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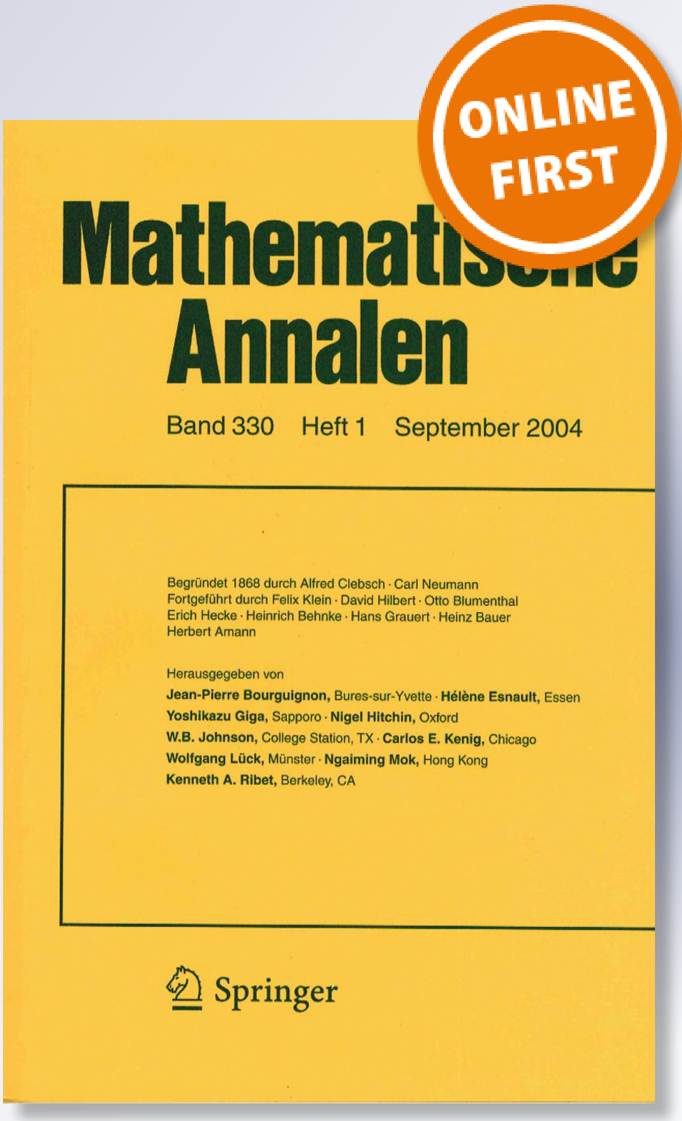

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Topological polylogarithms and p-adic interpolation of L-values of totally real fields

Alexander Beilinson[1] · **Guido Kings**[2] · **Andrey Levin**[3]

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Abstract We develop the topological polylogarithm which provides an integral version of Nori's Eisenstein cohomology classes for $\mathrm{GL}_n(\mathbb{Z})$ and yields classes with values in an Iwasawa algebra. This implies directly the integrality properties of special values of L-functions of totally real fields and a construction of the associated p-adic L-function. Using a result of Graf, we also apply this to prove some integrality and p-adic interpolation results for the Eisenstein cohomology of Hilbert modular varieties.

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✉ Guido Kings
guido.kings@ur.de

Alexander Beilinson
sasha@math.uchicago.edu

Andrey Levin
alevin57@gmail.com

1 Department of Mathematics, University of Chicago, Chicago, IL 60637, USA

2 Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

3 National Research University HSE, 7 Vavilova Str., Moscow, Russia

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1 Introduction

In the beginning of the 1990s Nori [13] and Sczech [14] almost simultaneously and independently developed the so called Eisenstein cohomology classes for $\mathrm{GL}_n(\mathbb{Z})$ with rational coefficients and showed that one can get the Klingen–Siegel theorem, about the rationality of zeta values of totally real fields at negative integers, as a direct consequence.

The approach by Nori involves the de Rham complex and is therefore restricted to rational coefficients. Sczech's construction is analytic in nature and he gets rational cohomology classes in the end by studying Dedekind sums.

In this paper we present a different approach, depending on the topological polylogarithm, which is very much inspired by Nori's beautiful construction, but works with almost arbitrary coefficients. Moreover, the cohomology class we construct has values in the formal completion of the group ring of a finitely generated free abelian group and is hence exactly the Iwasawa algebra if one considers p-adic coefficients.

The main idea of our construction can be explained as follows. Nori's cohomology classes should be really considered not as classes on the locally symmetric space associated to $\mathrm{GL}_n(\mathbb{Z})$ but rather on the universal family of topological metrized tori above it. On this universal family these classes are completely determined by a residue condition, so that the comparison map between de Rham and singular cohomology in Nori's approach becomes unnecessary. In particular, Nori's construction interpreted in this way works for almost arbitrary coefficients.

From our construction we get directly the integrality results of Deligne–Ribet and the p-adic interpolation of the special L-values of totally real fields. In fact we get directly cohomology classes with values in the Iwasawa algebra. We also explain a new result, building upon results of Graf [7], about the integrality and p-adic interpolation of Eisenstein cohomology classes for Hilbert modular varieties.

In recent years the question of finding integral versions of Sczech's Eisenstein cocycle or of Shintani's construction received considerable interest. Charollois and Dasgupta were able to refine Sczech's construction to the integral level in [4]. But because of problems with their smoothing construction they could only prove part of the integrality result of Deligne–Ribet for the L-values. Another approach to p-adic interpolation is via the Shintani cocycle of Hill [8] and Solomon, which was refined to give p-adic interpolation by Spiess [18] and subsequently Steele [19].

Our approach is completely different from all the above and relies only on the cohomological properties of the so called logarithm sheaf. Being purely topological, we do not need to choose any extra data as for the Shintani cocycle (which depends on a Shintani decomposition) nor do we have any severe restrictions on the coefficients as our approach is not analytic at all.

Moreover, in a geometric situation, where one replaces the tori by abelian varieties, a completely parallel story for other cohomology theories (and even for motivic cohomology) can be developed. This leads to p-adic interpolation of motivic cohomology classes and hence of non-critical L-values (see [10] and the applications of this theory in [11]). This gives a further argument for pursuing the approach by the topological polylogarithm.

The essential results on p-adic interpolation of L-values of this paper were obtained many years ago in the nineties but were never published. The only exception is the case of the Riemann zeta function which is covered in [2]. The newer results on the Eisenstein classes of Hilbert modular varieties and their p-adic interpolation depends on the purely topological construction of the Harder's Eisenstein cohomology in the thesis of Graf (forthcoming [7]).

2 A topological interpretation of the generating function for Bernoulli polynomials

To motivate and explain the constructions in this paper in the simplest case, we review here the topological construction of the Bernoulli polynomials and their p-adic interpolation which already appeared in [2, Section 2.3. and 2.5]. This is the case $n = 1$ of the construction in this paper.

Recall the generating function for the Bernoulli polynomials

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{z^k}{k!}$$

with $B_0(x) = 1$. In our approach we consider the function

$$f(x,z) := \frac{e^{xz}}{1-e^z} = -\frac{1}{z} + \sum_{k=0}^{\infty} -\frac{B_{k+1}(x)}{k+1}\frac{z^k}{k!}$$

for $x \in \mathbb{R}$ with values in the power series $z^{-1}\mathbb{R}[[z]]$. Note that $-\frac{B_{k+1}(x)}{k+1} = \zeta(x,-k)$ for $0 < x \leq 1$ where $\zeta(x,s) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^s}$ is the Hurwitz zeta function. The function $f(x,z)$ satisfies the differential equation $\frac{d}{dx} f(x,z) - zf(x,z) = 0$ and has the residue property $f(0,z) - f(1,z) = 1$. We will construct a local system on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ which has a connection of the form $\frac{d}{dx} - z$ and single out certain horizontal sections by residue conditions. These sections we call topological polylogarithms because their definition is analogue to the one of polylogarithms over $\mathbf{G}_m$. The function $f(x,z)$ will appear as the special value at $x \in S^1$ of such a polylogarithm. The crucial point of our approach to rationality and integrality of L-values is the fact that this local system is of purely topological nature and hence can be constructed with arbitrary coefficients.

We give now some more details, to explain our method in this simplest case. Fix a commutative ring A (think of $A = \mathbb{Z}_p, \mathbb{Q}, \mathbb{R}$) and consider the group ring $A[\mathbb{Z}]$ which can be identified with the ring of Laurent polynomials $A[t,t^{-1}]$. The completion with respect to the augmentation ideal $J = (t-1)$ of $A[\mathbb{Z}]$ is

$$R = R_A(\mathbb{Z}) = \varprojlim_k A[t,t^{-1}]/(t-1)^k \cong A[[t-1]].$$

The group $\mathbb{Z}$ acts on R by multiplication with t. If $\mathbb{Q} \subset A$ one has a canonical isomorphism

$$\exp^* : R \cong A[[z]] \qquad t \mapsto e^z = \sum_{k \geq 0} \frac{z^k}{k!}$$

under which the $\mathbb{Z}$-action becomes multiplication with e^z. We now use the $\mathbb{Z}$ action on R to define a local system $\mathscr{L}og$ on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ by letting $\mathscr{L}og$ be the sheaf of sections of the quotient

$$(\mathbb{R} \times R)/\mathbb{Z}$$

where $n \in \mathbb{Z}$ acts as $(x, r)n := (x + n, t^{-n} r)$. A global section of $\mathscr{L}og$ is then a map $f : \mathbb{R} \to R$ which satisfies $f(x + n) = t^{-n} f(x)$.

For $A = \mathbb{R}$ we can consider the $\mathscr{C}^\infty$ bundle $\mathscr{L}og^\infty$ associated to $\mathscr{L}og$, which is a vector bundle with fibre $\mathbb{R}[[z]]$. This has $x \mapsto e^{-xz}$ as a global section, which induces an isomorphism of $\mathscr{L}og^\infty$ with the sheaf of $\mathscr{C}^\infty$-sections of $S^1 \times \mathbb{R}[[z]]$. The connection ∇ on $\mathscr{L}og^\infty$ corresponds under this isomorphism to $\nabla_0 := d - z$, because $\nabla e^{-xz} = -ze^{-xz} dx$.

To describe the residue condition, we use the localization sequence in cohomology. Let $D \subset \mathbb{R}/\mathbb{Z}$ be a non-empty set of torsion sections, to fix ideas we take $D = \frac{1}{c}\mathbb{Z}/\mathbb{Z}$ the c-torsion sections for an integer $c \geq 2$. From the explicit description of the $\mathbb{Z}$-action one computes $H^0(\mathbb{Z}, R) = 0$ and $H^1(\mathbb{Z}, R) \cong A$, hence

$$H^0(S^1, \mathscr{L}og) = 0 \qquad H^1(S^1, \mathscr{L}og) \cong A.$$

Consider the localization sequence for $D \subset \mathbb{R}/\mathbb{Z}$

$$0 \to H^0(S^1 \backslash D, \mathscr{L}og) \xrightarrow{\text{res}} H^1_D(S^1, \mathscr{L}og) \to A \to 0.$$

We can identify $H^1_D(S^1, \mathscr{L}og) \cong \bigoplus_{d \in D} \mathscr{L}og_d$. It is important to note that for c invertible in A, one has a canonical isomorphism $\mathscr{L}og_d \cong R$ (the reason is that the c-multiplication is an isomorphism of R, see Proposition 3.5 below). With this identification the localization sequence reads

$$0 \to H^0(S^1 \backslash D, \mathscr{L}og) \to \bigoplus_{d \in D} R \to A \to 0.$$

Obviously, $A[D] := \bigoplus_{d \in D} A \subset \bigoplus_{d \in D} R$ and for every $\alpha \in A[D]^0 := \ker(A[D] \to A)$ we get a unique section

$$\mathrm{pol}_\alpha \in H^0(S^1\backslash D, \mathscr{L}og)$$

which has residue $\alpha(d)$ at d. If $x \in S^1$ is a non-zero f-torsion point, with f invertible in A, we can evaluate at x and get

$$\mathrm{pol}_\alpha(x) \in A[[t-1]].$$

If, for example, $A = \mathbb{Z}[1/cf]$, then by embedding A into $\mathbb{Q}$, one obtains that the coefficients of $\mathrm{pol}_\alpha(x)$ in $\mathbb{Q}[[z]]$ are in A. This gives the desired integrality.

To compute these sections explicitly, we first consider a slight variant of pol_α. Let $\mathbb{Q} \subset A$ and $L_A^* := Az^{-1}$ be the rank one A-module with trivial $\mathbb{Z}$-action. Then we can have the localization sequence for $L_A^* \otimes \mathscr{L}og$ and $D = \{0\}$

$$0 \to H^0(S^1\backslash\{0\}, L_A^* \otimes \mathscr{L}og) \to L_A^* \otimes R \to L_A^* \to 0$$

and we ask for a section $\mathrm{pol} \in H^0(S^1\backslash\{0\}, L_A^* \otimes \mathscr{L}og)$ with residue

$$z^{-1} \otimes z \in L_A^* \otimes R \cong z^{-1}A[[z]].$$

Now let $A = \mathbb{R}$, then $x \mapsto e^{\{x\}z}$, where $\{x\}$ is the fractional part of x, is a horizontal section of $z^{-1}\mathscr{L}og^\infty$ over $S^1\backslash\{0\}$ with residue $1-e^z$ at $x=0$. Thus pol is the section

$$\mathrm{pol}(x) = \frac{e^{\{x\}z}}{1-e^z} = f(\{x\}, z)$$

of $z^{-1}\mathbb{R}[[z]]$. From the definitions one sees that for $\alpha \in A[D]^0$ one has

$$\mathrm{pol}_\alpha(x) = \sum_{d\in D} \alpha(d)\,\mathrm{pol}(x-d).$$

If $D = \frac{1}{c}\mathbb{Z}/\mathbb{Z}$ are the c-torsion points, we can define an special element $\alpha_{[c]}$ of $A[D]^0$ by $\alpha(0) = c-1$ and $\alpha(d) = -1$ for $d \neq 0$. Then one gets

$$\begin{aligned}
\mathrm{pol}_{\alpha_{[c]}}(x) &= cf(\{x\}, z) - \sum_{a=0}^{c-1} f\left(\left\{x - \frac{a}{c}\right\}, z\right)\\
&= cf(\{x\}, z) - f(\{cx\}, c^{-1}z)\\
&= \sum_{k=0}^{\infty}(c\zeta(x,-k) - c^{-k}\zeta(cx,-k))\frac{z^k}{k!}
\end{aligned}$$

and, as explained above, the coefficients $c\zeta(x,-k) - c^{-k}\zeta(cx,-k)$ are in $\mathbb{Z}[1/cf]$ if x is a non-zero f-torsion point. A closer analysis shows that it is not necessary to assume f invertible in A, from which one obtains the usual integrality properties of $c\zeta(x,-k) - c^{-k}\zeta(cx,-k)$. For more details on this see the proof of Corollary 5.8.

3 The topological polylogarithm

3.1 Group rings of lattices

We consider free abelian groups L of finite rank n, which we call *lattices*. Let A be a commutative ring and $A[L]$ the group ring of L with coefficients in A. We write $\delta\colon L \to A[L]^\times$, $\ell \mapsto \delta_\ell$ for the universal group homomorphism. In particular, $\ell \in L$ acts on $A[L]$ by multiplication with δ_ℓ so that $\delta_\ell(\ell') = (\ell + \ell')$. Let $L_A := A \otimes_{\mathbb{Z}} L$.

Definition 3.1 The completion of $A[L]$ with respect to the augmentation ideal J is denoted by

$$R := R(L) := \varprojlim_k A[L]/J^k.$$

We write $I := JR$ and consider R with the filtration defined by the $I^k R$ and the induced L-action $\delta : L \to R^\times$, i.e. ℓ acts by multiplication with δ_ℓ. In particular, I is stable under the L-action. We write $R^{(k)} := R/I^{k+1} \cong A[L]/J^{k+1}$ and R_A if we need to express the dependence on A.

Remark 3.2 Let $T(L^\vee) := \operatorname{Spec} A[L] = \underline{\operatorname{Hom}}(L, \mathbf{G}_m)$ be the algebraic torus with character group L over $\operatorname{Spec} A$. The augmentation ideal $J := \ker(A[L] \to A)$ defines the unit section of the smooth map $T(L^\vee) \to \operatorname{Spec} A$ and hence is a regular ideal. Note also that it is stable under the L-action. Then $\operatorname{Spf} R$ is the formal group associated to $T(L^\vee)$. In particular, if $\ell_1, \ldots, \ell_n$ is a basis of L, then R is a power series ring in the $\delta_{\ell_1} - 1, \ldots, \delta_{\ell_n} - 1$.

Lemma 3.3 *There is an isomorphism*

$$\operatorname{Sym}^\cdot L_A \cong \operatorname{gr}_I^\cdot R = \bigoplus_{k \geq 0} I^k/I^{k+1}. \tag{3.1}$$

The induced action of L on $\operatorname{gr}_I^\cdot R$ is trivial.

Proof As L is abelian we have an isomorphism $L_A \cong J/J^2 \cong I/I^2$, which sends $1 \otimes \ell$ to $\delta_\ell - 1 \bmod J^2$. As J and hence I is a regular ideal, the induced map $\operatorname{Sym}^\cdot L_A \to \operatorname{gr}_I^\cdot R$ is an isomorphism. If $a \in I^k$ then $(\delta_\ell - 1) \cdot a \equiv 0 \bmod I^{k+1}$, so that $\delta_\ell \cdot a \equiv a \bmod I^{k+1}$, which implies that L acts trivially on $\operatorname{gr}_I^\cdot R$. □

The formation of R is functorial in L: For each homomorphism $\varphi\colon L \to L'$ we have an A-algebra homomorphism $A[L] \to A[L']$ compatible with the augmentation, which induces

$$\varphi_R\colon R \to R',$$

where $R' := R(L')$, which respects the filtrations by I and I', i.e., maps I^k to $(I')^k$.

Definition 3.4 A homomorphism of lattices $\varphi\colon L \to L'$ is called an *isogeny*, if it is injective with finite cokernel. For an isogeny φ we denote by $\deg\varphi := \#(L'/\varphi(L))$ the *degree* of φ.

Proposition 3.5 *Let $\varphi\colon L \to L'$ be an isogeny with $\deg\varphi$ invertible in A, then*

$$\varphi_R\colon R \to R'$$

is an isomorphism.

Proof Both rings R, R' are complete and separated so that it suffices to show that

$$\mathrm{gr}_I\,\varphi_R\colon\ \mathrm{Sym}\,L_A \to \mathrm{Sym}\,L'_A$$

is an isomorphism. The A-module I^k/I^{k+1} is generated by products of elements of the form $(\delta_\ell - 1)^r$ and φ_R maps these to $(\delta_{\varphi(\ell)} - 1)^r$. This shows that $\mathrm{gr}_I\,\varphi_R = \mathrm{Sym}\,\varphi_A$, where $\varphi_A : L_A \to L'_A$ is the induced map. If $\deg\varphi$ is invertible in A, φ_A and hence $\mathrm{gr}_I\,\varphi$ is an isomorphism. □

For the closer investigation of R we need the completion of the divided power algebra of L_A.

Definition 3.6 Let $\Gamma L_A = \bigoplus_{k\geq 0}\Gamma_k L_A$ be the graded divided power algebra of L_A. For $\ell \in L_A$ we write $\ell^{[k]}$ for the k-th divided power of ℓ and write

$$\widehat{\Gamma} L_A := \varprojlim_r \Gamma L_A/I^{[r]}$$

where $I := \Gamma_+ L_A$ is the augmentation ideal. We define an L-action on $\widehat{\Gamma} L_A$ by $\delta : L \to \widehat{\Gamma} L_A^\times$, $\ell \mapsto \sum_{k\geq 0}\ell^{[k]}$.

Note that one has $\ell^n = n!\ell^{[n]}$ and the formula

$$(\ell + \ell')^{[k]} = \sum_{m=0}^{k} \ell^{[m]}\ell'^{[k-m]} \tag{3.2}$$

in ΓL_A, which shows that δ is a group homomorphism. As $\Gamma(L_A \oplus L_A) \cong \Gamma L_A \otimes_A \Gamma L_A$ the diagonal makes the algebra ΓL_A into a graded Hopf algebra. Its (graded) dual is $(\Gamma L_A)^* \cong \mathrm{Sym}\,L_A^*$, where L_A^* is the A-dual of L_A. As L_A is free one has also a canonical isomorphism

$$\Gamma L_A \cong \mathrm{TSym}\,L_A \tag{3.3}$$

with the Hopf algebra of the symmetric tensors. The isomorphism $L_A \cong \Gamma_1 L_A$ induces an A-algebra homomorphism

$$\widehat{\mathrm{Sym}} L_A \to \widehat{\Gamma} L_A, \tag{3.4}$$

where $\widehat{\mathrm{Sym}} L_A$ is the completion of $\mathrm{Sym}\, L_A$ at its augmentation ideal. Explicitly, if we choose a basis $\ell_1, \ldots, \ell_n$ of L_A, this homomorphism is given by

$$\ell_1^{k_1} \cdots \ell_n^{k_n} \mapsto k_1! \cdots k_n! \ell_1^{[k_1]} \cdots \ell_n^{[k_n]}. \tag{3.5}$$

From this description it is clear that (3.4) is an isomorphism if A is a $\mathbb{Q}$-algebra.

Proposition 3.7 *There is an A-algebra homomorphism*

$$\exp^* \colon R \to \widehat{\Gamma} L_A$$

mapping I^k to $(\Gamma_+ L_A)^{[k]}$, which is functorial for isogenies and compatible with the L-actions. We write $\exp_k^* \colon R \to \Gamma_k L_A$ for the composition with the projection to $\Gamma_k L_A$.

Proof Consider the group homomorphism $\delta \colon L \to (\widehat{\Gamma} L_A)^\times$. This induces an A-algebra homomorphism $A[L] \to \widehat{\Gamma} L_A$ which maps $(\delta_\ell - 1)^r$ into $(\widehat{\Gamma}_+ L_A)^{[r]}$ and hence J^r to $(\widehat{\Gamma}_+ L_A)^{[r]}$. Taking completions, this induces the desired A-algebra homomorphism $\exp^* \colon R \to \widehat{\Gamma} L_A$. □

Remark 3.8 The map $\exp^*$ is induced from the exponential map of the formal group $\mathrm{Spf}\, R$. For this one should think of $\mathrm{Spf}\, \widehat{\Gamma} L_A$ as the divided power formal neighbourhood of 0 in the Lie algebra $L_A^* := \mathrm{Hom}_A(L_A, A)$ of $\mathrm{Spf}\, R$. The homomorphism $\exp^*$ has also the following description. Let $\mathbb{H} := \varinjlim_r \mathrm{Hom}_A(R/I^r, A)$ be the bigebra of translation invariant differential operators on $\mathrm{Spf}\, R$. Then one has $R \cong \mathrm{Hom}_A(\mathbb{H}, A)$ and one has a map $\mathcal{U}(L_A^*) \to \mathbb{H}$ of the universal enveloping algebra of the Lie algebra L_A^* to $\mathbb{H}$. If we observe that $\mathcal{U}(L_A^\vee) \cong \mathrm{Sym}\, L_A^*$ we get an A-algebra homomorphism

$$R \cong \mathrm{Hom}_A(\mathbb{H}, A) \to \mathrm{Hom}_A(\mathcal{U}(L_A^\vee), A) \cong \widehat{\Gamma} L_A$$

which coincides with $\exp^*$.

Proposition 3.9 *If A is a $\mathbb{Q}$-algebra, then*

$$\exp^* \colon R \to \widehat{\Gamma} L_A$$

is an isomorphism.

Proof Identify $\mathrm{gr}_I\, R \cong \mathrm{Sym}\, I/I^2 \cong \mathrm{Sym}\, L_A$. Then we claim that the associated graded of $\exp^* \colon R \to \widehat{\Gamma} L_A$

$$\mathrm{gr}_I \exp^* \colon \mathrm{Sym}\, L_A \to \Gamma L_A$$

coincides with the canonical map. But $L_A \cong I/I^2$ is generated by $\delta_\ell - 1$ which maps to $(\sum_{k \geq 0} \ell^{[k]}) - 1$. This is congruent to $\ell^{[1]}$ modulo $(\Gamma_+ L_A)^{[2]}$. Thus $\mathrm{gr}_I \exp^*$ must be induced from the isomorphism $L_A \cong \Gamma_1 L_A$ by the universal property of Sym L_A. In particular, if A is a $\mathbb{Q}$-algebra then $\mathrm{gr}_I \exp^*$ is an isomorphism. As R and $\widehat{\Gamma} L_A$ are complete and separated, this implies that $\exp^*$ is an isomorphism. □

Usually it is more convenient to work with the power series ring $\widehat{\mathrm{Sym}} L_A$ than with $\widehat{\Gamma} L_A$.

Corollary 3.10 *Let A be a $\mathbb{Q}$-algebra, then the isomorphism*

$$\exp^*\colon R \cong \widehat{\Gamma} L_A \cong \widehat{\mathrm{Sym}} L_A$$

is induced by the group homomorphism $\exp\colon L \to \widehat{\mathrm{Sym}} L_A$ *which maps* $\ell \mapsto \sum_{k\geq 0} \frac{\ell^{\otimes k}}{k!}$.

Proof This is clear as $\frac{\ell^{\otimes k}}{k!} \mapsto \ell^{[k]}$ under $\widehat{\mathrm{Sym}} L_A \cong \widehat{\Gamma} L_A$. □

3.2 Iwasawa algebras of lattices

This section is not needed for the construction of the topological polylogarithm, but it is needed later in the construction of the p-adic measures.

Fix a prime number p. In this section A will be a p-adically complete and separated ring.

Definition 3.11 The *Iwasawa algebra* $A[[L_{\mathbb{Z}_p}]]$ is the completed group ring

$$A[[L_{\mathbb{Z}_p}]] := \varprojlim_r A[L/p^r L]$$

where the projective limit is taken with respect to $A[L/p^{r+1}L] \to A[L/p^r L]$.

The A-algebra R is canonically isomorphic to the Iwasawa algebra.

Proposition 3.12 *The map $\delta\colon L \to R^\times$ induces a continuous A-algebra isomorphism*

$$A[[L_{\mathbb{Z}_p}]] \xrightarrow{\cong} R.$$

Proof Consider the composition $L \xrightarrow{\delta} A[L]^\times \to (A[L]/(p, J)^{r+1})^\times$. By induction on r one sees that $\delta_{p^r\ell} - 1 = \delta_\ell^{p^r} - 1 \in (p, I)^{r+1}$. This implies that this composition factors through $L/p^r L$ and one gets by the universal property of the group ring an A-algebra homomorphism $A[L/p^r L] \to A[L]/(p, J)^{r+1}$, such that the composition $A[L] \to A[L/p^r L] \to A[L]/(p, J)^{r+1}$ is the quotient map. This induces a continuous homomorphism

$$A[[L_{\mathbb{Z}_p}]] \to \varprojlim_r A[L]/(p, J)^{r+1}$$

which is an isomorphism on the subring $A[L]$. As $A[L]$ is a dense subring on both sides and both rings $A[[L_{\mathbb{Z}_p}]]$ and $\varprojlim_r A[L]/(p, J)^{r+1}$ are complete and separated, the homomorphism itself must be an isomorphism. It remains to show that

$$R \cong \varprojlim_r A[L]/(p, J)^{r+1},$$

i.e., that R is (p, I)-adically complete and separated. As $(p, I)^{2r} \subset (p)^r + I^r \subset (p, I)^r$ the (p, I)-adic topology on the finitely generated A-module $R/I^r \cong A[L]/J^r$ coincides with the (p)-adic one. Hence the R/I^r are complete in the (p, I)-adic topology, so that also R is (p, I)-adically complete. As $\bigcap_{r\geq 0}(p)^r = 0$ and $\bigcap_{r\geq 0} I^r = 0$ it is also separated. □

Definition 3.13 Let A be a p-adically complete and separated ring. Then we call

$$\mathrm{mom}\colon A[[L_{\mathbb{Z}_p}]] \overset{\cong}{\to} R \xrightarrow{\exp^*} \widehat{\Gamma} L_A \cong \widehat{\mathrm{TSym} L_A}$$

the *moment map*. The projection onto its k-th component

$$\mathrm{mom}^k\colon A[[L_{\mathbb{Z}_p}]] \overset{\cong}{\to} R \to \mathrm{TSym}^k L_A$$

we call the *k-th moment map*.

To explain the name "moment map" recall that $A[[L_A]]$ can be interpreted as the algebra of measures on $L_{\mathbb{Z}_p}$.

Definition 3.14 Let $\mathscr{C}(L_{\mathbb{Z}_p})$ be the continuous A-valued functions on $L_{\mathbb{Z}_p}$. An A-valued measure is an A-linear map $\mu\colon \mathscr{C}(L_{Zp}) \to A$. We write

$$\mathrm{Meas}(L_{\mathbb{Z}_p}, A) := \mathrm{Hom}_A(\mathscr{C}(L_{\mathbb{Z}_p}), A)$$

for the space of all A-valued measures.

It is well-known that $\mathrm{Meas}(L_{\mathbb{Z}_p}, A)$ is a ring under convolution of measures which is canonically isomorphic to $A[[L_{\mathbb{Z}_p}]]$.

Proposition 3.15 *Identify* $\mathrm{Meas}(L_{\mathbb{Z}_p}, A) \cong A[[L_{\mathbb{Z}_p}]]$ *and let*

$$\mathrm{Meas}(L_{\mathbb{Z}_p}, A) \cong R \xrightarrow{\mathrm{mom}^k} \mathrm{TSym}^k L_{\mathbb{Z}_p}$$

be the composition of the isomorphism in Proposition 3.12 with the k-th moment map. If we interpret the A-dual $(\mathrm{TSym}\, L_A)^* \cong \mathrm{Sym}\, L_A^*$ *as polynomial functions* $x_1^{k_1} \cdots x_n^{k_n}$ *on* L_A, *then*

$$\mathrm{mom}^k(\mu) = \sum_{k_1+\cdots+k_n=k} \mu\left(x_1^{k_1}\cdots x_n^{k_n}\right) \ell_1^{[k_1]}\cdots \ell_n^{[k_n]}$$

where $\mu(x_1^{k_1}\cdots x_n^{k_n})$ *are the moments of the measure* μ.

The proposition follows by a direct calculation, as we do not need it, we skip the proof.

3.3 Torsors and locally constant sheaves

We follow the principle "right action on spaces, left action on cohomology".

Let G be a group and $\pi : X \to S$ be a right G-torsor. For a left G-module M we define a G action on $X \times M$ by $(x, m)g := (xg, g^{-1}m)$ and write as usual

$$X \times^G M := X \times M/G$$

for the orbits of G on $X \times M$.

Definition 3.16 For a left G-module M, we define the locally constant sheaf $\widetilde{M}$ to be the sheaf of sections of $X \times^G M$ over S (where M has the discrete topology). If the G-action is trivial then $\widetilde{M}$ is the constant sheaf $\underline{M}$.

The sections over $U \subset S$ open of the sheaf $\widetilde{M}$ are explicitly given by

$$\widetilde{M}(U) = \{f : \pi^{-1}(U) \to M \mid f(ug) = g^{-1}f(u) \text{ for all } g \in G, u \in \pi^{-1}(U)\}. \tag{3.6}$$

If X is simply connected, then the functor

$$\begin{aligned} \{G\text{-modules}\} &\to \{\text{locally constant sheaves on } S\} \\ M &\mapsto \widetilde{M} \end{aligned} \tag{3.7}$$

is an equivalence of categories. The inverse functor is $\mathscr{F} \mapsto \Gamma(X, \pi^*\mathscr{F})$. We apply this in the case of lattices.

Definition 3.17 Let L be a lattice. We write $V := \mathbb{R} \otimes L$ where $\ell \in L$ acts from the right on V by $v \mapsto v + \ell$. We denote by

$$T := T(L) := V/L$$

the associated compact real torus.

Over T we have the fundamental L-torsor V

$$0 \to L \to V \xrightarrow{\pi} T \to 0 \tag{3.8}$$

with $\pi^{-1}(0) = L$.

Definition 3.18 Let $R_1^\times := (1+I)^\times \subset R^\times$ be the subgroup of 1 units. The $R_1^\times$-torsor $\mathrm{Log}^\times$ on T is the push-out of the sequence (3.8) with $\delta\colon L \to R_1^\times$, so that one has an exact sequence of abelian groups

$$0 \to R_1^\times \to \mathrm{Log}^\times \xrightarrow{\mathrm{pr}_1} T \to 0. \tag{3.9}$$

Note that we also have $\mathrm{Log}^\times := V \times^L R_1^\times$. The $R_1^\times$-torsor $\mathrm{Log}^\times$ is obviously rigidified over $0 \in T$ by $1 \in R_1^\times$. By [15, Expose VII, Proposition 1.3.5] the group structure on $\mathrm{Log}^\times$ can be uniquely recovered from its $R_1^\times$-torsor structure together with its rigidification 1 of its fibre $\mathrm{Log}_0^\times$ in $0 \in T$.

3.4 The logarithm sheaf

We will consider local systems on the compact torus

$$T := T(L) := V/L.$$

Proposition 3.19 *There exists a local system $\mathscr{L}og = \mathscr{L}og_T$ on T of free rank one $\underline{R}$-modules, such that the L-action $L \to \mathrm{Aut}(0^*\mathscr{L}og) = R^\times$ coincides with $\delta\colon L \to R^\times$. Let $\mathbf{1} \in 0^*\mathscr{L}og$ be a generator, then the pair $(\mathscr{L}og, \mathbf{1})$ is unique up to unique isomorphism.*

Proof Uniqueness: Let $(\mathscr{L}, s)$ be another pair with the properties of $\mathscr{L}og$. Then there exists a unique L-equivariant isomorphism $\alpha\colon 0^*\mathscr{L}og \cong 0^*\mathscr{L}$ with $\alpha(\mathbf{1}) = s$. Hence there is a unique isomorphism of local systems $\mathscr{L}og \cong \mathscr{L}$.

Existence: We give two constructions. For the first consider R as L-module via $\delta\colon L \to R^\times$ and define $\mathscr{L}og := \widetilde{R}$. As generator $\mathbf{1} \in 0^*\mathscr{L}og = R$ we choose the element $1 \in R$.

For the second let $\pi_!\underline{A}$ be the direct image with compact supports of the constant sheaf $\underline{A}$ on V. The sheaf $\pi_!\underline{A}$ is a local system of $\underline{A[L]}$-modules of rank one and $0^*\pi_!\underline{A} = A[L]$ has $1 \in A[L]$ as generator. Hence we can take

$$\mathscr{L}og := \underline{R} \otimes_{\underline{A[L]}} \pi_!\underline{A} \tag{3.10}$$

with the induced generator $\mathbf{1} \in 0^*\mathscr{L}og$. □

Definition 3.20 We call $(\mathscr{L}og, \mathbf{1})$ the *logarithm sheaf* and we let

$$\mathrm{Log} := V \times^L R$$

so that $\mathscr{L}og$ is the sheaf of sections of Log.

Proposition 3.21 *The logarithm sheaf $(\mathscr{L}og, \mathbf{1})$ has the following properties.*

(1) *Consider the filtration $I^k\mathscr{L}og := \widetilde{I^k}$ on $\mathscr{L}og$. Then there is a unique identification of local systems of $\mathrm{gr}_I^{\cdot} R = \mathrm{Sym}\, L_A$ modules*

$$\operatorname{gr}^{\cdot}_I \mathscr{L}og \cong \operatorname{Sym}^{\cdot} \underline{L_A}$$

that maps $\mathbf{1} \bmod I\mathscr{L}og$ *to* $1 \in \operatorname{Sym}^0 L_A = A$.

(2) *Let* $\varphi\colon L \to L'$ *be a homomorphism of lattices and* $\varphi\colon T \to T'$ *be the induced map, then one has an homomorphism of local systems*

$$\varphi_{\mathscr{L}og}\colon \mathscr{L}og_T \to \varphi^*\mathscr{L}og_{T'},$$

which is compatible with the filtrations and respects the generators $\mathbf{1}, \mathbf{1}'$.

(3) *If* $\varphi\colon L \to L'$ *is an isogeny and* $\deg\varphi$ *invertible in* A, *then*

$$\varphi_{\mathscr{L}og}\colon \mathscr{L}og_T \to \varphi^*\mathscr{L}og_{T'},$$

is an isomorphism.

(4) *Let* $+\colon T \times T \to T$ *be the group structure on the torus, then one has a unique isomorphism*

$$\operatorname{pr}_1^* \mathscr{L}og \otimes_{\underline{R}} pr_2^*\mathscr{L}og \cong +^*\mathscr{L}og,$$

under which $\mathbf{1}\otimes\mathbf{1} \mapsto \mathbf{1}$, *i.e.*, $\mathscr{L}og$ *is a* character sheaf.

(5) *Consider the* $R_1^\times$*-torsor of local sections of* $\mathscr{L}og$ *that are modulo* $I\mathscr{L}og$ *equal to* $1 \in \underline{A}$. *Then there is a canonical isomorphism of this* $R_1^\times$*-torsor with* $\operatorname{Log}^\times$ *such that* $\mathbf{1} \mapsto 1$. *Under this isomorphism the group structure on* $\operatorname{Log}^\times$ *is given by the product induced by the isomorphism in* (4).

Proof (1) follows immediately from Lemma 3.3 and the functoriality of the functor $M \mapsto \widetilde{M}$. For (2) note that $\varphi^*\mathscr{L}og_{T'}$ are the sections of $V \times^L R'$, where L acts via $\varphi\colon L \to L'$ and $\delta'\colon L' \to (R')^\times$ on R'. Then (3) follows from (2) and Proposition 3.5. The assertion (4) follows from the isomorphism $0^*(\operatorname{pr}_1^* \mathscr{L}og \otimes_{\underline{R}} pr_2^*\mathscr{L}og) \cong 0^*(+^*\mathscr{L}og)$. Finally, as $\mathscr{L}og = \widetilde{R}$, the torsor in (5) is $\widetilde{R_1^\times}$ and there is a unique isomorphism with $\operatorname{Log}^\times$ sending $\mathbf{1}$ to 1. From the remark after Definition 3.18 it follows that the group structure on $\operatorname{Log}^\times$ is induced by the isomorphism in (4). □

3.5 Trivializations of the logarithm sheaf

Definition 3.22 Let $H \subset T$ be a subgroup. A *multiplicative trivialization* of $\mathscr{L}og$ on H is a collection of generators $1_h \in \mathscr{L}og_h$ for all $h \in H$ such that $1_h \bmod I\mathscr{L}og_h$ equals $1 \in A$ and $1_h \otimes 1_{h'} = 1_{h+h'}$ under the isomorphism in Proposition 3.21 for all $h, h' \in H$.

We give two alternative descriptions of a multiplicative trivialization. First consider the group extension

$$0 \to R_1^{\times} \to \mathrm{Log}^{\times} \xrightarrow{\mathrm{pr}_1} T \to 0$$

from Definition 3.18. A multiplicative trivialization is a group homomorphism $\varrho\colon H \to \mathrm{Log}^{\times}$ which is a section of pr_1. In particular, the set of all multiplicative trivializations of $\mathscr{L}og$ is a $\mathrm{Hom}(H, R_1^{\times})$-torsor.

For a second description consider the right translation action $+\colon T \times H \to T$. A multiplicative trivialization is an extension of this H-action to $\mathrm{Log}^{\times}$, i.e., a map $\mathrm{Log}^{\times} \times H \to \mathrm{Log}^{\times}$ satisfying the usual condition for an H-action, such that one has a commutative diagram

$$\begin{array}{ccc} \mathrm{Log}^{\times} \times H & \longrightarrow & \mathrm{Log}^{\times} \\ {\scriptstyle \mathrm{pr}_1 \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ T \times H & \xrightarrow{+} & T. \end{array}$$

Given a multiplicative trivialization $\varrho\colon H \to \mathrm{Log}^{\times}$ the map $+\colon \mathrm{Log}^{\times} \times H \to \mathrm{Log}^{\times}$ is the composition of ϱ with the group structure $\mathrm{Log}^{\times} \times \mathrm{Log}^{\times} \to \mathrm{Log}^{\times}$.

Definition 3.23 Denote by $T^{\mathrm{tors}} := L_{\mathbb{Q}}/L \subset T$ the subgroup of torsion elements in T and by $T^{(A)} \subset T^{\mathrm{tors}}$ the subgroup of elements whose order is invertible in A.

Proposition 3.24 *There exists a unique multiplicative trivialization ϱ_{can} of $\mathscr{L}og$ over $T^{(A)}$. It is compatible with isogenies and for $t \in T[N] \subset T^{(A)}$ it is explicitly given by the isomorphism*

$$t^*\mathscr{L}og \cong t^*[N]^*\mathscr{L}og \cong 0^*[N]^*\mathscr{L}og \cong 0^*\mathscr{L}og,$$

where the outer isomorphisms are the pull-backs of Proposition 3.21(3) *and the middle one comes from* $[N] \circ t = [N] \circ 0$.

Proof Uniqueness: Let N be an integer which is invertible in A. It suffices to show that ϱ_{can} is uniquely determined on the N-torsion points $T[N]$. But the multiplicative trivializations on $T[N]$ form an $\mathrm{Hom}(T[N], R_1^{\times})$-torsor. But $R_1^{\times}$ has a filtration by $(1 + I^r)^{\times}$ such that $\mathrm{gr}^{>0} R_1^{\times} \cong \mathrm{Sym}^{>0} L_A$, which has no N-torsion as N is invertible in A. This implies that $\mathrm{Hom}(T[N], R_1^{\times}) = 0$.

Existence: Let $\varrho\,|_{T[N]}$ be the inverse of $\mathrm{Log}^{\times}[N] \cong T[N]$. By construction these isomorphisms are compatible for different N. □

3.6 Cohomology of the logarithm sheaf

All unlabelled tensor products in this section and the following ones are taken over $\mathbb{Z}$.

Let L be a lattice of rank n. Recall that one has a canonical isomorphism of algebras $H_\cdot(L, \mathbb{Z}) \cong \Lambda^\cdot L$. We define

$$\lambda := \lambda(L) := \Lambda^n L = H_n(L, \mathbb{Z}). \tag{3.11}$$

Theorem 3.25 *Let L be lattice of rank n. One has*

$$H^i(T, \mathscr{L}og) \cong \begin{cases} 0 & \textit{for } i \neq n \\ H^n(T, \underline{A}) & \textit{for } i = n \end{cases}$$

induced by the map $\mathscr{L}og \to \mathscr{L}og/I\mathscr{L}og = \underline{A}$. In particular, the cap-product induces an isomorphism

$$H^n(T, \mathscr{L}og \otimes \lambda) \cong A.$$

Proof From $\mathscr{L}og = \underline{R} \otimes_{\underline{A[L]}} \pi_! \underline{A}$ and because R is $A[L]$-flat one gets $H^i(T, \mathscr{L}og) \cong H^i(T, \pi_! \underline{A}) \otimes_{\underline{A[L]}} \underline{R}$. As

$$H^i(T, \pi_! \underline{A}) \cong H^i_c(V, \underline{A}) \cong \begin{cases} 0 & i \neq n \\ A & i = n \end{cases}$$

this implies the vanishing result. The homomorphism $H^n(T, \mathscr{L}og) \to H^n(T, \underline{A})$ induced by $\mathscr{L}og \to \underline{A}$ is surjective and because both groups are isomorphic to A it must be an isomorphism. The cap-product gives $H^n(T, \mathscr{L}og \otimes \lambda) \cong H^n(T, \underline{A}) \otimes H_n(L, \mathbb{Z}) \cong A$. □

Corollary 3.26 *Let $D \subset T$ be a finite and non-empty subset. Then for $i \neq n-1$*

$$H^i(T \backslash D, \mathscr{L}og \otimes \lambda) = 0$$

and one has a short exact sequence

$$0 \to H^{n-1}(T \backslash D, \mathscr{L}og \otimes \lambda) \xrightarrow{\text{res}} \mathscr{L}og \mid_D \xrightarrow{\sigma_D} A \to 0,$$

where $\mathscr{L}og \mid_D = \bigoplus_{d \in D} \mathscr{L}og_d$ is the restriction of $\mathscr{L}og$ to D and σ_D is the sum of the maps $\mathscr{L}og_d \to \mathscr{L}og_d/I\mathscr{L}og_d = A$.

Proof Consider the localization sequence for the closed subset $D \subset T$

$$\cdots \to H^i(T, \mathscr{L}og \otimes \lambda) \to H^i(T \backslash D, \mathscr{L}og \otimes \lambda) \to H^{i+1}_D(T, \mathscr{L}og \otimes \lambda) \to \cdots$$

For each $d \in D$ choose an open neighbourhood U_d such that $\mathscr{L}og$ is constant on U_d and the U_d for different d are disjoint. Then by excision

$$H^{i+1}_D(T, \mathscr{L}og \otimes \lambda) \cong \bigoplus_{d \in D} H^{i+1}_{\{d\}}(U_d, \mathscr{L}og \mid_{U_d} \otimes \lambda).$$

As $\mathscr{L}og \mid_{U_d}$ is constant and hence isomorphic to $\mathscr{L}og_d$, one has a canonical isomorphism

$$H^n_{\{d\}}(U_d, \mathscr{L}og\,|_{U_d} \otimes\lambda) \cong \mathscr{L}og_d$$

and $H^{i+1}_{\{d\}}(U_d, \mathscr{L}og\,|_{U_d} \otimes\lambda) = 0$ for $i+1 \neq n$ (see [12, Proposition 3.2.3]). □

3.7 Equivariant cohomology of the logarithm sheaf

We describe an equivariant version of the above construction.

Let $\Gamma \to \mathrm{GL}(L)$ be a group action on L. We write $L \rtimes \Gamma$ for the semi-direct product with multiplication

$$(\ell, \gamma)(\ell', \gamma') = (\ell + \gamma\ell', \gamma\gamma').$$

To follow our principle, we let $(l, \gamma) \in L \rtimes \Gamma$ act from the right on $v \in V$ by $v(\ell, \gamma) := \gamma^{-1}v + \gamma^{-1}\ell$. In particular, the L-torsor $\pi : V \to T$ is Γ-equivariant. From this we deduce a right action of Γ on Log by

$$\mathrm{Log} \times\Gamma \to \mathrm{Log};\quad ((v, r), \gamma) \mapsto (\gamma^{-1}v, \varphi_{\gamma^{-1}}(r)) \tag{3.12}$$

so that $\mathscr{L}og$ is a Γ-equivariant sheaf. We want to compute the Γ-equivariant cohomology $H^i(T, \Gamma; \mathscr{L}og \otimes \lambda)$ but for later needs, we compute a slightly more general cohomology group.

Theorem 3.27 *Let $D \subset T$ be a finite non-empty subset stabilized by Γ and M an $A[\Gamma]$-module, which is flat over A. Then*:

(1) *There are isomorphisms*

$$H^i(T, \Gamma; \underline{M} \otimes_{\underline{A}} \mathscr{L}og \otimes \lambda) \cong H^{i-n}(\Gamma, M)$$

and

$$H^i(T\backslash D, \Gamma; \underline{M} \otimes_{\underline{A}} \mathscr{L}og \otimes \lambda) \cong H^{i-n+1}(\Gamma, H^{n-1}(T\backslash D, \underline{M} \otimes_{\underline{A}} \mathscr{L}og \otimes \lambda)).$$

(2) *One has a long exact sequence*

$$\cdots \to H^i(T\backslash D, \Gamma; \underline{M} \otimes_{\underline{A}} \mathscr{L}og \otimes \lambda) \xrightarrow{\mathrm{res}} H^{i-n+1}(\Gamma, M \otimes_A \mathscr{L}og\,|_D) \xrightarrow{\sigma_D} H^{i-n+1}(\Gamma, M) \to \cdots$$

Proof This is follows from the spectral sequence

$$H^i(\Gamma, H^j(X, \underline{M} \otimes_{\underline{A}} \mathscr{L}og \otimes \lambda)) \Rightarrow H^{i+j}(X, \Gamma; \underline{M} \otimes_{\underline{A}} \mathscr{L}og \otimes \lambda)$$

for $X = T, T\backslash D$, the isomorphism $H^j(X, \underline{M}\otimes_{\underline{A}}\mathscr{L}og\otimes\lambda) \cong M\otimes_A H^j(X, \mathscr{L}og\otimes\lambda)$ of $A[\Gamma]$-modules, Theorem 3.25 and Corollary 3.26. □

As a special case, we get:

Corollary 3.28 *One has* $H^i(T\backslash D, \Gamma; \underline{M} \otimes_{\underline{A}} \mathscr{L}og \otimes \lambda) = 0$ *for* $i < n-1$ *and a canonical isomorphism*

$$\mathrm{res}\colon H^{n-1}(T\backslash D, \Gamma; \underline{M} \otimes_{\underline{A}} \mathscr{L}og \otimes \lambda) \cong \ker(M \otimes_A \mathscr{L}og\,|_D \xrightarrow{\sigma_d} M)^{\Gamma}.$$

3.8 The topological polylogarithm and Eisenstein classes

Definition 3.29 For $D \subset T$ finite and non-empty we define

$$A[D]^0 := \ker\left(\bigoplus_{d\in D} A \xrightarrow{\Sigma} A\right),$$

where Σ is the summation map $(a_d)_{d\in D} \mapsto \Sigma_{d\in D} a_d$. We view the elements $\alpha \in A[D]^0$ as functions $\alpha\colon D \to A$. We also set

$$R[D]^0 := \ker\left(\bigoplus_{d\in D} R \xrightarrow{\sigma_D} A\right)$$

where σ_D is the sum of the augmentations $R \to R/IR = A$.

Suppose that $D \subset T^{(A)}$ and that Γ stabilizes D. Then the trivialization ϱ_{can} from Proposition 3.24 induces an isomorphism $R[D]^0 \cong \ker(\mathscr{L}og\,|_D \xrightarrow{\sigma_D} A)$, so that we get

$$(A[D]^0)^{\Gamma} \subset (R[D]^0)^{\Gamma} \cong \ker(\mathscr{L}og\,|_D \xrightarrow{\sigma_D} A)^{\Gamma}.$$

We apply this to Corollary 3.28 in the case $M = A$:

Definition 3.30 For $D \subset T^{(A)}$, stabilized by Γ and $\alpha \in (A[D]^0)^{\Gamma}$ the unique cohomology class

$$\mathrm{pol}_{\alpha} \in H^{n-1}(T\backslash D, \Gamma; \mathscr{L}og \otimes \lambda)$$

with $\mathrm{res}(\mathrm{pol}_{\alpha}) = \alpha$ is called the *topological polylogarithm* associated with α.

Remark 3.31 Note that $(A[D]^0)^{\Gamma} \neq 0$ in general: Let N be invertible in A and $D = T[N]$ be the N-torsion points of T. Then D is stable under Γ and $N^n\delta_0 - \sum_{d\in T[N]} \delta_d \in (A[D]^0)^{\Gamma}$.

Let $t \in T\backslash D$ be any point stabilized by Γ. Then the pull-back of pol_{α} along t is a cohomology class

$$t^*\,\mathrm{pol}_{\alpha} \in H^{n-1}(\Gamma, \mathscr{L}og_t \otimes \lambda). \tag{3.13}$$

If $t \in T^{(A)}$, we can use the trivialization ϱ_{can} to identify $\mathscr{L}og_t \cong R$.

Definition 3.32 Let $D \subset T^{(A)}$ and $t \in T^{(A)} \backslash D$ be both stabilized by Γ, then for $\alpha \in (A[D]^0)^\Gamma$ the class

$$\mathrm{Eis}_\alpha(t) := t^* \operatorname{pol}_\alpha \in H^{n-1}(\Gamma, R \otimes \lambda)$$

is called the *Eisenstein class* associated to t and α. If we identify $\widehat{\Gamma} L_A \cong \widehat{\mathrm{TSym}} L_A$ then we also write

$$\mathrm{Eis}^k_\alpha(t) := \exp^*_k(\mathrm{Eis}_\alpha(t)) \in H^{n-1}(\Gamma, \mathrm{TSym}^k L_A \otimes \lambda) \tag{3.14}$$

for the k-th component of $\exp^*(\mathrm{Eis}_\alpha(t))$.

The following special case of the above definition was considered by Nori and Sczech.

Definition 3.33 Let $D \subset T^{(A)}$ be a finite non-empty subset such that $0 \notin D$. The Eisenstein operator of Nori and Sczech is the map

$$\begin{aligned} (A[D]^0)^\Gamma &\to H^{n-1}(\Gamma, R \otimes \lambda) \\ \alpha &\mapsto \mathrm{Eis}_\alpha(0). \end{aligned}$$

Remark 3.34 Let us explain how our approach is related to Nori's construction. Nori uses $A = \mathbb{Q}$ and considers the singular chain complex C of $V \backslash S$ where $S := \bigcup_{d \in D} d + L$ is a finite union of cosets of L, where D are representatives of torsion points in T. Let $\widetilde{C}$ be the kernel of the augmentation $C \to \mathbb{Q}$. On the other hand he considers a complex $\widetilde{D}$ which is quasi-isomorphic to $\Lambda^n \otimes \widehat{\mathrm{Sym}} L_{\mathbb{Q}}[n-1]$. The stabilizer of this union of cosets of L is an arithmetic subgroup of $V \rtimes \mathrm{GL}(V)$, which he calls π and which has L as a normal subgroup. Write $\pi = L \rtimes \Gamma$. Nori is looking for π-equivariant maps in the derived category from $\widetilde{C}$ to $\Lambda^n L_{\mathbb{Q}} \otimes \widehat{\mathrm{Sym}} L_{\mathbb{Q}}[n-1]$. Taking the local system defined by $\widehat{\mathrm{Sym}} L_{\mathbb{Q}}$ on $L \backslash (V \backslash S) = T \backslash D$ we get $\mathscr{L}og$, so that such a π-equivariant map is the same as a Γ-equivariant cohomology class in $H^{n-1}(T \backslash D, \Gamma; \mathscr{L}og \otimes \lambda)$. In this sense our construction is just a reinterpretation of Nori's in terms of sheaf theory. In contrast to Nori we work with the completion of the group ring $A[L]$ and not with $\widehat{\mathrm{Sym}} L_{\mathbb{Q}}$. This change is crucial if one wants to have integral coefficients.

In the case where t is an N-torsion point, but N not invertible in A, one can define also an Eisenstein class depending on N. This is often useful for integrality questions. The isogeny $[N]\colon L \subset L' := \frac{1}{N} L$ induces

$$[N]_{\mathscr{L}og}\colon \mathscr{L}og_t \to \mathscr{L}og'_0 = R'.$$

Definition 3.35 Let $D \subset T^{(A)}$ and $t \in T \backslash D$ an N-torsion point with N not necessarily invertible in A. Let $[N]\colon L \subset L' := \frac{1}{N} L$ be the natural inclusion. Assume that D and t are stabilized by Γ. Then we let

$$ {}_N\mathrm{Eis}_\alpha(t) := [N]_{\mathscr{L}og} t^* \mathrm{pol}_\alpha \in H^{n-1}(\Gamma, R' \otimes \lambda) $$

and write

$$ {}_N\mathrm{Eis}^k_\alpha(t) := \exp^*_k({}_N\mathrm{Eis}_\alpha(t)) \in H^{n-1}(\Gamma, \mathrm{TSym}^k L'_A \otimes \lambda). $$

Remark 3.36 (On integrality of $\mathrm{Eis}^k_\alpha(t)$) From the definitions it is clear that if we consider ${}_N\mathrm{Eis}^k_\alpha(t)$ as a class with coefficients in $A[1/N]$, i.e. in $H^{n-1}(\Gamma, \mathrm{TSym}^k L'_{A[1/N]} \otimes \lambda)$, then it coincides with the $[N]_{\mathscr{L}og}(\mathrm{Eis}^k_\alpha(t)) = N^k \mathrm{Eis}^k_\alpha(t)$. In particular, $N^k \mathrm{Eis}^k_\alpha(t)$ is a class with coefficients in A.

3.9 A variant of the polylogarithm I

For the study of the general Eisenstein distribution later the polylogarithm defined so far is not flexible enough. In this section we discuss the required slight generalization of the polylogarithm.

Let $E \subset T$ be a finite subset then $\mathscr{L}og\,|_E$ has an $A[E]$-module structure

$$ A[E] \otimes \mathscr{L}og\,|_E \to \mathscr{L}og\,|_E \tag{3.15} $$

given on a stalk $e \in E$ by multiplication with the value $f(e)$ for $f \in A[E]$. Assume that $E \subset T^{\mathrm{tors}}$, $E \cap D = \emptyset$ and suppose that Γ stabilizes E and D. Let $M = A[E]$, then from Corollary 3.28 we get the isomorphism

$$ \mathrm{res}\colon H^{n-1}(T\backslash D, \Gamma; \underline{A}[E] \otimes_{\underline{A}} \mathscr{L}og \otimes \lambda) \cong \ker(A[E] \otimes_A \mathscr{L}og\,|_D \xrightarrow{\sigma_D} A[E])^\Gamma. \tag{3.16} $$

From the definition of $A[D]^0$ we get

$$ (A[E] \otimes_A A[D]^0)^\Gamma \subset \ker(A[E] \otimes_A \mathscr{L}og\,|_D \xrightarrow{\sigma_D} A[E])^\Gamma. $$

Definition 3.37 We define for $h \in (A[E] \otimes_A A[D]^0)^\Gamma$ the polylogarithm pol_h to be the class

$$ \mathrm{pol}_h \in H^{n-1}(T\backslash D, \Gamma; \underline{A}[E] \otimes_{\underline{A}} \mathscr{L}og \otimes \lambda) $$

which corresponds to h under the isomorphism (3.16).

The restriction of pol_h to E is a class in $H^{n-1}(\Gamma, A[E] \otimes_A \mathscr{L}og\,|_E \otimes \lambda)$ and the image under the map from (3.15) gives a class

$$ \mathrm{Eis}_h \in H^{n-1}(\Gamma, \mathscr{L}og\,|_E \otimes \lambda). \tag{3.17} $$

Definition 3.38 For $E \subset T^{\text{tors}}$ and $D \subset T^{(A)}$ with $E \cap D = \emptyset$ and such that Γ stabilizes E and D, we define the map

$$\text{Eis}\colon (A[E] \otimes_A A[D]^0)^\Gamma \to H^{n-1}(\Gamma, \mathscr{L}og\,|_E \otimes \lambda)$$

by $h \mapsto \text{Eis}_h$.

Remark 3.39 A more intuitive way to think about Eis_h is as follows. Suppose that Γ stabilizes each point of E. Then we can view $h \in (A[E] \otimes_A A[D]^0)^\Gamma$ as a map $h\colon E \to (A[D]^0)^\Gamma$, $e \mapsto h_e$ with $h_e(d) := h(e, d)$. With this notation one has $\text{Eis}_h = \sum_{e \in E} e^* \text{pol}_{h_e}$, with pol_{h_e} as defined in Definition 3.30.

3.10 A variant of the polylogarithm II

The polylogarithm pol_α has the advantage of being defined for arbitrary coefficients and it has good trace compatibilities as we will show in the next section. The disadvantage is that it depends on functions α of degree zero. The variant pol discussed below can be evaluated on each non-zero torsion point but works only for $\mathbb{Q}$-algebras A. It is also this version of the polylogarithm which plays the dominant role in the literature on the motivic polylogarithm.

We specialize Corollary 3.28 to the case $D := \{0\}$ and $M = L_A^* := \text{Hom}_A(L_A, A)$. Then we get

$$\text{res}\colon H^{n-1}(T \backslash \{0\}, \Gamma; \underline{L}_A^* \otimes_{\underline{A}} \mathscr{L}og \otimes \lambda) \cong (L_A^* \otimes_A I)^\Gamma, \tag{3.18}$$

where $I \subset R$ is the augmentation ideal. If A is a $\mathbb{Q}$-algebra we have an isomorphism $\exp^*\colon R \cong \widehat{\text{Sym}}\, L_A$ and we have a canonical class

$$\varpi \in L_A^* \otimes_A L_A \subset L_A^* \otimes_A I \tag{3.19}$$

corresponding to $\text{id}\colon L_A \to L_A$. Obviously, $\varpi \in (L_A^* \otimes I)^\Gamma$.

Definition 3.40 Let A be a $\mathbb{Q}$-algebra, then the polylogarithm pol is the class

$$\text{pol} \in H^{n-1}(T \backslash \{0\}, \Gamma; \underline{L}_A^* \otimes_A \mathscr{L}og \otimes \lambda)$$

corresponding to ϖ under the isomorphism (3.18).

The contraction $L_A^* \otimes_A \text{Sym}^k L_A \to \text{Sym}^{k-1} L_A$ induces a map

$$\text{contr}\colon L_A^* \otimes_A R \to R. \tag{3.20}$$

Furthermore, the multiplication $L_A \otimes_A R \to R$ induces

$$\text{mult}\colon R \to L_A^* \otimes_A R \tag{3.21}$$

and it is straightforward to show that $\operatorname{contr}\circ\operatorname{mult} = \operatorname{id}$. The map mult extends to a homomorphism of sheaves

$$\operatorname{mult}\colon \mathscr{L}og \to \underline{L}_A^* \otimes_A \mathscr{L}og \tag{3.22}$$

Let $t \in T^{\mathrm{tors}}\backslash\{0\}$ be stabilized by Γ. Then ϱ_{can} allows us to identify $t^*\mathscr{L}og \cong R$.

Definition 3.41 Let A be a $\mathbb{Q}$-algebra and $t \in T^{\mathrm{tors}}\backslash\{0\}$ be stabilized by Γ. The class

$$\operatorname{Eis}(t) := \operatorname{contr}(t^* \operatorname{pol}) \in H^{n-1}(\Gamma, R \otimes \lambda)$$

is called the *Eisenstein class* associated to t. We also write

$$\operatorname{Eis}^k(t) := \exp_k^*(\operatorname{Eis}(t)) \in H^{n-1}(\Gamma, \operatorname{Sym}^k L_A \otimes \lambda).$$

Let us discuss one special case of the relation between $\operatorname{Eis}^k(t)$ and the class $\operatorname{Eis}_\alpha^k(t)$ defined in Definition 3.32, which will be used later (compare also [10, 12.4.4]).

Definition 3.42 Let $\varphi\colon L \to L'$ be an isogeny and define the function on $D := L'/\varphi(L)$

$$\alpha_\varphi := (\deg \varphi)\delta_0 - \sum_{d\in D} \delta_d.$$

Consider

$$\operatorname{mult}(\operatorname{pol}_{\alpha_\varphi}) \in H^{n-1}(T\backslash\varphi^{-1}(0), \Gamma; \underline{L}_A^* \otimes_A \mathscr{L}og \otimes \lambda)$$

then using the isomorphisms $\mathscr{L}og \cong \varphi^*\mathscr{L}og'$ and $L_A \cong L'_A$ (because A is a $\mathbb{Q}$-algebra) one also has ($D = \varphi^{-1}(0)$)

$$\varphi^* \operatorname{pol}' \in H^{n-1}\left(T\backslash D, \Gamma; \underline{L}_A^* \otimes_A \mathscr{L}og \otimes \lambda\right).$$

Finally, $\operatorname{pol}|_{T\backslash D}$, the restriction of pol to $T\backslash D$, gives a class in the same group.

Proposition 3.43 *One has the equality*

$$\operatorname{mult}(\operatorname{pol}_{\alpha_\varphi}) = (\deg \varphi) \operatorname{pol}|_{T\backslash D} - \varphi^* \operatorname{pol}'.$$

in $H^{n-1}(T\backslash D, \Gamma; \underline{L}_A^* \otimes_A \mathscr{L}og \otimes \lambda)$.

Proof From Theorem 3.27 we have an isomorphism

$$\operatorname{res}\colon H^{n-1}\left(T\backslash D, \Gamma; \underline{L}_A^* \otimes_{\underline{A}} \mathscr{L}og \otimes \lambda\right) \cong (L_A^* \otimes_A R[D]^0)^\Gamma.$$

We have $\operatorname{res}(\operatorname{mult}(\operatorname{pol}_{\alpha_\varphi})) = (\deg\varphi)\delta_0\varpi - \sum_{d\in D}\delta_d\varpi$ and $\operatorname{res}((\deg\varphi)\operatorname{pol}|_{T\backslash D}) = (\deg\varphi)\delta_0\varpi$. Moreover, $\operatorname{res}(\varphi^*\operatorname{pol}') = \sum_{d\in D}\delta_d\varpi$, which proves the claim. □

Corollary 3.44 *For $k \geq 0$ the relation of Eisenstein classes*

$$\mathrm{Eis}^k_{\alpha_\varphi}(t) = (\deg \varphi)\mathrm{Eis}^k(t) - \mathrm{Eis}'^k(\varphi(t))$$

holds in $H^{n-1}(\Gamma, \mathrm{TSym}^k L_A \otimes \lambda)$, where we have used the isomorphism

$$\mathrm{Sym}^k L_A \cong \mathrm{TSym}^k L_A \cong \mathrm{TSym}^k L'_A$$

to consider $\mathrm{Eis}'^k(\varphi(t))$ as a class in this cohomology group.

Proof One has

$$\begin{aligned}\mathrm{Eis}^k_{\alpha_\varphi}(t) &= \exp^*_k(\mathrm{contr} \circ \mathrm{mult}(t^* \mathrm{pol}_{\alpha_\varphi})) \\ &= (\deg \varphi) \exp^*_k \circ \mathrm{contr}\left(t^* \mathrm{pol}\,|_{T \setminus D} - t^* \varphi^* \mathrm{pol}'\right) \\ &= (\deg \varphi)\mathrm{Eis}^k(t) - \mathrm{Eis}'^k(\varphi(t)).\end{aligned}$$

□

3.11 Trace compatibility

The polylogarithm classes are compatible with respect to isogenies $\varphi_T : T' \to T$ (note that in this section we interchange the role of L and L'). This is a geometric incarnation of the distribution property of Eisenstein series.

We use the following set up: Let L, L' be lattices of rank n with actions by Γ and let $\varphi\colon L' \to L$ be an isogeny compatible with the Γ-action. Then one has a group homomorphism $(\varphi, \mathrm{id})\colon L' \rtimes \Gamma \to L \rtimes \Gamma$.

We consider finite non-empty subsets $D \subset T^{(A)}$ and $D' \subset T'^{(A)}$ such that $\varphi_T(D') \subset D$. One has a cartesian square

$$\begin{array}{ccc} \varphi^{-1}(D) & \longrightarrow & T' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ D & \longrightarrow & T \end{array} \tag{3.23}$$

Proposition 3.45 *Let M be an $A[\Gamma]$-module, then there is a trace map*

$$\mathrm{Tr}_\varphi\colon H^{n-1}(T' \setminus D', \Gamma; \underline{M} \otimes_{\underline{A}} \mathscr{L}og_{T'} \otimes \lambda') \to H^{n-1}(T \setminus D, \Gamma; \underline{M} \otimes_{\underline{A}} \mathscr{L}og_T \otimes \lambda)$$

such that the diagram

$$\begin{array}{ccc} H^{n-1}(T'\backslash D', \Gamma; \underline{M} \otimes_{\underline{A}} \mathscr{L}og_{T'} \otimes \lambda') & \xrightarrow[\cong]{\text{res}} & \ker(M \otimes_A R'[D'] \to M)^{\Gamma} \\ \downarrow{\scriptstyle \mathrm{Tr}_{\varphi}} & & \downarrow{\scriptstyle \varphi_R} \\ H^{n-1}(T\backslash D, \Gamma; \underline{M} \otimes_{\underline{A}} \mathscr{L}og_{T} \otimes \lambda) & \xrightarrow[\cong]{\text{res}} & \ker(M \otimes_A R[D] \to M)^{\Gamma} \end{array}$$

commutes.

Proof As $\varphi\colon T' \to T$ is a topological submersion and a finite map we have $\varphi^*(\underline{M} \otimes_{\underline{A}} \mathscr{L}og_T) \otimes \lambda' \cong \varphi^!(\underline{M} \otimes_{\underline{A}} \mathscr{L}og_T) \otimes \lambda$ (see [12, Section 3.3]). In particular, the trace map $R\varphi_!\varphi^!(\underline{M} \otimes_{\underline{A}} \mathscr{L}og_T) \to \underline{M} \otimes_{\underline{A}} \mathscr{L}og_T$ induces a map

$$\varphi_!\varphi^*(\underline{M} \otimes_{\underline{A}} \mathscr{L}og_T) \otimes \lambda' \to \underline{M} \otimes_{\underline{A}} \mathscr{L}og_T \otimes \lambda.$$

This gives

$$\begin{aligned} H^{n-1}(T'\backslash D', \underline{M} \otimes_{\underline{A}} \mathscr{L}og_{T'} \otimes \lambda') &\xrightarrow{\varphi \mathscr{L}og} H^{n-1}(T'\backslash D', \varphi^*(\underline{M} \otimes_{\underline{A}} \mathscr{L}og_T) \otimes \lambda') \\ &\xrightarrow{\text{restr}} H^{n-1}(T'\backslash \varphi^{-1}(D), \varphi^*(\underline{M} \otimes_{\underline{A}} \mathscr{L}og_T) \otimes \lambda') \\ &\xrightarrow{\cong} H^{n-1}(T\backslash D, \varphi_!\varphi^*(\underline{M} \otimes_{\underline{A}} \mathscr{L}og_T) \otimes \lambda') \\ &\to H^{n-1}(T\backslash D, \underline{M} \otimes_{\underline{A}} \mathscr{L}og_T \otimes \lambda), \end{aligned}$$

where we have used that $\varphi\colon T'\backslash\varphi^{-1}(D) \to T\backslash D$ is finite, so that $\varphi_! = \varphi_*$. The result follows from Theorem 3.27 and the diagram commutes because of the cartesian square (3.23) and [12, 3.1.9]. □

Remark 3.46 In this paper we consider only the trace compatibility for isogenies. We remark that a similar statement holds also in the more general case of a submersion. This was used in [9] to compute the residue of the Eisenstein classes on Hilbert modular varieties.

We discuss now the consequences of this proposition for the different notions of polylogarithm we have defined.

Corollary 3.47 *In the situation of Definition 3.30 one has for $\alpha \in (A[D']^0)^{\Gamma}$*

$$\mathrm{Tr}_{\varphi}(\mathrm{pol}'_{\alpha}) = \mathrm{pol}_{\varphi_*(\alpha)}$$

where $\varphi_*(\alpha)$ is the function $\varphi_*(\alpha)(d) = \sum_{d' \in \varphi^{-1}(d)} \alpha(d')$.

Proof This is immediate from the definition, Proposition 3.45 and the fact that the restriction of φ to the subspace $(A[D']^0)^{\Gamma} \subset (R[D']^0)^{\Gamma}$ is given by the formula in the corollary. □

The following generalization of the trace compatibility is used later in the general study of Eisenstein distributions. In the situation of Proposition 3.45 assume in addition that one has finite non-empty subsets $E \subset T^{\text{tors}}$, $E' \subset T'^{\text{tors}}$ with $\varphi_T(E') \subset E$ and $E \cap D = \emptyset$, $E' \cap D' = \emptyset$. We have

$$\begin{array}{ccccc} E' \amalg D' & \longrightarrow & \varphi^{-1}(E) \amalg \varphi^{-1}(D) & \longrightarrow & T' \\ & & \varphi \downarrow & & \downarrow \varphi \\ & & E \amalg D & \longrightarrow & T \end{array} \tag{3.24}$$

We assume that Γ stabilizes $E \cup D$ and $E' \cup D'$. Then the trace map Tr_φ induces a homomorphism $A[E'] \to A[E]$, which we call φ (it is the same as $\varphi\colon A[D'] \to A[D]$). Let

$$\varphi \circ \mathrm{Tr}_\varphi\colon H^{n-1}(T' \setminus D', \Gamma; \underline{A}[E'] \otimes_{\underline{A}} \mathscr{L}og_{T'} \otimes \lambda') \\ \to H^{n-1}(T \setminus D, \Gamma; \underline{A}[E] \otimes_{\underline{A}} \mathscr{L}og_T \otimes \lambda)$$

be the composition of the trace map Tr_φ with the map induced by φ. Recall the Eisenstein operator

$$\mathrm{Eis}\colon (A[E] \otimes_A A[D]^0)^\Gamma \to H^{n-1}(\Gamma, \mathscr{L}og\,|_E \otimes \lambda)$$

from Definition 3.38. The trace compatibility for pol_h has the following consequence for Eis.

Corollary 3.48 *Let $E' = \varphi^{-1}(E)$ and $\varphi^*\colon A[E] \to A[E']$ be the map $f \mapsto f \circ \varphi$. Then for $h \in (A[E] \otimes_A A[D']^0)^\Gamma$ one has*

$$\mathrm{Tr}_\varphi(\mathrm{Eis}'_{(\varphi^* \otimes \mathrm{id})(h)}) = \mathrm{Eis}_{(\mathrm{id} \otimes \varphi_*)(h)}.$$

Proof This is immediate from the definition of Eis and the commutative diagram

$$\begin{array}{ccccc} A[E] \otimes \mathscr{L}og'\,|_{E'} & \xrightarrow{\varphi^* \otimes \mathrm{id}} & A[E'] \otimes \mathscr{L}og'\,|_{E'} & \xrightarrow{(3.15)} & \mathscr{L}og'\,|_{E'} \\ \downarrow{\scriptstyle \mathrm{id} \otimes \mathrm{Tr}_\varphi} & & & & \downarrow{\scriptstyle \mathrm{Tr}_\varphi} \\ A[E] \otimes \mathscr{L}og\,|_E & & \xrightarrow{(3.15)} & & \mathscr{L}og\,|_E. \end{array}$$

□

4 Explicit formulas

In this section we give an explicit formula for the topological polylogarithm. The computations were essentially done by Nori [13] and we present them here in a slightly different form.

In this section we always consider $A = \mathbb{C}$ so that we can identify $R \cong \widehat{\mathrm{Sym}}^{\cdot} L_{\mathbb{C}}$.

4.1 The continuous trivialization of the logarithm sheaf

Let $\mathscr{P}$ be the space of positive definite symmetric bilinear forms on V, which we also consider as translation invariant metrics on V. Let V^* be the $\mathbb{R}$-dual of V, then we consider $\mathscr{P} \subset \mathrm{Hom}(V, V^*)$ with its induced right Γ-action: $B[\gamma](v, w) := B(\gamma v, \gamma w)$.

We let $L \rtimes \Gamma$ act on $\mathscr{P} \times V$ by

$$(B, v)(\ell, \gamma) := (B[\gamma], \gamma^{-1}(v + \ell)).$$

Note that the action of Γ on L factors through $\mathrm{GL}(L)$, which acts almost discretely on $\mathscr{P}$ and that $\mathscr{P}$ is contractible. We consider the sheaf $\mathscr{L}og$ over $(\mathscr{P} \times \mathrm{Log})/\Gamma$ as the sheaf of sections of $\mathscr{P} \times V/L \rtimes \Gamma$. Let $D \subset T^{(A)}$ be a finite non-empty subset, which does not contain 0. By general principles from equivariant cohomology we have

$$H^{n-1}(T \backslash D, \Gamma, \mathscr{L}og \otimes \lambda) \cong H^{n-1}(\mathscr{P} \times (V \backslash \pi^{-1}(D))/L \rtimes \Gamma, \mathscr{L}og \otimes \lambda).$$

We need to set up some more notation.

Recall that $R^{(k)} = R/I^{k+1}$ and that it has an induced action of L. Recall also that $\mathscr{L}og^{(k)}$ is the associated local system and that we denote by $\underline{R}^{(k)}$ the constant sheaf associated to $R^{(k)}$. The canonical maps $R^{(k+1)} \to R^{(k)}$ makes these local systems into pro-local systems.

We write $\mathscr{L}og^{\infty}$ for the pro-bundle defined by $\mathscr{C}^{\infty} \otimes \mathscr{L}og^{(k)}$ and similar for $\underline{R}^{\infty}$. More generally, we consider the pro-bundles of sheaves of $\mathscr{C}^{\infty}$-differential forms $\Omega^i \widehat{\otimes} \mathscr{L}og$ defined by $\Omega^i \otimes \mathscr{L}og^{(k)}$ and currents $\widehat{\Omega}^i \widehat{\otimes} \mathscr{L}og$.

On $\mathscr{L}og^{\infty}$ we have the connection $\nabla := d \otimes \mathrm{id}$ and on $\underline{R}^{\infty}$ we have the connection $\nabla_0 := d \otimes \mathrm{id}$.

Definition 4.1 Let κ be the $\underline{R}^{\infty}$-valued 1-form on T

$$\kappa \in V^* \otimes V \subset V^* \widehat{\otimes} R \subset \Gamma(T, \Omega^1_T \widehat{\otimes} \underline{R}),$$

which corresponds to the identity map $\mathrm{id} \in \mathrm{Hom}(V, V) \cong V^* \otimes V$.

Obviously, κ is Γ invariant.

Lemma 4.2 *The sheaf $\mathscr{L}og^{\infty}$ admits a unique continuous multiplicative trivialization ϱ_{cont} on T. The section ϱ_{cont} is $\mathscr{C}^{\infty}$, compatible with N-multiplication, and one has*

$$\nabla(\varrho_{\mathrm{cont}}) = -\kappa \varrho_{\mathrm{cont}}.$$

In particular, $\varrho_{\mathrm{cont}} |_{T^{\mathrm{tors}}} = \varrho_{\mathrm{can}}$.

Proof The set of continuous multiplicative trivializations with the property in the lemma is a torsor under $\mathrm{Hom}_{\mathrm{cont}}(T, (1 + I)^{\times})$, which is trivial because T is compact. This shows the uniqueness of ϱ_{cont}. For the existence we consider

$$V \to V \times R^{\times} \quad v \mapsto (v, \exp(-v))$$

where $\exp(-v) := \sum_{k\geq 0} \frac{(-v)^{\otimes k}}{k!}$. This is a section by (3.6), is compatible with N-multiplication and has the desired property $d\exp(-v) = -\kappa\exp(-v)$. $\square$

4.2 Green's currents and the topological polylogarithm

We use ϱ_{cont} from Lemma 4.2 to identify $\varrho_{\mathrm{cont}}\colon \underline{R}^\infty \cong \mathscr{L}og^\infty$. The connection ∇ of $\mathscr{L}og^\infty$ corresponds to $\nabla_0 - \kappa$ under this identification. In particular, we can compute the equivariant cohomology of $\mathscr{L}og$ as

$$H^i(T\backslash D, \Gamma; \mathscr{L}og \otimes \lambda) = H^i((\Omega^\cdot(\mathscr{P} \times (T\backslash D))\widehat{\otimes} R \otimes \lambda)^\Gamma, \nabla).$$

For the construction of a cohomology class representing the topological polylogarithm pol_α, we will first construct a certain Green's-current. To define these, we need two notations: Let $\lambda^* := \mathrm{Hom}_{\mathbb{Z}}(\lambda, \mathbb{Z})$, then the volume form on T is defined to be the section

$$\mathrm{vol} \in \lambda^* \otimes \lambda \subset \Omega^n(T) \otimes \lambda \tag{4.1}$$

corresponding to the isomorphism $\lambda \cong \lambda$. Let $\delta_{\mathscr{P}\times\{0\}}$ be the delta function of $\mathscr{P} \times \{0\} \subset \mathscr{P} \times T$. We consider this as an element in $\widehat{\Omega}^n(T) \otimes \lambda$ by multiplying it with vol.

Definition 4.3 A *Green's-current* is an $n-1$-current $\mathscr{G} \in (\widehat{\Omega}^{n-1}(\mathscr{P} \times T)\widehat{\otimes} R \otimes \lambda)^\Gamma$, which is smooth on $\mathscr{P} \times (T\backslash\{0\})$, and such that

$$\nabla(\mathscr{G}) = \delta_{\mathscr{P}\times\{0\}}\,\mathrm{vol} - \mathrm{vol}$$

in $(\widehat{\Omega}^n(\mathscr{P} \times T)\widehat{\otimes} R \otimes \lambda)^\Gamma$.

With a method due essentially to Nori we prove in the next section (see Corollary 4.14):

Theorem 4.4 *A Green's-current as in Definition 4.3 exists.*

Here we explain how we get a representative of pol_α with the help of $\mathscr{G}$. The group T acts on the complex $\widehat{\Omega}^\cdot(\mathscr{P} \times T)\widehat{\otimes} R$ by translation.

Definition 4.5 Let $D \subset T^{\mathrm{tors}}$ be finite and non-empty and $\mathscr{G}$ be a Green's-current. Let τ_d be the translation by $d \in T$ and $\alpha = \sum_{d\in D} \alpha_d 1_d \in \mathbb{C}[D]^0$. Then we define the $n-1$-current

$$\mathscr{G}(\alpha) := \sum_{d\in D} \alpha_d \tau_{-d}^* \mathscr{G},$$

which is smooth on $\mathscr{P} \times (T\backslash D)$.

With this notation we can formulate the main result in this section.

Theorem 4.6 *If $D \subset T^{\mathrm{tors}}$ and $\alpha \in (\mathbb{C}[D]^0)^\Gamma$, then the restriction of $\mathcal{G}(\alpha)$ to $\mathcal{P} \times (T \backslash D)$ is a smooth Γ-invariant closed $n-1$-form, which represents* pol_α.

Proof As α and $\mathcal{G}$ are Γ-invariant, the same holds for $\mathcal{G}(\alpha)$. By definition $\nabla(\mathcal{G}(\alpha)) = \sum_{d \in D} \alpha_d \delta_d$, which implies that the restriction of $\mathcal{G}(\alpha)$ to $\mathcal{P} \times (T \backslash D)$ is closed and that $\mathrm{res}(\mathcal{G}(\alpha)) = \alpha$. With Corollary 3.28 we see that $\mathcal{G}(\alpha)$ represents pol_α. □

We also want to construct a current, which represents the variant of the polylogarithm pol from Definition 3.40. Let $\ell_1, \ldots, \ell_n$ be a basis of L and $\mu_1, \ldots, \mu_n$ be the dual basis of V^*. Define the closed form $\eta := \frac{1}{n} \sum_{j=1}^n (-1)^j \mu_j d\mu_1 \wedge \cdots \wedge \widehat{d\mu_j} \wedge \cdots \wedge d\mu_n$, then a straightforward computation shows

$$-\kappa\eta = \varpi \,\mathrm{vol},$$

where $\varpi \in L_A^* \otimes_A L_A$ is element from (3.19).

Theorem 4.7 *Let $\widetilde{\mathcal{G}} := \varpi\mathcal{G} + \eta$, then*

$$\nabla(\widetilde{\mathcal{G}}) = \delta_{\mathcal{P}\times\{0\}} \varpi \,\mathrm{vol}$$

and $\widetilde{\mathcal{G}}$ represents $\mathrm{pol} \in H^{n-1}(T\backslash\{0\}, \Gamma; \underline{L}_A^* \otimes_A \mathcal{L}og \otimes \lambda)$ *defined in Definition* 3.40.

Proof This follows from the formula

$$(\nabla_0 - \kappa)(\varpi\mathcal{G} + \eta) = \varpi(\delta_{\mathcal{P}\times\{0\}} \,\mathrm{vol} - \mathrm{vol}) - \kappa\eta = \varpi\delta_{\mathcal{P}\times\{0\}} \,\mathrm{vol}.$$

□

4.3 Explicit construction of a Green's current

The idea for the construction of the Green's current presented in this section goes essentially back to Nori [13]. One rewrites $\mathcal{G}$ as a Fourier-series and considers the resulting differential equations for the coefficients. This differential equation can be solved by inverting a differential operator.

We write

$$L^* := \mathrm{Hom}(L, \mathbb{Z}) \subset V^*$$

for the dual lattice of L and $\langle , \rangle \colon V \times V^* \to \mathbb{R}$ for the evaluation map. Further we let $i := \sqrt{-1}$ be a square-root of -1. Any current $\mathcal{G} \in \widehat{\Omega}^{n-1}(\mathcal{P} \times T)\widehat{\otimes} R \otimes \lambda$ has a Fourier-series

$$\mathcal{G}(B, v) = \sum_{\mu \in L^*} E_\mu(B) e^{2\pi i \langle v, \mu\rangle}$$

where $E_\mu(B) \in \Omega^{n-1}(\mathscr{P} \times T)\widehat{\otimes} R \otimes \lambda$ are R-valued differential forms, which are constant in the T direction. We write

$$E_\mu(B) = E_\mu^0 + \cdots + E_\mu^{n-1}$$

and $E_\mu^a \in \Omega^a(\mathscr{P}) \otimes \Lambda^{n-1-a} V^* \widehat{\otimes} R \otimes \lambda$ is the component in bidegree $(a, n-1-a)$ of E_μ.

Lemma 4.8 *Suppose that $\mathscr{G}$ is a Green's-current as in Definition 4.3 and*

$$\mathscr{G}(B, v) = \sum_{\mu \in L^*} E_\mu(B) e^{2\pi i \langle v, \mu \rangle}$$

its Fourier-series. If we assume that $E_0 = 0$, then the differential equation $\nabla(\mathscr{G}) = \delta_{\mathscr{P}\times\{0\}} \operatorname{vol} - \operatorname{vol}$ amounts to

$$dE_\mu + (2\pi i \mu - \kappa) E_\mu = \operatorname{vol} \quad \textit{for all } \mu \neq 0, \tag{4.2}$$

i.e.,

$$(2\pi i \mu - \kappa) E_\mu^0 = \operatorname{vol} \quad \textit{and} \quad dE_\mu^a + (2\pi i \mu - \kappa) E_\mu^{a+1} = 0. \tag{4.3}$$

Here we view $\mu \in V^* \subset \Lambda^{\cdot} V^* \widehat{\otimes} R$ as an R-valued 1-form, so that $(2\pi i \mu - \kappa) \in \Lambda^{\cdot} V^* \widehat{\otimes} R$.

Proof Immediate calculation using $\nabla = \nabla_0 - \kappa$ and the fact that the Fourier series of $\delta_{\mathscr{P}\times\{0\}} \operatorname{vol}$ is $\sum_{\mu \in L^*} e^{2\pi i \langle v, \mu \rangle} \operatorname{vol}$. □

We will now forget the fact that μ comes from the lattice L^* and try to find a natural solution of (4.2) for any $0 \neq \mu \in V^*$. For this we consider the half-space $V_{\mu>0} := \{v \in V \mid \langle v, \mu \rangle > 0\}$. We consider B as an isomorphism $B\colon V \cong V^*$, so that we have a map

$$v_\mu\colon \mathscr{P} \to V_{\mu>0} \quad B \mapsto B^{-1}(\mu). \tag{4.4}$$

We will construct E_μ as the v_μ-pull-back of a natural $n-1$-form $E_{(\mu)}$ on $V_{\mu>0}$.

Define the commutative DG-algebra $\mathcal{A} := \Omega^{\cdot}(V) \otimes \Lambda^{\cdot} V^*$ with differential $d(\omega \otimes \xi) = d\omega \otimes \xi$. On $\mathcal{A}$ we have the derivation θ of degree -1, which is zero on $\Omega^{\cdot}(V)$ and maps $\mu \in V^* \subset \Lambda^{\cdot} V^*$ to the linear function $\mu_V \in \mathscr{C}^\infty(V)$ with $\mu_V(v) := \mu(v)$. The DG-algebra $\mathcal{A}$ contains the subalgebra

$$\Lambda^{\cdot}(V^* \oplus V^*) \cong \Lambda^{\cdot} V^* \otimes \Lambda^{\cdot} V^* \subset \Omega^{\cdot}(V) \otimes \Lambda^{\cdot} V^*$$

and we let $\Delta\colon \Lambda V^* \to \Lambda^{\cdot} V^* \otimes \Lambda^{\cdot} V^*$ be the algebra homomorphism induced by the diagonal map $V^* \to V^* \oplus V^*$.

Let $\ell_1, \ldots, \ell_n$ be a basis of L and $\mu_1, \ldots, \mu_n$ be the dual basis of V^*. Then

$$\text{vol} := \mu_1 \wedge \cdots \wedge \mu_n \otimes \ell_1 \wedge \cdots \wedge \ell_n \in \Lambda^n V^* \otimes \lambda$$

and $\kappa = \sum_{j=1}^n d\mu_{j,V} \otimes \ell_j \in \Omega^1(V)\widehat{\otimes} R$.

Definition 4.9 We let $\psi := \Delta(\text{vol}) \in \mathcal{A}^n \otimes \lambda$ and write $\psi = \sum_{a=0}^n \psi^a$ with $\psi^a \in \Omega^a(V) \otimes \Lambda^{n-a} V^* \otimes \lambda$. Then we define

$$\nu^a := \theta(\psi^a) \in \Omega^a(V) \otimes \Lambda^{n-1-a} V^* \otimes \lambda.$$

We note that the forms ν^a have the following explicit description. Let $\omega_i := d\mu_{i,V} \in \Omega^1(V)$, $\omega_I := \Lambda_{i\in I}\omega_i$ for any subset $I \subset \{1, \ldots, n\}$ and define similarly μ_I. Then

$$\nu^a = \sum_{|I|=a} \sum_{j=1}^n \mu_{j,V}\omega_I \otimes \mu_{I^c\setminus\{j\}},$$

where I^c is the complement of I. The forms ν^a have the following properties:

Lemma 4.10 *For $\xi \in V^*$ one has the formulae*

$$\begin{aligned} d\nu^a &= (a+1)\psi^{a+1} \\ \xi \wedge \psi^a &= -d\xi_V \wedge \psi^{a-1} \\ \xi \wedge \nu^a &= \xi_V \psi^a - d\xi_V \wedge \nu^{a-1} \end{aligned}$$

In particular, if one writes $\kappa_V := \sum_{j=1}^n \mu_{j,V} \otimes \ell_j$, so that $d\kappa_V = \kappa$, one has

$$\kappa \wedge \nu^a = \kappa_V \psi^a - d\kappa_V \wedge \nu^{a-1}.$$

Proof For a form $\omega \in \mathcal{A}$ denote by $\omega^a \in \Omega^a(V) \otimes \Lambda^{n-a} V^*$ its a-part. For $\xi \in V^*$ one has $\Delta(\xi) = d\xi_V + \xi$ and hence $d\theta(\Delta(\xi)^1) = 0$ and $d\theta(\Delta(\xi)^0) = d\xi_V = \Delta(\xi)^1$. Write $\text{vol}_n = \text{vol}_{n-1} \wedge \mu_n$. Then

$$\Delta(\text{vol}_n)^a = \Delta(\text{vol}_{n-1})^{a-1} \wedge \Delta(\mu_n)^1 + \Delta(\text{vol}_{n-1})^a \wedge \Delta(\mu_n)^0.$$

Applying $d\theta$ and induction on n gives

$$\begin{aligned} d\nu^a = {} & a\Delta(\text{vol}_{n-1})^a \wedge \Delta(\mu_n)^1 + (a+1)\Delta(\text{vol}_{n-1})^{a+1} \wedge \Delta(\mu_n)^0 \\ & + \Delta(\text{vol}_{n-1})^a \wedge \Delta(\mu_n)^1. \end{aligned}$$

This shows the first equation. The second follows from $\Delta(\xi)\wedge\Delta(\text{vol}) = \Delta(\xi\wedge\text{vol}) = 0$ and $\Delta(\xi) = d\xi_V + \xi$ and the third by applying θ to it. The formula for κ follows from the third equation using the explicit formulae for κ and κ_V. □

Write $\mathcal{A}_{(\mu)} := \Omega^{\cdot}(V_{>\mu}) \otimes \Lambda^{\cdot} V^*$, then μ_V is invertible in $\mathcal{A}_{(\mu)}$. The element $\kappa_V := \sum_{j=1}^n \mu_{j,V} \otimes \ell_j \in \mathscr{C}^\infty(V)\widehat{\otimes} R$ is topologically nilpotent, so that $\mu_V - \kappa_V$ is invertible in $\mathscr{C}^\infty(V_{>\mu})\widehat{\otimes} R$. Define

$$E^a_{(\mu)} := (-1)^a a!(2\pi i \mu_V - \kappa_V)^{-a-1} \nu^a \quad E_{(\mu)} := \sum_{a=0}^{n-1} E^a_{(\mu)}. \tag{4.5}$$

Lemma 4.11 *The formulae*

$$(2\pi i \mu - \kappa) E^0_{(\mu)} = \psi^0 = \mathrm{vol}$$
$$d E^a_{(\mu)} + (2\pi i \mu - \kappa) E^{a+1}_{(\mu)} = 0$$

hold. In particular, $E_\mu := v_\mu^* E_{(\mu)}$ *satisfies the differential equation* (4.2). *Moreover, for* $\gamma \in \Gamma$ *one has*

$$\gamma^* E_\mu = E_{\mu \circ \gamma^{-1}}.$$

Proof From Lemma 4.10 we have

$$(2\pi i \mu - \kappa)\nu^{a+1} = (2\pi i \mu_V - \kappa_V)\psi^{a+1} - d(2\pi i \mu_V - \kappa_V) \wedge \nu^a$$
$$d((2\pi i \mu_V - \kappa_V)^{-a-1}\nu^a) = (a+1)((2\pi i \mu_V - \kappa_V)^{-a-1}\psi^{a+1} - (2\pi i \mu_V - \kappa_V)^{-a-2} d(2\pi i \mu_V - \kappa_V)\nu^a)$$

which show that the differential equations are satisfied. For the action of γ note that $v_\mu \circ \gamma = \gamma^{-1} \circ v_{\mu\circ\gamma^{-1}}$. As vol and θ are Γ-invariant one has $(\gamma^{-1})^* \nu^a = \nu^a$. The map $\kappa_V : V \to R$ is the canonical inclusion and obviously Γ-invariant. Therefore $(\gamma^{-1})^* E_{(\mu)} = E_{(\mu\circ\gamma^{-1})}$ and the formula follows. □

Lemma 4.12 *Let* $E^a_\mu := v_\mu^*(E^a_{(\mu)})$ *and consider* $B^{-1} : V^* \cong V$ *as bilinear form on* V^**, then one has explicitly*

$$E^a_\mu = (-1)^a a! \frac{v_\mu^*(\nu^a)}{(2\pi i B^{-1}(\mu,\mu) - B^{-1}(\mu))^{a+1}}$$
$$= (-1)^a \frac{(k+a)!}{k!} \sum_{k\geq 0} \frac{B^{-1}(\mu)^{\otimes k}}{(2\pi i B^{-1}(\mu,\mu))^{k+a+1}} v_\mu^*(\nu^a)$$

(*where we let* $B^{-1}(\mu)^{\otimes 0} := 1$) *and*

$$v_\mu^*(\nu^a) = \sum_{|I|=a} \sum_{j=1}^n B^{-1}(\mu_j, \mu) \Lambda_{i\in I} d B^{-1}(\mu_i, \mu) \otimes \mu_{I^c \setminus \{j\}} \otimes \ell_1 \wedge \cdots \wedge \ell_n.$$

Proof Direct computation. □

We are going to show that the series $\sum_{\mu\in L^*\backslash\{0\}} E^a_\mu e^{2\pi i\langle v,\mu\rangle}$ defines a current on $\mathscr{P}\times T$, which is smooth on $\mathscr{P}\times(T\backslash\{0\})$. The following proof is due to Levin (see the Appendix of [3]). Let $P\colon \mathscr{P}\times V\to\mathbb{C}$ be a $\mathscr{C}^\infty$ function, which is a homogeneous polynomial of degree g in the V-variables. We consider the series of distributions

$$\mathscr{K}_s(B,v,P) := \sum_{\mu\in L^*\backslash\{0\}} \frac{P(B,\mu)}{B^{-1}(\mu,\mu)^{s+g/2}} e^{2\pi i\langle v,\mu\rangle}$$

and we are interested in the convergence and the analyticity in s. We have the following result:

Theorem 4.13 *Let $v\neq 0$ then $\mathscr{K}_s(B,v,P)$ is a smooth distribution for all $s\in\mathbb{C}$.*

Proof We give the essential steps of the proof.

The first step is to show that the series $\mathscr{K}_s(B,v,P)$ defines a (tempered) distribution on $\mathscr{P}\times T$ for all $s\in\mathbb{C}$. We may assume that B varies in a compact subset of $\mathscr{P}$. A Fourier series defines a distribution, if the coefficients grow less than a polynomial of fixed degree $N\geq 0$. But $\frac{B^{-1}(\mu)^{\otimes k}}{(2\pi i B^{-1}(\mu,\mu))^{k/2+s}}$ satisfies this requirement if $s\geq -N$.

The second step is to remark that the map $s\mapsto\mathscr{K}_s(B,v,P)$ is analytic. This follows because for each test function ψ the series $\mathscr{K}_s(B,v,P)(\psi)$ converges absolutely and uniformly on every compact subset of $\mathbb{C}$ (same proof as for Dirichlet series, one also has to use that weakly analytic functions with values in the dual of a Frechet space are actually analytic).

Next we note that $\mathscr{K}_s(B,v,P)$ converges as a sequence of functions absolutely and uniformly for $\Re(s)\geq n/2+\epsilon$ with $\epsilon>0$. The resulting analytic function on the half plane $\Re(s)>n/2$ can be analytically continued with the standard procedure known from the analytic continuation of the zeta functions: One writes $\mathscr{K}_s(B,v,P)$ as the Mellin-transform of a theta series as in [17, Chapter I, Paragraph 5] and uses the Poisson summation formula to obtain the analytic continuation $\widetilde{\mathscr{K}_s}(B,v,P)$ of $\mathscr{K}_s(B,v,P)$. To see that the function $\widetilde{\mathscr{K}_s}(B,v,P)$ has no poles one uses [17, Chapter I,5, Theorem 3]. Note that our polynomial function P is homogeneous so that its value at $v=0$ is 0 if the degree $g>0$. For $g=0$ the polynomial is constant and it is here that the assumption $v\neq 0$ enters to guarantee that $\widetilde{\mathscr{K}_s}(B,v,P)$ has no pole.

Finally we remark that the principle of analytic continuation holds for tempered distributions, so that we can conclude that $\widetilde{\mathscr{K}_s}(B,v,P)=\mathscr{K}_s(B,v,P)$ for all $s\in\mathbb{C}$. This shows the assertion of the theorem. □

The next corollary finishes the proof of Theorem 4.4.

Corollary 4.14 *The series*

$$\mathscr{G}(B,v)=\sum_{a=0}^{n-1}\sum_{\mu\in L^*\backslash\{0\}} E^a_\mu e^{2\pi i\langle v,\mu\rangle}$$

defines a $R\otimes\lambda$-valued, Γ-invariant current on $\mathscr{P}\times T$, which is smooth on $\mathscr{P}\times(T\backslash\{0\})$ and satisfies the differential equation

$$\nabla(\mathscr{G})=\delta_{\mathscr{P}\times\{0\}}\operatorname{vol}-\operatorname{vol}.$$

In particular, $\mathscr{G}$ is a Green's-current.

Proof We have

$$\sum_{\mu\in L^*\backslash\{0\}}\frac{B^{-1}(\mu)^{\otimes k}e^{2\pi i\langle v,\mu\rangle}}{(2\pi i B^{-1}(\mu,\mu))^{k+a+1}}v_\mu^*(v^a)=\frac{1}{(2\pi i)^{k+a1}}\sum_{k\geq 0}\mathscr{K}_{(k+a+1)/2}(B,v,P)\omega$$

where $\omega\in\Omega^a(\mathscr{P})\otimes\Lambda^{n-1-a}V^*$ is a smooth differential form, which does not depend on μ and P is a polynomial of degree $k+a+1$ in the V-variables. Hence, by Theorem 4.13 the left hand side defines a current on $\mathscr{P}\times T$, which is smooth on $\mathscr{P}\times(T\backslash\{0\})$. By Lemma 4.11 we have $\gamma^*(E_\mu e^{2\pi i\langle v,\mu\rangle})=E_{\mu\circ\gamma^{-1}}e^{2\pi i\langle v,\mu\circ\gamma^{-1}\rangle}$, which shows that $\mathscr{G}(B,v)$ is Γ-invariant. The differential equation is an immediate consequence of Lemma 4.11. □

5 Applications to L-values of totally real fields and Eisenstein cohomology of Hilbert modular varieties

We discuss the relation between the topological polylogarithm and special values of partial L-functions of totally real fields. This is due to Nori and Szcech but we need the explicit formulae for the p-adic interpolation. The second application shows the relation of the topological polylogarithm to Eisenstein cohomology for Hilbert modular varieties. This is a new result due to Graf and the detailed relationship will appear in his thesis [7]. We discuss here the p-adic interpolation of his construction.

5.1 Values of partial L-functions of totally real fields

In this section F will be a totally real field of degree n over $\mathbb{Q}$ and ring of integers $\mathcal{O}_F$. Let $L\subset F\otimes\mathbb{R}$ be a fractional ideal and h be an element which is non-zero and torsion in $T(L)=F\otimes\mathbb{R}/L$. We define

$$\mathcal{O}_h^{+,\times}:=\{u\in\mathcal{O}_F^\times\mid uh\equiv h\bmod L\text{ and }u\text{ totally positive}\}.$$

We consider the partial zeta function for $\Re(s)>1$

$$\zeta(h,L,s):=\sum_{\alpha\in(h+L)^+/\mathcal{O}_h^{+,\times}}\mathrm{N}\alpha^{-s}. \tag{5.1}$$

where $(h+L)^+$ are the totally positive elements in $h+L$. A sign character $\varepsilon\colon(F\otimes\mathbb{R})^\times\to\{\pm1\}$ is a character, which is trivial on $(F\otimes\mathbb{R})^{+,\times}$ the connected component of 1 in $(F \otimes \mathbb{R})^\times$. Writing $(F \otimes \mathbb{R})^\times \cong \prod_\tau \mathbb{R}^\times$ the sign character is the product of n sign characters of $\mathbb{R}^\times$. We denote by $|\varepsilon|$ the number of non-trivial sign characters in this product. We then consider also more generally the partial zeta functions

$$\zeta(\varepsilon, h, L, s) := \sum_{\alpha \in (h+L)/\mathcal{O}_h^{+,\times}} \frac{\varepsilon(\alpha)}{|\mathrm{N}\,\alpha|^s} \tag{5.2}$$

and we have the identity

$$\sum_\varepsilon \zeta(\varepsilon, h, L, s) = 2^n \zeta(h, L, s). \tag{5.3}$$

Proposition 5.1 *The function $\zeta(\varepsilon, h, L, s)$ admits an holomorphic continuation to $\mathbb{C}\backslash\{1\}$ and satisfies the functional equation*

$$\zeta(\varepsilon, h, L, 1-s) = (\cos(\pi(s+1)/2))^{|\varepsilon|}(\cos(\pi s/2))^{n-|\varepsilon|} \frac{2^n i^{|\varepsilon|}\Gamma(s)^n}{(2\pi)^{ns}\,\mathrm{vol}(L)} \times \sum_{\mu \in L^*\backslash\{0\}/\mathcal{O}_h^{+,\times}} \frac{\varepsilon(\mu) e^{2\pi i\langle h,\mu\rangle}}{|\mathrm{N}\,\mu|^s}$$

where $L^* = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ is the dual lattice.

Proof This is a standard result. A sketch of the proof can be found in [16] (for $h = 1$). The case of general h is the same. Alternatively the result can be deduced from [6, Theorem 3.12]. □

Corollary 5.2 *Let sgn^{k+1} be the sign character $\mathrm{sgn}^{k+1}(\mu) := \frac{N(\mu)^{k+1}}{|N(\mu)|^{k+1}}$. Then for any integer $k \geq 0$, the value $\zeta(\varepsilon, h, L, -k)$ is 0 except for $\varepsilon = \mathrm{sgn}^{k+1}$. In particular, $\zeta(\mathrm{sgn}^{k+1}, h, L, -k) = 2^n \zeta(h, L, -k)$ and one has*

$$\zeta(h, L, -k) = \frac{(k!)^n}{(2\pi i)^{n(k+1)}\,\mathrm{vol}(L)} \sum_{\mu \in L^*\backslash\{0\}/\mathcal{O}_h^{+,\times}} \frac{e^{2\pi i\langle h,\mu\rangle}}{\mathrm{N}\,\mu^{k+1}}.$$

Proof This is an easy consequence of Proposition 5.1 and the location of the zeroes of the functions $(\cos(\pi(s+1)/2))^{|\varepsilon|}$ and $(\cos(\pi s/2))^{n-|\varepsilon|}$. □

5.2 The evaluation map

We keep the notation of the previous section, i.e., F is a totally real field of degree n over $\mathbb{Q}$ with ring of integers $\mathcal{O}_F$, $L \subset F \otimes \mathbb{R}$ is a fractional ideal and we consider the torus $T := F \otimes \mathbb{R}/L$.

Let $\mathcal{O}_F^{+,\times}$ be the group of totally positive units in $\mathcal{O}_F$. Note that this is a free abelian group of rank $n-1$. Let $D \subset T^{(A)}$ be a finite non-empty set of torsion points, $t \in T^{(A)} \setminus D$ and $\alpha \in (A[D]^0)^\Gamma$. We consider the Eisenstein class

$$\mathrm{Eis}_\alpha(t) \in H^{n-1}(\Gamma, R \otimes \lambda(L))$$

from Definition 3.32, where $\Gamma \subset \mathcal{O}_F^{+,\times}$ is the stabilizer of D and t. Note that Γ acts through the norm and hence trivially on λ. The cap-product with $H_{n-1}(\Gamma, \mathbb{Z})$ induces a homomorphism

$$H^{n-1}(\Gamma, R \otimes \lambda(L)) \otimes H_{n-1}(\Gamma, \mathbb{Z}) \to H_0(\Gamma, R \otimes \lambda) = R_\Gamma \otimes \lambda \tag{5.4}$$

where R_Γ are the Γ-coinvariants. For the actual evaluation we choose coordinates for T, which at the same time allow us to trivialize $\lambda(L) \otimes \mathbb{R}$ and to give a generator for $H_{n-1}(\Gamma, \mathbb{Z})$.

Let $\{\tau_1, \ldots, \tau_n\}$ be the different embeddings of F into $\mathbb{R}$, so that we have an isomorphism $F \otimes \mathbb{R} \cong \mathbb{R}^n$. On $\mathbb{R}^n$ we use the standard orientation. For each $x \in F \otimes \mathbb{R}$ we write $x_i := \tau_i(x)$.

If we identify $\lambda = H_n(L, \mathbb{Z}) \cong H_n(F \otimes \mathbb{R}/L, \mathbb{Z})$, then the fundamental class of $F \otimes \mathbb{R}/L$ provides us with a generator of λ.

Let $(F \otimes \mathbb{R})^{+,\times}$ be the totally positive and invertible elements in $F \otimes \mathbb{R}$. This is the connected component of the identity in $(F \otimes \mathbb{R})^\times$. The norm of $F/\mathbb{Q}$ defines a homomorphism $\mathrm{N}\colon (F \otimes \mathbb{R})^\times \to \mathbb{R}^\times$ and we denote by

$$(F \otimes \mathbb{R})^1 := \ker\left((F \otimes \mathbb{R})^{+,\times} \xrightarrow{\mathrm{N}} \mathbb{R}^{+,\times}\right)$$

the subgroup of elements of norm 1. Then $\Gamma \subset \mathcal{O}_F^{+,\times} \subset (F \otimes \mathbb{R})^1$ and one has a canonical isomorphism

$$H_{n-1}(\Gamma, \mathbb{Z}) \cong H_{n-1}((F \otimes \mathbb{R})^1/\Gamma, \mathbb{Z})$$

with the homology of $(F \otimes \mathbb{R})^1/\Gamma$. The logarithm $\log\colon (F \otimes \mathbb{R})^{+,\times} \xrightarrow{\cong} F \otimes \mathbb{R} \cong \mathbb{R}^n$ induces an orientation on $(F \otimes \mathbb{R})^{+,\times}$. Using the standard orientation on $\mathbb{R}^{+,\times}$ this induces also an orientation on $(F \otimes \mathbb{R})^1$: For this consider the map $\mathbb{R}^{+,\times} \to (F \otimes \mathbb{R})^{+,\times}$, $t \mapsto 1 \otimes t^{1/n}$ which is a section of the norm map. Then we can write $\mathbb{R}^{+,\times} \times (F \otimes \mathbb{R})^1 \cong (F \otimes \mathbb{R})^{+,\times}$, which provides us with the desired orientation. We use the fundamental class of $(F \otimes \mathbb{R})^1/\Gamma$ as a generator of $H_{n-1}(\Gamma, \mathbb{Z})$.

Definition 5.3 With the above notations and generators we define the *evaluation map* to be the homomorphism induced by (5.4)

$$\mathrm{ev}\colon H^{n-1}((F \otimes \mathbb{R})^1/\Gamma, R) \to R_\Gamma.$$

Note that ev is defined for any coefficient ring A. In the case $A = \mathbb{R}$ or $\mathbb{C}$ the isomorphism $F \otimes \mathbb{R} \cong \mathbb{R}^n$ induces

$$\lambda(L) \otimes \mathbb{R} \cong \lambda(\mathbb{Z}^n) \otimes \mathbb{R}$$

and we define $\mathrm{vol}(L) \in \mathbb{R}$, such that $\mathrm{vol}(L)\lambda(\mathbb{Z}^n)$ corresponds to the lattice $\lambda(L)$ under this isomorphism. Then the evaluation is given explicitly by

$$\mathrm{ev}(\eta) = \mathrm{vol}(L)^{-1} \int_{(F\otimes\mathbb{R})^1/\Gamma} \eta \tag{5.5}$$

for a differential form $\eta \in H^{n-1}((F \otimes \mathbb{R})^1/\Gamma, R \otimes \lambda(L))$.

We give a more explicit description of $(R_{\mathbb{C}})_\Gamma$. The isomorphism $L_{\mathbb{R}} \cong \mathbb{R}^n$ allows us to identify $R_{\mathbb{C}} \cong \widehat{\mathrm{Sym}L_{\mathbb{C}}}$ with the power series ring $\mathbb{C}[[z_1, \dots, z_n]]$. The action of $\Gamma \subset \mathcal{O}_F^{+,\times}$ on $L \otimes \mathbb{R}$ decomposes into a direct sum of homomorphisms $\tau_i : \Gamma \to \mathbb{R}^\times$, such that $u \in \Gamma$ acts as $\tau_i(u)z_i$ on z_i.

Lemma 5.4 *Let $w := z_1 \cdots z_n$ be the product of the z_i's, then*

$$R_{\mathbb{C}}^\Gamma = \mathbb{C}[[w]] \subset \mathbb{C}[[z_1, \dots, z_n]] = R_{\mathbb{C}}$$

and the projection $p_\Gamma \colon R_{\mathbb{C}} \to (R_{\mathbb{C}})_\Gamma$ induces an isomorphism $(R_{\mathbb{C}})^\Gamma \cong (R_{\mathbb{C}})_\Gamma$.

Proof On each monomial $z_1^{k_1} \cdots z_n^{k_n}$ the element $u \in \Gamma$ acts via $\tau(u)^{k_1} \cdots \tau_n(u)^{k_n}$, so that the action of $\Gamma \otimes \mathbb{R}$ on $R_{\mathbb{C}}$ is semi-simple. In particular, $R_{\mathbb{C}}^\Gamma = R_{\mathbb{C}}^{\Gamma\otimes\mathbb{R}}$ is a direct summand. Moreover, as each trivial Γ-representation has to factor through the norm, Γ acts trivially exactly on w^k for integers $k \geq 0$. □

Remark 5.5 Let A be any $\mathbb{Q}$-algebra and let $\Lambda^{\cdot} := \Lambda^{\cdot} \mathrm{Hom}(\Gamma, \mathbb{Q}) = H^{\cdot}(\Gamma, \mathbb{Q})$. Then the projection $p_\Gamma \colon R_A \to (R_A)_\Gamma$ yields isomorphisms

$$H^p(\Gamma, R_A) \cong H^p(\Gamma, (R_A)_\Gamma) = (R_A)_\Gamma \otimes \Lambda^p.$$

5.3 The topological polylogarithm and L-values of totally real fields

Theorem 4.6 implies that the class of $\mathrm{Eis}_\alpha(t)$ is represented by $\sum_{d\in D} \alpha(d)(t-d)^*\mathscr{G}$, where we consider $t-d$ as a torsion section of the torus family $((F \otimes \mathbb{R})^1 \times T)/\Gamma \to (F \otimes \mathbb{R})^1/\Gamma$. Note that for any torsion section h

$$h^*\mathscr{G} \in H^{n-1}((F \otimes \mathbb{R})^1/\Gamma, R_{\mathbb{C}} \otimes \lambda(L))$$

because $\nabla(h^*\mathscr{G}) = h^*(\delta_{(F\otimes\mathbb{R})^1/\Gamma\times\{0\}} \mathrm{vol} - \mathrm{vol}) = 0$. We can now formulate the main result in this section.

Theorem 5.6 *Let $h \in T = F \otimes \mathbb{R}/L$ be a non-zero torsion section and let $\Gamma = \mathcal{O}_h^{+,\times}$ be the stabilizer of h in $\mathcal{O}_F^{+,\times}$. Identify $(R_{\mathbb{C}})_\Gamma \cong \mathbb{C}[[w]]$ as in Lemma 5.4. Then one has*

$$\mathrm{ev}(h^*\mathscr{G}) = (-1)^{n-1} \sum_{k \geq 0} \zeta(h, L, -k) \frac{w^k}{(k!)^n}.$$

Equivalently, using the isomorphism $\widehat{\mathrm{Sym}} L_{\mathbb{C}} \cong \widehat{\mathrm{TSym}} L_{\mathbb{C}}$, we get

$$\exp_k^* \circ \mathrm{ev}(h^*\mathscr{G}) = \zeta(h, L, -k) z_1^{[k]} \cdots z_n^{[k]} \in \mathrm{TSym}^k L_{\mathbb{C}}.$$

Proof From Corollary 4.14 we know that $\mathscr{G}(B, v) = \sum_{a=0}^{n-1} \sum_{\mu \in L^* \setminus \{0\}} E_\mu^a e^{2\pi i \langle v, \mu \rangle}$ and by definition $h^* E_\mu^a = 0$ for $a \neq n-1$.

For the evaluation we use the following explicit embedding of $(F \otimes \mathbb{R})^1 \to \mathscr{P}$. For $q = (q_1, \ldots, q_n) \in (F \otimes \mathbb{R})^1$ we consider the form $B_q \in \mathscr{P}$ on $\mathbb{R}^n$, defined by

$$B_q(v, w) := \sum_{j=1}^n q_j^{-1} v_j w_j. \tag{5.6}$$

Then the map $v_\mu \colon \mathscr{P} \to V$ is given by $v_\mu(B_q) = (q_1 \mu_1, \ldots, q_n \mu_n)$ and writing $R = \mathbb{C}[[z_1, \ldots, z_n]]$ the map $\kappa_V \colon V \to R$ is given by $\kappa_V(v) = \sum_{j=1}^n v_j z_j$. We want to compute the integral

$$\begin{aligned}\mathrm{ev}(h^*\mathscr{G}) &= \mathrm{vol}(L)^{-1} \int_{(F\otimes\mathbb{R})^1/\Gamma} h^*\mathscr{G} \\ &= \frac{(-1)^{n-1}}{\mathrm{vol}(L)} \sum_{\mu \in L^* \setminus \{0\}/\Gamma} e^{2\pi i \langle h, \mu \rangle} \int_{(F\otimes\mathbb{R})^1} \frac{(n-1)! v_\mu^*(\nu^{n-1})}{\left(\sum_{j=1}^n 2\pi i \mu_j^2 q_j - \mu_j q_j z_j\right)^n}.\end{aligned} \tag{5.7}$$

Using $\mathrm{N}(q) = q_1 \cdots q_n = 1$ we get

$$v_\mu^*(\nu^{n-1}) \mid_{(F\otimes\mathbb{R})^1} = \mathrm{N}(\mu) \sum_{j=1}^n (-1)^{j-1} d \log q_1 \wedge \cdots \wedge \widehat{d \log q_j} \wedge \cdots d \log q_n.$$

Let $y_1, \ldots, y_n$ be the coordinate functions of $\mathbb{R}^n$ and let $t := (\mathrm{N}\, y)^{1/n}$, so that $y_j = t q_j$. Then $d \log t = \frac{1}{n} \sum_{j=1}^n d \log y_j$, which gives

$$\mathrm{N}\,\mu \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n} = \frac{dt}{t} v_\mu^*(\nu^{n-1}) \mid_{(F\otimes\mathbb{R})^1}. \tag{5.8}$$

We write

$$\frac{(n-1)!}{\left(\sum_{j=1}^{n} 2\pi i\mu_j^2 q_j - \mu_j q_j z_j\right)^n} = \int_{\mathbb{R}^{+,\times}} e^{-t\left(\sum_{j=1}^{n} 2\pi i\mu_j^2 q_j - \mu_j q_j z_j\right)} t^n \frac{dt}{t}$$

and we substitute this and (5.8) into (5.7). Using the exact sequence

$$0 \to (F\otimes\mathbb{R})^1 \to (F\otimes\mathbb{R})^{+,\times} \xrightarrow{\mathrm{N}} \mathbb{R}^{+,\times} \to 0,$$

we have to compute the integral

$$\begin{aligned}
&\int_{(F\otimes\mathbb{R})^{+,\times}} e^{-\left(\sum_{j=1}^{n} 2\pi i\mu_j^2 q_j - \mu_j q_j z_j\right)} \mathrm{N}\, y \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n} \\
&= \prod_{j=1}^{n} \int_{\mathbb{R}^{+,\times}} e^{-y_j\mu_j(2\pi i\mu_j - z_j)} y_j \frac{dy_j}{y_j} \\
&= \prod_{j=1}^{n} \frac{1}{\mu_j(2\pi i\mu_j - z_j)} \\
&= \frac{1}{\mathrm{N}\,\mu^2} \sum_{\ell\geq 0} \frac{1}{(2\pi i)^{\ell+n}} \prod_{\ell_1+\cdots+\ell_n=\ell} \frac{z_1^{\ell_1}\cdots z_n^{\ell_n}}{\mu_1^{\ell_1}\cdots\mu_n^{\ell_n}}.
\end{aligned}$$

If we apply the projection $p_\Gamma\colon \mathbb{C}[[z_1,\ldots,z_n]] \to \mathbb{C}[[w]]$ only the monomials for $\ell = nk$ of the form $(\frac{w}{\mathrm{N}\,\mu})^k$ survive and we get

$$p_\Gamma \int_{(F\otimes\mathbb{R})^1/\Gamma} \frac{(n-1)!v_\mu^*(\nu^{n-1})}{\left(\sum_{j=1}^{n} 2\pi i\mu_j^2 q_j - \mu_j q_j z_j\right)^n} = \sum_{k\geq 0} \frac{w^k}{(2\pi i)^{n(k+1)}\,\mathrm{N}\,\mu^{k+1}}.$$

This gives

$$\begin{aligned}
\mathrm{ev}(h^*\mathscr{G}) &= (-1)^{n-1}\,\mathrm{vol}(L)^{-1} \sum_{k\geq 0} \left(\sum_{\mu\in L^*\setminus\{0\}/\Gamma} \frac{e^{2\pi i\langle h,\mu\rangle}}{\mathrm{N}\,\mu^{k+1}} \right) \frac{w^k}{(2\pi i)^{n(k+1)}} \\
&= (-1)^{n-1} \sum_{k\geq 0} \zeta(h, L, -k) \frac{w^k}{(k!)^n}
\end{aligned}$$

where in the last line we have used Corollary 5.2. The formula for $\exp_k^*(\mathrm{ev}(h^*\mathscr{G}))$ follows from the fact that $w^k = (z_1\cdots z_n)^k \mapsto (k!)^n z_1^{[k]}\cdots z_n^{[k]}$ under the isomorphism $\mathrm{Sym}^k L_{\mathbb{C}} \cong \mathrm{TSym}^k L_{\mathbb{C}}$. □

From the theorem we immediately get all the known rationality, integrality and p-adic interpolation properties of $\zeta(h, L, s)$. For this we use the following principle: For any subring $A \subset \mathbb{C}$ consider the natural inclusion $R_A \subset R_{\mathbb{C}}$. Then we have a commutative diagram

$$\begin{array}{ccc} H^{n-1}(\Gamma, R_A \otimes \lambda) & \xrightarrow{\text{ev}} & (R_A)_\Gamma \\ \downarrow & & \downarrow \\ H^{n-1}(\Gamma, R_{\mathbb{C}} \otimes \lambda) & \xrightarrow{\text{ev}} & (R_{\mathbb{C}})_\Gamma \end{array} \tag{5.9}$$

and any class coming from $H^{n-1}(\Gamma, R_A \otimes \lambda)$ has to have coefficients in A under the evaluation map.

We express the next results in a more classical language. Let $\mathfrak{f}$ and $\mathfrak{b}$ be an integral ideals. Then the partial zeta function of the ray class of $\mathfrak{b}$ modulo $\mathfrak{f}$ is defined to be

$$\zeta(\mathfrak{b}, \mathfrak{f}, s) = \sum_{\mathfrak{g}} N\mathfrak{g}^{-s}$$

where the sum is taken over all integral ideals in the ray class modulo $\mathfrak{f}$ defined by $\mathfrak{b}$. These $\mathfrak{g}$ are all of the form $\mathfrak{g} = \mathfrak{b}\mu$, with $\mu \in (1 + \mathfrak{f}\mathfrak{b}^{-1})^+ = (1 + L)^+$, where we write $L = \mathfrak{f}\mathfrak{b}^{-1}$, so that

$$\zeta(\mathfrak{b}, \mathfrak{f}, s) := \mathrm{N}\,\mathfrak{b}^{-s}\zeta(1, \mathfrak{f}\mathfrak{b}^{-1}, s). \tag{5.10}$$

It follows directly from the definition that for $\mathfrak{f}' \subset \mathfrak{f}$

$$\sum_{\mathfrak{b}' \bmod \mathfrak{f}', \mathfrak{b}' \equiv \mathfrak{b} \bmod \mathfrak{f}} \zeta(\mathfrak{b}', \mathfrak{f}', s) = \left(\prod_{\mathfrak{p}|\mathfrak{f},\ \mathfrak{p}\nmid\mathfrak{f}'} \frac{1}{1 - N\mathfrak{p}^{-s}} \right) \zeta(\mathfrak{b}, \mathfrak{f}, s). \tag{5.11}$$

Corollary 5.7 (Klingen–Siegel) *Let $h \in F \otimes \mathbb{R}/L$ be a non-zero torsion point, then for $k \geq 0$ one has $\zeta(h, L, -k) \in \mathbb{Q}$. In particular,*

$$\zeta(\mathfrak{b}, \mathfrak{f}, -k) \in \mathbb{Q}$$

for all $k \geq 0$, if $\mathfrak{f} \neq \mathcal{O}_F$ and for all $k \geq 1$ if $\mathfrak{f} = \mathcal{O}_F$.

Proof Recall from Theorem 4.7 that the class from Definition 3.41

$$\mathrm{Eis}(h) = \mathrm{contr}(h^* \mathrm{pol}) \in H^{n-1}(\Gamma, R_{\mathbb{Q}} \otimes \lambda)$$

is represented by $\mathrm{contr}(h^*(\varpi\mathscr{G} + \eta) =) = \mathrm{contr}(\varpi(h^*\mathscr{G}))$. By definition of mult in (3.21) we have $\varpi(h^*\mathscr{G}) = \mathrm{mult}(h^*\mathscr{G})$, so that $\mathrm{Eis}(h)$ is represented by $h^*\mathscr{G}$ (recall that $\mathrm{contr} \circ \mathrm{mult} = \mathrm{id}$). It follows from Theorem 5.6 and the above principle that

$$\exp_k^*(\mathrm{ev}(\mathrm{Eis}(h))) = (-1)^{n-1}\zeta(h, L, -k)z_1^{[k]} \cdots z_n^{[k]}$$

has coefficients in $\mathbb{Q}$. If $\mathfrak{f} \neq \mathcal{O}_F$ then 1 is a non-zero $\mathfrak{f}$-torsion point and the result follows immediately from (5.10). If $\mathfrak{f} = \mathcal{O}_F$ choose any prime ideal $\mathfrak{f}' = \mathfrak{p}$ and then the result follows from (5.11). □

With our methods we immediately obtain the following integrality result.

Corollary 5.8 (Deligne–Ribet, Cassou-Nougès [5,6]) *Let $\mathfrak{c}$ be an integral ideal coprime to $\mathfrak{f}\mathfrak{b}^{-1}$. Then for $k \geq 0$ one has*

$$\mathrm{N}\,\mathfrak{c}\zeta(1, \mathfrak{f}\mathfrak{b}^{-1}, -k) - \zeta(1, \mathfrak{c}^{-1}\mathfrak{f}^{-1}\mathfrak{b}, -k) \in \mathbb{Z}\left[\frac{1}{\mathrm{N}\,\mathfrak{c}}\right].$$

In particular, for $\mathfrak{f} \neq \mathcal{O}_F$ and $k \geq 0$ one has

$$(\mathrm{N}\,\mathfrak{c})^{1+k}\zeta(\mathfrak{b}, \mathfrak{f}, -k) - \zeta(\mathfrak{b}\mathfrak{c}, \mathfrak{f}, -k) \in \mathrm{N}\,\mathfrak{b}^k\mathbb{Z}\left[\frac{1}{\mathrm{N}\,\mathfrak{c}}\right].$$

If $\mathfrak{f} = \mathcal{O}_F$ the same result holds for $k \geq 1$.

Proof Let $L := \mathfrak{f}\mathfrak{b}^{-1}$ and choose $f \in \mathfrak{f}$ totally positive coprime to $\mathfrak{c}\mathfrak{b}$. Then

$$\zeta\left(\frac{1}{f}, f^{-1}L, s\right) = Nf^s\zeta(1, L, s)$$

which implies $Nf^k\zeta(\frac{1}{f}, f^{-1}L, -k) = \zeta(1, L, -k)$. Write $L' := f^{-1}L$ and consider the isogeny $[\mathfrak{c}]\colon L' \to L'\mathfrak{c}^{-1}$ of degree $\mathrm{N}\,\mathfrak{c}$. Let $A := \mathbb{Z}[\frac{1}{\mathrm{N}\,\mathfrak{c}}]$, so that $\ker[\mathfrak{c}] \subset T^{(A)}$. Then the function $\alpha_{[\mathfrak{c}]} \in A[\ker[\mathfrak{c}]]^0$ defined in (3.20) gives an element

$${}_f\mathrm{Eis}_{\alpha_{[\mathfrak{c}]}}\left(\frac{1}{f}\right) \in H^{n-1}\left(\Gamma, R_{\mathbb{Z}\left[\frac{1}{\mathrm{N}\,\mathfrak{c}}\right]} \otimes \lambda\right).$$

From Corollary 3.44 and from Remark 3.36 we get that

$$Nf^k\mathrm{Eis}^k_{\alpha_{[\mathfrak{c}]}}\left(\frac{1}{f}\right) = Nf^k\left((\mathrm{N}\,\mathfrak{c})\mathrm{Eis}^k\left(\frac{1}{f}\right) - \mathrm{Eis}'^k\left([\mathfrak{c}]\left(\frac{1}{f}\right)\right)\right)$$

has coefficients in $\mathbb{Z}[\frac{1}{\mathrm{N}\,\mathfrak{c}}]$. From the proof of Corollary 5.7 we deduce that

$$\begin{aligned}&\mathrm{ev}\left(Nf^k\mathrm{Eis}^k_{\alpha_{[\mathfrak{c}]}}(1)\right)\\ &= (-1)^{n-1}Nf^k\left((\mathrm{N}\,\mathfrak{c})\zeta\left(\frac{1}{f}, L', -k\right) - \zeta\left(\frac{1}{f}, L'\mathfrak{c}^{-1}, -k\right)\right)z_1^{[k]}\cdots z_n^{[k]}\end{aligned}$$

so that

$$\mathrm{N}\,\mathfrak{c}\zeta(1, \mathfrak{f}\mathfrak{b}^{-1}, -k) - \zeta(1, \mathfrak{c}^{-1}\mathfrak{f}^{-1}\mathfrak{b}, -k) \in \mathbb{Z}\left[\frac{1}{\mathrm{N}\,\mathfrak{c}}\right].$$

Multiplying by the integer $\mathrm{N}(\mathfrak{bc})^k$ gives the result as stated for $\mathfrak{f} \neq \mathcal{O}_F$. The case $\mathfrak{f} = \mathcal{O}_F$ follows again from (5.11). □

Finally, we deduce the p-adic interpolation of the zeta values. This result is due to Deligne–Ribet [6] and, with different methods, to Barsky [1], Cassou-Noguès [5].

Fix a prime number p, an integral ideal $\mathfrak{c}$ prime to p and let $A = \mathbb{Z}_p$. Recall from Proposition 3.12 that $R_{\mathbb{Z}_p}$ is isomorphic to the Iwasawa algebra $\mathrm{Meas}(L_{\mathbb{Z}_p}, \mathbb{Z}_p)$. Consider the polynomial function $w^k\colon L_{\mathbb{Z}_p} \to \mathbb{Z}_p$ defined by the element $w^k \in \mathrm{Sym}^k L^*_{\mathbb{Z}_p}$, which maps $a_1\ell_1 + \cdots + a_n\ell_n$ to $(a_1 \cdots a_n)^k$. Let $\Gamma \subset \mathcal{O}_F^\times$ be a subgroup of finite index. Then the moment map

$$\mathrm{mom}^k\colon (R_{\mathbb{Z}_p})_\Gamma \to (\mathrm{TSym}^k L_{\mathbb{Z}_p})_\Gamma \cong \mathbb{Z}_p z_1^{[k]} \cdots z_n^{[k]} \tag{5.12}$$

maps $\mu \mapsto \mu(w^k) z_1^{[k]} \cdots z_n^{[k]}$. One has a commutative diagram

$$\begin{array}{ccc} (R_{\mathbb{Z}[1/c]})_\Gamma & \xrightarrow{\mathrm{mom}^k} & (\mathrm{TSym}^k L_{\mathbb{Z}[1/c]})_\Gamma \\ \downarrow & & \downarrow \\ (R_{\mathbb{Z}_p})_\Gamma & \xrightarrow{\mathrm{mom}^k} & (\mathrm{TSym}^k L_{\mathbb{Z}_p})_\Gamma \end{array}$$

and the image of $\mathrm{Eis}_{\alpha_{[\mathfrak{c}]}}$ under the upper horizontal map were computed in Corollary 5.8. We keep the lattice $L = \mathfrak{f}\mathfrak{b}^{-1}$ and consider the function $\alpha_{[\mathfrak{c}]}$ for the isogeny $[\mathfrak{c}]\colon L \to L\mathfrak{c}^{-1}$ as defined in Definition 3.42.

Corollary 5.9 (p-adic interpolation, Deligne-Ribet, Cassou-Noguès, Barsky [1,5,6]) *Let p be a prime number and $\mathfrak{f}$ be divisible by all primes above p and $\Gamma := \mathcal{O}_1^{+,\times}$ be the stabilizer of* 1 *in the totally positive units. Then the element*

$$\mathrm{ev}(\mathrm{Eis}_{\alpha_{[\mathfrak{c}]}}(1)) \in (R_{\mathbb{Z}_p})_\Gamma$$

is a measure whose value on w^k is $(-1)^{n-1}((\mathrm{N}\,\mathfrak{c})\zeta(1, \mathfrak{f}\mathfrak{b}^{-1}, -k) - \zeta(1, \mathfrak{f}\mathfrak{b}^{-1}\mathfrak{c}^{-1}, -k))$.

Proof This is immediate from the above remarks and Corollary 5.8. □

5.4 Eisenstein distributions and measures

In this section we let $A = \mathbb{Z}_p$ so that we can identify $R = \mathbb{Z}_p[[L_{\mathbb{Z}_p}]]$ (see Proposition 3.12). Denote by $T^{(p)} := T^{(\mathbb{Z}_p)}$ the subgroup of T^{tors} of elements of order prime to p. For any $\mathbb{Z}_p$-module M we consider the $\mathbb{Z}_p$-module

$$M[[\Gamma\backslash L_{\mathbb{Z}_p}]] := \mathrm{Meas}(\Gamma\backslash L_{\mathbb{Z}_p}, M) := \varprojlim_r M[\Gamma\backslash(L/p^r L)]$$

of M-valued distributions on $\Gamma\backslash L_{\mathbb{Z}_p}$. Here we have set $M[\Gamma\backslash(L/p^r L)] := M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma\backslash(L/p^r L)]$. Note that these are measures in the ordinary sense, if M is a finitely generated $\mathbb{Z}_p$-module, otherwise these are just distributions.

Proposition 5.10 *For every function $g \in (\mathbb{Z}_p[T^{(p)} \setminus \{0\}]^0)^\Gamma$ there is an $H^{n-1}(\Gamma, R \otimes \lambda)$-valued distribution*

$$\mu^g_{L,\mathrm{Eis}} \in \mathrm{Meas}(\Gamma \backslash L_{\mathbb{Z}_p}, H^{n-1}(\Gamma, R \otimes \lambda))$$

on $\Gamma \backslash L_{\mathbb{Z}_p}$.

Proof We are going to construct elements $\mu^g_{r,L,\mathrm{Eis}} \in M[\Gamma \backslash (L/p^r L)]$ in a compatible way. The distribution $\mu^g_{r,L,\mathrm{Eis}}$ assigns to a Γ-invariant function f on $L/p^r L$ an element in $H^{n-1}(\Gamma, R \otimes \lambda)$. This we define as follows. The isogeny $[p^r]\colon T_{p^r L} \to T_L$ associated to the inclusion $p^r L \subset L$ yields an isomorphism $[p^r]\colon T^{(p)}_{p^r L} \cong T^{(p)}_L$, which allows to consider the function $g \in (\mathbb{Z}_p[T^{(p)}_L \setminus \{0\}]^0)^\Gamma$ as an element $g_r \in (\mathbb{Z}_p[T^{(p)}_{p^r L} \setminus \{0\}]^0)^\Gamma$. Then Definition 3.38 gives an element

$$\mathrm{Eis}(f \otimes g_r) \in H^{n-1}(\Gamma, \mathscr{L}og_{p^r L} |_{L/p^r L} \otimes \lambda_{p^r L}),$$

where we view $L/p^r L \subset T_{p^r L}$ as the kernel of the isogeny $[p^r]$. The trace map associated to $[p^r]$ induces

$$\begin{aligned}\mathrm{Tr}_{[p^r]}\colon H^{n-1}(\Gamma, \mathscr{L}og_{p^r L} |_{L/p^r L} \otimes \lambda_{p^r L}) \to H^{n-1}&(\Gamma, \mathscr{L}og_L |_{\{0\}} \otimes \lambda_L)\\ &= H^{n-1}(\Gamma, R \otimes \lambda)\end{aligned}$$

and we define

$$\mu^g_{r,L,\mathrm{Eis}}(f) := \mathrm{Tr}_{[p^r]}(\mathrm{Eis}(f \otimes g_r)).$$

As $g_r = [p^r]^*(g)$ it follows from Corollary 3.48 that the $\mu^g_{r,L,\mathrm{Eis}}(f)$ indeed define a distribution on $\Gamma \backslash L_{\mathbb{Z}_p}$. □

Proposition 5.11 *Recall that $R = \mathbb{Z}_p[[L_{\mathbb{Z}_p}]]$. There is a canonical homomorphism*

$$H^{n-1}(\Gamma, \mathbb{Z}_p[[L_{\mathbb{Z}_p}]] \otimes \lambda) \to \mathrm{Meas}(\Gamma \backslash L_{\mathbb{Z}_p}, H^{n-1}(\Gamma, \mathbb{Z}_p \otimes \lambda)).$$

Proof The pairing between distributions and functions on $L/p^r L$ is a map

$$\mathbb{Z}_p[L/p^r L] \times \mathbb{Z}_p[L/p^r L] \to \mathbb{Z}_p$$

so that the cup-product defines a pairing

$$H^0(\Gamma, \mathbb{Z}_p[L/p^r L]) \otimes_{\mathbb{Z}_p} H^{n-1}(\Gamma, \mathbb{Z}_p[L/p^r L] \otimes \lambda) \to H^{n-1}(\Gamma, \mathbb{Z}_p \otimes \lambda)$$

and hence a homomorphism

$$H^{n-1}(\Gamma, \mathbb{Z}_p[L/p^r L] \otimes \lambda) \to H^{n-1}(\Gamma, \mathbb{Z}_p \otimes \lambda) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma \backslash (L/p^r L)].$$

Composing this homomorphism with the projection

$$H^{n-1}(\Gamma, \mathbb{Z}_p[[L_{\mathbb{Z}_p}]] \otimes \lambda) \to H^{n-1}(\Gamma, \mathbb{Z}_p[L/p^r L] \otimes \lambda)$$

and passing to the projective limit give the desired homomorphism. □

Using this proposition the Eisenstein distribution gives rise to an element in

$$\begin{aligned} &\mathrm{Meas}(L_{\mathbb{Z}_p}/\Gamma, \mathrm{Meas}(\Gamma\backslash L_{\mathbb{Z}_p}, H^{n-1}(\Gamma, \mathbb{Z}_p \otimes \lambda))) \\ &\cong \mathrm{Meas}(\Gamma\backslash L_{\mathbb{Z}_p} \times \Gamma\backslash L_{\mathbb{Z}_p}, H^{n-1}(\Gamma, \mathbb{Z}_p \otimes \lambda)) \end{aligned} \tag{5.13}$$

where the isomorphism comes from the Fubini theorem about integration on a product space. Note that in the case where $\Gamma \subset \mathrm{Aut}(L)$ is an arithmetic subgroup, the $\mathbb{Z}_p$-module $H^{n-1}(\Gamma, \mathbb{Z}_p \otimes \lambda)$ is finitely generated, so that the Eisenstein distribution becomes a measure on $\Gamma\backslash L_{\mathbb{Z}_p} \times \Gamma\backslash L_{\mathbb{Z}_p}$.

The next theorem shows that the Eisenstein distribution does not give anything essentially new.

Theorem 5.12 *For any $g \in (\mathbb{Z}_p[T^{(p)} \setminus \{0\}]^0)^\Gamma$ the Eisenstein measure*

$$\mu^g_{L,\mathrm{Eis}} \in \mathrm{Meas}(\Gamma\backslash L_{\mathbb{Z}_p} \times \Gamma\backslash L_{\mathbb{Z}_p}, H^{n-1}(\Gamma, \mathbb{Z}_p \otimes \lambda))$$

is supported on the diagonal $\Gamma\backslash L_{\mathbb{Z}_p} \xrightarrow{\Delta} \Gamma\backslash L_{\mathbb{Z}_p} \times \Gamma\backslash L_{\mathbb{Z}_p}$.

Proof The proof is formal and one has just to unravel the definition of $\mu^g_{L,\mathrm{Eis}}$. It certainly suffices to show this for $\mu^g_{r,L,\mathrm{Eis}}$, i.e., to work with $L_r := L/p^r L$ and $A_r := \mathbb{Z}/p^r\mathbb{Z}$. Let $L' := p^r L$ and $t \in L$. Then t defines a p^r-torsion point on T' because this group is just L_r. We assume in this proof that Γ acts trivially on L_r otherwise one has to replace t by some linear combination of p^r-torsion sections. We consider the Eisenstein measure

$$\mu^g_{r,L,\mathrm{Eis}} \colon A_r[L_r]^\Gamma \to H^{n-1}(\Gamma, A_r[L_r] \otimes \lambda)$$

as a map from the Γ-invariant functions on L_r to $H^{n-1}(\Gamma, A_r[L_r] \otimes \lambda)$. By its construction $\mu^g_{r,L,\mathrm{Eis}}$ evaluated at $\delta_t \in A_r[L_r]$ is given by $\mathrm{Tr}_{[p^r]}(\mathrm{Eis}(\delta_t \otimes [p^r]^*(g)))$, where $[p^r]$ is the isogeny $[p^r] \colon L' \to L$. We have

$$\mathrm{Eis}(\delta_t \otimes [p^r]^*(g)) \in H^{n-1}(\Gamma, t^*\mathscr{L}og' \otimes \lambda').$$

We claim that $t^*\mathscr{L}og' \cong \mathrm{Meas}(t + L'_{\mathbb{Z}_p}, A_r)$. This follows because $\mathscr{L}og' \cong \pi_! A_r[L'] \otimes_{A_r} R'$ by the construction of $\mathscr{L}og'$ and because $t^*\pi_! A_r[L'] = A_r[t + L']$, where we denote by $A_r[t+L']$ the free A_r-module on $t+L'$. Taking the tensor product with R completes this $\mathbb{Z}_p[L']$-module and gives the desired isomorphism.

The isogeny $[p^r]$ induces a map $[p^r]_{\mathscr{L}og}\colon t^*\mathscr{L}og' \to 0^*\mathscr{L}og$ which identifies with

$$[p^r]_*\colon \operatorname{Meas}(t+L'_{\mathbb{Z}_p}, A_r) \to \operatorname{Meas}(L_{\mathbb{Z}_p}, A_r).$$

If we compose this with $\operatorname{Meas}(L_{\mathbb{Z}_p}, A_r) \to \operatorname{Meas}(L_r, A_r) = A_r[L_r]$ we see that the image of $\operatorname{Meas}(t+L'_{\mathbb{Z}_p}, A_r)$ in $A_r[L_r]$ is given by $A_r\delta_t$. In particular, if we consider the pairing

$$A_r[L_r]^\Gamma \otimes H^{n-1}(\Gamma, A_r[L_r]\otimes\lambda) \to H^{n-1}(\Gamma, A_r\otimes\lambda)$$

we see that $\delta_s \otimes \mu^g_{r,L,\mathrm{Eis}}(\delta_t) \mapsto 0$ for $s \neq t$. If we rewrite the Eisenstein measure as

$$\mu^g_{r,L,\mathrm{Eis}}\colon A_r[L_r]^\Gamma \times A_r[L_r]^\Gamma \to H^{n-1}(\Gamma, A_r\otimes\lambda)$$

this just means that $\mu^g_{r,L,\mathrm{Eis}}(\delta_t\otimes\delta_s) = 0$ for $t \neq s$, i.e., that $\mu^g_{r,L,\mathrm{Eis}}$ is supported on the diagonal. □

Corollary 5.13 *The measure* $\mu^g_{L,\mathrm{Eis}} \in \operatorname{Meas}(\Gamma\backslash L_{\mathbb{Z}_p}, H^{n-1}(\Gamma, A[[L_{\mathbb{Z}_p}]]\otimes\lambda))$ *is completely determined by its image*

$$\overline{\mu}^g_{L,\mathrm{Eis}} \in \operatorname{Meas}(\Gamma\backslash L_{\mathbb{Z}_p}, H^{n-1}(\Gamma, A\otimes\lambda))$$

under the augmentation map $A[[L_{\mathbb{Z}_p}]] \to A$. *It is also completely determined by its value on the constant function* 1 *on* $L_{\mathbb{Z}_p}/\Gamma$:

$$\mu^g_{L,\mathrm{Eis}}(1) \in H^{n-1}(\Gamma, A[[L_{\mathbb{Z}_p}]]\otimes\lambda).$$

Proof This is just a reformulation of the theorem. □

In the case of totally real fields, this has the following consequence.

Corollary 5.14 *Let* F *be a totally real field with ring of integers* $\mathcal{O}_F$ *and* $\Gamma := \mathcal{O}_1^{+,\times}$ *be the stabilizer of* 1 *in the totally positive units. Let* $\mathfrak{f}, \mathfrak{b}$ *be integral ideals with* $\mathfrak{f}$ *divisible by all primes above* p *and* $L = \mathfrak{f}\mathfrak{b}^{-1}$. *Then the measure*

$$\operatorname{ev}(\mathrm{Eis}_{\alpha[\mathfrak{c}]}(1)) \in (R_{\mathbb{Z}_p})_\Gamma$$

from Corollary 5.9 coincides with the measure $\overline{\mu}^{\alpha[\mathfrak{c}]}_{L,\mathrm{Eis}}$.

5.5 Relation with Eisenstein cohomology on Hilbert modular varieties

The Eisenstein cohomology classes constructed in this paper are very special. Even for GL_2 it is not clear how to get all Eisenstein cohomology classes starting from the polylogarithm. We point out that this amounts to a purely topological construction of the Eisenstein cohomology considered by Harder, which is transcendental. It is therefore surprising that in the case of Hilbert modular varieties Graf [7] succeeded to get all of Harder's Eisenstein cohomology classes starting from the topological polylogarithm.

In this section we explain (a slight variant of) the construction of Graf with the purpose of proving some integrality and p-adic interpolation properties of the Eisenstein cohomology classes. For more details and the actual comparison with Harder's Eisenstein cohomology we refer to the thesis of Graf [7]. We remark that Harder's applications to special values of L-functions are in the case of GL_2 over fields which are not totally real. This means that applications to special values of L-functions in the case treated here, if any, still have to be found.

We mention that meanwhile the construction of Graf has been generalized to the motivic setting by the second author of this article (forthcoming).

Let again $F/\mathbb{Q}$ be a totally real field of degree n with ring of integers $\mathcal{O}_F$. We define for any fractional ideal $\mathfrak{a}$ of F the group

$$\mathrm{GL}_2^+(\mathcal{O}_F,\mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F) \mid a, d \in \mathcal{O}_F, b \in \mathfrak{a}, c \in \mathfrak{a}^{-1}, ad - bc \in \mathcal{O}_F^{+,\times} \right\}.$$

We identify the centre of $\mathrm{GL}_2^+(\mathcal{O}_F,\mathfrak{a})$ with $\mathcal{O}_F^\times$ and we let $\mathrm{PGL}_2(\mathcal{O}_F,\mathfrak{a})$ be the quotient. The group $\mathrm{GL}_2^+(\mathcal{O}_F,\mathfrak{a})$ acts on $(F\otimes\mathbb{R})^2 \cong F\otimes\mathbb{C}$ from the right and stabilizes the lattice

$$L := \mathcal{O}_F \cdot 1 + \mathfrak{a}\cdot\sqrt{-1} \subset F\otimes\mathbb{C}.$$

We consider the torus $T := F\otimes\mathbb{C}/L$ of real dimension $2n$ and an integer $N > 1$ which is invertible in A. We let

$$D := T[N]\backslash\{0\} \subset T \tag{5.14}$$

be the N-torsion subgroup without the 0-section and denote by $\Gamma \subset \mathrm{GL}_2^+(\mathcal{O}_F,\mathfrak{a})$ the stabilizer of D. Let $\Delta := \Gamma \cap \mathcal{O}_F^\times$ be the intersection of Γ with the centre. Then $\Gamma \subset \mathrm{GL}_2^+(\mathcal{O}_F,\mathfrak{a})$ and $\Delta \subset \mathcal{O}_F^\times$ are subgroups of finite index and we define $\Gamma' := \Gamma/\Delta \subset \mathrm{PGL}_2(\mathcal{O}_F,\mathfrak{a})$, so that we have an exact sequence

$$0 \to \Delta \to \Gamma \to \Gamma' \to 0.$$

Remark 5.15 To have a geometric perspective on this, we define $\mathrm{GL}_2^+(F\otimes\mathbb{R}) := \{(\omega_1,\omega_2) \in (F\otimes\mathbb{C})^2 \mid \Im(\frac{\omega_2}{\omega_1}) > 0\}$. Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ acts on $(\omega_1,\omega_2) \in \mathrm{GL}_2^+(F\otimes\mathbb{R})$ by right multiplication $(\omega_1,\omega_2)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\lambda \in (F\otimes\mathbb{C})^\times$ acts by left multiplication $\lambda(\omega_1,\omega_2) = (\lambda\omega_1,\lambda\omega_2)$. The map $(\omega_1,\omega_2) \mapsto \tau := \frac{\omega_2}{\omega_1}$ identifies the quotient $(F\otimes\mathbb{C})^\times\backslash\mathrm{GL}_2^+(F\otimes\mathbb{R})$ with the upper half plane $F\otimes\mathbb{H} := \{\tau \in F\otimes\mathbb{C} \mid \Im\tau \text{ totally positive}\}$. Note that the map $\mathrm{GL}_2^+(F\otimes\mathbb{R}) \to F\otimes\mathbb{H}$ is compatible with the homomorphism $\Gamma \to \Gamma'$.

The Eisenstein class of Definition 3.33 provides us for any ring A in which N is invertible with a map

$$\mathrm{Eis}\colon (A[D]^0)^\Gamma \to H^{2n-1}(\Gamma, R \otimes \lambda). \tag{5.15}$$

We explain how to get cohomology classes in other degrees starting from this class. Consider the Hochschild–Serre spectral sequence

$$H^{2n-1-p}(\Gamma', H^p(\Delta, R \otimes \lambda)) \Rightarrow H^{2n-1}(\Gamma, R \otimes \lambda).$$

As Δ has cohomological dimension $n-1$ we have an edge morphism

$$H^{2n-1}(\Gamma, R \otimes \lambda) \to H^n(\Gamma', H^{n-1}(\Delta, R \otimes \lambda)). \tag{5.16}$$

If we compose this with the cap-product with $H_{n-1}(\Delta, \mathbb{Z})$ we get a map

$$\mathrm{Eis}\colon (A[D]^0)^\Gamma \otimes H_{n-1}(\Delta, \mathbb{Z}) \to H^n(\Gamma', (R \otimes \lambda)_\Delta).$$

In order to get also the Eisenstein classes in other cohomological degrees consider $\mathcal{O}_N^{+,\times} := \{u \in \mathcal{O}_F^{+,\times} \mid u \equiv 1 \bmod N\}$ and the determinant map $\Gamma \xrightarrow{\det} \mathcal{O}_N^{+,\times}$. For the rest of the section we use the following notation:

$$\Lambda^\cdot := \Lambda^\cdot \operatorname{Hom}(\mathcal{O}_N^{+,\times}, \mathbb{Z}) = H^\cdot(\mathcal{O}_N^{+,\times}, \mathbb{Z}). \tag{5.17}$$

Then the map det gives rise to a ring homomorphism

$$\det{}^*\colon \Lambda^\cdot \to H^\cdot(\Gamma, \mathbb{Z})$$

so that $H^\cdot(\Gamma, \mathbb{Z})$ becomes a $\Lambda^\cdot$-module. Therefore (5.15) yields the map

$$\mathrm{Eis}\colon (A[D]^0)^\Gamma \otimes \Lambda^p \to H^{2n-1+p}(\Gamma, R \otimes \lambda).$$

A further composition with the edge morphism and the cap-product with $H_{n-1}(\Delta, \mathbb{Z})$ gives:

Definition 5.16 For each $0 \le p \le n-1$ we define the Eisenstein cohomology operator in degree $n+p$ to be the map

$$\mathrm{Eis}\colon A[D]^0 \otimes \Lambda^p \otimes H_{n-1}(\Delta, \mathbb{Z}) \to H^{n+p}(\Gamma', (R \otimes \lambda)_\Delta).$$

Composing with $\exp_k^*$ gives

$$\mathrm{Eis}^k\colon A[D]^0 \otimes \Lambda^p \otimes H_{n-1}(\Delta, \mathbb{Z}) \to H^{n+p}(\Gamma', (\mathrm{TSym}^k L_A)_\Delta \otimes \lambda).$$

Remark 5.17 For A a $\mathbb{Q}$-algebra, one can show that $(\mathrm{TSym}^k L_A)_\Delta = 0$ if k is not a multiple of n and non-trivial otherwise.

Choose generators for Λ^p and $H_{n-1}(\Delta,\mathbb{Z}) \cong \Lambda^{n-1}\Delta$, then we get directly from the construction the following integrality result for the Eisenstein cohomology:

Proposition 5.18 *Let $\alpha \in \mathbb{Z}[\frac{1}{N}][D]^0$ then with the above generators*

$$\mathrm{Eis}^k_\alpha \in H^{n+p}\left(\Gamma', \left(\mathrm{TSym}^k L_{\mathbb{Z}\left[\frac{1}{N}\right]}\right)_\Delta \otimes \lambda\right).$$

Keeping the generators and putting $A = \mathbb{Z}_p$ we get also a p-adic interpolation result. Recall Proposition 3.12 that for $A = \mathbb{Z}_p$ one has an isomorphism $A[[L_{\mathbb{Z}_p}]] \cong R$.

Proposition 5.19 *With the above notations, for each $\alpha \in \mathbb{Z}_p[D]^0$ the class*

$$\mathrm{Eis}_\alpha \in H^{n+p}(\Gamma', (A[[L_{\mathbb{Z}_p}]])_\Delta \otimes \lambda)$$

has the interpolation property that

$$\mathrm{mom}^k(\mathrm{Eis}_\alpha) = \mathrm{Eis}^k_\alpha \in H^{n+p}(\Gamma', (\mathrm{TSym}^k L_{\mathbb{Z}_p})_\Delta \otimes \lambda). \tag{5.18}$$

Proof This is clear from the construction. □

Remark 5.20 The consequences for special values of L-functions and p-adic L-functions still have to be explored. We remark that in [9] the constant terms of the Eisenstein classes of the polylogarithm was computed in terms of L-functions for totally real fields. This should generalize to the Eisenstein classes constructed by Graf.

Proposition 5.21 (Graf [7]) *If A is a $\mathbb{Q}$-algebra then the product map*

$$\bigoplus_p H^{2n-1-p}(\Gamma', R^\Delta \otimes \lambda) \otimes \Lambda^p \to H^{2n-1}(\Gamma, R \otimes \lambda)$$

is an isomorphism.

Proof One has $\Lambda^{\cdot} \otimes \mathbb{Q} \cong H^{\cdot}(\Delta, \mathbb{Q})$. As in Remark 5.5 the projection $p_\Delta : R \to R_\Delta$ then gives rise to isomorphisms

$$H^p(\Delta, R \otimes \lambda) \cong H^p(\Delta, R_\Delta \otimes \lambda) \cong R_\Delta \otimes \lambda \otimes \Lambda^p$$

and the result follows from the Hochschild–Serre spectral sequence.

Remark 5.22 In his thesis Graf decomposes the topological polylogarithm according to the isomorphism in the above proposition and shows that the resulting cohomology classes give all the Eisenstein cohomology constructed by Harder. For this he explicitly computes these Eisenstein classes.

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