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Nikita Malykhin, Philip Ushchev

Clustering or co-agglomeration? A love-for-variety approach

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Nikita Malykhin[1] Philip Ushchev[2]

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Abstract

We develop a simple partial-equilibrium model of endogenous city structure formation. No production externalities are at work, the only two forces shaping the spatial configurations of the city being love for variety (on the consumer side) and seeking for a better access to the market (on the firm side). We show that, unlike in existing models of a similar nature, our model generates clustering rather than co-agglomeration. Namely, if there are few firms relative to the urban population size, then firms tend to cluster at the city center, while consumers choose to reside on the outskirts. Otherwise, the opposite holds. Although a continuum of equilibrium city structures may emerge, we show that all spatial equilibria are segregated. In addition, the market outcome features spatial price dispersion, even though our framework does not involve imperfect information and search costs on the consumer side.

Keywords: urban structure, monopolistic competition, agglomeration, clustering, quadratic preferences, segregated spatial equilibrium, price dispersion.

JEL classification: R12, R14, D43, L13

[1]National Research University Higher School of Economics (Moscow, Russia). Center for Market Studies and Spatial Economics. Research fellow. Email: nikmalykhin@gmail.com

[2]National Research University Higher School of Economics (Moscow, Russia). Center for Market Studies and Spatial Economics. Senior research fellow. E-mail: fuschev@hse.ru

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Introduction

The broad diversity of urban land-use patterns is well documented and represents a salient feature of modern cities. Uncovering the nature of dominant forces molding the spatial structures of cities is therefore an important issue in urban economics. Fujita and Thisse (2013) singled out two dominant approaches to endogenizing urban structure. In the first approach, developed by Fujita and Ogawa (1980), Fujita and Ogawa (1982), and more recently in Lucas and Rossi-Hansberg (2002), firms are involved in perfect competition and enjoy spatial production externalities. These externalities form an agglomeration force which drives firms closer to the city center. The second approach, used by De Palma et al. (1985) and Fujita (1988), rests on assuming imperfect competition, and stresses the role of demand side, product differentiation, and shopping costs as the key factors shaping the urban landscape. This line of inquiry, however, has not been pursued much since. One reason for that perhaps lies in the technical complexities of the approach developed in Fujita (1988), which become insurmountable, unless one assumes a very specific utility function.

Meanwhile, as stressed by Schiff (2015), "easy access to an impressive variety of goods may be one of the most attractive features of urban living". That being said, variety-seeking consumer behavior is among the key forces driving city structure formation. Moreover, recent theoretical work on monopolistic competition (Behrens and Murata, 2007; Zhelobodko et al., 2012; Bertoletti and Etro, 2016) and its applications to economic geography and international trade (Ottaviano et al., 2002; Melitz and Ottaviano, 2008; Mrazova and Neary, 2013; Bertoletti and Epifani, 2014), based on assuming variable elasticity of substitution on the consumer side, has spurred interest in studying the impact of versatile patterns of variety-loving consumer behavior on market competition and spatial distribution of economic activities. We therefore find it timely to revisit the question of *how the demand-side properties of urban economies affect the city structure under monopolistic competition*.

To achieve this, we consider a simple partial-equilibrium model of endogenous city structure formation. Our model is similar to that developed by Fujita (1988), but differs in two respects: we work with linear-quadratic preferences a lá Ottaviano et al. (2002) instead of using the entropy-type utility, while transportation costs are assumed to be quadratic rather than linear. We prefer to work with this type of setting because it yields an analytically tractable model of spatial monopolistic competition with variable markups.

Our main findings may be summarized as follows. First, we show that *only segregated land-use equilibria exist*. This differs from Picard and Tabuchi (2013), who find that firms and consumers tend to co-agglomerate, meaning that the most appealing urban locations are neither pure residential areas, nor pure business areas, but accommodate both. In other words, we find that, unlike in Fujita (1988), consumers and firms never share land. This may be viewed as a form of *clustering*, a type of spatial economic behavior for which Kerr and Kominers (2015) provide strong evidence in a different context. Moreover, we argue that the non-existence of mixed equilibria is not a vulnerable result stemming from restrictive properties of the quadratic preference specification. As discussed at the end of Section 2, the same holds at least as long as both preferences and transportation costs are described by analytic functions, which is quite a broad family.

Second, we derive conditions which lead to an equilibrium spatial configuration where firms cluster at the city centre, while consumers reside at the city outskirts. The key factor of such city structures is the interaction between two forces: (i) the desire of firms to locate as close as possible to the mass of consumers, i.e. to cover the largest possible market area; (ii) variety-loving consumer behavior: individuals choose to locate closer to retailing districts in order to reduce shopping costs. We also show that multiple spatial equilibria involving more complex city structures may emerge.

Third, we show that, unlike in De Palma et al. (1985) and Fujita (1988), the market outcome features spatial price dispersion. This may seem surprising at first glance, for unlike Wolinsky (1983) and Schulz and Stahl (1996), where incomplete information on the consumer side works as an agglomeration force, our framework involves no search costs. The only source of price dispersion in our model is the locational advantage of more centrally located firms over the others: since more accessible outlets have more market power, they charge higher prices.

The paper is structured as follows. Section 2 sets the baseline model and shows that only segregated equilibria exist. Section 3 provides a characterization of spatial patterns in the baseline model. Section 4 provides extensions of the baseline model for cases of linear (rather than quadratic) transportation costs, and a non-zero substitution term in the utility function. Section 5 concludes.

The model

We consider a one-dimensional city X made up of N consumers, M firms, and absentee landlords. For conciseness, we will call a consumer located at $x \in X$ an *x-consumer*, and a firm located at $y \in X$ a *y-firm*.

Transportation costs, which we assume to be a quadratic function of distance[4], are fully borne by consumers. For this reason, in what follows we will also call these costs *shopping costs*.

The spatial distributions of firms and consumers across the city are described, respectively, by the densities $m(y), y \in X$, and $n(x), x \in X$, which are non-negative and satisfy the following balance conditions:

$$\int_X m(y)\mathrm{d}y = M, \qquad \int_X n(x)\mathrm{d}x = N. \tag{1}$$

These densities are endogenously determined by market interactions between consumers and firms, which eventually shape the land use pattern in the city (see Section 3 for more details).

Following most literature in urban economics, we also assume that both $n(x)$ and $m(y)$ are *symmetric* with respect to the origin:

$$n(x) = n(-x) \text{ for all } x \in X, \qquad m(y) = m(-y) \text{ for all } y \in X. \tag{2}$$

[4]Ever since d'Aspremont et al. (1979), it has been well known that the functional form of transportation cost may dramatically alter the market outcome. In Section 4, we discuss how assuming linear shopping cost changes our main results.

Consumers

Consumers share identical linear-quadratic preferences. More precisely, the utility function of an x-consumer is given by

$$U(z; q(x,y), y \in X) = z + \alpha \int_X q(x,y)m(y)\mathrm{d}y - \frac{\beta}{2} \int_X [q(x,y)]^2 m(y)\mathrm{d}y \tag{3}$$

where z is the outside good consumption level and $q(x,y)$ is the x-consumer's consumption level of a variety produced by a y-firm.

The utility function (3) differs from the one used in the literature since Ottaviano et al. (2002) by the absence of a substitution term. We show in Section 4 that our main results remain valid if we allow for non-zero substitutability across varieties.

Each x-consumer seeks to maximize utility (3) subject to the budget constraint:

$$\int_X [p(y) + t(x-y)^2]\, q(x,y)m(y)dy + R(x) + z = Y. \tag{4}$$

Here Y is consumer's income, t is half the marginal transportation cost per unit of distance, while $p(y)$ is the price charged by a y-firm. Finally, $R(x)$ is the rental price of land at location x. We describe how $R(x)$ is set in Section 3.

Using the budget constraint, the utility function of an x-consumer may be represented as follows:[5]

$$\begin{aligned} V(x) = {} & \alpha \int_X q(x,y)m(y)\mathrm{d}y - \tfrac{\beta}{2} \int_X [q(x,y)]^2 m(y)\mathrm{d}y \\ & - \int_X [p(y) + t(x-y)^2] q(x,y)m(y)dy - R(x) + Y, \end{aligned} \tag{5}$$

The choke price α is assumed to be sufficiently high to rule out corner solutions of the consumer's program. To be precise, we show in Appendix 1 that the following condition renders this solution interior for each consumer:

$$\sqrt{\frac{\alpha - c}{2t}} > M + N. \tag{6}$$

The intuition behind (6) is easy to comprehend. There are two potential hindrances for consumers to purchase all the available varieties: either their willingness-to-pay α is too low, or high transportation costs restrain them from visiting remote shopping districts. Condition (6) rules out both these possibilities.

Note that condition (6) is a sufficient, but not necessary condition. We provide below weaker conditions of the same sort, which are *necessary and sufficient* for the consumer program to have an interior solution in the special but relevant case of *segregated spatial equilibria*. As stated in

[5] To be precise, such representation requires the standard assumption that each consumer's income Y is high enough to guarantee positive consumption level of the numeraire.

the Introduction, this is the case we focus on in this paper.

Given that (6) holds, the individual demands of an x-consumer are linear:

$$q(x,y) = \frac{\alpha - t(x-y)^2 - p(y)}{\beta}. \tag{7}$$

As implied by (7), an increase in the distance between an x-consumer and a y-firm triggers a parallel downward shift of the corresponding individual demand curve. In other words, a larger distance means a lower choke price, while the slope of the demand schedule is the same, both across consumers and varieties.

Firms

It follows from (7) that the aggregate demand $Q(y)$ faced by a y-firm is given by

$$Q(y) = \frac{N}{\beta}\left[\alpha - t\delta(y) - p(y)\right], \tag{8}$$

where $\delta(y)$ is the *mean-squared distance from a y-firm to the whole mass of consumers* across the city:

$$\delta(y) \equiv \frac{1}{N}\int_X (x-y)^2 n(x)dx. \tag{9}$$

The intuition behind $\delta(y)$ is easy to grasp: it is a reverse measure of a y-firm's market access. Indeed, a lower $\delta(y)$ means that location y is "closer" to the whole mass of consumers, i.e. that a y-firm enjoys better access to the market. To further clarify this idea, we use the symmetry condition (2) and restate (9) as follows:

$$\delta(y) = y^2 + \delta_0, \tag{10}$$

where δ_0 is the *population dispersion*, defined as the mean-squared distance of the whole population of consumers from the central location $y = 0$:

$$\delta_0 \equiv \frac{1}{N}\int_X x^2 n(x)\mathrm{d}x. \tag{11}$$

Observe that, as implied by (10), the further location y is from the city centre $x = 0$, the higher is $\delta(y)$. In other words, *more centrally located firms have better access to the market*.

Each firm's production technology exhibits a constant marginal cost c and a fixed cost f, both expressed in terms of the numéraire. In addition, a y-firm pays rent $R(y)$. Hence, the profit of a y-firm is given by

$$\pi(y) = \frac{N}{\beta}\left[\alpha - t\delta(y) - p(y)\right]\left[p(y) - c\right] - R(y) - f, \tag{12}$$

Maximizing (12) with respect to $p(y)$ yields the following expression for the profit-maximizing price of a y-firm:

$$p^*(y) = \frac{1}{2}\left[\alpha + c - t\delta(y)\right]. \tag{13}$$

Plugging (13) into (12), we obtain the profit earned by a y-firm:

$$\pi^*(y) = \frac{N}{4\beta}\left[\alpha - c - t\delta(y)\right]^2 - f - R(y). \tag{14}$$

Equations (13) and (14) show that a lower value of $\delta(y)$ implies a higher price and a higher profit for y-firms. It is well-known that the mean-squared error of a distribution with a finite variance reaches a minimum at the mean. Hence, $\delta(y)$ achieves a minimum when y equals the mean of the spatial distribution of consumers, i.e. at the central point $y = 0$ of the city. Furthermore, the difference in prices between firms located, respectively, in y_1 and $y_2 \in X$, $0 < y_1 < y_2$ equals

$$p^*(y_1) - p^*(y_2) = \frac{t}{2}\left[\delta(y_2) - \delta(y_1)\right]. \tag{15}$$

Using (13) and (15) yields the following result.

Proposition 1.

(i) *More centrally located firms charge higher prices and earn higher profits*.

(ii) *A reduction in transportation cost t leads to higher prices and higher profits for all firms, and reduces spatial price differentials.*

Proposition 1 shows that a firm's pricing strategy depends on the firm's location. This impact is fully captured by the product of transportation cost t and the mean-squared distance $\delta(y)$ of a y-firm from the whole urban population. Indeed, as shown by (15), a higher transportation cost denerates more spatial price dispersion. Whatever the equilibrium distribution of firms, the market outcome shows spatial price dispersion.[6] This feature distinguishes our results from those obtained in similar contexts (De Palma et al., 1985; Fujita, 1988) and echoes the industrial organization literature where imperfectly informed consumers incur search costs (Wolinsky, 1983; Schulz and Stahl, 1996). Claim (ii) of Proposition 1 also highlights that infrastructural improvements are beneficial for all firms.

An object dual to $\delta(y)$ is the mean-squared distance $\sigma(x)$ from an x-consumer to the whole mass of firms across the city,[7] defined by

$$\sigma(x) \equiv \frac{1}{M}\int_X (x-y)^2 m(y)\mathrm{d}y. \tag{16}$$

[6]Formally, this conclusion requires that the support of the distribution of firms contains at least three points. This requirement is always fulfilled in our model, where any distribution of firms is given by a density. Hence, its support must contain an open interval.

[7]Note that $\delta(y)$ and $\sigma(x)$ have the same flavor as the "accessibility measures" introduced by Picard and Tabuchi (2013).

Because $m(y)$ is a symmetric density, we have $\sigma(x) = \sigma_0 + x^2$, where

$$\sigma_0 \equiv \frac{1}{M} \int_X y^2 m(y) \mathrm{d}y \tag{17}$$

is the mean-squared distance of the whole population of firms from the central location $x = 0$.

Combining (13) with (7) and using (10) implies that the quantity $q^*(x,y)$ of a variety supplied at y demanded by an x-consumer is given by

$$q^*(x,y) = \frac{1}{\beta}\left[\frac{\alpha - c}{2} - t(x-y)^2 + \frac{t}{2}\delta(y)\right] \tag{18}$$

Landlords and land use pattern formation

The land market works in the standard way: absentee landlords choose between renting land to consumers and to firms, depending on whose bid is higher. We build on Fujita and Ogawa (1982) in introducing the *consumer bid rent function* $\Psi(x, U^*)$, which is defined as the maximum rental price a consumer would agree to pay for locating at x, conditional on having the utility level at least as high as U^*. In other words, it must be that

$$\Psi[x, V(x)] = R(x),$$

where $V(x)$ is the utility level (5) gained by an x-consumer.

Similarly, $\Phi(y, \pi^*)$ stands for the *firm bid rent function*, i.e. it shows the maximum rent $R(y)$, which guarantees $\pi^*(y) \geq \pi^*$. Equivalently, $\Phi(y, \pi^*)$ must satisfy

$$\Phi[y, \pi^*(y)] = R(y). \tag{19}$$

Because landlords seek to maximize their income, it must be that the rent at $x \in X$ is given by:

$$R^*(x) = \max\{\Psi(x, U^*), \Phi(x, \pi^*), R_a\}, \text{ for all } x \in X$$

where R_a is the land use opportunity cost (e.g. agricultural rent).

Following Fujita (1988), we define a *spatial equilibrium* as a dyad $\{n(\cdot), m(\cdot)\}$, which satisfies the following conditions:

(i) for all $x, y \in X$,

$$n(x) > 0 \Rightarrow \Psi(x, U^*) = R^*(x),$$
$$m(y) > 0 \Rightarrow \Phi(y, \pi^*) = R^*(y),$$

(ii) for all $x \in X$,

$$R^*(x) \geq R_a \Rightarrow n(x) + m(x) = 1,$$
$$R^*(x) < R_a \Rightarrow n(x) + m(x) = 0$$

Let $[a,b]$ be a non-empty interval of the real line X. We call $[a,b]$

- a *residential district* if the equalities $n(x) = 1$ and $m(x) = 0$ hold for all $x \in [a,b]$, but fail to hold over an arbitrary open neighbourhood of $[a,b]$;
- a *shopping district* if the equalities $n(x) = 0$ and $m(x) = 1$ hold for all $x \in [a,b]$, but fail to hold over an arbitrary open neighbourhood of $[a,b]$;
- a *mixed district* if the inequalities $0 < m(x) < 1$ and $0 < n(x) < 1$ hold for all $x \in [a,b]$, but fail to hold over an arbitrary open neighbourhood of $[a,b]$.

We call a spatial equilibrium *segregated* if it does not involve mixed districts. Otherwise, we call a spatial equilibrium *mixed.* A mixed equilibrium is *pooled* if the city involves only one district.

Spatial equilibria: preliminary results

We now show that potential variety of spatial configurations can be substantially reduced. To do so, we first establish a technical result which yields a clue to characterizing the urban patterns.

Lemma 1.

(i) *The consumer bid rent function* $\Psi(x,U^*)$ *is a polynomial of degree* 4 *with respect to x, which involves only even degrees of x.*

(ii) *The firm bid rent function* $\Phi(y,\pi^*)$ *is a polynomial of degree* 4 *with respect to y, which involves only even degrees of y.*

Proof. Plugging (18) into (5), solving (5) for $R(x)$ and using (19), we obtain after rearranging:

$$\Psi(x,U^*) = \frac{Mt^2}{2\beta}\left[\left(x^2 + \frac{5\sigma_0 - \delta_0}{2} - \frac{\alpha - c}{2t}\right)^2 + K\right] - U^* + Y, \tag{20}$$

where K is independent of x, while δ_0 and σ_0 are given, respectively, by equations (11) and (17). See Appendix 3 for details.

Along the same lines, we derive the firm bid rent function $\Phi(y,\, \pi^*)$. To do that, we solve (14) for $R(y)$ and apply (19), which yields

$$\Phi(y,\, \pi^*) = \frac{Nt^2}{4\beta}\left(y^2 + \delta_0 - \frac{\alpha - c}{t}\right)^2 - f - \pi^*. \tag{21}$$

Equations (20) and (21) imply directly both claims of the Lemma. □

Using Lemma 1, we come to the following Proposition.

Proposition 2.

(i) *Mixed spatial equilibria do not exist.*

(ii) *Any segregated equilibrium involves either three or five districts.*

Proof. (i) Assume that, on the contrary, a mixed spatial equilibrium exists. In such an equilibrium, $\Psi(x,U^*)$ and $\Phi(x,\pi^*)$ must identically coincide over some non-degenerate interval $[a,b]\in X$. This can happen if and only if these functions coincide over the whole urban space $\left[-\frac{M+N}{2},\frac{M+N}{2}\right]$, for they are polynomial in x by Lemma 1. In other words, any mixed spatial equilibrium is a pooled equilibrium. Moreover, because the coefficients by x^4 in the expressions for $\Psi(x,U^*)$ and $\Phi(x,\pi^*)$ are, respectively, $Mt^2/(2\beta)$ and $Nt^2/(4\beta)$, a pooled equilibrium may arise only when $N=2M$, which is a zero-measure case, hence it can be ruled out without loss of generality. This completes the proof of part (i).

(ii) By Lemma 1, the functions $\Psi(x,U^*)$ and $\Phi(x,\pi^*)$ may have either two, or four, or no intersection points over $\left[-\frac{M+N}{2},\frac{M+N}{2}\right]$. In the latter case the city accommodates either no consumers or no firms, which cannot be true in equilibrium due to (1). □

The following comment is in order. It may seem at first sight that the non-existence of mixed equilibria is a mere theoretical curiosity which stems from the very specific functional forms of preferences and transportation costs. We believe, however, that our choice of preference and shopping cost specifications does not undermine our main results. First, both linear-quadratic preferences and quadratic transportation costs have been extensively used in the literature. Although we work here with an additive separable version of the linear-quadratic utility used by Ottaviano et al. (2002), i.e. when γ, we show in Section 4.2 that our main results still hold when there is substitutability across varieties, i.e. when $\gamma>0$. Second, the argument used in the proofs of Lemma 1 and Proposition 2 keeps its relevance whenever $\Psi(x,U^*)$ and $\Phi(x,\pi^*)$ are *analytic functions* in x. This is due to the uniqueness of analytic continuation (Lang, Ch. 5, p. 160), which implies that whenever two analytic functions defined over a connected domain coincide over a non-degenerate subinterval of this domain, they have to coincide in the whole domain. Because $\Psi(x,U^*)$ and $\Phi(x,\pi^*)$ are obtained, respectively, from consumer and firm FOC, sufficient conditions for them to be analytic in x are that (i) the cost of transporting a unit of the differentiated good from x to y is given by $T(x-y)$, where the function $T(\cdot)$ is analytic, and (ii) preferences of an x-consumer are given by

$$U(z;q(x,y),y\in X)=z+\int_X u\left[q(x,y)\right]m(y)\mathrm{d}y, \tag{22}$$

where the sub-utility function $u(\cdot)$ is increasing, concave, and analytic. Equation (22) embraces a wide variety of preference specifications used in the literature, including (i) the CES under $u(q)=q^\rho$ with $0<\rho<1$, (ii) the CARA (Behrens and Murata, 2007) under $u(q)=1-\exp(-\alpha q)$ (with $\alpha>0$), and (iii) the Stone-Geary preferences giving rise to the linear expenditure system (Simonovska, 2015) under $u(q)=\ln(1+\beta q)$.[8] To sum up, claim (i) of Proposition 2 holds far beyond the linear-quadratic setting.[9]

[8] The function $\ln(1+\beta q)$ is analytic only for $q<1/\beta$, so we have to assume β sufficiently large.

[9] This is not true, of course, regarding claim (ii).

Spatial equilibria

In this section, we study the conditions for various spatial configurations of the city to emerge. Recall that $n(\cdot)$ and $m(\cdot)$ are, respectively, the densities describing the distributions of consumers and firms across the urban space. Due to Proposition 2, we may focus on segregated equilibria without loss of generality.

Central business area

Consider first the case of a segregated spatial equilibrium in which the central area accommodates firms, while peripheral areas are used for housing. Such an equilibrium involves one business district, $[-\frac{M}{2}, \frac{M}{2}]$, and two symmetric residential districts, $[-\frac{M+N}{2}, -\frac{M}{2})$ and $(\frac{M}{2}, \frac{M+N}{2}]$.

When does an equilibrium of this type exist? In this case, the equilibrium population density $n(x)$ must be given by

$$n(x) = \begin{cases} 1, & \text{if } x \in [-\frac{M+N}{2}, -\frac{M}{2}) \cup (\frac{M}{2}, \frac{M+N}{2}], \\ 0, & \text{otherwise,} \end{cases} \tag{23}$$

while the equilibrium density of firms is

$$m(y) = \begin{cases} 1, & \text{if } y \in [-\frac{M}{2}, \frac{M}{2}], \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

Combining (23) – (24) with (9), we find that the mean squared distance is given by

$$\delta(y) = y^2 + \frac{1}{12}\left(3M^2 + 3MN + N^2\right). \tag{25}$$

The sufficient condition (6) for the consumer program to have an interior solution can be considerably relaxed. Namely, as shown in Appendix 2, each consumer purchases the whole range of available varieties if and only if:

$$\frac{\alpha - c}{t} > M^2 + \frac{5}{4}MN + \frac{5}{12}N^2. \tag{26}$$

Plugging (25) into (21), we find that the firm bid rent function takes the form:

$$\Phi(y, \pi^*) = \frac{Nt^2}{4\beta}\left[y^4 - 2\left(\frac{\alpha - c}{t} - \frac{3M^2 + 3MN + N^2}{12}\right)y^2\right] + \frac{Nt^2}{4\beta}\left(\frac{\alpha - c}{t} - \frac{3M^2 + 3MN + N^2}{12}\right)^2 - \pi^* - f. \tag{27}$$

Also, we show in Appendix 4 that the consumer bid rent function may be expressed as follows:

$$\Psi(x, U^*) = \frac{Mt^2}{2\beta}\left[x^4 - \left(\frac{\alpha - c}{t} - \frac{2M^2 - 3MN - N^2}{12}\right)x^2\right] + k - U^* + Y, \tag{28}$$

where k depends on the parameters of the model, but not on x.

When does a segregated spatial equilibrium involving shopping area in the centre exist? The answer is most conveniently given in terms of the *relative number of firms* defined by $\mu \equiv M/N$.

Proposition 3. *Assume that* (26) *holds. Then, there exists a threshold value* $\bar{\mu} \in (0,1)$ *of* μ*, such that*[10]

(i) if $0 < \mu \leq \bar{\mu}$*, then a segregated spatial equilibrium with a shopping area in the center exists;*

(ii) if $\bar{\mu} < \mu < 1$*, then such a spatial equilibrium exists if and only if the following inequality holds:*

$$\frac{\alpha - c}{t} > \frac{N^2}{24} \frac{-10\mu^3 + 15\mu^2 + 8\mu + 2}{1 - \mu}; \tag{29}$$

(iii) if $\mu \geq 1$*, no such spatial equilibrium exists.*

Proof. See Appendix 5. □

Note that (29) has the same nature as (6) and (26): *either consumers have a high willingness-to-pay, or transportation costs are low, or both.*

Central residential area

At the other extreme is the segregated spatial configuration where the central area is residential. In this case the urban space is composed by one residential district, $[-\frac{N}{2}, \frac{N}{2}]$, and two symmetric business districts, $[-\frac{M+N}{2}, -\frac{N}{2})$ and $(\frac{N}{2}, \frac{M+N}{2}]$.

When does an equilibrium of this type exist? In this case, the equilibrium population density $n(x)$ must be given by

$$n(x) = \begin{cases} 1, & \text{if } x \in [-\frac{N}{2}, \frac{N}{2}], \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

while the equilibrium density of firms is

$$m(y) = \begin{cases} 1, & \text{if } y \in [-\frac{M+N}{2}, -\frac{N}{2}) \cup (\frac{N}{2}, \frac{M+N}{2}], \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

Plugging (68) into (21), we find that the firm bid rent function takes the form:

$$\Phi(y, \pi^*) = \frac{Nt^2}{4\beta} \left[y^4 - 2\left(\frac{\alpha - c}{t} - \frac{N^2}{12} \right) y^2 \right] + \frac{Nt^2}{4\beta} \left(\frac{\alpha - c}{t} - \frac{N^2}{12} \right)^2 - \pi^* - f. \tag{32}$$

We show in Appendix 6 that the consumer bid rent function boils down to

$$\Psi(x, U^*) = \frac{Mt^2}{2\beta} \left[x^4 - \left(\frac{\alpha - c}{t} - \frac{5M^2 + 15MN + 14N^2}{12} \right) x^2 \right] + \kappa - U^* + Y, \tag{33}$$

[10]Surprisingly, $\bar{\mu}$ is fully independent of the parameters of the model. The numerical value of $\bar{\mu}$ equals approximately 0.732982. See Appendix 5 for details.

where κ is independent on x.

Setting $\nu \equiv N/M$ to be the relative number of consumers, we obtain the following result.

Proposition 4. *A segregated spatial equilibrium with a residential area in the centre and shopping areas at the outskirts exists if and only if*

(i) $\nu < 1$*, and*

(ii) the following inequality holds:

$$\frac{\alpha - c}{t} > \frac{M^2}{24} \frac{16 + 39\nu + 34\nu^2 - 8\nu^3}{1 - \nu}.$$

Proof. See Appendix 7. □

To sum up, a segregated spatial equilibrium in which the central area is residential exists when the number of consumers per firm is relatively low, meaning that competition among firms is tough.

More complex city structures and multiple equilibria

As implied by Propositions 3 and 4, the spatial equilibria having the simple structure considered above do not always exist. In particular, this is the case when $M = N$, in which neither consumers nor firms have unambiguous advantage in competing for land.

A full characterization of the spatial configurations in our model is long and tedious. Therefore, we focus here on the case when $M = N$. Although this case is very special, it is sufficient to show that competition for central locations between consumers and firms (embodied in the bidding process) may result in multiple equilibria.

Proposition 5. *Assume that* $M = N$*. Then,*

(i) there exists a continuum of segregated spatial equilibria with business areas given by

$$\left[-(1+\theta)\frac{M}{2}, -\theta\frac{M}{2}\right] \cup \left[\theta\frac{M}{2}, (1+\theta)\frac{M}{2}\right],$$

where $\theta \in \left(0, \frac{\sqrt{7/3}-1}{2}\right)$*, and*

(ii) no other spatial equilibria exist.

Proof. That no equilibria of the types discussed in Sections 3.1 and 3.2 exist is implied directly by Propositions 3 and 4. For the rest of the proof, see Appendix 8. □

Proposition 5 highlights multiplicity of equilibria, which frequently shows up in spatial models (Fujita and Ogawa, 1982; Mossay and Picard, 2011). Note, however, that for a five-district equilibrium with a residential (shopping) downtown area to exist, the bid-rent differential must be a U-shaped (bell-shaped) function of x^2 over $[0, (M+N)^2/4]$. Hence, a *necessary* (but not sufficient) condition for a five-district equilibrium with a residential (business) downtown area to exist is that $2M > N$ ($2M < N$). In particular, neither of the five-district spatial configurations exists under $2M = N$. See Appendix 8 for the details. Relating this result to Propositions 3 and 4, we conclude that, under $\mu = 1/2$, the three-district spatial configuration with the shopping area downtown is *the unique equilibrium*.

Extensions

In this section, we provide two extensions of the baseline model considered above. First, we study the implications of replacing quadratic shopping costs with linear shopping costs. Second, we introduce non-zero substitutability across varieties by considering the non-additive quadratic utility a la Ottaviano et al. (2002).

Linear transportation costs

That changes in the specification of transportation costs may dramatically affect the behavior of equilibria in models of imperfect competition where space matters is a well-known issue first raised by d'Aspremont et al. (1979). Therefore, we discuss here the implications of assuming a *linear* shopping cost $t|x-y|$. In this case, it is readily verified that, given the density $n(\cdot)$ describing the distribution of individuals across the city, the profit-maximizing price of a y-firm given by

$$p^*(y) = \frac{1}{2}\left[\alpha + c - t\Delta(y)\right]. \tag{34}$$

where $\Delta(y)$ is the mean distance from firm's location y to the whole population of consumers:

$$\Delta(y) \equiv \frac{1}{N}\int_X |x-y| n(x)\mathrm{d}x.$$

It is well-known that $\Delta(y)$ is minimized in the median of the distribution given by $n(\cdot)$ (Stroock, 2011). However, unlike the mean, a median may be non-unique. This implies the following result.

Proposition 6. *Assume that the spatial equilibrium is segregated and involves a central shopping area. Then, all firms in this shopping area charge the same price.*

Proposition 6 shows that, when the shopping cost is linear, the equilibrium outcome no longer exhibits spatial price dispersion across the central area. This is not the case, however, when the city's central district is residential.

Non-zero substitution term

Up to now, we have been assuming that preferences are additive. We now relax this assumption by introducing a substitution term into the utility function, like in Ottaviano et al. (2002). In other words, preferences are now given by

$$\mathscr{U}(z;\mathbf{q}) \equiv z + \alpha\int_X q(x,y)m(y)\mathrm{d}y - \frac{\beta}{2}\int_X [q(x,y)]^2 m(y)\mathrm{d}y - \frac{\gamma}{2}\left[\int_X q(x,y)m(y)dy\right]^2,$$

where $\gamma \in (0,\beta)$ captures substitutability across varieties of the differentiated good, hence the degree of competitive toughness in the market.

The inverse demand of an x-consumer for a variety produced by a y-firm is given by

$$p(y) = \alpha - \beta q(x,y) - \gamma \mathbb{Q}(x) - t(x-y)^2, \tag{35}$$

where $\mathbb{Q}(x) \equiv \int_X q(x,y)m(y)dy$ is the *consumption index* of an x-consumer, while $\sigma(x)$ is defined by (16) and shows the mean-squared distance of consumer's location x to the whole population of firms.

Multiplying both parts of (35) by $m(y)$ and integrating with respect to y across X, we obtain

$$P = M\alpha - (\beta + \gamma M)\mathbb{Q}(x) - Mt\sigma(x), \tag{36}$$

where P is the *price index* defined by

$$P \equiv \int_X p(y)m(y)dy, \tag{37}$$

while $\sigma(x)$ stands for the mean-squared distance from an x-consumer to the whole mass of firms given by (16).

Using (36), we obtain the following expression for $\mathbb{Q}(x)$:

$$\mathbb{Q}(x) = \frac{M\alpha}{\beta + \gamma M} - \frac{P}{\beta + \gamma M} - \frac{Mt}{\beta + \gamma M}\sigma(x), \tag{38}$$

The first term in (38), $M\alpha/(\beta + \gamma M)$, shows that the x-consumer's aggregate willingness to pay increases with M, though less than proportionally to M. The second term, $-P/(\beta + \gamma M)$, captures the negative effect of an increase in the price index on an x-consumer's total consumption. Finally, the third term, $-Mt\sigma(x)/(\beta + \gamma M)$, keeps track of the impact of consumer's location x on the volume of consumption.

Solving (35) with respect to $q(x,y)$ and using (38), we find that the individual demand of an x-consumer for a variety produced by a y-firm is given by

$$q(x,y) = \frac{1}{\beta}\left[\frac{\alpha\beta}{\beta + \gamma M} + \frac{\gamma M}{\beta + \gamma M}\left(t\sigma(x) + \frac{P}{M}\right) - p(y) - t(x-y)^2\right]. \tag{39}$$

The aggregate demand faced by a y-firm is then given by

$$Q(y) = \frac{N}{\beta}\left[\frac{\alpha\beta}{\beta + \gamma M} + \frac{\gamma M}{\beta + \gamma M}\left(t\theta + \frac{P}{M}\right) - p(y) - t\delta(y)\right]. \tag{40}$$

where $\delta(y)$ is the mean-squared distance between firm's location y and the population of consumers defined by (9), while θ is the overall mean-squared distance between consumers and firms:[11]

$$\theta \equiv \frac{1}{MN}\int_X \int_X (x-y)^2 n(x)m(y)\mathrm{d}x\mathrm{d}y. \tag{41}$$

The presence of the price aggregate P in the right-hand side of (40) indicates that the market of the differentiated good is *no longer a collection of monopolists*, but that the market structure

[11] Using symmetry of both densities, it can be shown that $\theta = \delta_0 + \sigma_0$.

is now *truly monopolistic competition*. Indeed, the market demands faced by firms now depend directly on the aggregate of the choices of other players.[12] Because there is a continuum of firms, each firm is negligible to the market. Hence, firms lack the ability to strategically manipulate the value of the market aggregate P, which they treat parametrically. As a consequence, the profit-maximizing price $\hat{p}(y,P)$ of a y-firm can be expressed as

$$\hat{p}(y,P) = \frac{1}{2}\left[\frac{\alpha\beta}{\beta+\gamma M} + c + \frac{\gamma M}{\beta+\gamma M}\left(t\theta + \frac{P}{M}\right) - t\delta(y)\right]. \tag{42}$$

Integrating (42) with respect to y across X and using (37), we come to the following fixed-point condition for the average price-index P/M:

$$\frac{P}{M} = \frac{1}{2}\left[c + \frac{\beta(\alpha - t\theta)}{\beta+\gamma M} + \frac{\gamma M}{\beta+\gamma M}\frac{P}{M}\right],$$

solving which for P/M yields

$$\frac{P}{M} = c + \frac{\beta(\alpha - c - t\theta)}{2\beta+\gamma M}. \tag{43}$$

Plugging (43) into (42), we obtain the profit-maximizing price set by a y-firm:

$$p^*(y) = c + \left[\frac{(\alpha-c)\beta}{2\beta+\gamma M} + \frac{\gamma M}{4\beta+2\gamma M}\theta t - \frac{t}{2}\delta(y)\right]. \tag{44}$$

The bracketed term in (44) is the markup of a y-firm. The first term in brackets, $(\alpha - c)\beta/(2\beta+\gamma M)$, captures the non-spatial component of the markup, which decreases in both the number M of firms and the degree γ of substitutability across varieties, reflecting the standard *competition effect*. The second term, $[\gamma M/(4\beta+2\gamma M)]\cdot\theta t$, represents the *global shopping cost effect*, which increases with transportation cost but is independent of firm's location y. This effect is in line with the common wisdom of spatial competition theory: local monopoly power increases over the space when the shopping costs increase. Finally, the third term, $t\delta(y)/2$, captures the firm's *location effect*: moving further away from the center to a location with a higher $\delta(y)$ results in an erosion of firm's monopoly power.

It is legitimate to ask how markups change in response to an increase in toughness of competition γ and to a reduction in transportation cost t. The answer is given by the following result.

Proposition 7.

(i) The markups of all firms increase/decrease with γ if and only if the following inequality holds/does not hold:

$$c + t\theta > \alpha.$$

(ii) The markup of a y-firm increases in response to a reduction in t if and only if the following condition is fulfilled:

[12] See Anderson et al. (2015) for a discussion of the relationship between monopolistic competition and aggregative games.

$$\delta(y) > \frac{\gamma M}{2\beta + \gamma M}\theta. \tag{45}$$

The intuition behind (45) is as follows. When $\gamma > 0$, infrastructural imrpovements are always beneficial for less centrally located firms (i.e. those with high values of $\delta(y)$), but may be detrimental for more centrally located firms (i.e. those with low values of $\delta(y)$). In other words, *higher substitutability across varieties dampens the role of locational advantage*.

How do spatial equilibria look like when $\gamma > 0$? In the same vein as in Section 2, it can be shown using (39) and (44) that Lemma 1, hence Proposition 2, hold for the case when $\gamma > 0$ without any changes. Thus, we again end up with *only segregated spatial equilibria*.

Concluding remarks

We have developed a model that contributes to studying agglomeration forces which Fujita and Thisse (2013) describe as "created through market interactions between firms and consumers". We have shown that clustering may occur without any production externalities, fully driven by the demand-side factors. Furthermore, we have demonstrated that imperfectly informed consumers and search costs are not essential ingredients for spatial price dispersion.

Our results imply that equilibrium patterns depend crucially on the assumptions imposed on variety-loving consumer behavior. These considerations, emphasized by the growing literature on variable markups, suggest a new agenda for empirical urban economists, for little has been done as yet to study the impact of these factors on urban structure.

We believe that our model is flexible enough to study industrial specialization of cities. Indeed, the approach proposed in this paper reveals the fundamental role of modelling assumptions on the type of resulting spatial equilibria (see the discussion at the end of Section 2). Therefore, we find it potentially interesting to blend our setting with that developed by Helsley and Strange (2014) in order to obtain further clear-cut theoretical results regarding the conditions of clustering. Another possible line of further inquiry is to study whether the array of city structures arising in equilibrium becomes richer if we allow for heterogeneities across consumers not only in locations, but also in tastes (as in Tarasov, 2014; Osharin et al. 2014). We leave these tasks for future research.

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Appendix

Appendix 1: (6) is sufficient for consumer's interior solution.

In what follows, we use $\operatorname{supp}\phi(\cdot)$ to denote the support of a function $\phi : \mathbb{R} \to \mathbb{R}$. Using (7), we find that

$$\alpha - t(x-y)^2 - p(y) > 0 \tag{46}$$

is necessary and sufficient to hold for all $x \in \operatorname{supp} n(\cdot)$ and for all $y \in \operatorname{supp} m(\cdot)$, for the solution of each consumer's program to be interior.

By definition of the spatial equilibrium, the urban space is confined to $[-\frac{M+N}{2}, \frac{M+N}{2}]$, no matter what the equilibrium land use pattern is. Using (13), we may restate (46) as follows:

$$\frac{\alpha - c}{t} > 2(x-y)^2 - \delta(y) \text{ for all } x, y \in \left[-\frac{M+N}{2}, \frac{M+N}{2}\right]. \tag{47}$$

Observe that, because $x, y \in [-\frac{M+N}{2}, \frac{M+N}{2}]$, we have (i) $(x-y)^2 < (M+N)^2$, and (ii) $\delta(y) \geq 0$, regardless of a particular shape of the population density $n(x)$. Hence, if (6) holds, then (47) holds. In other words, (6) is sufficient for each consumer's program to possess an interior solution. Q.E.D.

Appendix 2 . Deriving (26) and (73).

Consider first the case of such segregated spatial equilibrium where the central area is a shopping district, while the outskirt areas are residential. In this case, we have $\operatorname{supp} n(\cdot) = [-\frac{M+N}{2}, -\frac{M}{2}] \cup [\frac{M}{2}, \frac{M+N}{2}]$ and $\operatorname{supp} m(\cdot) = [-\frac{M}{2}, \frac{M}{2}]$. Combining this with (25) and using symmetry, we find that (47) boils down to

$$\frac{\alpha - c}{t} > \lambda(x,y) - \frac{3M^2 + 3MN + N^2}{12}, \tag{48}$$

where $\lambda(x,y) \equiv 2x^2 - 4xy + y^2$. A necessary and sufficient condition for (48) to hold for all $x \in [\frac{M}{2}, \frac{M+N}{2}]$ and for all $y \in [-\frac{M}{2}, \frac{M}{2}]$ is

$$\frac{\alpha - c}{t} > \lambda(\hat{x},\hat{y}) - \frac{3M^2 + 3MN + N^2}{12}, \tag{49}$$

where $(\hat{x},\hat{y})$ is a global maximizer of $\lambda(x,y)$ over $[\frac{M}{2}, \frac{M+N}{2}] \times [-\frac{M}{2}, \frac{M}{2}]$. Because all admissible values of x are strictly positive, $\hat{y}$ has to be non-positive. Indeed, otherwise $\lambda(x,-\hat{y}) > \lambda(x,\hat{y})$ for any x, which is a contradiction. Moreover, for any given $x_0 \in [\frac{M}{2}, \frac{M+N}{2}]$, $\lambda(x_0,y)$ is a strictly decreasing function of y over $[-\frac{M}{2}, 0]$. Hence, we have $\hat{y} = -M/2$. As for $\hat{x}$, it is a maximizer for

$\lambda(x,-M/2) = 2x^2 + 2Mx + M^2/4$ over $x \in [\frac{M}{2}, \frac{M+N}{2}]$. Because $\lambda(x,-M/2)$ is increasing in x, we have $\hat{x} = (M+N)/2$. Plugging $\hat{x}$ and $\hat{y}$ into (49) yields (26).

The proof of (73) is fully analogous to the above proof of (26).

Appendix 3: deriving $\Psi(x,U^*)$:

Using (5) yields

$$\Psi(x,U^*) = \alpha \int_X q(x,y)m(y)\mathrm{d}y - \frac{\beta}{2}\int_X [q(x,y)]^2 m(y)\mathrm{d}y - \int_X [p(y)+t(x-y)^2]q(x,y)m(y)dy - U^* + Y.$$

Combining this with (7), we obtain:

$$\begin{aligned}\Psi(x,U^*) = \; & \frac{\alpha}{\beta}\int_X \left(\frac{\alpha-c}{2} - t(x-y)^2 + \frac{t}{2}\delta(y)\right) m(y)\mathrm{d}y - \frac{1}{2\beta}\int_X \left(\frac{\alpha-c}{2} - t(x-y)^2 + \frac{t}{2}\delta(y)\right)^2 m(y)\mathrm{d}y \\ & -\frac{1}{\beta}\int_X \left[\frac{1}{2}(\alpha + c - t\delta(y)) + t(x-y)^2\right]\left(\frac{\alpha-c}{2} - t(x-y)^2 + \frac{t}{2}\delta(y)\right) m(y)dy \\ & -U^* + Y,\end{aligned}$$

which yields (20) after rearranging.

Appendix 4. Deriving (28).

Integrating the right-hand side of (16), we obtain

$$\sigma(x) = x^2 + \frac{M^2}{12}, \tag{50}$$

while carrying out double integration in (41) yields

$$\theta = \frac{M^2}{3} + \frac{MN}{4} + \frac{N^2}{12}. \tag{51}$$

Plugging (23) – (25) and (50) – (51) into (20) and setting

$$k \equiv \frac{Mt^2}{8\beta}\left[\left(\frac{\alpha-c}{t}\right)^2 + \frac{(M+N)(2M+N)}{6}\left(\frac{\alpha-c}{t}\right) + \frac{24M^4 + 60M^3N + 65M^2N^2 + 30MN^3 + 5N^4}{720}\right], \tag{52}$$

we obtain (28).

Appendix 5. Proof of Proposition 2.

Setting

$$\widetilde{\Phi}(y) \equiv \frac{Nt^2}{4\beta}\left[\frac{\alpha-c}{t} - \frac{1}{12}\left(3M^2 + 3MN + N^2\right) - y^2\right]^2, \tag{53}$$

$$\widetilde{\Psi}(x) \equiv \frac{Mt^2}{2\beta}\left[x^4 - \left(\frac{\alpha - c}{t} - \frac{2M^2 - 3MN - N^2}{12}\right)x^2\right] + k, \tag{54}$$

where k is given by (52), we may state the existence conditions for a segregated spatial equilibrium with firms in the center as follows:

$$\widetilde{\Psi}\left(\frac{M+N}{2}\right) - U^* + Y = R_a, \tag{55}$$

$$\widetilde{\Phi}\left(\frac{M}{2}\right) - \pi^* - f = \widetilde{\Psi}\left(\frac{M}{2}\right) - U^* + Y, \tag{56}$$

$$\widetilde{\Phi}(x) - \pi^* - f > \widetilde{\Psi}(x) - U^* + Y \quad \text{for all } x : 0 < x < \frac{M}{2}, \tag{57}$$

$$\widetilde{\Phi}(x) - \pi^* - f < \widetilde{\Psi}(x) - U^* + Y \quad \text{for all } x : \frac{M}{2} < x < \frac{M+N}{2}. \tag{58}$$

As seen from (53) – (54), $\widetilde{\Phi}(x)$ and $\widetilde{\Psi}(x)$ are both quadratic functions of solely x^2. Hence, (57) – (58) may be restated as follows:

$$\widetilde{\Phi}(0) - \pi^* - f > \widetilde{\Psi}(0) - U^* + Y \tag{59}$$

$$\widetilde{\Phi}\left(\frac{M+N}{2}\right) - \pi^* - f < \widetilde{\Psi}\left(\frac{M+N}{2}\right) - U^* + Y \tag{60}$$

From (55) and (56) we obtain:

$$U^* = \widetilde{\Psi}\left(\frac{M+N}{2}\right) + Y - R_a, \tag{61}$$

$$\pi^* = \widetilde{\Phi}\left(\frac{M}{2}\right) - \widetilde{\Psi}\left(\frac{M}{2}\right) + \widetilde{\Psi}\left(\frac{M+N}{2}\right) - R_a - f. \tag{62}$$

Using (61) – (62), we find that (59) – (60) boil down to:

$$\widetilde{\Phi}(0)-\widetilde{\Psi}(0)>\widetilde{\Phi}(\frac{M}{2})-\widetilde{\Psi}(\frac{M}{2}), \tag{63}$$

$$\widetilde{\Phi}(\frac{M+N}{2})-\widetilde{\Psi}(\frac{M+N}{2})<\widetilde{\Phi}(\frac{M}{2})-\widetilde{\Psi}(\frac{M}{2}). \tag{64}$$

Restating (63) and (64) as

$$\widetilde{\Phi}(0)-\widetilde{\Phi}\left(\frac{M}{2}\right)>\widetilde{\Psi}(0)-\widetilde{\Psi}\left(\frac{M}{2}\right),$$
$$\widetilde{\Phi}\left(\frac{M}{2}\right)-\widetilde{\Phi}\left(\frac{M+N}{2}\right)>\widetilde{\Psi}\left(\frac{M}{2}\right)-\widetilde{\Psi}\left(\frac{M+N}{2}\right),$$

and using (53) – (54), we find that shown abowe inequalities hold if and only if

$$24(N-M)\frac{\alpha-c}{t}>-10M^3+15M^2N+8MN^2+2N^3$$
$$24(N-M)\frac{\alpha-c}{t}>-16M^3+6M^2N+8MN^2+5N^3$$

A crucial role is played by total numbers of consumers and firms. In case $N>M$ we get existence condition:

$$\begin{cases} \dfrac{\alpha-c}{t}>\dfrac{-10M^3+15M^2N+8MN^2+2N^3}{24(N-M)} \\ \dfrac{\alpha-c}{t}>\dfrac{-16M^3+6M^2N+8MN^2+5N^3}{24(N-M)} \end{cases} \tag{65}$$

Otherwise,

$$\begin{cases} \dfrac{\alpha-c}{t}<\dfrac{-10M^3+15M^2N+8MN^2+2N^3}{24(N-M)} \\ \dfrac{\alpha-c}{t}<\dfrac{-16M^3+6M^2N+8MN^2+5N^3}{24(N-M)} \end{cases} \tag{66}$$

To sum up, a segregated spatial equilibrium with a shopping district in the center exists if and only if

(A) $\mu<1$, where $\mu\equiv M/N$;

(B) (26) and (65) hold. Using μ, we can restate (B) as follows:

$$\frac{24}{N^2}\frac{\alpha-c}{t}>\frac{1}{1-\mu}\max\{G_1(\mu),G_2(\mu),G_3(\mu)\},\quad \text{for all } \mu\in(0,1),$$

where

$$G_1(\mu) \equiv \left(24\mu^2 + 30\mu + 10\right)(1-\mu), \qquad G_2(\mu) \equiv -10\mu^3 + 15\mu^2 + 8\mu + 2,$$

$$G_3(\mu) \equiv -16\mu^3 + 6\mu^2 + 8\mu + 5.$$

Figure 1A plots $G_1(\mu)$, $G_2(\mu)$, and $G_3(\mu)$

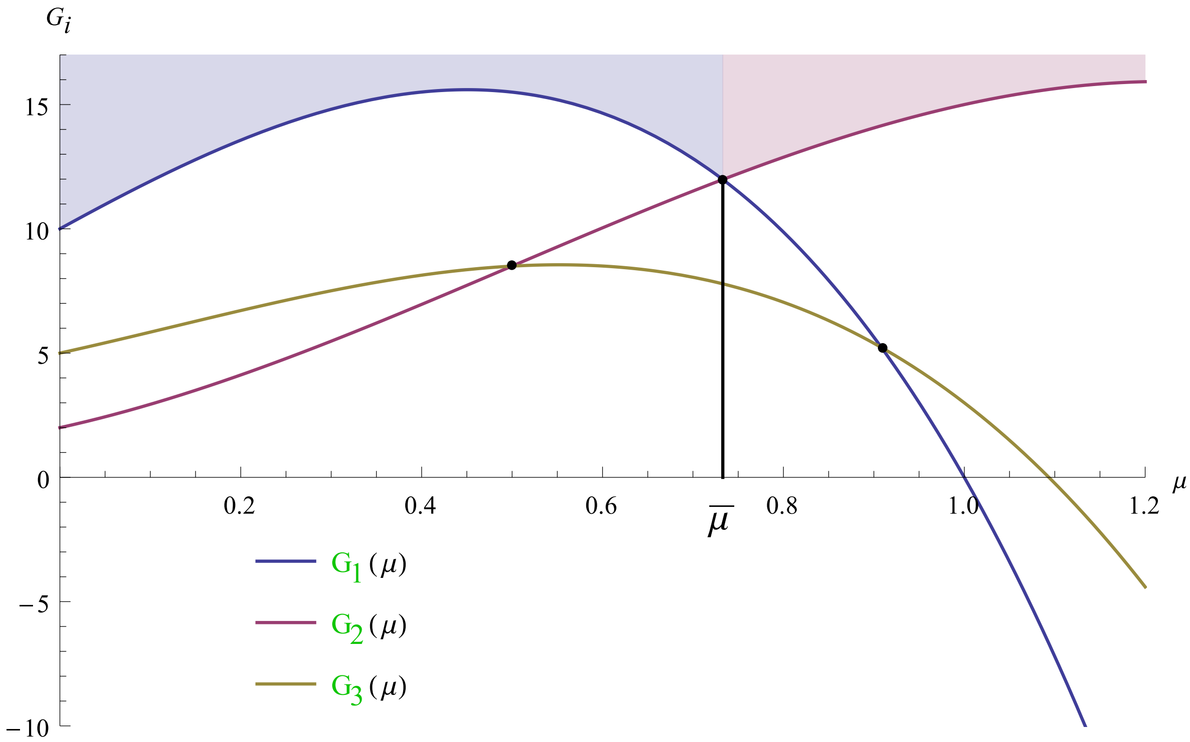

Figure 1A. The plots of $G_1(\mu)$, $G_2(\mu)$, and $G_3(\mu)$. $N > M$ case.

Solving the equation $G_1(\mu) = G_2(\mu)$ numerically yields a unique solution $\bar{\mu} \approx 0.732982$ over $(0,1)$. As seen from Figure 1A, when $0 < \mu < \bar{\mu}$, we have

$$G_1(\mu) = \max\{G_1(\mu), G_2(\mu), G_3(\mu)\},$$

hence, (B) boils down to (26). However, Figure 1A also shows that, when $\bar{\mu} < \mu < 1$, (B), we have

$$G_2(\mu) = \max\{G_1(\mu), G_2(\mu), G_3(\mu)\}.$$

In this case, (B) amounts to (29). Combining this with (A) completes the proof of Proposition 2.

Notice that when we have $N < M$, analogously, using $\mu \equiv M/N$, (26) and (66) form following conditions on the existence of the equilibrium:

$$\frac{24}{N^2}\frac{\alpha - c}{t} > \frac{1}{1-\mu} G_1(\mu),$$

$$\frac{24}{N^2}\frac{\alpha - c}{t} < \frac{1}{1-\mu}\max\{G_2(\mu), G_3(\mu)\}$$

for all $\mu \geq 1$.

Figure 2A illustrates that there is no solution for the case when $\mu \geq 1$:

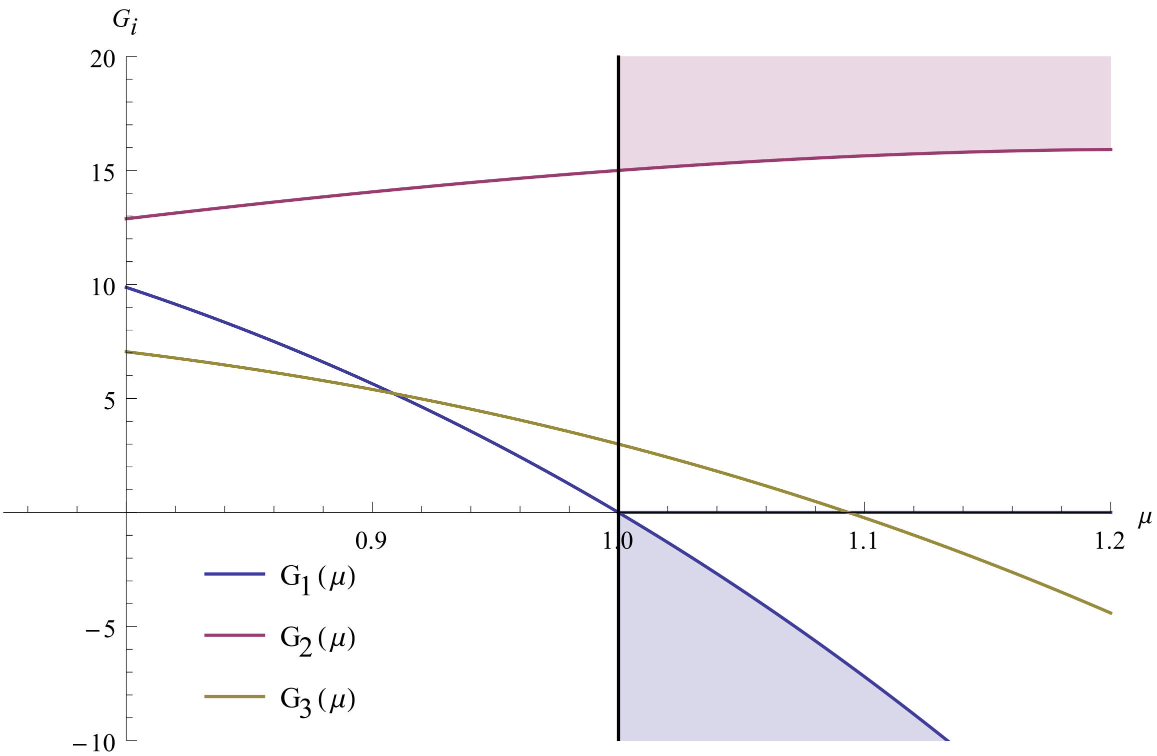

Figure 2A. The plots of $G_1(\mu)$, $G_2(\mu)$, and $G_3(\mu)$. $N < M$ case.

Appendix 6. Deriving (32).

Integrating the right-hand sides of (16) and (9), we obtain

$$\sigma(x) = x^2 + \frac{M^2 + 3MN + 3N^2}{12}, \tag{67}$$

$$\delta(y) = y^2 + \frac{N^2}{12} \tag{68}$$

while carrying out double integration in (41) yields

$$\theta = \frac{M^2}{12} + \frac{MN}{4} + \frac{N^2}{3}. \tag{69}$$

Plugging (31) – (68) and (67) – (69) into (20) and setting

$$\kappa \equiv \frac{Mt^2}{8\beta}\left[\left(\frac{\alpha - c}{t}\right)^2 - \frac{(M+N)(2M+N)}{6}\left(\frac{\alpha - c}{t}\right) + \frac{9M^4 + 45M^3N + 80M^2N^2 + 60MN^3 + 20N^4}{720}\right], \tag{70}$$

we obtain (33).

Appendix 7. Proof of Proposition 4.

One can derive

$$\widetilde{\Phi}(y) = \frac{Nt^2}{4\beta}\left[y^4 - 2\left(\frac{\alpha - c}{t} - \frac{N^2}{12}\right)y^2\right] + \frac{Nt^2}{4\beta}\left(\frac{\alpha - c}{t} - \frac{N^2}{12}\right)^2 \tag{71}$$

$$\widetilde{\Psi}(x) = \frac{Mt^2}{2\beta}\left[x^4 - \left(\frac{\alpha - c}{t} - \frac{5M^2 + 15MN + 14N^2}{12}\right)x^2\right] + \kappa \tag{72}$$

Very much in the spirit of proving (26) (see Appendix 2), it can be shown that

$$\frac{\alpha - c}{t} > \frac{1}{2}M^2 + 2MN + \frac{5}{3}N^2 \tag{73}$$

is a necessary and sufficient condition for the choices of consumers to be interior.

The same bunch of conditions for the existence of an equilibrium with consumers in the center is as follows:

$$\widetilde{\Phi}(\frac{M+N}{2}) - \pi^* = R_a \tag{74}$$

$$\widetilde{\Phi}(\frac{N}{2}) - \pi^* = \widetilde{\Psi}(\frac{N}{2}) - U^* \tag{75}$$

$$\widetilde{\Phi}(0) - \pi^* < \widetilde{\Psi}(0) - U^* \tag{76}$$

$$\widetilde{\Phi}(\frac{M+N}{2}) - \pi^* > \widetilde{\Psi}(\frac{M+N}{2}) - U^* \tag{77}$$

Which results in:

$$\pi^* = \widetilde{\Phi}(\frac{M+N}{2}) - R_a$$

$$U^* = \widetilde{\Psi}(\frac{N}{2}) - \widetilde{\Phi}(\frac{N}{2}) + \widetilde{\Phi}(\frac{M+N}{2}) - R_a$$

From the last equations follows:

$$\widetilde{\Psi}(0)-\widetilde{\Phi}(0)>\widetilde{\Psi}(\frac{N}{2})-\widetilde{\Phi}(\frac{N}{2})>\widetilde{\Psi}(\frac{M+N}{2})-\widetilde{\Phi}(\frac{M+N}{2}) \tag{78}$$

Using (78), we derive the existence conditions for a segregated spatial equilibrium with consumer's center. Deriving (71)–(72), we can show that (78) holds if the following inequalities hold simultaneously:

$$24(M-N)\frac{\alpha-c}{t}>10M^3+30M^2N+34MN^2-5N^3 \tag{79}$$

$$24(M-N)\frac{\alpha-c}{t}>16M^3+39M^2N+34MN^2-8N^3 \tag{80}$$

Then, if (73), (79), and (80) simultaneously hold, $\nu<1$, where $\nu\equiv N/M$.

For a segregated equilibrium with residential central area to exist, (73), (79), and (80) must hold simultaneously. When $\nu<1$ (where $\nu\equiv N/M$), this is equivalent to

$$(1-\nu)\frac{\alpha-c}{t}>\frac{M^2}{24}\max\{H_1(\nu),H_2(\nu),H_3(\nu)\},$$

where

$$H_1(\nu)\equiv\left(40\nu^2+48\nu+12\right)(1-\nu),\qquad H_2(\nu)\equiv-5\nu^3+34\nu^2+30\nu+10,$$

$$H_3(\nu)\equiv-8\nu^3+34\nu^2+39\nu+16.$$

When $\nu>1$, simultaneous holding of (73), (79), and (80) may be stated as

$$\frac{M^2}{24}\max\{H_2(\nu),H_3(\nu)\}<(1-\nu)\frac{\alpha-c}{t}<\frac{M^2}{24}H_1(\nu).$$

Figure 2B shows that there is no solution for the case when $\nu\geq1$:

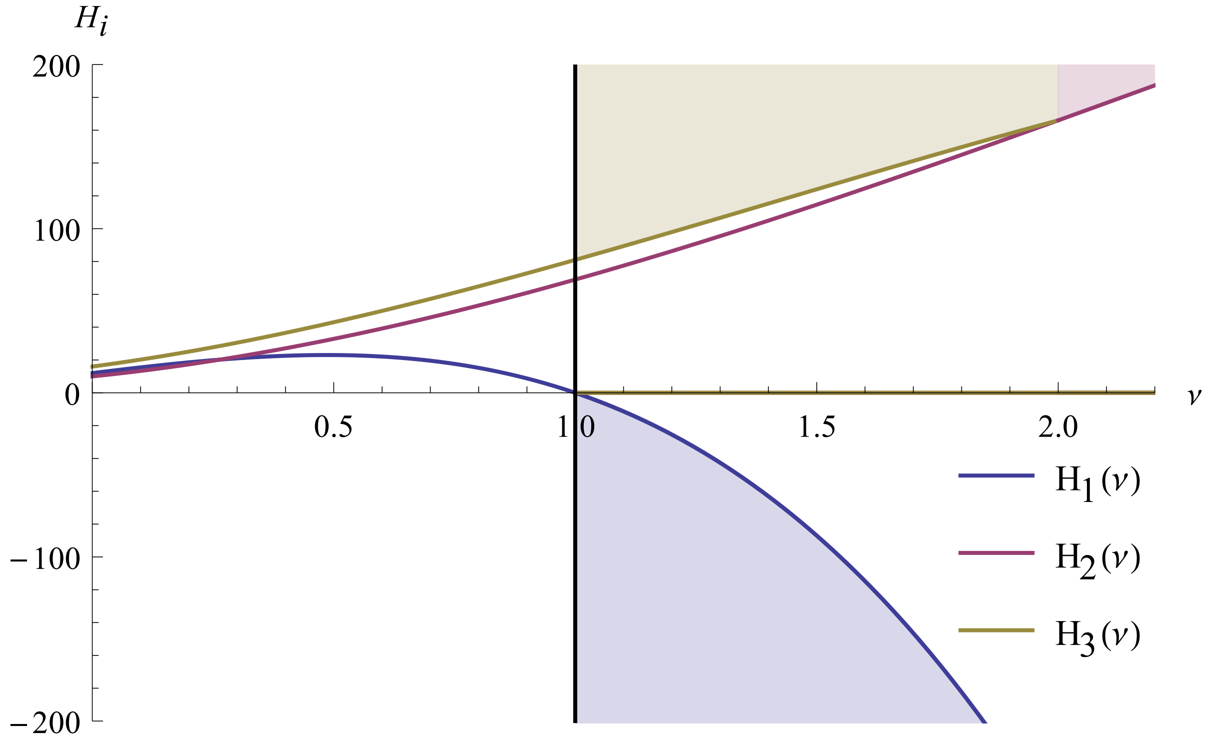

Figure 2B. The graphs of $H_1(\nu)$, $H_2(\nu)$, and $H_3(\nu)$.

Appendix 8: Proof of Proposition 5.

First, observe that

$$M\sigma_0 = \frac{(M+N)^3}{12} - N\delta_0. \tag{81}$$

Second, we also have

$$\frac{N^2}{12} \leq \delta_0 \leq \frac{N^2}{12} + \frac{M}{4}(M+N), \tag{82}$$

which is implied by definition (11) of δ_0 and the fact that a more concentrated (dispersed) population implies a lower (higher) value of δ_0.

Finally, set $\xi \equiv x^2 \in [0, (M+N)^2/4]$. Combining this with (20) and (21) implies that the *bid-rent differential* is given (up to a positive affine transformation) by:

$$\Delta(\xi) \equiv 2M\left(\xi + \frac{5\sigma_0 - \delta_0}{2} - \frac{\alpha - c}{2t}\right)^2 - N\left(\xi + \delta_0 - \frac{\alpha - c}{t}\right)^2.$$

5 districts, business downtown. For this configuration to emerge in equilibrium, the following inequalities must hold:

$$\Delta'(0) > 0 > \Delta'\left(\frac{(M+N)^2}{4}\right).$$

Since $\Delta(\xi)$ is a quadratic function, it is straightforward to check that this cannot be true when $2M > N$. This proves part (ii) of Proposition 5.

5 districts, residential downtown. This configuration emerges in equilibrium iff $\Delta(\xi)$ is U-shaped over $[0, (M+N)^2/4]$. Hence, the necessary and sufficient conditions for such an equilibrium to exist are given by the following system of inequalities:

$$\begin{cases} \Delta'(0) < 0 < \Delta'\left(\frac{(M+N)^2}{4}\right), \\ \frac{N^2}{12} < \delta_0 < \frac{N^2}{12} + \frac{M}{4}(M+N). \end{cases}$$

Using (81) – (82), this system of inequalities can be shown to hold iff *at least one* of the following chains of inequalities (or maybe both of them) is satisfied:

$$5(M+N)^3 < N^2(M+6N) + 12(M-N)\frac{\alpha - c}{t} < 5(M+N)^3 + 3(2M-N)(M+N)^2,$$

$$5(M+N)^3 < N^2(M+6N) + 3M(M+N)(M+6N) + 12(M-N)\frac{\alpha - c}{t} < 5(M+N)^3 + 3(2M-N)(M+N)^2.$$

Note that, when $M = N$, the second chain of inequalities amounts to

$$40N^3 < 49N^3 < 52N^3,$$

hence it trivially holds. Therefore, a 5-district equilibrium configuration exists for $M = N$ *regardless of the value of* $(\alpha - c)/t$. Moreover, a continuum of such equilibria exists. To see this, we check by direct computation for which symmetric distributions of population (82) holds under $M = N$. This yields part (i) of Proposition 5.□

Contact details:
Nikita Malykhin
National Research University Higher School of Economics (Moscow, Russia).
Center for Market Studies and Spatial Economics, research-assistant.
E-mail: nikmalykhin@gmail.com

Philip Ushchev
National Research University Higher School of Economics (Moscow, Russia).
Center for Market Studies and Spatial Economics, senior research fellow.
E-mail: fuschev@hse.ru