

$$[E_{ij}, F_{mn}] = 0, \quad ((i, j), (m, n)) \in \mathcal{C}_{II} \cup \mathcal{C}_V \cup \mathcal{C}_{VI}, \quad (3.15)$$

$$[E_{ij}, F_{mn}] = \kappa_q F_{jn} E_{im} q^{-H_{mj}}, \quad ((i, j), (m, n)) \in \mathcal{C}_{IV}. \quad (3.16)$$

Note that in (3.16) the root vectors at the right hand side commute, $[F_{jn}, E_{im}] = 0$ for the given values of the indices, and, besides,

$$q^{-H_{mj}} F_{jn} = q^{-1} F_{jn} q^{-H_{mj}}, \quad q^{-H_{mj}} E_{im} = q E_{im} q^{-H_{mj}}.$$

Interchanging the pairs of indices (i, j) and (m, n) of the root vectors in the above commutation relations, additionally to (3.13)–(3.16) we obtain

$$[E_{mn}, F_{ij}] = -q q^{H_{ij}} E_{jn} = -E_{jn} q^{H_{ij}}, \quad ((i, j), (m, n)) \in \mathcal{C}_I, \quad (3.17)$$

$$[E_{mn}, F_{ij}] = q^{-1} q^{H_{mn}} F_{im} = F_{im} q^{H_{mn}}, \quad ((i, j), (m, n)) \in \mathcal{C}_{III}, \quad (3.18)$$

$$[E_{mn}, F_{ij}] = 0, \quad ((i, j), (m, n)) \in \mathcal{C}_{II} \cup \mathcal{C}_V \cup \mathcal{C}_{VI}, \quad (3.19)$$

$$[E_{mn}, F_{ij}] = -\kappa_q F_{im} E_{jn} q^{H_{mj}}, \quad ((i, j), (m, n)) \in \mathcal{C}_{IV}. \quad (3.20)$$

Again, the root vectors at the right hand side of (3.20) commute for the given values of the indices, $[F_{im}, E_{jn}] = 0$, and similarly to the preceding we have

$$q^{H_{mj}} E_{jn} = q^{-1} E_{jn} q^{H_{mj}}, \quad q^{H_{mj}} F_{im} = q F_{im} q^{H_{mj}}.$$

Now, it is easy to demonstrate that equations (3.2) and (3.5)–(3.20) are sufficient to rewrite any monomial as a sum of ordered monomials of the form (3.1). In the case of the quantum group $U_q(\mathfrak{sl}_{l+1})$ we obtain the same result using the ordered monomials of the form (3.1) with $X \in \mathfrak{h}_{l+1}$.

4. Defining Verma $U_q(\mathfrak{gl}_{l+1})$ -module relations

Given $\lambda \in \mathfrak{k}_{l+1}^*$, denote by \tilde{V}^λ the corresponding Verma $U_q(\mathfrak{gl}_{l+1})$ -module. This is a highest weight module with the highest weight vector v^λ satisfying the relations

$$E_i v^\lambda = 0, \quad i = 1, \dots, l, \quad q^X v^\lambda = q^{\langle \lambda, X \rangle} v^\lambda, \quad X \in \mathfrak{k}_{l+1}, \quad \lambda \in \mathfrak{k}_{l+1}^*. \quad (4.1)$$

Below we identify the highest weight λ with the set of its components

$$\lambda_i = \langle \lambda, K_i \rangle.$$

We denote by $\tilde{\pi}^\lambda$ the representation of $U_q(\mathfrak{gl}_{l+1})$ corresponding to \tilde{V}^λ . The structure and properties of \tilde{V}^λ and $\tilde{\pi}^\lambda$ for $l = 1$ and $l = 2$ are considered in much detail in our papers [19–22, 28]. Here we deal with the case of general l .

Denote by \mathbf{m} the $l(l+1)/2$ -tuple of non-negative integers m_{ij} , $(i, j) \in \Lambda_l$, arranged in the colexicographical order of (i, j) . More explicitly,

$$\mathbf{m} = (m_{12}, m_{13}, m_{23}, \dots, m_{1j}, \dots, m_{j-1,j}, \dots, m_{1,l+1}, \dots, m_{l,l+1}).$$

The vectors

$$v_{\mathbf{m}} = F_{12}^{m_{12}} F_{13}^{m_{13}} F_{23}^{m_{23}} \dots F_{1,j}^{m_{1,j}} \dots F_{j-1,j}^{m_{j-1,j}} \dots F_{1,l+1}^{m_{1,l+1}} \dots F_{l,l+1}^{m_{l,l+1}} v_0,$$

where for consistency we denote $v_0 = v^\lambda$, form a basis of \tilde{V}^λ . The relations describing the action of the generators $q^{\nu K_i}$, E_i and F_i of the quantum group $U_q(\mathfrak{gl}_{l+1})$ on a general basis vector $v_{\mathbf{m}}$ is what we exactly mean under defining $U_q(\mathfrak{gl}_{l+1})$ -module relations.

We first obtain how the generators $q^{\nu K_i}$ act on the basis vectors. Using (3.2) and taking into account the second relation of (4.1), we derive

$$q^{\nu K_i} v_{\mathbf{m}} = q^{\nu(\lambda_i - \sum_{j=1}^l \sum_{r=1}^j \sum_{s=j+1}^{l+1} c_{ij} m_{rs})} v_{\mathbf{m}}, \quad i = 1, \dots, l+1.$$

Rearranging the summations in the exponential at the right hand side of the above equation, the same result can be written as

$$q^{\nu K_i} v_{\mathbf{m}} = q^{\nu(\lambda_i - \sum_{s=2}^{l+1} \sum_{r=1}^{s-1} \sum_{j=r}^{s-1} m_{rs} c_{ij})} v_{\mathbf{m}}, \quad i = 1, \dots, l+1.$$

Recalling the exact form (2.2) of the quantities c_{ij} , we can make the above formulas more explicit and eligible for further use. We have

$$q^{\nu K_i} v_{\mathbf{m}} = q^{\nu(\lambda_i + \sum_{k=1}^{i-1} m_{ki} - \sum_{k=i+1}^{l+1} m_{ik})} v_{\mathbf{m}}, \quad i = 1, \dots, l+1. \quad (4.2)$$

Then, for $q^{\nu H_i}$, $i = 1, \dots, l$, where $H_i = K_i - K_{i+1}$, we obtain

$$q^{\nu H_i} v_{\mathbf{m}} = q^{\nu[\lambda_i - \lambda_{i+1} + \sum_{k=1}^{i-1} (m_{ki} - m_{k,i+1}) - 2m_{i,i+1} - \sum_{k=i+2}^{l+1} (m_{ik} - m_{i+1,k})]} v_{\mathbf{m}}. \quad (4.3)$$

To define the action of the generators $F_k = F_{k,k+1}$ on the basis vectors we need subsidiary formulas following from relations (3.9)–(3.12). These are

$$F_{k,k+1} F_{ik}^{m_{ik}} = q^{-m_{ik}} F_{ik}^{m_{ik}} F_{k,k+1} + [m_{ik}]_q F_{ik}^{m_{ik}-1} F_{i,k+1}$$

for $i = 1, \dots, k-1$ and $k = 2, \dots, l$, and

$$F_{j,k+1} F_{i,k+1}^{m_{i,k+1}} = q^{m_{i,k+1}} F_{i,k+1}^{m_{i,k+1}} F_{j,k+1}, \quad i < j = 2, \dots, k.$$

Applying these formulas for all $k = 1, \dots, l$, we obtain

$$F_{i,i+1} v_{\mathbf{m}} = q^{-\sum_{k=1}^{i-1} (m_{ki} - m_{k,i+1})} v_{\mathbf{m} + \epsilon_{i,i+1}} + \sum_{j=1}^{i-1} q^{-\sum_{k=1}^{j-1} (m_{ki} - m_{k,i+1})} [m_{ji}]_q v_{\mathbf{m} - \epsilon_{ji} + \epsilon_{j,i+1}}. \quad (4.4)$$

Here and below $\mathbf{m} + \nu \epsilon_{ij}$ means shifting by ν the entry m_{ij} in the $l(l+1)/2$ -tuple \mathbf{m} .

To define the action of the generators $E_k = E_{k,k+1}$, $k = 1, \dots, l$, on the basis vectors we mainly need the following subsidiary formulas obtained from equations (3.13)–(3.16) and (3.17)–(3.20):

$$\begin{aligned}
E_{k,k+1} F_{k,k+1}^{m_{k,k+1}} &= F_{k,k+1}^{m_{k,k+1}} E_{k,k+1} + [m_{k,k+1}]_q F_{k,k+1}^{m_{k,k+1}-1} [H_k - m_{k,k+1} + 1]_q, \\
E_{k,k+1} F_{kj}^{m_{kj}} &= F_{kj}^{m_{kj}} E_{k,k+1} - q^{m_{kj}-2} [m_{kj}]_q F_{kj}^{m_{kj}-1} F_{k+1,j} q^{-H_k}, \\
E_{k,k+1} F_{i,k+1}^{m_{i,k+1}} &= F_{i,k+1}^{m_{i,k+1}} E_{k,k+1} + [m_{i,k+1}]_q F_{ik} F_{i,k+1}^{m_{i,k+1}-1} q^{H_k}
\end{aligned}$$

for all possible values of i, j and k . These equations, supplied with (4.3), allow us to derive the desirable formula. We obtain

$$\begin{aligned}
E_{i,i+1} v_m &= [\lambda_i - \lambda_{i+1} - \sum_{j=i+2}^{l+1} (m_{ij} - m_{i+1,j}) - m_{i,i+1} + 1]_q [m_{i,i+1}]_q v_{m-\epsilon_{i,i+1}} \\
&\quad + q^{\lambda_i - \lambda_{i+1} - 2m_{i,i+1} - \sum_{j=i+2}^{l+1} (m_{ij} - m_{i+1,j})} \sum_{j=1}^{i-1} q^{\sum_{k=j+1}^{l-1} (m_{ki} - m_{k,i+1})} [m_{j,i+1}]_q v_{m-\epsilon_{j,i+1}+\epsilon_{ji}} \\
&\quad - \sum_{j=i+2}^{l+1} q^{-\lambda_i + \lambda_{i+1} - 2 + \sum_{k=j}^{l+1} (m_{ik} - m_{i+1,k})} [m_{ij}]_q v_{m-\epsilon_{ij}+\epsilon_{i+1,j}}
\end{aligned} \tag{4.5}$$

for all $i = 1, \dots, l$.

To construct representations of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ we will also need in section 5 the action of the specific root vectors $F_{1,l+1}$ and $E_{1,l+1}$ on the basis vectors v_m . Using the formulas

$$F_{1,l+1} F_{lj}^{m_{lj}} = q^{m_{lj}} F_{lj}^{m_{lj}} F_{1,l+1}, \quad j = 2, \dots, l,$$

and

$$F_{1,l+1} F_{ij}^{m_{ij}} = F_{ij}^{m_{ij}} F_{1,l+1}, \quad (i, j) \in \Lambda_l,$$

following from equations (3.9)–(3.12), we obtain

$$F_{1,l+1} v_m = q^{\sum_{i=2}^l m_{li}} v_{m+\epsilon_{1,l+1}}. \tag{4.6}$$

The corresponding formula for the action of $E_{1,l+1}$ is given in the appendix.

Note that \tilde{V}^λ and $\tilde{\pi}^\lambda$ are infinite dimensional for the general weights $\lambda \in \mathfrak{k}_{l+1}^*$. However, if the weights are such that $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$ for all $i = 1, \dots, l$, there is a maximal submodule, such that the respective quotient module is finite dimensional. We denote such $U_q(\mathfrak{gl}_{l+1})$ -module and the corresponding representation by V^λ and π^λ , respectively. The reduction to the special linear case from the general linear one can be achieved simply by replacing relation (4.2) by (4.3) in the above module relations.

5. Quantum loop algebras $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ and some their representations

For the construction and investigation of quantum integrable systems one often uses finite and infinite dimensional representations of the quantum loop algebra $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$. The relevant representations are usually obtained from Verma representations of the quantum group $U_q(\mathfrak{gl}_{l+1})$. Let us describe the corresponding procedure.

To define the quantum loop algebra $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ it is convenient to start with the definition of the quantum group $U_q(\tilde{\mathcal{L}}(\mathfrak{sl}_{l+1}))$. Remind that the Lie algebra $\tilde{\mathcal{L}}(\mathfrak{sl}_{l+1})$ is a special extension of the loop algebra $\mathcal{L}(\mathfrak{sl}_{l+1})$ by a one-dimensional center $\mathbb{C}c$ [29]. For any $i = 1, \dots, l$ denote by h_i the image of the Cartan elements H_i of \mathfrak{sl}_{l+1} under the natural embedding of \mathfrak{sl}_{l+1} into $\tilde{\mathcal{L}}(\mathfrak{sl}_{l+1})$. We will also use the notation

$$\tilde{\mathfrak{h}}_{l+1} = \mathbb{C}c \oplus \bigoplus_{i=1}^l \mathbb{C}h_i$$

for the ‘Cartan subalgebra’ of $\tilde{\mathcal{L}}(\mathfrak{sl}_{l+1})$. Introducing the element $h_0 = c - \sum_{i=1}^l h_i$, we obtain a more symmetric expression:

$$\tilde{\mathfrak{h}}_{l+1} = \bigoplus_{i=0}^l \mathbb{C}h_i.$$

We define the ‘simple roots’ α_i , $i = 0, 1, \dots, l$, of $\tilde{\mathcal{L}}(\mathfrak{sl}_{l+1})$ as the elements of $\tilde{\mathfrak{h}}_{l+1}^*$ satisfying the equation

$$\langle \alpha_i, h_j \rangle = \tilde{a}_{ji},$$

where \tilde{a}_{ij} are the entries of the generalized Cartan matrix of an affine Lie algebra of type $A_l^{(1)}$.

The quantum group $U_q(\tilde{\mathcal{L}}(\mathfrak{sl}_{l+1}))$ is a unital associative \mathbb{C} -algebra generated by the elements e_i, f_i , $i = 0, 1, \dots, l$, and q^x , $x \in \tilde{\mathfrak{h}}_{l+1}$, satisfying certain defining relations. These are the following commutation relations:

$$\begin{aligned} q^0 &= 1, & q^{x_1} q^{x_2} &= q^{x_1 + x_2}, \\ q^x e_i q^{-x} &= q^{\langle \alpha_i, x \rangle} e_i, & q^x f_i q^{-x} &= q^{-\langle \alpha_i, x \rangle} f_i, \\ [e_i, f_j] &= \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \end{aligned}$$

supplemented by the Serre relations:

$$\sum_{k=0}^{1-\tilde{a}_{ij}} (-1)^k (e_i)^{(1-\tilde{a}_{ij}-k)} e_j (e_i)^{(k)} = 0, \quad \sum_{k=0}^{1-\tilde{a}_{ij}} (-1)^k (f_i)^{(1-\tilde{a}_{ij}-k)} f_j (f_i)^{(k)} = 0.$$

Here we use the notation $(e_i)^{(n)} = (e_i)^n / [n]_q!$ and $(f_i)^{(n)} = (f_i)^n / [n]_q!$.

The quantum group $U_q(\tilde{\mathcal{L}}(\mathfrak{sl}_{l+1}))$ does not have any finite dimensional representations with $q^{\nu c}$ acting nontrivially [30, 31]. Therefore, to construct quantum integrable systems with finite dimensional state space one should use representations with trivial action of $q^{\nu c}$. In fact, it appears more convenient to use the quantum loop algebra $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ defined as the quotient

$$U_q(\mathcal{L}(\mathfrak{sl}_{l+1})) = U_q(\tilde{\mathcal{L}}(\mathfrak{sl}_{l+1})) / \langle q^{\nu c} - 1 \rangle_{\nu \in \mathbb{C}}.$$

We consider the quantum loop algebra $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ in terms of the same generators and defining relations as $U_q(\tilde{\mathcal{L}}(\mathfrak{sl}_{l+1}))$, with the additional relations

$$q^{\nu c} = 1, \quad \nu \in \mathbb{C}^\times. \quad (5.1)$$

As the quantum group $U_q(\mathfrak{gl}_{l+1})$, also the quantum loop algebra $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ is a Hopf algebra with respect to appropriately defined co-multiplication, antipode and counit.

To construct representations of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$, not only finite dimensional but also infinite dimensional ones, it is common to use the Jimbo’s homomorphism ε from the quantum loop algebra $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ to the quantum group $U_q(\mathfrak{gl}_{l+1})$ defined by the equations [23]

$$\varepsilon(q^{\nu h_0}) = q^{\nu(K_{l+1} - K_1)}, \quad \varepsilon(q^{\nu h_i}) = q^{\nu(K_i - K_{i+1})}, \quad (5.2)$$

$$\varepsilon(e_0) = F_{1, l+1} q^{K_1 + K_{l+1}}, \quad \varepsilon(e_i) = E_{i, i+1}, \quad (5.3)$$

$$\varepsilon(f_0) = E_{1,l+1} q^{-K_1-K_{l+1}}, \quad \varepsilon(f_i) = F_{i,i+1}, \quad (5.4)$$

where i runs from 1 to l . If π is a representation of $U_q(\mathfrak{gl}_{l+1})$ then $\pi \circ \varepsilon$ is a representation of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$.

In fact, in applications to the theory of quantum integrable systems, one usually considers families of representations parametrized by a complex parameter called a spectral parameter. We introduce a spectral parameter with the help of a family of mappings $\Gamma_\zeta: U_q(\mathcal{L}(\mathfrak{sl}_{l+1})) \rightarrow U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$, $\zeta \in \mathbb{C}^\times$. Explicitly, Γ_ζ is defined by its action on the generators as

$$\Gamma_\zeta(q^x) = q^x, \quad \Gamma_\zeta(e_i) = \zeta^{s_i} e_i, \quad \Gamma_\zeta(f_i) = \zeta^{-s_i} f_i,$$

where s_i are arbitrary integers. We denote the total sum of these integers by s . Now, for any representation φ of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ we define the corresponding family φ_ζ of representations as

$$\varphi_\zeta = \varphi \circ \Gamma_\zeta.$$

Of our special interest are the families of representations $(\tilde{\varphi}^\lambda)_\zeta$ and $(\varphi^\lambda)_\zeta$ related to infinite and finite dimensional representations $\tilde{\pi}^\lambda$ and π^λ of $U_q(\mathfrak{gl}_{l+1})$, because they play a special role in the theory of quantum integrable systems. These families are defined as

$$(\tilde{\varphi}^\lambda)_\zeta = \tilde{\pi}^\lambda \circ \varepsilon \circ \Gamma_\zeta, \quad (\varphi^\lambda)_\zeta = \pi^\lambda \circ \varepsilon \circ \Gamma_\zeta.$$

Let us consider the corresponding defining $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ -module relations.

First, using (4.2) and (5.2), we obtain

$$q^{\nu h_0} v_m = q^{\nu[\lambda_{l+1}-\lambda_1+\sum_{i=2}^l(m_{1i}+m_{i,l+1})+2m_{1,l+1}]} v_m, \quad (5.5)$$

$$q^{\nu h_i} v_m = q^{\nu[\lambda_i-\lambda_{i+1}+\sum_{k=1}^{i-1}(m_{ki}-m_{k,i+1})-2m_{i,i+1}-\sum_{k=i+2}^{l+1}(m_{ik}-m_{i+1,k})]} v_m. \quad (5.6)$$

Further, taking into account that

$$q^{K_1+K_{l+1}} v_m = q^{\lambda_1+\lambda_{l+1}-\sum_{i=2}^l(m_{1i}-m_{i,l+1})} v_m$$

and using (4.6), we obtain from the first equation of (5.3) the module relation for e_0 ,

$$e_0 v_m = \zeta^{s_0} q^{\lambda_1+\lambda_{l+1}+\sum_{i=2}^l m_{i,l+1}} v_{m+\epsilon_{1,l+1}}. \quad (5.7)$$

The module relations for e_i , $i = 1, \dots, l$, follow from the second equation of (5.3) with account of equation (4.5),

$$\begin{aligned} e_i v_m &= \zeta^{s_i} [\lambda_i - \lambda_{i+1} - \sum_{j=i+2}^{l+1} (m_{ij} - m_{i+1,j}) - m_{i,i+1} + 1]_q [m_{i,i+1}]_q v_{m-\epsilon_{i,i+1}} \\ &\quad + \zeta^{s_i} q^{\lambda_i-\lambda_{i+1}-2m_{i,i+1}-\sum_{j=i+2}^{l+1}(m_{ij}-m_{i+1,j})} \sum_{j=1}^{i-1} q^{\sum_{k=j+1}^{i-1}(m_{ki}-m_{k,i+1})} [m_{j,i+1}]_q v_{m-\epsilon_{j,i+1}+\epsilon_{ji}} \\ &\quad - \zeta^{s_i} \sum_{j=i+2}^{l+1} q^{-\lambda_i+\lambda_{i+1}-2+\sum_{k=j}^{l+1}(m_{ik}-m_{i+1,k})} [m_{ij}]_q v_{m-\epsilon_{ij}+\epsilon_{i+1,j}}. \end{aligned} \quad (5.8)$$

Using (4.4) and the second relation from (5.4), we obtain the action of the generators f_i on the basis vectors v_m :

$$\begin{aligned} f_i v_m &= \zeta^{-s_i} q^{-\sum_{j=1}^{i-1} (m_{ji} - m_{j,i+1})} v_{m+\epsilon_{i,i+1}} \\ &+ \zeta^{-s_i} \sum_{j=1}^{i-1} q^{-\sum_{k=1}^{j-1} (m_{ki} - m_{k,i+1})} [m_{ji}]_q v_{m-\epsilon_{ji}+\epsilon_{j,i+1}}. \end{aligned} \quad (5.9)$$

One can also obtain the action of f_0 on v_m using equations (A.1) and (4.2) and the first one of the maps given by (5.4).

Now, relations (5.5)–(5.8), and (5.9) together with an expression for $f_0 v_m$ constructed as just mentioned above, form the basic $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ -module relations.

6. Degenerations of the shifted $U_q(\mathfrak{b}_+)$ -modules

From the point of view of quantum integrable systems, representations of the Borel subalgebras of the quantum groups under consideration are most interesting. There are two standard Borel subalgebras of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$, the positive Borel subalgebra $U_q(\mathfrak{b}_+)$ generated by e_i , $i = 0, 1, \dots, l$ and q^x , $x \in \mathfrak{h}_{l+1}$, and the negative Borel subalgebra $U_q(\mathfrak{b}_-)$ generated by f_i , $i = 0, 1, \dots, l$ and q^x , $x \in \mathfrak{h}_{l+1}$. We restrict ourselves to the case of the positive Borel subalgebra used in our consideration of universal integrability objects [19–22].

Certainly, the restriction of any representation of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ to $U_q(\mathfrak{b}_+)$ is a representation of $U_q(\mathfrak{b}_+)$. In particular, we can consider the restriction of the representations $(\tilde{\varphi}^\lambda)_\zeta$ and $(\varphi^\lambda)_\zeta$. Here relations (5.5)–(5.8) constitute the corresponding $U_q(\mathfrak{b}_+)$ -module relations. The representations $(\tilde{\varphi}^\lambda)_\zeta$ and $(\varphi^\lambda)_\zeta$ are used for the construction of very important integrability objects called transfer operators. Besides, there are no less important integrability objects called Q -operators. The representations used for the construction of the Q -operators are essentially different. However, for the quantum integrable systems related to the quantum groups $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ the latter can be obtained from the former as certain degenerations, see, for example, [18] and [21, 22].

The degenerations in question are obtained by sending each difference $\lambda_i - \lambda_{i+1}$, $i = 1, \dots, l$, to positive or negative infinity. However, looking at (5.5) and (5.6) we see that it gives either infinity or zero for the action of the corresponding elements $q^{\nu h_i}$. To overcome this difficulty we use the notion of a shifted representation.

Let φ be a representation of $U_q(\mathfrak{b}_+)$ and $\xi \in \tilde{\mathfrak{h}}_{l+1}^*$. Then the relations

$$\varphi[\xi](e_i) = \varphi(e_i), \quad \varphi[\xi](q^x) = q^{\langle \xi, x \rangle} \varphi(q^x) \quad (6.1)$$

define a representation $\varphi[\xi]$ of $U_q(\mathfrak{b}_+)$ called a shifted representation. Note that due to (5.1) the element ξ must satisfy the equation

$$\langle \xi, c \rangle = 0.$$

We see that the only difference between the shifted and initial representations appears in the factor $q^{\langle \xi, x \rangle}$.

Now we consider the $U_q(\mathfrak{b}_+)$ -module defined by relations (5.5)–(5.8), and perform there a shift according to definition (6.1), with ξ specified by the equations

$$\langle \xi, h_0 \rangle = -\lambda_{l+1} + \lambda_1, \quad \langle \xi, h_i \rangle = -\lambda_i + \lambda_{i+1}, \quad i = 1, \dots, l.$$

Obviously, this shift has an effect only on the relations describing the action of the generators $q^{\nu h_i}$, $i = 0, 1, \dots, l$. Now we have

$$\begin{aligned} q^{\nu h_0} v_m &= q^{\nu(\sum_{j=2}^l (m_{lj} + m_{j,l+1}) + 2m_{1,l+1})} v_m, \\ q^{\nu h_i} v_m &= q^{\nu(\sum_{j=1}^{i-1} (m_{ji} - m_{j,i+1}) - 2m_{i,i+1} - \sum_{j=i+2}^{l+1} (m_{ij} - m_{i+1,j}))} v_m. \end{aligned}$$

Since these relations do not contain λ_i , $i = 1, \dots, l+1$, we can allow the differences $\lambda_i - \lambda_{i+1}$, $i = 1, \dots, l$, to go in these relations to positive or negative infinity. To be concrete, we are going to consider the limit

$$\lambda_i - \lambda_{i+1} \rightarrow -\infty, \quad i = 1, \dots, l. \quad (6.2)$$

The representations which can be obtained with other choices can be obtained then with the help of automorphisms of $U_q(\mathfrak{b}_+)$.

It is clear that some problems with the limit (6.2) for relations (5.7) and (5.8) remain. Let us define a new basis in the representation space formed by the vectors

$$w_m = c_m v_m,$$

where

$$c_m = q^{\sum_{i=1}^l [\lambda_i - \lambda_{i+1} + 1 + (2\lambda_{l+1} - l)s_i/s]} \sum_{j=1}^i \sum_{k=i+1}^{l+1} m_{jk}.$$

Here we note that

$$c_{m+\nu\epsilon_{ij}} = q^{\nu \sum_{k=i}^{j-1} [\lambda_k - \lambda_{k+1} + 1 + (2\lambda_{l+1} - l)s_k/s]} c_m.$$

Recall that s denotes the total sum of the integers s_i , $i = 0, 1, \dots, l$. Using this equation, we obtain from (5.7)–(5.8) the following $U_q(\mathfrak{b}_+)$ -module relations in the new basis,

$$e_0 w_m = \tilde{\zeta}^{s_0} q^{\sum_{i=2}^l m_{i,l+1}} w_{m+\epsilon_{1,l+1}}$$

and

$$\begin{aligned} e_i w_m &= \tilde{\zeta}^{s_i} \kappa_q^{-1} \left(q^{2(\lambda_i - \lambda_{i+1}) + 2 - \sum_{j=i+2}^{l+1} (m_{ij} - m_{i+1,j}) - m_{i,i+1}} \right. \\ &\quad \left. - q^{\sum_{j=i+2}^{l+1} (m_{ij} - m_{i+1,j}) + m_{i,i+1}} \right) [m_{i,i+1}]_q w_{m-\epsilon_{i,i+1}} \\ &\quad + \tilde{\zeta}^{s_i} q^{2(\lambda_i - \lambda_{i+1}) + 1 - 2m_{i,i+1} - \sum_{j=i+2}^{l+1} (m_{ij} - m_{i+1,j})} \\ &\quad \times \sum_{j=1}^{i-1} q^{\sum_{k=j+1}^{i-1} (m_{ki} - m_{k,i+1})} [m_{j,i+1}]_q w_{m-\epsilon_{j,i+1} + \epsilon_{ji}} \\ &\quad - \tilde{\zeta}^{s_i} \sum_{j=i+2}^{l+1} q^{\sum_{k=j}^{l+1} (m_{ik} - m_{i+1,k}) - 1} [m_{ij}]_q w_{m-\epsilon_{ij} + \epsilon_{i+1,j}}, \end{aligned}$$

where $\tilde{\zeta}$ is the new spectral parameter defined as

$$\tilde{\zeta} = q^{(2\lambda_{l+1} - l)/s} \zeta.$$

Now we can consider the infinite limit (6.2). The final result is a degeneration of the shifted $U_q(\mathfrak{b}_+)$ -module described by the relations

$$q^{\nu h_0} w_m = q^{\nu(\sum_{i=2}^l (m_{li} + m_{i,l+1}) + 2m_{1,l+1})} w_m, \quad (6.3)$$

$$q^{\nu h_i} w_m = q^{\nu(\sum_{j=1}^{i-1}(m_{ji}-m_{j,i+1})-2m_{i,i+1}-\sum_{j=i+2}^{l+1}(m_{ij}-m_{i+1,j}))} w_m \quad (6.4)$$

and

$$e_0 w_m = \tilde{\zeta}^{s_0} q^{\sum_{i=2}^l m_{i,l+1}} w_{m+\epsilon_{1,l+1}}, \quad (6.5)$$

$$\begin{aligned} e_i w_m = & -\tilde{\zeta}^{s_i} \kappa_q^{-1} q^{\sum_{j=i+2}^{l+1}(m_{ij}-m_{i+1,j})+m_{i,i+1}} [m_{i,i+1}]_q w_{m-\epsilon_{i,i+1}} \\ & - \tilde{\zeta}^{s_i} \sum_{j=i+2}^{l+1} q^{\sum_{k=j}^{l+1}(m_{ik}-m_{i+1,k})-1} [m_{ij}]_q w_{m-\epsilon_{ij}+\epsilon_{i+1,j}}. \end{aligned} \quad (6.6)$$

These are our main $U_q(\mathfrak{b}_+)$ -module relations on the basis of which we will make all the subsequent constructions.

7. Factoring out the submodules

We denote by ρ'' the representation of $U_q(\mathfrak{b}_+)$ defined by the relations

$$\begin{aligned} q^{\nu h_0} v_m &= q^{\nu(\sum_{j=2}^l(m_{1j}+m_{j,l+1})+2m_{1,l+1})} v_m, \\ q^{\nu h_i} v_m &= q^{\nu(\sum_{j=1}^{i-1}(m_{ji}-m_{j,i+1})-2m_{i,i+1}-\sum_{j=i+2}^{l+1}(m_{ij}-m_{i+1,j}))} v_m, \\ e_0 v_m &= q^{\sum_{i=2}^l m_{i,l+1}} v_{m+\epsilon_{1,l+1}}, \\ e_i v_m &= -\kappa_q^{-1} q^{\sum_{j=i+2}^{l+1}(m_{ij}-m_{i+1,j})+m_{i,i+1}} [m_{i,i+1}]_q v_{m-\epsilon_{i,i+1}} \\ &\quad - \sum_{j=i+2}^{l+1} q^{\sum_{k=j}^{l+1}(m_{ik}-m_{i+1,k})-1} [m_{ij}]_q v_{m-\epsilon_{ij}+\epsilon_{i+1,j}}, \end{aligned}$$

where $i = 1, \dots, l$. The $U_q(\mathfrak{b}_+)$ -module corresponding to ρ'' is denoted by W'' . It is clear that the representation described by (6.3)–(6.6) is $(\rho'')_{\tilde{\zeta}}$.

The representation ρ'' is reducible. Indeed, let us define an $l(l-1)/2$ -tuple \mathbf{p} of nonnegative integers

$$\mathbf{p} = (p_{12}, p_{13}, p_{23}, \dots, p_{1j}, \dots, p_{j-1,j}, \dots, p_{1,l}, \dots, p_{l-1,l}), \quad (7.1)$$

such that

$$\sum_{k=1}^i (-1)^{i-k} p_{kj} \geq 0, \quad 1 \leq i < j \leq l.$$

The subspaces generated by the vectors v_m with the indices restricted by the inequalities

$$m_{i-1,j} + m_{ij} \leq p_{ij}, \quad 1 \leq i < j \leq l,$$

where we assume that $m_{0j} = 0$, are invariant with respect to the action of the quantum group $U_q(\mathfrak{b}_+)$. We denote such $U_q(\mathfrak{b}_+)$ -submodule by $W''_{\mathbf{p}}$.

Now we introduce a partial order for the $l(l-1)/2$ -tuples \mathbf{p} of the form (7.1) by assuming that $\mathbf{p}' < \mathbf{p}$ if $p'_{ij} < p_{ij}$ for all possible $i = 1, \dots, l-1$ and $j = 2, \dots, l$.

Further, we denote by ρ' the representation of $U_q(\mathfrak{b}_+)$ defined by the relations

$$q^{\nu h_0} v_m = q^{\nu(2m_1 + \sum_{j=2}^l m_j)} v_m, \quad (7.2)$$

$$q^{\nu h_i} v_{\mathbf{m}} = q^{\nu(m_{i+1}-m_i)} v_{\mathbf{m}}, \quad i = 1, \dots, l-1, \quad (7.3)$$

$$q^{\nu h_l} v_{\mathbf{m}} = q^{-\nu(2m_l + \sum_{i=1}^{l-1} m_i)} v_{\mathbf{m}}, \quad (7.4)$$

$$e_0 v_{\mathbf{m}} = q^{\sum_{j=2}^l m_j} v_{\mathbf{m}+\epsilon_1}, \quad (7.5)$$

$$e_i v_{\mathbf{m}} = -q^{m_i-m_{i+1}-1} [m_i]_q v_{\mathbf{m}-\epsilon_i+\epsilon_{i+1}}, \quad i = 1, \dots, l-1, \quad (7.6)$$

$$e_l v_{\mathbf{m}} = -\kappa_q^{-1} q^{m_l} [m_l]_q v_{\mathbf{m}-\epsilon_l}, \quad (7.7)$$

where \mathbf{m} denotes now the l -tuple of nonnegative integers (m_1, \dots, m_l) , and $\mathbf{m} + \nu\epsilon_i$ means the respective shift of m_i in this ordered tuple. The corresponding $U_q(\mathfrak{b}_+)$ -module is denoted by W' . We see that there are isomorphisms

$$W_p'' / \bigcup_{p' \leq p} W_{p'}'' \cong W'[\xi_p],$$

where the shift ξ_p is determined by the relations

$$\begin{aligned} \langle \xi_p, h_0 \rangle &= \sum_{j=2}^l p_{1j}, \\ \langle \xi_p, h_i \rangle &= \sum_{k=1}^{i-1} \sum_{j=1}^k (-1)^{k-j} (p_{ji} - p_{j,i+1}) - 2 \sum_{j=1}^i (-1)^{i-j} p_{j,i+1} \\ &\quad - 2 \sum_{k=i+2}^l \sum_{j=1}^i (-1)^{i-j} p_{kj} + \sum_{k=i+2}^l p_{i+1,k}, \quad i = 1, \dots, l-1, \\ \langle \xi_p, h_l \rangle &= \sum_{j=1}^{l-1} \sum_{k=1}^j (-1)^{j-k} p_{kl}. \end{aligned}$$

It should be noted here that the integers m_i in the new multi-index \mathbf{m} in the module relations (7.2)–(7.7) are nothing but the former $m_{i,l+1}$, $i = 1, \dots, l$, survived the reduction to the factor module. We have denoted $m_{i,l+1}$ shortly by m_i after the reduction, and used a similar simplification also for the shift units $\epsilon_{i,l+1}$, for which we have reserved the notation ϵ_i .

8. Interpretation in terms of q -oscillators

Degenerations of the shifted $U_q(\mathfrak{b}_+)$ -modules have a useful interpretation in terms of the so-called q -oscillators [18–22]. The q -oscillator algebra Osc_q is defined as a unital associative \mathbb{C} -algebra with generators $b^\dagger, b, q^{\nu N}$, $\nu \in \mathbb{C}$, and relations

$$\begin{aligned} q^0 &= 1, & q^{\nu_1 N} q^{\nu_2 N} &= q^{(\nu_1 + \nu_2)N}, \\ q^{\nu N} b^\dagger q^{-\nu N} &= q^\nu b^\dagger, & q^{\nu N} b q^{-\nu N} &= q^{-\nu} b, \\ b^\dagger b &= [N]_q, & b b^\dagger &= [N+1]_q, \end{aligned}$$

see, for example, section 5.1 of [32] and references therein. Here, we again consider the deformation parameter to be $q = \exp \hbar$, where \hbar is a complex number, such that q is not a root of unity.

There are two standard representations of the q -oscillator algebra. They are constructed as follows. One can see that the relations

$$q^{\nu N} v_m = q^{\nu m} v_m, \\ b^\dagger v_m = v_{m+1}, \quad b v_m = [m]_q v_{m-1},$$

supplied with the assumption $v_{-1} = 0$, endow the free vector space generated by the set $\{v_0, v_1, \dots\}$ with the structure of an Osc_q -module. We denote this Osc_q -module by W^+ and the corresponding representation by χ^+ . The other representation of Osc_q is defined by the relations

$$q^{\nu N} v_m = q^{-\nu(m+1)} v_m, \\ b v_m = v_{m+1}, \quad b^\dagger v_m = -[m]_q v_{m-1},$$

where it is assumed again that $v_{-1} = 0$. Similarly as before, these relations endow the free vector space generated by the set $\{v_0, v_1, \dots\}$ with the structure of an Osc_q -module. This Osc_q -module and the corresponding representation are denoted by W^- and χ^- , respectively. However, since the automorphism of Osc_q

$$b \rightarrow b^\dagger, \quad b^\dagger \rightarrow -b, \quad q^{\nu N} \rightarrow q^{-\nu(N+1)}$$

relates these representations, it is actually sufficient to use only one of them.

We consider the tensor product of l copies of the q -oscillator algebra, $\text{Osc}_q \otimes \dots \otimes \text{Osc}_q = (\text{Osc}_q)^{\otimes l}$, and denote

$$b_i = 1 \otimes \dots \otimes b \otimes \dots \otimes 1, \quad b_i^\dagger = 1 \otimes \dots \otimes b^\dagger \otimes \dots \otimes 1, \\ q^{\nu N_i} = 1 \otimes \dots \otimes q^{\nu N} \otimes \dots \otimes 1,$$

where b , b^\dagger and $q^{\nu N}$ occupy only the i -th place of the respective tensor products.

Let us consider the $U_q(\mathfrak{b}_+)$ -module W' and the corresponding representation ρ' given by relations (7.2)–(7.7). Supply W' with the structure of $(\text{Osc}_q)^{\otimes l}$ -module assuming that

$$q^{\nu N_i} v_m = q^{\nu m_i} v_m, \\ b_i^\dagger v_m = v_{m+\epsilon_i}, \quad b_i v_m = [m_i]_q v_{m-\epsilon_i}.$$

Now, the module relations (7.2)–(7.7) can be written in terms of the q -oscillators as follows:

$$q^{\nu h_0} v_m = q^{\nu(2N_1 + \sum_{j=2}^l N_j)} v_m, \\ q^{\nu h_i} v_m = q^{\nu(N_{i+1} - N_i)} v_m, \quad i = 1, \dots, l-1, \\ q^{\nu h_l} v_m = q^{-\nu(2N_l + \sum_{j=1}^{l-1} N_j)} v_m, \\ e_0 v_m = b_1^\dagger q^{\sum_{j=2}^l N_j} v_m, \\ e_i v_m = -b_i b_{i+1}^\dagger q^{N_i - N_{i+1} - 1} v_m, \quad i = 1, \dots, l-1, \\ e_l v_m = -\kappa_q^{-1} b_l q^{N_l} v_m.$$

It is natural now to define a homomorphism $\rho : U_q(\mathfrak{b}_+) \rightarrow \text{Osc}_q^{\otimes l}$ by the relations

$$\rho(q^{\nu h_0}) = q^{\nu(2N_1 + \sum_{j=2}^l N_j)}, \quad \rho(q^{\nu h_i}) = q^{\nu(N_{i+1} - N_i)}, \quad \rho(q^{\nu h_l}) = q^{-\nu(2N_l + \sum_{j=1}^{l-1} N_j)}, \\ \rho(e_0) = b_1^\dagger q^{\sum_{j=2}^l N_j}, \quad \rho(e_i) = -b_i b_{i+1}^\dagger q^{N_i - N_{i+1} - 1}, \quad \rho(e_l) = -\kappa_q^{-1} b_l q^{N_l},$$

where $i = 1, \dots, l-1$. In a sense, the homomorphism ρ plays the role of the Jimbo's homomorphism ε . To get a representation of $U_q(\mathfrak{b}_+)$, one chooses some representation of $(\text{Osc}_q)^{\otimes l}$, and then takes the composition of this representation with ρ . In particular, for the representation ρ' one has

$$\rho' = (\chi^+ \otimes \dots \otimes \chi^+) \circ \rho \circ \Gamma_\zeta.$$

More representations can be obtained using twisting by the automorphisms of $U_q(\mathfrak{b}_+)$. These are the representations used for the construction of the Q -operators.

9. Conclusions

We have analysed the Verma modules over the quantum group $U_q(\mathfrak{gl}_{l+1})$ for arbitrary values of l . The explicit expressions for the action of the generators on the elements of the natural basis have been obtained. The corresponding representations of the quantum loop algebras $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ have been constructed. This has allowed us to find certain representations of the positive Borel subalgebras of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ as degenerations of the shifted representations. These are the representations used in the construction of the so-called Q -operators in the theory of quantum integrable systems. The interpretation of the corresponding simple quotient modules in terms of representations of the q -deformed oscillator algebra has been given. The obtained results can be used for the investigation of quantum integrable systems in the spirit of the papers [15–18] and [19–22]. We expect also applications to the higher rank generalization of the quantum group approach to the construction of correlation functions for integrable spin chain and conformally invariant models [33–37].

The q -oscillator representations of the Borel subalgebras of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ are closely related to the prefundamental representations introduced by D. Hernandez and M. Jimbo [38]. The explicit relation for the case of $U_q(\mathcal{L}(\mathfrak{sl}_2))$ and $U_q(\mathcal{L}(\mathfrak{sl}_3))$ was found in the paper [28].

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Appendix

Acting by $E_{1,l+1}$ on the basis vectors

Here we derive the action of the root vector $E_{1,l+1}$ on the basis of the Verma $U_q(\mathfrak{gl}_{l+1})$ -module defined in section 4. To this end, we first obtain from equations (3.5)–(3.12) and (3.13)–(3.20) the following subsidiary formulas:

$$\begin{aligned} E_{i,l+1} F_{ij}^{m_{ij}} &= F_{ij}^{m_{ij}} E_{i,l+1} - q^{-m_{ij}+1} [m_{ij}]_q F_{ij}^{m_{ij}-1} E_{j,l+1} q^{H_{ij}}, \quad 1 \leq i < j \leq l, \\ E_{i,l+1} F_{i,l+1}^{m_{i,l+1}} &= F_{i,l+1}^{m_{i,l+1}} E_{i,l+1} + [m_{i,l+1}]_q F_{i,l+1}^{m_{i,l+1}-1} [H_{i,l+1} - m_{i,l+1} + 1]_q, \quad 1 \leq i \leq l, \\ E_{j,l+1} F_{ik}^{m_{ik}} &= F_{ik}^{m_{ik}} E_{j,l+1} - \kappa_q [m_{ik}]_q F_{ij} F_{ik}^{m_{ik}-1} E_{k,l+1} q^{H_{jk}}, \quad 1 \leq i < j < k \leq l, \\ E_{j,l+1} F_{i,l+1}^{m_{i,l+1}} &= F_{i,l+1}^{m_{i,l+1}} E_{j,l+1} + [m_{i,l+1}]_q F_{ij} F_{i,l+1}^{m_{i,l+1}-1} q^{H_{j,l+1}}, \quad 1 \leq i < j \leq l, \\ E_{i,l+1} F_{j,l+1}^{m_{j,l+1}} &= F_{j,l+1}^{m_{j,l+1}} E_{i,l+1} + q^{m_{j,l+1}} [m_{j,l+1}]_q F_{j,l+1}^{m_{j,l+1}-1} E_{ij} q^{-H_{j,l+1}}, \quad 1 \leq i < j \leq l, \end{aligned}$$

In the first equation of these five we take into account that $E_{j,l+1} F_{ij} = F_{ij} E_{j,l+1}$ and

$$q^{H_{ij}} F_{ij} = q^{-2} F_{ij} q^{H_{ij}}, \quad E_{j,l+1} q^{H_{ij}} = q^{-2} q^{H_{ij}} E_{j,l+1}.$$

These commutations are needed to move the root vectors $E_{i,l+1}$, $1 \leq i \leq l$, to the right, where they annihilate the highest weight vector v_0 . Due to the last relation a root vector E_{ij} with $1 \leq i < j \leq l$ appears. However, it commutes with the remaining root vectors F_{ij} and annihilates v_0 .

Besides, one needs relations which enable the root vectors F_{ij} arising during the commutations to move back to the left to occupy a proper position according to the prescribed ordering. Such relations are

$$\begin{aligned} F_{jk}^{m_{jk}} F_{ij} &= q^{-m_{jk}} F_{ij} F_{jk}^{m_{jk}} + [m_{jk}]_q F_{ik} F_{jk}^{m_{jk}-1}, \quad 1 \leq i < j < k \leq l+1, \\ F_{jn}^{m_{jn}} F_{ik} &= F_{ik} F_{jn}^{m_{jn}} + \kappa_q q^{m_{jn}-1} [m_{jn}]_q F_{in} F_{jk} F_{jn}^{m_{jn}-1}, \quad 1 \leq i < j < k < n \leq l+1. \end{aligned}$$

To describe the final result, let us introduce some additional notations. First of all for $0 \leq k \leq l-1$ denote

$$\Lambda_{l,k} = \{(i_0 = 1, i_1, \dots, i_k, i_{k+1} = l+1) \in \mathbb{N}^{\times(k+2)} \mid i_0 < i_1 < \dots < i_k < i_{k+1}\}.$$

Note that the first and last elements of the tuples entering $\Lambda_{l,k}$ are fixed. However, it is convenient to consider the tuples of this form. Up to this extension, the set Λ_l introduced in section 3 coincides with $\Lambda_{l,2}$. To make the formulas more compact we denote an element $(i_0, i_1, \dots, i_k, i_{k+1})$ of $\Lambda_{l,k}$ as \mathbf{i} . Further, we use the notation

$$\Psi_{l,k} = \{(\mathbf{i}, \mathbf{j}) \in \Lambda_{l,k} \times \Lambda_{l,k} \mid j_{a-1} < i_a \leq j_a, a = 1, \dots, k\}.$$

Now we can write

$$\begin{aligned} E_{1,l+1} v_m &= \sum_{k=0}^{l-1} \sum_{\substack{(\mathbf{i}, \mathbf{j}) \in \Psi_{l,k} \\ i_k = j_k}} a_{\mathbf{m}|\mathbf{k}, \mathbf{i}, \mathbf{j}} v_{\mathbf{m} - \epsilon_{j_0 j_1} + \epsilon_{i_1 j_1} - \dots - \epsilon_{j_{k-1} j_k} + \epsilon_{i_k j_k} - \epsilon_{j_k j_{k+1}}} \\ &\quad + \sum_{k=1}^{l-1} \sum_{\substack{(\mathbf{i}, \mathbf{j}) \in \Psi_{l,k} \\ i_k \neq j_k}} b_{\mathbf{m}|\mathbf{k}, \mathbf{i}, \mathbf{j}} v_{\mathbf{m} - \epsilon_{j_0 j_1} + \epsilon_{i_1 j_1} - \dots - \epsilon_{j_{k-1} j_k} + \epsilon_{i_k j_k} - \epsilon_{j_k j_{k+1}}}. \end{aligned} \quad (\text{A.1})$$

Here we assume that $\epsilon_{ii} = 0$. The explicit form of the coefficients $a_{\mathbf{m}|\mathbf{k}, \mathbf{i}, \mathbf{j}}$ and $b_{\mathbf{m}|\mathbf{k}, \mathbf{i}, \mathbf{j}}$ is

$$\begin{aligned} a_{\mathbf{m}|\mathbf{k}, \mathbf{i}, \mathbf{j}} &= (-1)^k \kappa_q^{\gamma_{i,j}} q^{\lambda_1 - \lambda_{j_k} - \sum_{j=2}^{l+1} m_{1j} + m_{j_k, l+1} - \sum_{a=1}^k \delta_{\mathbf{m}|\mathbf{a}, \mathbf{i}, \mathbf{j}} + k - \gamma_{i,j}} \\ &\quad \times [\lambda_{j_k} - \lambda_{l+1} - \sum_{j=j_k+1}^l m_{i,j} - \sum_{i=i_k+1}^l m_{i, l+1} + 1]_q \prod_{a=1}^{k+1} [m_{i_{a-1} j_a}]_q, \\ b_{\mathbf{m}|\mathbf{k}, \mathbf{i}, \mathbf{j}} &= (-1)^k \kappa_q^{\gamma_{i,j}-1} q^{\lambda_1 - \lambda_{l+1} - \sum_{j=2}^{l+1} m_{1j} - \sum_{a=1}^k \delta_{\mathbf{m}|\mathbf{a}, \mathbf{i}, \mathbf{j}} + k - \gamma_{i,j}} \prod_{a=1}^{k+1} [m_{i_{a-1} j_a}]_q, \end{aligned}$$

where we denote

$$\gamma_{i,j} = \sharp(\{a = 1, \dots, k \mid i_a \neq j_a\})$$

and

$$\delta_{\mathbf{m}|a,i,j} = \sum_{j=j_{a-1}+1}^{j_a-1} m_{i_{a-1}j} + \sum_{i=i_{a-1}+1}^{i_a} m_{ij_a}.$$

It is also assumed that $m_{ii} = 0$.

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