

On a scenario of onset of strongly dissipative mixed dynamics

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Abstract

In this paper we present the scenario of the occurrence of strongly dissipative mixed dynamics in two-dimensional reversible diffeomorphisms, using as an example the system describing a motion of two point vortices under the influence of wave perturbation and shear flow. For mixed dynamics of this type the chaotic attractor intersects with the chaotic repeller, but their intersection forms a “thin” set. The main stage of this scenario is the appearance of homoclinic structures for a symmetric saddle orbit which arise after crisis of a homoclinic attractor and repeller.

1 Introduction

On three types of chaos

Until recently, it was believed that chaos in dynamical systems can be only of two forms: conservative and dissipative chaos. Conservative (Hamiltonian) chaos is typical for systems preserving the phase volume, while, the dissipative chaos is often observed in systems in which the phase volume is compressed. It follows from the well-known theorem of Conley [1] that in any dynamical system defined on a compact manifold there exists an attractor and a repeller. There are many different definitions of an attractor. Here, as usual, by an attractor we will mean a stable closed invariant set of a system,

and be a repeller – an attractor under the time reversal. However, following Conley [1], Ruelle [2], and Hurley [3] under stability we will mean the so-called stability under permanent acting perturbations, which is also called *total stability* or *Lyapunov stability by ε -orbits*, see more details in [4].

We note, that such defined attractors (repellers) are in a good agreement with numerics and, what is important, allow to distinguish three types of chaos: conservative, dissipative and *mixed dynamics* [4]. So, if we denote the attractor in the system by \mathcal{A} and the repeller by \mathcal{R} then the condition $\mathcal{A} = \mathcal{R}$ holds for conservative chaos (moreover in this case, the attractor / repeller is the whole phase space), while the condition $\mathcal{A} \cap \mathcal{R} = \emptyset$ takes always place for dissipative chaos. The mixed dynamics is characterized by the fact, that the attractor and the repeller intersect, but do not coincide [4], i.e. $\mathcal{A} \cap \mathcal{R} \neq \emptyset$, and $\mathcal{A} \neq \mathcal{R}$. It is worth noticing that, from a logical point of view, this is the third and final possibility and, thus, a fourth type chaos does not exist.

For the first time the term “mixed dynamics” for this phenomenon was introduced in [5] (and later was used, e.g. in the papers, [6, 7]). However, the phenomenon associated with the possibility of intersecting attractors and repellers was discovered earlier in the paper [8] where it was proved that near any two-dimensional diffeomorphism with a non-transverse heteroclinic cycle containing 2 saddle points with Jacobians less and greater than one¹ there exist open regions (Newhouse domains) in which there are dense diffeomorphisms with infinitely many stable, completely unstable, and saddle periodic orbits whose closures have non-empty intersections, which, in fact, means that attractors are not separable from repellers. Later, in the paper [9], an analogous theorem was proved in the case when a diffeomorphism f with such a non-transverse heteroclinic cycle is *reversible*, i.e., when f and f^{-1} are conjugated by means of some involution h for which $h \circ h = id$. However, in contrast to the general case, *symmetric periodic orbits* of conservative type (with the Jacobian $J = 1$), such as elliptic and area-preserving saddles, also arise here in addition to periodic sinks and sources. Note, that such symmetric periodic orbits intersect the set $Fix(h)$ of fixed points of the involution h (i.e. the set of such points x that $h(x) = x$). Such orbits appear due to the *reversible Newhouse phenomenon* [9].

Mixed dynamics in reversible systems

¹Here, under non-transverse heteroclinic cycle we mean a contour containing 2 saddle periodic points for which one pair of stable and unstable invariant manifolds intersects transversally and the other pair have a heteroclinic tangency.

The phenomenon, when the dissipative dynamics in the system coexist with the conservative dynamics, was previously discovered in the papers [10, 11, 12], where it was found that the phase space for reversible two-dimensional diffeomorphisms can be divided into invariant domains with conservative dynamics² and invariant domains containing pairs of attractor–repeller. The possibility of intersecting an attractor and a repeller in physical systems was (as we know) first found numerically in [13], where for the model of four coupled rotators was observed that the attractor and the repeller “may overlap”. Later, in [14] such “overlapping” was explained by the emergence of the mixed dynamics in this model. Moreover, in [14] two different scenarios of the occurrence of mixed dynamics were also described. According to the first scenario, mixed dynamics appear in a “soft” manner, due to a sequence of local and global bifurcations of symmetry breaking. According to the second scenario, mixed dynamics can arise by explosion due to the emergence of heteroclinic structures, which appear after crisis of a simple attractor and a simple repeller.

Among the studies of mixed dynamics in systems from applications, here we would like to note the papers [15, 16, 17] in which mixed dynamics were found in nonholonomic models of the Celtic stone, rubber Chaplygin top, and Suslov top respectively.

We note that in almost all cases listed above, mixed dynamics appears in reversible systems, which are obtained as a result of small perturbation of conservative systems. As a result of such perturbations the conservativity is destroyed, in general. Thus, the phase portraits of attractors and repellers will be slightly different but, due to reversibility, they remains to be symmetric with respect to $Fix(h)$ (see e.g. Fig. 1, where the phase portraits of the attractor and the repeller in the nonholonomic model of Suslov top [17] are presented).

Strongly dissipative mixed dynamics

In this paper we present the scenario of the appearance of strongly dissipative mixed dynamics in reversible two-dimensional diffeomorphisms, using as an example the system describing the motions of two point vortices perturbed by a wave and a shear flow [19]. For mixed dynamics of this type the attractor and the repeller are very different from each other, and their

²Probably, inside such invariant domains there exist inseparable sets of stable and completely unstable periodic orbits. But the dynamics in such areas looks conservative due to the fact that the absolute values of Jacobians for such orbits are very close to 1.

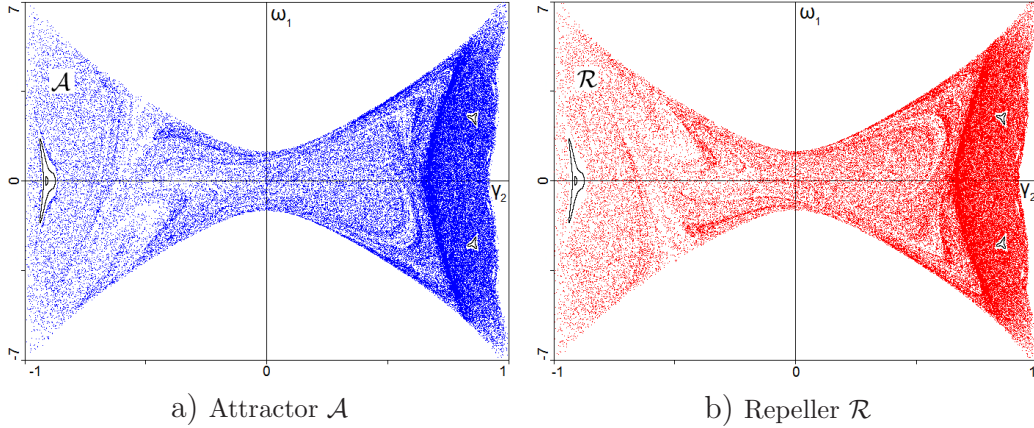


Figure 1: The phase portraits of the attractor \mathcal{A} and the repeller \mathcal{R} in the nonholonomic model of Suslov top [18]. The line $\omega_1 = 0$ forms the set of fixed points of the involution $h : \omega_1 \rightarrow -\omega_1$ in this model. It is clearly seen that $\mathcal{A} \cap \mathcal{R} \neq \emptyset, \mathcal{A} \neq \mathcal{R}$.

intersection forms a “thin” set (see. Fig. 2b). The main stage of this scenario is the appearance of homoclinic structures (for a symmetrical saddle orbit s_1) which arise after the crisis of a homoclinic attractor and repeller. Recall, that under the homoclinic attractor here we mean a chaotic attractor which contains a fixed point of a saddle type [23]. In the case under consideration, the attractor \mathcal{A} contains a saddle point with very small Jacobian J and the repeller \mathcal{R} – with very large Jacobian J^{-1} (for example, in the case presented in Fig. 2a, $J = 1/88$ for the attractor, while $J^{-1} = 88$ for the repeller). After the appearance of mixed dynamics, both the attractor and the repeller also contain a symmetrical saddle point s_1 with $J = 1$ (see. Fig. 2b).

We note that, in accordance with Theorem 1 from [4], the set belonging to the the intersection of an attractor and a repeller (also called the *reversible core*) is the limit of an infinite sequence of attractors, and also of an infinite sequence of repellers. Therefore, when the saddle point s_1 belongs to the reversible core, it can be argued that strongly dissipative mixed dynamics also contain countable sets of stable, as well as completely unstable periodic orbits.

However, in contrast to previously investigated cases ([15, 16, 17, 14]) where stable, as well as completely unstable periodic orbits, belonging to the regions with mixed dynamics, have Jacobians close to 1, strongly dissipative mixed dynamics contains periodic saddles, sinks and sources with various Jacobians (large, small, close, and equal to 1). In our opinion, the existence of

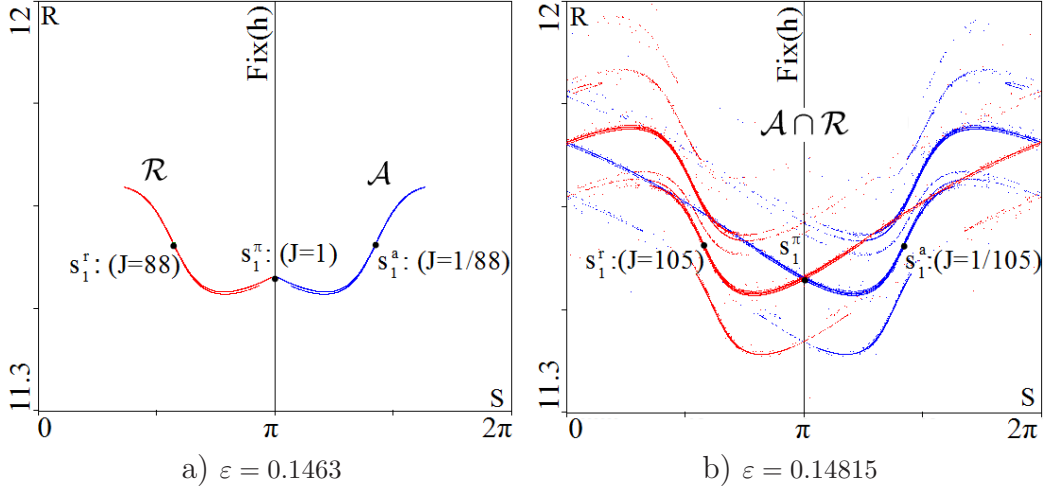


Figure 2: The phase portraits of the attractor \mathcal{A} and the repeller \mathcal{R} in the model of two point vortices perturbed by a wave and a shear flow. $\text{Fix}(h)$ – is the line of fixed points of the involution $h : \{S \rightarrow 2\pi - S, R \rightarrow R\}$, s_1 – is a symmetrical saddle point (with Jacobian $J = 1$). (a) $\mathcal{A} \cap \mathcal{R} = \emptyset$. Saddle point s_a with Jacobian $J = 1/88$ belongs to the attractor, while the saddle point s_r with $J = 88$ belongs to the repeller. (b) $\mathcal{A} \cap \mathcal{R} \neq \emptyset, \mathcal{A} \neq \mathcal{R}$. Mixed dynamics contain dissipative and also conservative periodic orbits.

such dissipative orbits is explained by the fact that the system strongly contracts volumes in some regions of phase space and strongly expands volumes in other (symmetrical) regions. We also suppose, that such a property of the system leads to a large difference between numerically obtained attractor and repeller after their intersection (when mixed dynamics appears)³.

The work is organized as follows: the scenario of the appearance of strongly dissipative mixed dynamics in two-dimensional reversible diffeomorphisms is described in Sec. 2, the example of the implementation of such a scenario in the equations describing the motions of two point vortices perturbed by a wave and a shear flow is given in Sec. 4, and the equations governing the motions in this model are presented in Sec. 3.

³We note, that in this case, theoretically, the attractor should be slightly different from the repeller due to appearance of “homoclinic tangle” [4]. But due to finiteness of the numerical calculation, the difference between attractor and repeller remains noticeable.

2 Scenario of the appearance of strongly dissipative mixed dynamics in reversible systems

Let us consider a one-parameter family of two-dimensional reversible diffeomorphisms $\bar{X} = F(X, \varepsilon)$ defined on a compact manifold and depending on the parameter ε . For simplicity we suppose that this family has a symmetric elliptic fixed point e for $\varepsilon < \varepsilon_0$ and this point (at $\varepsilon = \varepsilon_0$) undergoes reversible pitch-fork bifurcation [20] after which this point becomes a symmetric saddle point s and a pair of stable f^s and completely unstable f^u points (one point is symmetric to another) appears in a neighborhood of s (see Fig. 3, near $\varepsilon = \varepsilon_0$). We also suppose that, with further increase of ε , a *Feigenbaum-like attractor* AF [21] is born as a result of the cascade of period-doubling bifurcations of the point f^s . In the same way, a Feigenbaum-like repeller RF is born from f^u . We note, that after the first period doubling bifurcation the points f^s and f^u become saddle: we denote these fixed points as h_a and h_r .

Recall that immediately after the onset of chaotic dynamics through the cascade of period-doubling bifurcations, the Feigenbaum-like attractor consists of a set of components. With the change of the parameter the components of this attractor pairwise merge (as a result of the occurrence of heteroclinic intersections between manifolds of the saddle orbits belonging to the components of the attractor and manifolds of the saddle orbits lying between these components). Finally, two last components separated by h_a are merged and the *homoclinic Henon-like attractor* AH appears [22] (see Fig. 3a at $\varepsilon = \varepsilon_H$ and Fig. 3b). Symmetrically the homoclinic Henon-like repeller RH containing the fixed point h_r occurs.

With a further increase in ε , the Henon-like attractor AH becomes larger and approaches the boundary of its basin of attraction which is formed by the stable manifold W^s of the saddle point s (accordingly, the basin for the Henon-like repeller RH is bounded by the unstable manifold W^u of the same point s). Also we note that starting with a certain value of the parameter the pairs of manifolds W^s and W^u intersects transversally, i.e. formed the homoclinic structure. (see Fig. 3b).

When $\varepsilon = \varepsilon_{MD}$, the crisis of the attractor AH and the repeller RH appears (AH collides with the boundary of its basin of attraction W^s , while RH collides with W^u), after which both these sets capture the homoclinic

appears.

$$\begin{cases} \dot{R} = \frac{1}{2}AR \sin 2\varphi - \varepsilon \sin \varphi \sin S \sin(R \sin \varphi) \\ \dot{S} = -1 + \varepsilon \cos S \cos(R \sin \varphi) \\ \dot{\varphi} = \frac{\kappa}{R^2} + A \cos^2 \varphi - \frac{\varepsilon}{R} \cos \varphi \sin S \sin(R \sin \varphi). \end{cases} \quad (1)$$

Here $R \in (0, \infty)$, $S \in [0, 2\pi)$, $\varphi \in [0, 2\pi)$ are the phase variables and A, ε, κ are the parameters of the system. Note that this model is a generalization of the system describing the motions of two point vortices which interact with a potential wave [24, 25]⁶. In this system the parameter ε specifies the amplitude of a wave, A is the vorticity of the external flow, and κ is the sum of intensities of the vortices.

Note that the equations (1) are invariant with respect to the substitution

$$\begin{aligned} H_1 : \{R \rightarrow R, \quad S \rightarrow -S, \quad \varphi \rightarrow -\varphi, \quad t \rightarrow -t\}, \\ H_2 : \{R \rightarrow R, \quad S \rightarrow 2\pi - S, \quad \varphi \rightarrow 2\pi - \varphi, \quad t \rightarrow -t\}. \end{aligned} \quad (2)$$

Thus, the system under consideration is reversible and $H = H_1 \cup H_2$ is the involution of this system. Also we note, that the variable φ can be chosen as a secant in the system. In the case $\varphi = 0$ (or $\varphi = \pi$) the involution H defines the involution $h = h_1 \cup h_2$ for the Poincaré map of the system, where:

$$\begin{aligned} h_1 : \{R \rightarrow R, \quad S \rightarrow -S\} \\ h_2 : \{R \rightarrow R, \quad S \rightarrow 2\pi - S\}, \end{aligned} \quad (3)$$

and the set $Fix(h)$ of fixed points of this involution consists of two lines:

$$Fix(h) = \{S = 0\} \cup \{S = \pi\}.$$

Hereafter we choose the variable $\varphi = 0$ as a secant for the system and perform one-parameter analysis by varying ε , assuming that other parameters are fixed as follows:

$$A = 0.1, \kappa = 4.65.$$

⁶The author of the system is E.V. Vetchanin [19]. In contrast to the system investigated in [24, 25], here the vortices also perturbed by an external flow with constant vorticity A in addition to the interaction with a potential wave.

4 A description of a scenario of occurrence of strongly dissipative mixed dynamics in the vortex model under consideration

For $\varepsilon = 0$ the system (1) describing the motion of unperturbed vortices is integrable [26] and its phase space foliated into invariant tori. When $\varepsilon > 0$ some tori become resonance and the pair of symmetric saddle and elliptic orbits appear, see Fig. 4a. With a further increase in the parameter ε elliptic points e_i undergo symmetry breaking bifurcations due to which these points become symmetric saddles and in their neighborhood stable f_i^s and completely unstable f_i^u fixed points appear (see, Fig. 4b). Note, that for sufficiently large values of the parameter ε there are coexist 8 stable f_i^s (and also 8 completely unstable f_i^u) fixed points, see Fig. 4c.

With increasing of ε Henon-like attractors AH_i and Henon-like repellers RH_i are born from points f_i^s and f_i^u due to scenario presented in Sec. 2. Figure 4d shows the coexisting of two Henon-like attractors AH_1 and AH_2 with stable fixed (and periodic) points for $\varepsilon = 0.146$. In this Figure s_i^a and s_i^r – saddle point which appear after period doubling bifurcation of stable f_i^s and completely unstable points f_i^u , respectively. We note, that saddle points s_i^a have Jacobian less than 1, while saddle points s_i^r have Jacobian greater than one.

When $\varepsilon = \varepsilon_{cris1} \approx 0.14635$ the attractor AH_1 and the repeller RH_1 undergo crisis due to which these two sets begin to intersect. It is important to note, that the intersections of attractors and repellers in the system appear due to homoclinic bifurcations. Further we will describe such bifurcations in details, but first, let us recall two well-known facts related to an evolution of homoclinic attractors:

- in many cases the boundary of basins of attraction for homoclinic attractors is formed by stable manifolds of some saddle points;
- a homoclinic attractor is the closure of the unstable manifold of one of its saddle points.

Remark 1. *Both these properties hold for homoclinic Henon-like attractors. In particular, in the Henon map $\bar{x} = y, \bar{y} = M - bx - y^2$ with $|b| < 1$ the exact boundary of absorbing domain for the attractor is composed by one (or two, if $b < 0$) of stable manifolds of the saddle fixed point O_1 having*

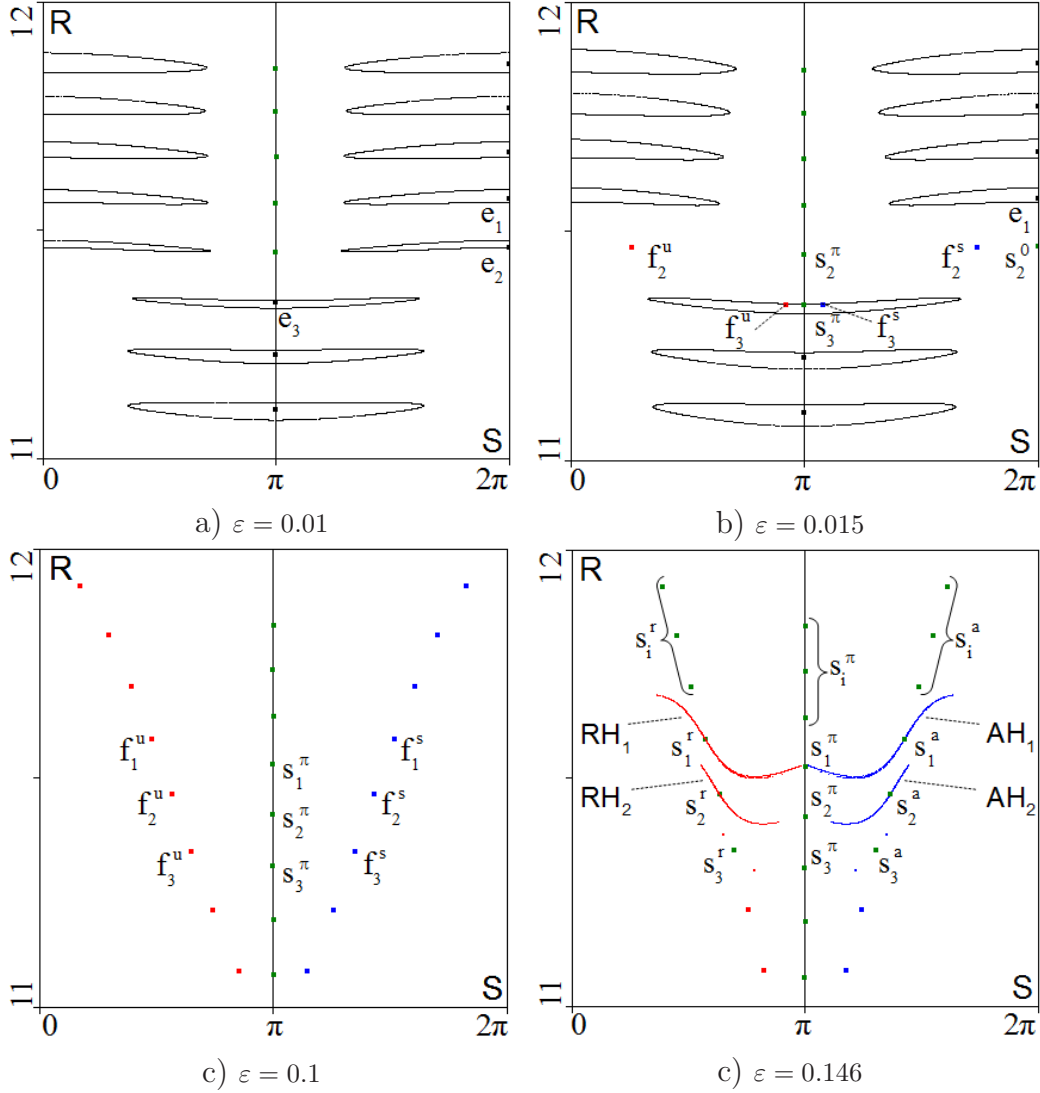


Figure 4: The phase spaces of the system (1) for different ε . (a) Resonant tori for small ε ; (b) elliptic points e_2 and e_3 undergo symmetry breaking bifurcations due to which symmetric saddles s_2^0 , s_3^π and also pairs (f_2^s, f_2^u) and (f_3^s, f_3^u) of stable and completely unstable points appear; (c) 8 stable f_i^s and 8 completely unstable f_i^u fixed points coexist in the system; (d) Henon-like attractors AH_1 and AH_2 (together with Henon-like repellers RH_1 and RH_2) born from stable points f_1^s , f_2^s (completely unstable points f_1^u , f_2^u).

a positive unstable multiplier, while the homoclinic attractor contains other

saddle fixed point O_2 with negative unstable multiplier. Here the homoclinic Henon-like attractor AH can be defined as a the prolongation of the point O_2 , what means that AH contains (or coincide with) the closure of the unstable manifold $W^U(O_2)$ ⁷.

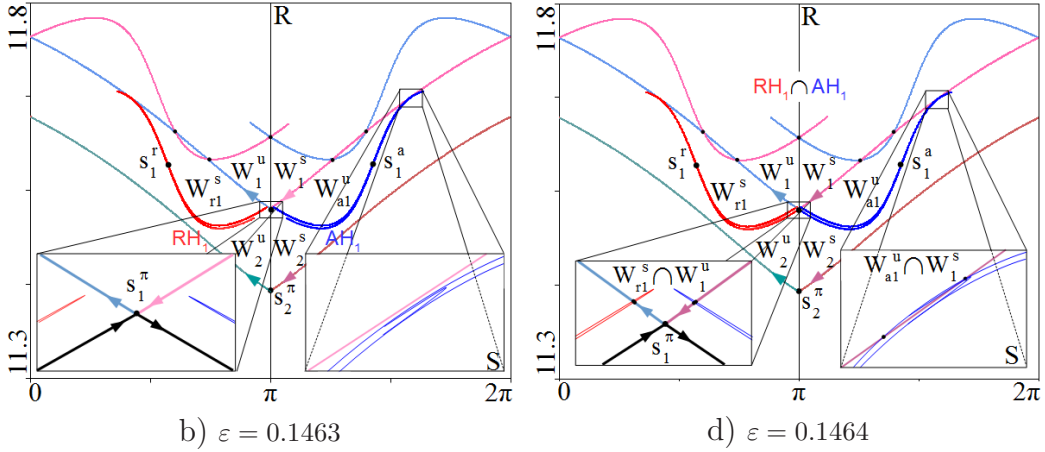


Figure 5: The location of manifolds of the saddle points s_1^π, s_2^π, s_1^a and s_1^r . W_{a1}^u is the unstable manifold of the saddle point s_1^a belonging to the attractor AH_1 ; W_1^s and W_2^s are the stable manifolds forming the boundary of the basin of attraction for this attractor. W_{r1}^s is the stable manifold of s_1^r belonging to the repeller RH_1 , while W_1^u and W_2^u are unstable manifolds forming the “basin of repulsion” for RH_1 .

In the case under consideration the attractor AH_1 belongs to the closure of the unstable manifold W_{a1}^u of the saddle fixed point s_1^a . The basin of attraction of AH_1 is bounded from above by the stable manifold W_1^s of the symmetric saddle point s_1^π , and from below – by the stable manifold W_2^s of the saddle point s_2^π (see Fig. 5a). We also note that the manifolds W_1^s and W_1^u intersect transversally.

When $\varepsilon > \varepsilon_{cris1}$, intersections between W_1^s and W_{a1}^u , as well as W_1^u and W_{r1}^s appear (see Fig. 5b when $\varepsilon = 0.1464$) and, as a result, the attractor AH_1 collides with the upper boundary of its basin of attraction while the repeller RH_1 collides with the upper boundary of its “basin of repulsion”. We note that after this collision the transversal intersection between W_1^s and

⁷Recall that the prolongation of a set Q is a closed, invariant stable set consisting of all points attainable from Q by ε^+ -orbits for arbitrarily small ε [4]. Here ε^+ -orbits are ε -orbits for forward iteration of the map; $\varepsilon > 0$.

W_1^u remains and thus we can state here that the attractor AH_1 intersects with the repeller RH_1 . But in this case such an intersection is not visible due to the existing of homoclinic Henon-like attractor AH_2 which attracts almost all trajectories from the neighborhood of AH_1 in forward time and RH_2 which attracts trajectories in backward time (here after the collision of AH_1 and RH_1 the unstable manifold W_{a1}^u also intersects with the stable manifold W_2^s of the symmetric saddle point s_2^π , see Fig. 6a, and this intersection gives a transition mechanism from AH_1 to AH_2).

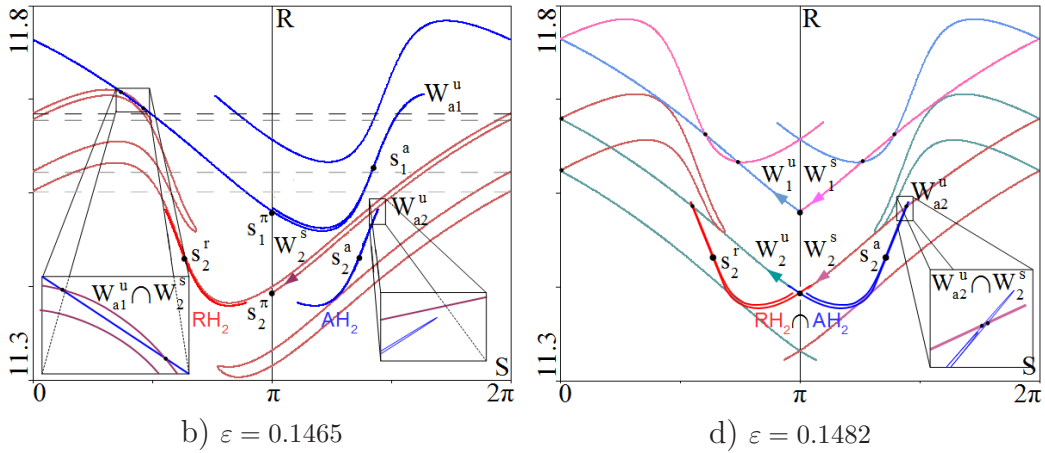


Figure 6: The location of manifold of the saddle points s_1^π, s_2^π, s_1^a and s_2^a . W_{a1}^u is the unstable manifold of the saddle point s_1^a belonging to the attractor AH_1 ; W_1^s and W_2^s are the stable manifolds forming the boundary of the basin of attraction for this attractor. W_{r1}^s is the stable manifold of s_1^π belonging to the repeller RH_1 , while W_1^u and W_2^u are unstable manifolds forming the “basin of repulsion” for RH_1 .

When $\varepsilon = \varepsilon_{cris2}$ the unstable manifold W_{a2}^u of s_2^a , belonging to the attractor AH_2 touches the stable manifold W_2^s which forms the boundary of the basin of attraction for AH_2 . The same non-transversal heteroclinic tangency appears between another pair of manifolds: W_{r2}^s and W_2^u , one of which compose the repeller RH_2 , while another forms the boundary of the basin for RH_2 . Thus, for $\varepsilon > \varepsilon_{cris2}$ the attractor AH_2 intersects with the repeller RH_2 , and moreover, due to homoclinic tangle, these two sets intersect with $AH_1 \cap RH_2$ and we have an intersection of two attractors (AH_1 and AH_2) with two repellers (RH_1 and RH_2).

But this intersection also is not visible due to existing of other Henon-like attractor AH_3 and Henon-like repeller RH_3 , which appear from f_3^s and f_3^u

and, in its turn, collide by the same way as AH_1 , RH_1 , AH_2 and RH_2 , giving more complex global connection between different homoclinic attractors and repellers, see Fig. 2b.

Such a joining of attractors and repellers ends at $\varepsilon > \varepsilon_{crisg} \approx 0.206$, when all 8 attractors AH_i and repellers RH_i undergo crisis and form a complex set consisting of the intersection of all these attractors and repellers. Since there are no other attractors in the neighborhood of this intersection, the mixed dynamics becomes visible⁸, see Fig. 7a.

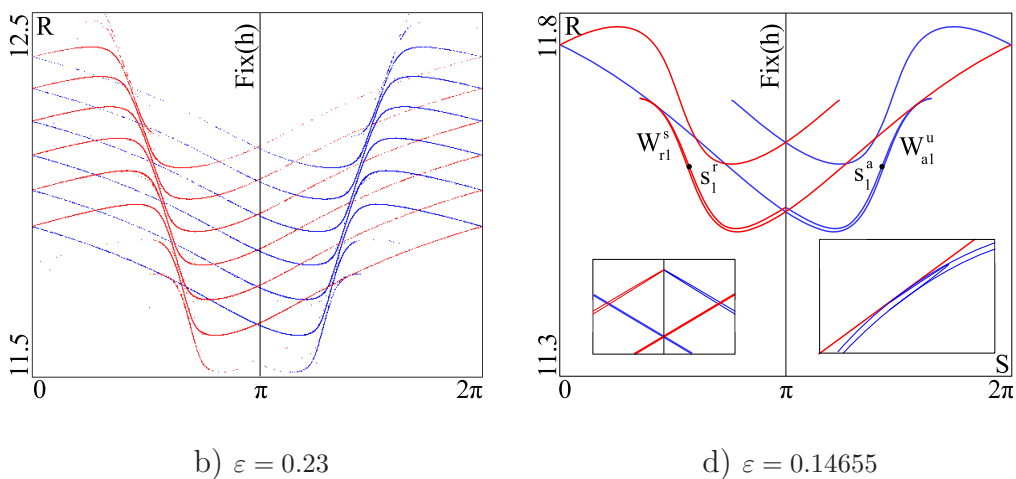


Figure 7: (a) Mixed dynamics after the collision of all homoclinic Henon-like attractors AH_i and all Henon-like repellers RH_i ; (b) non-transversal heteroclinic cycle containing a pair of non-conservative saddles s_1^a , s_1^r with $J < 1$ and $J > 1$, respectively.

Finally we note, that after each attractor-repeller collision heteroclinic cycles, containing a pair of non-conservative saddles s_i^a , s_i^r with $J < 1$ and $J > 1$, appears (see, for example, Fig. 7b, after the collision of AH_1 and RH_1 , when $\varepsilon = 0.14655$). In the accordance with theorems from [9, 7] the bifurcations of such cycles lead to a simultaneous birth of infinitely many periodic attractors, repellers and elliptic orbits. Moreover, the closures of

⁸It is possible, that mixed dynamics manifests itself before the collision of all 8 attractors and repellers. For example, Fig. 2b shows mixed dynamics which we obtain with 50000 forward and backward iterations after collision of 3 attractors AH_i , $i = 1, 2, 3$ and 3 repellers AH_i , $i = 1, 2, 3$. In this case it is difficult to understand either we have a coexisting of mixed dynamics with other (in particular, Henon-like) attractors and repeller or mixed dynamics here is transient chaos and finally trajectories evolve to other attractor.

these sets have non-empty intersection. From the other hand, at least one area-preserving saddle point s_i^π (or s_i^0) belongs to the intersection of AH_i and RH_i after their collision. As it is known from the paper [4], such points belong to a *reversible core*, which, by theorem 1 from this paper, contains a limit of an infinite sequence of attractors and repellers. So this theorem gives us another evidence of infiniteness and inseparability of attractors and repellers in the system.

5 Conclusion

In this paper, a phenomenological scenario of the emergence of mixed dynamics of a new, strongly dissipative, type is proposed for two-dimensional reversible maps. For this type of mixed dynamics the numerically obtained chaotic attractor and the chaotic repeller are very different from each other, and their intersection forms a “thin” set. The implementation of such a scenario is demonstrated on a system describing the motion of two point vortices perturbed by a wave and external shear flow. It is shown that mixed dynamics in this system contain both conservative and dissipative periodic orbits, while closures of orbits of different types have non-empty intersection. Moreover, in contrast to previously observed types of mixed dynamics, the Jacobians of dissipative orbits in this case can be essentially differ from 1. Thus, it can be argued that strongly dissipative mixed dynamics obviously can not be regarded as quasi-conservative chaos, but represents a new, third (along with conservative and dissipative) type of chaos. Moreover, we assume that this type of chaotic behavior is typical for strongly dissipative reversible systems.

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