GUARANTEED APPROACH FOR DETERMINING THE OPTIMAL DESIGN OF ACCELEROMETER UNIT CALIBRATION

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Abstract. The calibration problem is considered for the accelerometer unit at a high-precision test bench. Besides instrumental errors of the accelerometer unit itself, possible faults of the test bench (which are accumulated during its operation) are taken into account. One of the main problems is to choose the optimal design of the angular unit positions. The guaranteed approach is proposed to determine this optimal design.

1 Introduction

It is well known that the accelerometer unit of a strapdown inertial navigation system should be calibrated on a test bench [2], [3], [4], [5], [6], [7], [8], [10], [13], [14], [15], [16].

The unit errors are determined by the inaccuracies of scale coefficients, the misalignments of sensor axes and sensor biases. However, besides the errors mentioned, the high-precision test bench during the long-term operation can have the faults caused by the misalignments of its own test bench axes and the biases in the measurement of the angles of the test bench rotation about its own axes. Then, aside from the calibration problem itself, one has to solve the test bench functional diagnostic problem. This circumstance makes the state vector to be high-dimensional. An example of such test bench is the motion simulator produced by "Acutronic" [9].

When dimension is high the engineering intuition may fail. Therefore it is necessary to develop a mathematical problem statement for the calibration of the nominally high-precision test benches taking into account the peculiarities mentioned above. In the work, it is proposed to set and solve this mathematical problem basing on the guaranteed approach. This approach allows us to design the optimal angular positions of the high-precision test bench in the most simple way, to obtain the optimal calibration algorithms, and to calculate the maximum attainable accuracy for the desired parameters estimation.

The main result of the paper is the optimal design of experiments. By the estimates of the test bench parameters one can conclude on whether to carry out expensive maintenance work.

2 Problem statement

Consider a test bench with two degrees of freedom, which is consisted of two frames (gimble rings): outer an inner ones.

2.1 Ideal test bench scheme

Ideally, the base of the test bench is set exactly in the horizon. The outer axis is also located in the horizontal plane and has a known asimuthal orientation. With zero rotation angle i of the outer frame relative to the base, the inner axis coincides with the geographical vertical. The outer and inner axes intersect at a point M^b and are orthogonal. A reference system $M^b i$ is bind up to outer frame so that the first axis $M^b i_1$ coincides with the outer rotation axis, the third axis $M^b i_3$ coincides with the inner rotation axis $M^b j_3$, and the second axis $M^b i_2$ makes with the first and the third axes a right trihedron.

A right trihedron $M^b j$ is bound to the inner frame so that, under zero rotation angle of the inner frame relative to the outer frame j, this trihedron coincides with the trihedron $M^b i$ (and with an arbitrary j it has a common axis $M^b i_3 = M^b j_3$ with $M^b i$). A so-called faceplate is placed on the inner frame and the strapdown inertial navigation system with accelerometer unit is mounted on the faceplate so that the instrumental axes of the unit coincide with the axes of the trihedron $M^b j$. Setting the test bench in different positions it is required to determine the accelerometer unit parameters by the measurements of the angles (i, j) and the signals of the accelerometers.

2.2 The scheme of the test bench with errors

Assume that the base of the test bench is set not exactly in the horizon (e.g., due to a subsidence of foundation) In particular, the outer axis is not strictly horizontal. Assume also that the outer and the inner axes intersect at the point M^b but are not strictly orthogonal. Introduce a vertical plane $M^b V$ that passes through geographical vertical $M^b Z$ at the point M^b and the outer rotation axis of the test bench. Introduce also a normal vector $M^b n$ to the vertical plane $M^b V$ (which lies in the horizontal plane) that forms with $M^b Z$ and the outer axis a right (non-orthogonal) trihedron.

Suppose that, because of the non-ideality of the test bench base installation, the angle of the azimuthal orientation A of the vertical half-plane of the plane $M^b V$ containing the outer axis relative to the geographical trihedron (this angle is measured clockwise from the North direction when viewed from above) is known with small error ΔA. Besides, due to the non-ideality of the test bench installation, the outer axis is deflected from the horizontal plane by a small angle $\delta_2^{i_0}$ measured in the positive direction (counter-clockwise) around the normal $M^b n$.

Bind up the reference system $M^b i$ with the outer frame so that the first axis $M^b i_1$ coincides with the outer rotation axis, the third axis $M^b i_3$ lies in the plane formed by outer and inner axes (the third axis almost coincides with the inner axis), and the second axis $M^b i_2$ makes a right orthogonal trihedron with $M^b i_1$ and $M^b i_3$. Due to non-ideality of the test bench base installation in the horizon, the axis $M^b i_3$ coincides with the plane $M^b V$ with unknown small angle of the outer frame rotation relative to the base $i^* \neq 0$. The rotation angle of the outer frame (the trihedron $M^b i$) with respect to the test bench base i is measured with a constant error Δi.

Bind up the reference system $M^b j$ with the inner frame as follows. The axis $M^b j_3$ coincides with the inner rotation axis. In the plane formed by the outer and inner axes (or, what is the same, in the plane formed by the axes $M^b i_1$, $M^b i_3$), let us turn the trihedron $M^b i$ around the axis $M^b i_2$ in positive direction by a small angle $\delta_2^{j_0}$ so that the axis $M^b i_3$ coincides with the inner rotation axis $M^b j_3$. Thus the angle of the non-orthogonality of the outer and inner axes is determined by the small angle $delta_2^{j_0}$. Then with zero rotation angle of the inner frame relative to the outer frame $j = 0$ a new position of the axis $M^b i_1$ will define the axis $M^b j_1$, which we bind up with the inner frame. The axis $M^b j_2$ makes with the first and the third axes a right orthogonal trihedron. With arbitrary value of the angle j the trihedron $M^b j$ transits to a new position together with the inner frame. The rotation angle of the inner frame (the trihedron $M^b j$) relative to the outer frame j is measured with a constant error Δj.

The right orthogonal trihedron that is bind up with the faceplate (which is set on the inner frame) is rotated relative to the $M^b j$ by a small angle around an unknown axis. In its turn, the instrumental trihedron $M^b z$ (along which axes the accelerometer sensor axes must be located ideally) is rotated around the faceplate axes by another one small angle around another one unknown axis. As aresult the orientation of the instrumental trihedron $M^b z$ relative to the trihedron $M^b j$ is characterized by a small vector angle $(\delta_1^{zj}, \delta_2^{zj}, \delta_3^{zj})^T$.

Introduce also the following notation: 1) Γ_{11}, Γ_{22}, Γ_{33} – the errors of accelerometer scale coefficients; 2) Γ_{12}, Γ_{13}, Γ_{21}, Γ_{23}, Γ_{31}, Γ_{32} – the accelerometer misalignments; 3) $\Delta f_{z_1}^0$, $\Delta f_{z_2}^0$, $\Delta f_{z_3}^0$ – the constant accelerometer biases; 4) g' – nominal gravity force acceleration; 5) Δg – an error in the knowledge of gravity force acceleration.

Setting the test bench in different positions it is required to determine the accelerometer unit parameters and the constant test bench errors by the measurements of the angles (i, j) and the accelerometer signals.

After cumbersome calculation one can get the dependence of the accelerometer unit readings on the frame rotation angles. Then the calibration problem can be represented in the form that is traditional in estimation theory. Namely, in the issue we will assume that there are three groups of measurements:

$$
\begin{aligned}
\overset{(1)}{z}(i,j) &= \overset{(1)}{H}{}^T(i,j)\, q + \overset{(1)}{\varrho}(i,j), & \overset{(1)}{z}(i,j) &= \frac{a'_{z_1}(i,j)}{g'}, \\
\overset{(2)}{z}(i,j) &= \overset{(2)}{H}{}^T(i,j)\, q + \overset{(2)}{\varrho}(i,j), & \overset{(2)}{z}(i,j) &= \frac{a'_{z_2}(i,j)}{g'}, \\
\overset{(3)}{z}(i,j) &= \overset{(3)}{H}{}^T(i,j)\, q + \overset{(3)}{\varrho}(i,j), & \overset{(3)}{z}(i,j) &= \frac{a'_{z_3}(i,j)}{g'}.
\end{aligned}
\qquad (1)
$$

Here $\{i, j\}$ are angle parameters, which characterize the angular positions of the test bench and take the value from $[0, 2\pi]$; $\{\overset{(1)}{z}(i,j),\ \overset{(2)}{z}(i,j),\ \overset{(3)}{z}(i,j)\}$ are normalized readings of the three accelerometers $\{a'_{z_1}(i,j),\ a'_{z_2}(i,j),\ a'_{z_3}(i,j)\}$; $\{\overset{(1)}{H}(i,j),\ \overset{(2)}{H}(i,j),\ \overset{(3)}{H}(i,j)\}$ are known vectors from $\mathbb{R}^m$ $(m = 15)$; $q \in \mathbb{R}^m$ is the vector of unknown parameters, which is consisted of the accelerometer unit errors; $\{\overset{(1)}{\varrho}(i,j)$, $\overset{(2)}{\varrho}(i,j)$, $\overset{(3)}{\varrho}(i,j)\}$ are unknown measurement (dimensionless) errors.

The estimated parameters have the form:

$$
\begin{array}{lll}
q_1 = \delta_2^{i_0}, & q_2 = \Delta i + i^*, & q_3 = \delta_2^{j_0}, \\
q_4 = \Gamma_{13} - \delta_2^{zj}, & q_5 = \Gamma_{11} - \frac{\Delta g}{g'}, & q_6 = \Gamma_{12} - \Delta j + \delta_3^{zj}, \\
q_7 = \frac{\Delta f_{z1}^0}{g'}, & q_8 = \Gamma_{23} + \delta_1^{zj}, & q_9 = \Gamma_{21} + \Delta j - \delta_3^{zj}, \\
q_{10} = \Gamma_{22} - \frac{\Delta g}{g'}, & q_{11} = \frac{\Delta f_{z2}^0}{g'}, & q_{12} = \Gamma_{31} + \delta_2^{zj}, \\
q_{13} = \Gamma_{32} - \delta_1^{zj}, & q_{14} = \Gamma_{33} - \frac{\Delta g}{g'}, & q_{15} = \frac{\Delta f_{z3}^0}{g'}.
\end{array}
\tag{2}
$$

The vectors $\overset{(p)}{H}(i,j) \in \mathbb{R}^m$ $(m = 15)$ are defined by the expressions:

$$
\overset{(1)}{H}(i,j) = \begin{pmatrix} -\cos j \\ -\cos i \sin j \\ -\cos i \cos j \\ \cos i \\ \sin i \sin j \\ \sin i \cos j \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad
\overset{(2)}{H}(i,j) = \begin{pmatrix} \sin j \\ -\cos i \cos j \\ \cos i \sin j \\ 0 \\ 0 \\ 0 \\ 0 \\ \cos i \\ \sin i \sin j \\ \sin i \cos j \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad
\overset{(3)}{H}(i,j) = \begin{pmatrix} 0 \\ \sin i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sin i \sin j \\ \sin i \cos j \\ \cos i \\ 1 \end{pmatrix}.
\tag{3}
$$

Thus the formulas (1) describe the continuum of every possible readings of the accelerometer unit at all its conceivable positions. The problem is to determine all entries of the vector parameters q by means of the measurements $\overset{(p)}{z}(i,j)$, $p = 1, 2, 3$.

3 Guaranteed estimation method

In accordance with the guaranteed approach to estimation [11, 1] we assume that the measurements errors are bounded in absolute value by a parameter σ:

$$
|\overset{(1)}{\varrho}(i,j)| \le \sigma, \qquad |\overset{(2)}{\varrho}(i,j)| \le \sigma, \qquad |\overset{(3)}{\varrho}(i,j)| \le \sigma, \qquad i,j \in [0, 2\pi].
$$

Consider linear estimators for $l = a^T q$ of the form

$$\hat{l} = \sum_{p=1}^{3} \int \overset{(p)}{\Phi}(i,j) \overset{(p)}{z}(i,j)\, di\, dj \tag{4}$$

where $\overset{(p)}{\Phi}(i,j)$ are weight functions. For the estimate of the ν-th entry of the unknown parameter vector q the vector $a = e^{(\nu)}$, where $e^{(\nu)} \in \mathbb{R}^m$ consists of zeroes except for the unity at the ν-th place.

The quantity

$$\sup_{q \in \mathbb{R}^m,\, |\overset{(p)}{\varrho}(i,j)| \leq \sigma,\, p=1,2,3} |\hat{l} - l|$$

is called the guaranteed estimation error. With a chosen estimator this is a maximal value of the estimation error under all possible values of uncertain factors. Let us search the weight coefficients $\overset{(p)}{\Phi}(i,j)$ that minimize the guaranteed estimation error, i.e., from the solution of the following minmax problem:

$$\inf_{\overset{(p)}{\Phi}(i,j),\, p=1,2,3} \ \sup_{q \in \mathbb{R}^m,\, |\overset{(p)}{\varrho}(i,j)| \leq \sigma,\, p=1,2,3} |\hat{l} - l|.$$

This problem is called the optimal guaranteed estimation problem.

It can be shown that this problem reduces to the following variational problem of the form:

$$\inf_{\overset{(p)}{\Phi}(i,j),\, p=1,2,3} \sigma \sum_{p=1}^{3} \int |\overset{(p)}{\Phi}(i,j)|\, di\, dj \tag{5}$$

subject to the constraints (they are called unbiasedness conditions)

$$\sum_{p=1}^{3} \int \overset{(p)}{H}(i,j) \overset{(p)}{\Phi}(i,j)\, di\, dj = a. \tag{6}$$

If the vectors $\overset{(p)}{H}(i,j) \in \mathbb{R}^m$ are continuous, then the solution of (5), (6) exists and can be represented by the impulse function with m pulses (Dirac delta-functions) [1],[12]. In this case, the value $\sigma \sum_{p=1}^{3} \int |\overset{(p)}{\Phi}{}^0(i,j)|\, di\, dj$, where $\overset{(p)}{\Phi}{}^0(i,j)$ are the optimal coefficients, define the optimal guaranteed estimation error of the parameter l.

Note also that the involving of nonlinear estimators instead of (4) will not reduce the guaranteed estimation error [12].

4 Analytical solution of the calibration problem

Using the features of the calibration problem we will find a solution in analytical form. In order to do it, first note that if we have to estimate a parameter q_ν, then the corresponding entry of the vector a (with the number ν) in the unbiasedness condition

6) is equal to 1 and the other entries of the a are zero. Consider the ν-th row in the unbiasedness condition. Since the absolute values of all entries of the vectors $\overset{(p)}{H}(i,j)$ are less or equal to unity, it is obvious that for any estimator (subjected to the unbiasedness condition) the following relations hold: (for any $\nu = 1, \ldots, m$):

$$\sigma = \left| \sigma \sum_{p=1}^{3} \int \overset{(p)}{H}_{\nu}(i,j) \overset{(p)}{\Phi}(i,j)\, di\, dj \right| \leq \sigma \sum_{p=1}^{3} \int | \overset{(p)}{H}_{\nu}(i,j) \overset{(p)}{\Phi}(i,j) |\, di\, dj \tag{7}$$

$$\leq \sigma \max_{p} \max_{(i,j)} | \overset{(p)}{H}_{\nu}(i,j) | \cdot \sum_{p=1}^{3} \int | \overset{(p)}{\Phi}(i,j) |\, di\, dj = \sigma \sum_{p=1}^{3} \int | \overset{(p)}{\Phi}(i,j) |\, di\, dj.$$

This means that the optimal guaranteed estimation error for any parameter q_ν is not less than σ. So, if we can find an estimate with the guaranteed accuracy σ, then it is optimal one.

Investigate the formulas (3). As candidates for the optimal values of i, j we can take the angles that maximize the absolute value of the corresponding coefficient at the estimated parameter. Then it is easy to determine the following estimators for q_ν (here $\delta((i,j) - (i^0, j^0))$ is the Dirac delta-function centered at (i^0, j^0)):

$$\overset{(1)}{\Phi}_1(i,j) = \frac{1}{2}\left[\delta((i,j)-(-\pi/2,\,\pi))-\delta((i,j)-(\pi/2,\,0))\right], \qquad \overset{(2)}{\Phi}_1(i,j)=\overset{(3)}{\Phi}_1(i,j)=0,$$

$$\overset{(3)}{\Phi}_2(i,j) = \frac{1}{2}\left[\delta((i,j)-(\pi/2,\,0))-\delta((i,j)-(-\pi/2,\,\pi))\right], \qquad \overset{(1)}{\Phi}_2(i,j)=\overset{(2)}{\Phi}_2(i,j)=0,$$

$$\overset{(1)}{\Phi}_3(i,j) = \frac{1}{4}\left[-\delta((i,j)-(0,\,0))+\delta((i,j)-(0,\,\pi))+\right.$$

$$\left.+\,\delta((i,j)-(\pi,\,0))-\delta((i,j)-(\pi,\,\pi))\right], \qquad \overset{(2)}{\Phi}_3(i,j)=\overset{(3)}{\Phi}_3(i,j)=0,$$

$$\overset{(1)}{\Phi}_4(i,j) = \frac{1}{2}\left[\delta((i,j)-(0,\,\pi/2))-\delta((i,j)-(\pi,\,-\pi/2))\right], \qquad \overset{(2)}{\Phi}_4(i,j)=\overset{(3)}{\Phi}_4(i,j)=0,$$

$$\overset{(1)}{\Phi}_5(i,j) = \frac{1}{2}\left[\delta((i,j)-(\pi/2,\,\pi/2))-\delta((i,j)-(\pi/2,\,-\pi/2))\right], \quad \overset{(2)}{\Phi}_5(i,j)=\overset{(3)}{\Phi}_5(i,j)=0,$$

$$\overset{(1)}{\Phi}_6(i,j) = \frac{1}{2}\left[\delta((i,j)-(\pi/2,\,0))-\delta((i,j)-(-\pi/2,\,0))\right], \qquad \overset{(2)}{\Phi}_6(i,j)=\overset{(3)}{\Phi}_6(i,j)=0,$$

$$\overset{(1)}{\Phi}_7(i,j) = \frac{1}{2}\left[\delta((i,j)-(\pi/2,\,\pi/2))+\delta((i,j)-(\pi/2,\,-\pi/2))\right], \quad \overset{(2)}{\Phi}_7(i,j)=\overset{(3)}{\Phi}_7(i,j)=0,$$

$$\overset{(2)}{\Phi}_8(i,j) = \frac{1}{4}\left[\delta((i,j)-(0,\,\pi/2))-\delta((i,j)-(\pi,\,-\pi/2))+\right.$$

$$\left.+\,\delta((i,j)-(0,\,-\pi/2))-\delta((i,j)-(\pi,\,\pi/2))\right], \qquad \overset{(1)}{\Phi}_8(i,j)=\overset{(3)}{\Phi}_8(i,j)=0,$$

$$\overset{(2)}{\Phi}_9(i,j) = \frac{1}{2}\left[\delta((i,j)-(\pi/2,\,\pi/2))-\delta((i,j)-(-\pi/2,\,\pi/2))\right], \quad \overset{(1)}{\Phi}_9(i,j)=\overset{(3)}{\Phi}_9(i,j)=0,$$

$$\overset{(2)}{\Phi}_{10}(i,j) = \frac{1}{2}\left[\delta((i,j)-(\pi/2,\,0))-\delta((i,j)-(-\pi/2,\,0))\right], \qquad \overset{(1)}{\Phi}_{10}(i,j)=\overset{(3)}{\Phi}_{10}(i,j)=0,$$

$$\overset{(2)}{\Phi}_{11}(i,j) = \frac{1}{2}\left[\delta((i,j)-(\pi/2,\,0))+\delta((i,j)-(-\pi/2,\,0))\right], \qquad \overset{(1)}{\Phi}_{11}(i,j)=\overset{(3)}{\Phi}_{11}(i,j)=0,$$

$$\overset{(3)}{\Phi}_{12}(i,j) = \frac{1}{2}\left[\delta((i,j)-(\pi/2,\,\pi/2))-\delta((i,j)-(\pi/2,\,-\pi/2))\right], \quad \overset{(1)}{\Phi}_{12}(i,j)=\overset{(2)}{\Phi}_{12}(i,j)=0,$$

$$\overset{(3)}{\Phi}_{13}(i,j) = \frac{1}{2}\left[\delta((i,j)-(\pi/2,\,0))-\delta((i,j)-(\pi/2,\,\pi))\right], \qquad \overset{(1)}{\Phi}_{13}(i,j)=\overset{(2)}{\Phi}_{13}(i,j)=0,$$

$$\overset{(3)}{\Phi}_{14}(i,j) = \frac{1}{2}\left[\delta((i,j)-(0,\,0))-\delta((i,j)-(\pi,\,0))\right], \qquad \overset{(1)}{\Phi}_{14}(i,j)=\overset{2)}{\Phi}_{14}(i,j)=0,$$

$$\overset{(3)}{\Phi}_{15}(i,j) = \frac{1}{2}\left[\delta((i,j)-(0,\,0))+\delta((i,j)-(\pi,\,0))\right], \qquad \overset{(1)}{\Phi}_{15}(i,j)=\overset{(2)}{\Phi}_{15}(i,j)=0.$$

Clearly, the guaranteed estimation errors for all of these estimators are equal to σ. It was shown above that these guaranteed values are not less than σ. Consequently, the estimators mentioned above are optimal.

Thus applying the guaranteed approach to the estimation of desired parameters we obtained analytically the following formulas for the optimal guaranteed estimates and corresponding optimal design (the first argument in the measurements suggests the value of the angle i, and the second one suggests the value of the angle j):

$$
\begin{aligned}
\hat{q}_1 &= \frac{1}{2}\left[\overset{(1)}{z}(-\pi/2,\,\pi) - \overset{(1)}{z}(\pi/2,\,0)\right], \qquad (8)\\
\hat{q}_2 &= \frac{1}{2}\left[\overset{(3)}{z}(\pi/2,\,0) - \overset{(3)}{z}(-\pi/2,\,\pi)\right],\\
\hat{q}_3 &= \frac{1}{4}\left[-\overset{(1)}{z}(0,\,0) + \overset{(1)}{z}(0,\,\pi) + \overset{(1)}{z}(\pi,\,0) - \overset{(1)}{z}(\pi,\,\pi)\right],\\
\hat{q}_4 &= \frac{1}{2}\left[\overset{(1)}{z}(0,\,\pi/2) - \overset{(1)}{z}(\pi,\,-\pi/2)\right],\\
\hat{q}_5 &= \frac{1}{2}\left[\overset{(1)}{z}(\pi/2,\,\pi/2) - \overset{(1)}{z}(\pi/2,\,-\pi/2)\right],\\
\hat{q}_6 &= \frac{1}{2}\left[\overset{(1)}{z}(\pi/2,\,0) - \overset{(1)}{z}(-\pi/2,\,0)\right],\\
\hat{q}_7 &= \frac{1}{2}\left[\overset{(1)}{z}(\pi/2,\,\pi/2) + \overset{(1)}{z}(\pi/2,\,-\pi/2)\right],\\
\hat{q}_8 &= \frac{1}{4}\left[\overset{(2)}{z}(0,\,\pi/2) - \overset{(2)}{z}(\pi,\,-\pi/2) + \overset{(2)}{z}(0,\,-\pi/2) - \overset{(2)}{z}(\pi,\,\pi/2)\right],\\
\hat{q}_9 &= \frac{1}{2}\left[\overset{(2)}{z}(\pi/2,\,\pi/2) - \overset{(2)}{z}(-\pi/2,\,\pi/2)\right],\\
\hat{q}_{10} &= \frac{1}{2}\left[\overset{(2)}{z}(\pi/2,\,0) - \overset{(2)}{z}(-\pi/2,\,0)\right],\\
\hat{q}_{11} &= \frac{1}{2}\left[\overset{(2)}{z}(\pi/2,\,0) + \overset{(2)}{z}(-\pi/2,\,0)\right],\\
\hat{q}_{12} &= \frac{1}{2}\left[\overset{(3)}{z}(\pi/2,\,\pi/2) - \overset{(3)}{z}(\pi/2,\,-\pi/2)\right],\\
\hat{q}_{13} &= \frac{1}{2}\left[\overset{(3)}{z}(\pi/2,\,0) - \overset{(3)}{z}(\pi/2,\,\pi)\right],\\
\hat{q}_{14} &= \frac{1}{2}\left[\overset{(3)}{z}(0,\,0) - \overset{(3)}{z}(\pi,\,0)\right],\\
\hat{q}_{15} &= \frac{1}{2}\left[\overset{(3)}{z}(0,\,0) + \overset{(3)}{z}(\pi,\,0)\right].
\end{aligned}
$$

In these formulas for the optimal guaranteed estimates, the angles in parentheses defines the optimal calibration design, i.e., they characterize the optimal positions for the test bench. The total angular positions are as follows (their amount is equal to m=15):

$$(\pm\tfrac{\pi}{2},\pi),\ (\pm\tfrac{\pi}{2},0),\ (0,0),\ (0,\pi),\ (\pi,0),\ (\pi,\pi),\ (0,\pm\tfrac{\pi}{2}),\ (\pi,\pm\tfrac{\pi}{2}),\ (\pm\tfrac{\pi}{2},\tfrac{\pi}{2}),\ (\tfrac{\pi}{2},-\tfrac{\pi}{2}).$$

4.1 Estimates for misalignments

The estimates for misalignments of accelerometers are of interest. Up to terms of the second order of smallness, they are defined by the following formulas [1]:

$$\Gamma_{12}+\Gamma_{21}=q_6+q_9, \qquad \Gamma_{13}+\Gamma_{31}=q_4+q_{12} \qquad \Gamma_{23}+\Gamma_{32}=q_8+q_{13}.$$

One can show that the corresponding optimal (guaranteed) estimates have the form:

$$\widehat{(q_6+q_9)}=\hat q_6+\hat q_9, \qquad \widehat{(q_4+q_{12})}=\hat q_4+\hat q_{12} \qquad \text{and} \qquad \widehat{(q_8+q_{13})}=\hat q_8+\hat q_{13}. \quad (9)$$

Note that the relations (9) for the guaranteed approach in general case are not valid (in contrast to the least squares method). However, for the problem under consideration they are do valid albeit this is not so obvious. Let us prove this assertion.

Really, the vector a in the unbiasedness condition (6) with two units in the 6-th and 9-th place (and remaining zeros) corresponds to the estimate of q_6+q_9. Consider two these rows in the unbiasedness condition (with numbers 6 and 9). Summing and subtracting them (with preserving of other rows in (6)) we get the restrictions that are equivalent to the initial unbiasedness condiiton (6). By virtue of the structure of the vectors $\overset{(p)}{H}(i,j)$, which are defined by the formulas (3), when added we get a vector with the entries that all are less or equal to unity (in absolute value). This is because that in the corresponding places one of two add-in elements is zero. Therefore, similarly to (7), we get

$$2\sigma \le \sigma \sum_{p=1}^{3}\int |\overset{(p)}{\Phi}(i,j)|\,di\,dj.$$

Hence the optimal guaranteed estimation error for q_6+q_9 is not less than 2σ. If we show that the guaranteed estimation error of $\hat q_6+\hat q_9$ is not greater than 2σ, then this is an optimal guaranteed estimate for q_6+q_9.

Indeed, let the optimal estimates q_6 q_9 have the form

$$\hat q_6=\sum_{p=1}^{3}\int \overset{(p)}{\Phi}_6(i,j)\,\overset{(p)}{z}(i,j)\,di\,dj \qquad\qquad \hat q_9=\sum_{p=1}^{3}\int \overset{(p)}{\Phi}_9(i,j)\,\overset{(p)}{z}(i,j)\,di\,dj,$$

respectively, where $\overset{(p)}{\Phi}_6(i,j)$ and $\overset{(p)}{\Phi}_9(i,j)$ are optimal unbiased weight functions for q_6 q_9. Then as is easy to see

$$\hat q_6+\hat q_9-(e^{(6)}+e^{(9)})^T q$$
$$=\left(\sum_{p=1}^{3}\int \overset{(p)}{\Phi}_6(i,j)\,\overset{(p)}{H}(i,j)\,di\,dj-e^{(6)}+\sum_{p=1}^{3}\int \overset{(p)}{\Phi}_9(i,j)\,\overset{(p)}{H}(i,j)\,di\,dj-e^{(9)}\right)^T q \quad (10)$$

$$+\sum_{p=1}^{3}\int \overset{(p)}{\Phi}_6(i,j)\,\overset{(p)}{\varrho}(i,j)\,di\,dj+\sum_{p=1}^{3}\int \overset{(p)}{\Phi}_9(i,j)\,\overset{(p)}{\varrho}(i,j)\,di\,dj.$$

Since $\overset{(p)}{\Phi}_6(i,j)$ and $\overset{(p)}{\Phi}_9(i,j)$ are unbiased estimators, the expression in parentheses in (10) is zero. Therefore,

$$\sup_{q\in\mathbb{R}^m,\,|\overset{(p)}{\varrho}(i,j)|\le\sigma,\,p=1,2,3}\left|\hat{q}_6+\hat{q}_9-(e^{(6)}+e^{(9)})^T q\right|$$

$$=\sup_{|\overset{(p)}{\varrho}(i,j)|\le\sigma,\,p=1,2,3}\left|\sum_{p=1}^{3}\int \overset{(p)}{\Phi}_6(i,j)\,\overset{(p)}{\varrho}(i,j)\,di\,dj+\sum_{p=1}^{3}\int \overset{(p)}{\Phi}_9(i,j)\,\overset{(p)}{\varrho}(i,j)\,di\,dj\right|$$

$$\le\sum_{p=1}^{3}\int |\overset{(p)}{\Phi}_6(i,j)\,\overset{(p)}{\varrho}(i,j)|\,di\,dj+\sum_{p=1}^{3}\int |\overset{(p)}{\Phi}_9(i,j)\,\overset{(p)}{\varrho}(i,j)|\,di\,dj$$

$$\le\sigma\sum_{p=1}^{3}\int |\overset{(p)}{\Phi}_6(i,j)|\,di\,dj+\sigma\sum_{p=1}^{3}\int |\overset{(p)}{\Phi}_9(i,j)|\,di\,dj=2\,\sigma.$$

The same consideration is true for other two parameters in (9).

5 Conclusion

The paper is devoted to the application of the guaranteed approach to the calibration of the accelerometer unit of a strapdown inertial navigation system. The optimal design of calibration is constructed for the estimation of each unknown parameter. This method is supposed to apply for the accelerometer unit calibration by means of a two-axes test bench produced by "Acutronic" [9]. The obtained results are planned to be extended to the case of nonlinear models of the accelerometer unit when the unit parameters depend on the sign of signal measured. Clearly, our approach can be generalized to the calibration by means of three-axes test benches.

References

[1] Akimov P.A., Derevyankin A.V., Matasov A.I. *Guaranteed Approach and l_1-norm Approximation in the Problems of SDINS Parameter Estimation under Bench Testing.* Moscow: Moscow State Lomonosov University Press, 2012.

[2] Bolotin Yu.V., Golikov V.P., Larionov S.V., Trebukhov A.V. Algorithm for the calibration of a platform inertial navigation system. *Gyroscopy and Navigation*, 2008, No.3, pp.13-27.

[3] Cai Q., Yang G., Song N., Lin Y. Systematic calibration for ultra-high accuracy inertial measurement unit. *Sensors*, 2016, 16, (940).

[4] Dranitsyna E.V. Calibration of a mesurement unit by means of navigation solution of SDINS: a choice of test bench motion design. *Proceedings of the XXIV-th Saint Petersburg International Conference on Integrtated Inertial Navigation Systems.*. SPb.: 2017, pp.235-240.

[5] Egorov Yu.G., Popov E.A. The study of a minimal excess calibration programs for a triad of accelerometers. *Aerospace Instrumentation*, 2016, No.6, pp.3-8.

[6] Emel'yantsev G.I., Stepanov A.P. *Integrated inertial satellite systems for orientation and navigation.* SPb.: State Research Center of the Russian Federation Concern CRSI Elektropribor, JSC, 2016.

[7] Ermakov V.S., Dunaev D.A., Shirokov A.A., etc. Calibration of a strapdown inertial navigation and orientation systems. *Aerospace Engineering. Bulletin of the Perm State Technical University*, 2004, No.18, pp.25-30.

[8] Gusinskii V.Z., Lesyuchevskii V.M., Litmanovich Yu.A., Stolbov A.A. Calibration algorithm for a three-axes accelerometer unit destined for using in a strapdown inertial navigation system. *Gyroscopy and Navigation*, 2000, No.4(31), p.86.

[9] Internet source: www.acutronic.com/ru/produkcija/2-osevye-stendy.html.

[10] Izmailov E.A., Lepe S.N., Molchanov A.V., Polikovskii E.F. Scalar method of calibration and balancing for strapdown inertial navigation systems. *Proceedings of the XV-th Jubilee Saint Petersburg International Conference on Integrtated Inertial Navigation Systems.*. SPb.: 2008. pp.145-154.

[11] Lidov M.L. To a priori accuracy estimates of parameter determining by least squares method. *Cosmic Research,* 1964, vol. 2, No.5, pp.713-715.

[12] Matasov A.I. *Estimators for Unicertain Dynamic Systems.* Springer Science+Business Media, B.V. Netherlands, 2013.

[13] Moon-Sik Kim, Si-Bok Yu, Kwang-Soo Lee. Development of high-precision calibration method for inertial measurement unit. *International Journal of Precision Engineering and Manufacturing*, 2014, vol.15, No.3, pp.567-575.

[14] Panahandeh G., Skog I., Jansson M. Calibration of the accelerometer triad of an inertial measurement unit, maximum likelihood estimation and Cramer-Rao bound. *International Conference on indoor positioning and indoor navigation*, 2010, Zurich, Switzerland.

[15] Secer G., Barshan B. Improvements in deterministic error modeling and calibration of inertial sensors and magnitometers. *Sensors and Actuators A*, 2016, (247), pp.522-538.

[16] Vavilova N.B., Parusnikov N.A., Sazonov I.Yu. Calibration of strapdown inertial navigation systems by means of coarse one-axis test benches. *Modern problems of mathematics and mechanics. Applied reseach*, 2009, vol.1, pp.212-223.