

# Optimal transportation of processes with infinite Kantorovich distance: Independence and symmetry

Alexander V. Kolesnikov and Danila A. Zaev

---

**Abstract** We consider probability measures on  $\mathbb{R}^\infty$  and study optimal transportation mappings for the case of infinite Kantorovich distance. Our examples include (1) quasi-product measures and (2) measures with certain symmetric properties, in particular, exchangeable and stationary measures. We show in the latter case that the existence problem for optimal transportation is closely related to the ergodicity of the target measure. In particular, we prove the existence of the symmetric optimal transportation for a certain class of stationary Gibbs measures.

## 1. Introduction

Let us consider two Borel probability measures  $\mu, \nu$  on  $\mathbb{R}^d$ . The central result (Brenier's theorem) of the finite-dimensional optimal transportation theory establishes under fairly general assumptions the existence of the corresponding optimal transportation mapping  $T$ , which can be characterized by the following properties:

- (1)  $T = \nabla\varphi$ , where  $\varphi$  is a convex function,
- (2)  $\nu$  is the image of  $\mu$  under  $T$ :  $\nu = \mu \circ T^{-1}$ .

The mapping  $T$  exists, in particular, when both measures are absolutely continuous and have finite second moments. The second assumption can be replaced by the weaker assumption of the finiteness of the corresponding Kantorovich distance  $W_2(\mu, \nu)$  but it does not make much difference for the finite-dimensional problems. However, this difference becomes essential in the infinite-dimensional case.

---

*Kyoto Journal of Mathematics*, Vol. 57, No. 2 (2017), 293–324

DOI 10.1215/21562261-3821819, © 2017 by Kyoto University

Received December 1, 2015. Revised March 1, 2016. Accepted March 3, 2016.

*2010 Mathematics Subject Classification*: 49J27, 28C20.

Kolesnikov's work supported by the Russian Foundation for Basic Research project 17-01-00662, and the German Research Foundation project RO 1195/12-1. The article was prepared within the framework of the Academic Fund Program at the National Research University Higher School of Economics (HSE) in 2017–2018 (grant No. 17-01-0102), and by the Russian Academic Excellence Project "5-100."

Zaev's work supported in part by the Simons Foundation.

It is well known that the optimal transportation mapping  $T$  solves the so-called Monge problem, meaning that  $T$  minimizes the functional

$$\int_{\mathbb{R}^d} \|r(x) - x\|^2 d\mu(x)$$

among the mappings  $r: \mathbb{R}^d \mapsto \mathbb{R}^d$  pushing forward  $\mu$  onto  $\nu$ . Here  $\|\cdot\|$  is the standard Euclidean norm. The corresponding minimal value coincides with the squared Kantorovich distance  $W_2^2(\mu, \nu)$ .

Now let us consider a couple of measures on an infinite-dimensional linear space  $X$ . To avoid unessential technicalities, we will assume everywhere that  $X = \mathbb{R}^\infty$ . We deal throughout with the standard Hilbert norm

$$\|x\|^2 := \|x\|_{l^2}^2 = \sum_{i=1}^{\infty} x_i^2,$$

which takes infinite value almost everywhere with respect to most of the measures we are interested in.

What is a natural analogue of the Brenier theorem in this setting? To understand the situation better, let us consider the Gaussian model.

#### EXAMPLE 1.1

Let  $\gamma = \prod_{i=1}^{\infty} \gamma_i = \prod_{i=1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}} dx_i$  be the standard Gaussian product measure on  $\mathbb{R}^\infty$ , and let  $H = l^2$  be the corresponding Cameron–Martin space. More generally, one can consider any abstract Wiener space.

The optimal transportation problem is well understood for the case of measures  $\mu$  and  $\nu$  which are absolutely continuous with respect to  $\gamma$ . The most general results were obtained in [12]. (Another approach has been developed in [15].) In particular, for a broad class of probability measures  $f \cdot \gamma$  absolutely continuous with respect to  $\gamma$ , there exists a transportation mapping  $T(x) = x + \nabla\varphi(x)$  minimizing the cost

$$\int \|T(x) - x\|_{l^2}^2 d\gamma$$

and pushing forward  $\gamma$  onto  $f \cdot \gamma$ . Analogously, there exists a transportation mapping pushing forward  $f \cdot \gamma$  onto  $\gamma$ . The gradient operator  $\nabla$  is understood with respect to the  $\langle \cdot, \cdot \rangle_{l^2}$ -scalar product.

It is known (this follows from the so-called *Talagrand transportation inequality*) that under the assumption that  $\int f \log f d\gamma < \infty$  the Kantorovich distance between  $\gamma$  and  $f \cdot \gamma$  is finite:

$$W_2^2(\gamma, f \cdot \gamma) = \int \|T(x) - x\|_{l^2}^2 d\gamma < \infty.$$

In particular,  $\nabla\varphi(x) \in l^2$  for  $\gamma$ -almost all  $x$ . For more on optimal transportation on the Wiener space, the corresponding Monge–Ampère equation, regularity issues, and transportation on other infinite-dimensional spaces, see [4], [6], [8], [11], and [10].

In this article we study situations in which the Kantorovich distance between measures is a priori *infinite*. This makes it generally impossible to understand  $T$  as a solution to a certain minimization problem. Nevertheless, we have many good candidates that may be called “optimal transportation” in many particular cases. The following example motivates our study.

**EXAMPLE 1.2**

(1) Let  $\mu = \prod_{i=1}^{\infty} \mu_i(dx_i)$ ,  $\nu = \prod_{i=1}^{\infty} \nu_i(dx_i)$  be product probability measures. Assume that all  $\mu_i$ 's have densities. Then there exists a mass transportation mapping  $T$  pushing forward  $\mu$  onto  $\nu$  which has the form

$$T(x) = (T_1(x_1), \dots, T_i(x_i), \dots),$$

where  $T_i(x_i)$  is the 1-dimensional optimal transportation pushing forward  $\mu_i$  onto  $\nu_i$ .

(2) Let us consider the Gaussian measure  $\mu$  which is a pushforward image of the standard Gaussian measure  $\gamma$  under a linear mapping  $T(x) = Ax$  with  $A$  symmetric and positive. It is well known (and can be obtained from the law of large numbers) that  $\gamma$  and  $\mu$  are mutually singular even in the simplest case  $A = 2 \cdot \text{Id}$ . Because it is linear and given by a positive symmetric operator,  $T$  is “optimal.” Heuristically,

$$T(x) = \frac{1}{2} \nabla \langle Ax, x \rangle.$$

It is clear that in both cases  $T$  cannot be obtained as a minimizer of a functional of the type  $\int \|T(x) - x\|_2^2 d\mu$ .

We state now the central problem of this article.

**PROBLEM 1.3**

Let  $\mu$  and  $\nu$  be two probability measures on  $\mathbb{R}^{\infty}$ . When does there exist a transportation mapping  $T$  pushing forward  $\mu$  onto  $\nu$  which is “optimal” for the cost function  $c(x, y) = \|x - y\|_2^2$ ?

In this article we deal with two model situations.

**Quasiproduct measures**

We assume that both measures have densities with respect to product probability measures (see [5], [7], [8], [11], [10], and [18])

$$\begin{aligned} \mu &= f \cdot \mu_0, & \nu &= g \cdot \nu_0, \\ \mu_0 &= \prod_{i=1}^{\infty} \mu_i(dx_i), & \nu_0 &= \prod_{i=1}^{\infty} \nu_i(dx_i). \end{aligned}$$

Then the corresponding “optimal transportation” is a small perturbation of the diagonal mapping considered in Example 1.2.

### Symmetric measures

It is possible to give a meaning to the Monge–Kantorovich optimization problem if we restrict ourselves to a certain class of symmetric measures. In this article we consider two types of symmetry: exchangeable measures (invariant with respect to finite permutations of coordinates) and stationary measures on  $\mathbb{R}^\infty$  (invariant with respect to shifts of coordinates). Note that  $\|x - y\|_{l_2}^2$  is symmetric with respect to both types of symmetry. More generally, let  $G$  be a group of linear operators which acts on  $X = Y = \mathbb{R}^\infty$  and  $X \times Y: x \rightarrow gx, (x, y) \rightarrow (gx, gy), g \in G$ , and preserves the cost function  $c(x, y)$ . We assume that every basic vector  $e_j$  can be obtained from any other  $e_i$  by the action of this group: there exists  $g \in G$  such that  $e_i = ge_j$ . Note that under these assumptions all the coordinates are identically distributed. This leads us to the following definition: given  $G$ -invariant marginals  $\mu$  and  $\nu$ , we call  $\pi$  an *optimal* (symmetric, invariant) solution to the Monge–Kantorovich problem if  $\pi$  solves the Monge–Kantorovich problem

$$\int (x_1 - y_1)^2 d\pi \rightarrow \min$$

among all of the measures which are invariant with respect to  $G$ . If there exists a mapping  $T$  such that its graph  $\Gamma = \{x, T(x)\}$  satisfies  $m(\Gamma) = 1$ , we say that  $T$  is an *optimal transportation mapping* pushing forward  $\mu$  onto  $\nu$ .

The following counterexample, however, demonstrates that the optimal transportation may fail to exist for a quite simple reason.

#### EXAMPLE 1.4

Let  $\mu = \gamma$  be the standard Gaussian measure on  $\mathbb{R}^\infty$ , and let

$$\nu = \frac{1}{2}(\gamma + \gamma_2)$$

be the average of  $\gamma$  and its homothetic image  $\gamma_2 = \gamma \circ S^{-1}$ , where  $S(x) = 2x$ . There is no mass transportation  $T$  of  $\mu$  to  $\nu$  which commutes with any cylindrical rotation. Indeed, any such mapping must have the form  $T(x) = g(x)(x_1, x_2, \dots) = g(x) \cdot x$ , where  $g$  is invariant with respect to any “rotation,” in particular, with respect to any coordinate permutation. But any function  $g$  of this type is constant  $\gamma$ -almost everywhere. This is a corollary of the Hewitt–Savage zero-one law. It is clear that there is no mass transportation of this type for the given target measure.

There is a general principle behind this simple example. Recall that a measure  $\mu$  is called *ergodic* with respect to a group action  $G$  if for every  $G$ -invariant set  $A$  one has either  $\mu(A) = 1$  or  $\mu(A) = 0$ . It follows directly from the definition that there does not exist a bijective mass transportation  $T$  pushing forward  $\mu$  onto  $\nu$  such that  $T \circ g = g \circ T$  for every  $g \in G$ , provided  $\mu$  is  $G$ -ergodic but  $\nu$  is not.

This observation leads to the following problem.

#### PROBLEM 1.5

Let  $G$  be a group of linear operators acting on  $\mathbb{R}^\infty$  and preserving  $l_2$ -distance

(model example: group of shifts). Let  $\mu, \nu$  be *ergodic*  $G$ -invariant measures. When does there exist a transportation  $T: \mathbb{R}^\infty \mapsto \mathbb{R}^\infty$  pushing forward  $\mu$  onto  $\nu$ , which commutes with  $G$  and minimizes the Monge functional  $T \mapsto \int_{\mathbb{R}^\infty} (T_1(x) - x_1)^2 d\mu$ ?

Trivially, the ergodicity by itself is not sufficient for an affirmative answer to this problem. In addition to it, we need to have certain infinite-dimensional analogues of “absolute continuity” for the source measure  $\mu$ .

We believe that the symmetric transportation problem must have a deep and very interesting relation to ergodic theory. The second author [25] studied the interplay between ergodic decompositions and transportation theory. Another interesting connection has been established in [3]. It was shown that the Birkhoff ergodic theorem implies equivalence between optimality and the so-called *cyclical monotonicity property*. The related problems on optimal transportation in symmetric settings have been considered in [22] (stationary processes), in [23] (symmetric measures on graphs), and in [19], [20], and [9] (ergodic theory). Transportation problems with symmetries have been studied in [13] and [21]. Further development of the duality theory for transportation problem with linear restriction has been obtained in [26].

The article is organized as follows. In Section 2 we give preliminaries in transportation theory and ergodic theory, and we recall some important results on log-concave measures. In Section 3 we establish sufficient conditions for the existence of optimal transportation mappings which are obtained as almost everywhere limits of finite-dimensional approximations. The applications of this result are obtained in Section 4. Here we prove the existence of optimal transportation for a couple of measures having densities with respect to product measures. In Section 5 we discuss the invariant optimal transportation problem, consider examples, and prove some basic facts. In Section 6 we briefly discuss Kantorovich duality for a problem which is invariant with respect to the action of a group. In Section 7 we construct a nontrivial example of a symmetric optimal transportation  $T$ . Namely, we establish sufficient conditions for the existence of  $T$  pushing forward a stationary measure into the standard Gaussian measure. Finally, we apply this result to a certain class of Gibbs measures.

## 2. Preliminaries

### 2.1. Optimal transportation problem

#### PROBLEM 2.1 (KANTOROVICH PROBLEM)

Given two probability measures  $\mu$  and  $\nu$  on the spaces  $X$  and  $Y$ , respectively, and a cost function  $c: X \times Y \mapsto \mathbb{R} \cup \{+\infty\}$ , we are looking for the minimum of the functional

$$W_2^2(\mu, \nu) = \inf \left\{ \int \|x - y\|^2 dm : m \in P(\mu, \nu) \right\}$$

on the space  $P(\mu, \nu)$  of probability measures with fixed projections:  $\Pr_X m = \mu$ ,  $\Pr_Y m = \nu$ .

In the classical setup  $X = Y = \mathbb{R}^n$ ,  $c = |x - y|^2$ , the solution  $m$  is supported on the graph of a mapping  $T : \mathbb{R}^n \mapsto \mathbb{R}^n$  (see [2], [5], [24]):

$$m(\Gamma) = 1, \quad \text{where } \Gamma = \{(x, T(x)), x \in \mathbb{R}^d\}.$$

The functional  $W_2(\mu, \nu)$  is a distance in the space of probability measures. In what follows we call it the *Kantorovich distance*. The mapping  $T$  is called the *optimal transportation* of  $\mu$  onto  $\nu$ .

Another well-known fact which will be used throughout the article is the following relation, called the *Kantorovich duality*:

$$W_2(\mu, \nu) = -\frac{1}{2}J(\varphi, \psi),$$

where

$$J(\varphi, \psi) = \inf_{\varphi, \psi} \left\{ \int \left( \varphi(x) - \frac{|x|^2}{2} \right) d\mu + \int \left( \psi(y) - \frac{|y|^2}{2} \right) d\nu, \varphi(x) + \psi(y) \geq \langle x, y \rangle \right\},$$

and where the infimum is taken over couples of integrable Borel functions  $\varphi(x), \psi(y)$ . The function  $\varphi$  in the dual problem coincides with the potential generating the transportation mapping

$$T = \nabla \varphi.$$

## 2.2. Ergodic decomposition

Given a Borel transformation  $S : X \mapsto X$  of the space  $X$  we call a Borel probability measure  $\mu$  ergodic if any  $S$ -invariant measurable set  $A$  has the property  $\mu(A) = 1$  or  $\mu(A) = 0$ . A similar terminology is used if instead of a single mapping  $S$  we deal with a family  $G$  of transformations.

The ergodic  $G$ -invariant measures are extreme points of the set of all  $G$ -invariant measures; hence, any  $G$ -invariant measure can be represented as the average of  $G$ -invariant ergodic measures. The famous de Finetti theorem establishes decomposition of this type for a class of exchangeable measures, that is, measures invariant with respect to a permutation of a finite number of coordinates.

### THEOREM 2.2

*Let  $\mathcal{P}$  be the space of Borel probability measures on  $\mathbb{R}$  equipped with the weak topology. Then for every Borel exchangeable  $\mu$  on  $\mathbb{R}^\infty$  there exists a Borel probability measure  $\Pi$  on  $\mathcal{P}$  such that*

$$\mu(B) = \int m^\infty(B) \Pi(dm),$$

*for every Borel  $B \subset \mathbb{R}^\infty$ .*

Yet another example of the ergodic decomposition where a precise description is possible is given by rotationally invariant measures (see Example 5.9).

### 2.3. Log-concave measures and functional inequalities

We recall that a probability measure  $\mu$  on  $\mathbb{R}^n$  is called *log-concave* if it has the form  $e^{-V} \cdot \mathcal{H}^k|_L$ , where  $\mathcal{H}^k$  is the  $k$ -dimensional Hausdorff measure,  $k \in \{0, 1, \dots, n\}$ ,  $L$  is an affine subspace, and  $V$  is a convex function.

In what follows we consider uniformly log-concave measures. Roughly speaking, these are the measures with potential  $V$  satisfying

$$V(x) - V(y) - \langle \nabla V(y), x - y \rangle \geq \frac{K}{2} |x - y|^2,$$

which is equivalent to  $D^2V \geq K \cdot \text{Id}$  in the smooth (finite-dimensional) case. Here  $K$  is a positive constant.

More precisely, we say that a probability measure  $\mu$  is  *$K$ -uniformly log-concave* ( $K > 0$ ) if for any  $\varepsilon > 0$  the measure  $\hat{\mu} = \frac{1}{Z} e^{\frac{K-\varepsilon}{2}|x|^2} \cdot \mu$  is log-concave for a suitable renormalization factor  $Z$ . It is well known (see Borell [8]) that the projections of log-concave measures are log-concave. (This is in fact a corollary of the Brunn–Minkowski theorem.) It can be easily checked that the uniform log-concavity is preserved by projections as well. We can extend this notion to the infinite-dimensional case. Namely, we call a probability measure  $\mu$  on a locally convex space  $X$  *log-concave* ( $K$ -uniformly log-concave with  $K > 0$ ) if its images  $\mu \circ l^{-1}$ ,  $l \in X^*$ , under linear continuous functionals are all log-concave ( $K$ -uniformly log-concave with  $K > 0$ ).

Throughout the article we apply the following estimate (see [15], [17]), which generalizes the famous Talagrand transportation inequality.

#### THEOREM 2.3 (GENERALIZED TALAGRAND INEQUALITY)

Let  $m$  be a  $K$ -uniformly log-concave probability measure with some  $K > 0$ . Then for any couple of probability measures  $\mu = e^{-V} dx$ ,  $\nu = e^{-W} dx$  and the corresponding optimal mappings  $\nabla\varphi_\mu$ ,  $\nabla\varphi_\nu$ , pushing forward  $\mu$ ,  $\nu$  onto  $m$ , respectively, one has the estimate

$$\text{Ent}_\nu\left(\frac{\mu}{\nu}\right) = \int \log \frac{d\mu}{d\nu} d\mu = \int (W - V) d\mu \geq \frac{K}{2} \int |\nabla\varphi_\mu - \nabla\varphi_\nu|^2 d\mu.$$

Another result used in the article is Caffarelli's contraction theorem. Here is the version from [17, Theorem 7.4] (see also [16]).

#### THEOREM 2.4 (CAFFARELLI CONTRACTION THEOREM)

Let  $\nabla\Phi$  be the optimal transportation of the probability measure  $\mu = e^{-V} dx$  into  $\nu = e^{-W} dx$ . Assume that for some positive  $c, C$  one has  $D^2V \leq C \cdot \text{Id}$ ,  $D^2W \geq c \cdot \text{Id}$ . Then  $\nabla\Phi$  is Lipschitz with  $\|\nabla\Phi\|_{\text{Lip}} \leq \sqrt{\frac{C}{c}}$ .

The quantity  $\text{Ent}_\nu\left(\frac{\mu}{\nu}\right)$  is called the *relative entropy* or the *Kullback–Leibler distance* between  $\mu$  and  $\nu$ .

### 3. Sufficient condition for existence of limits of finite-dimensional optimal mappings

#### 3.1. Preliminary finite-dimensional estimates

Let  $\mu$  and  $\nu$  be probability measures on  $\mathbb{R}^d$ , and let  $T(x) = \nabla\varphi(x)$  be the optimal transportation mapping pushing forward  $\mu$  onto  $\nu$ . Let us denote by  $\mu_v$  the images of  $\mu$  under the shifts  $x \mapsto x + v$ ,  $v \in \mathbb{R}^d$ . It will be assumed throughout that the  $\mu_v$ 's have densities with respect to  $\mu$ :

$$\frac{d\mu_v}{d\mu} = e^{\beta v}.$$

LEMMA 3.1

For every  $p, q \geq 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\varepsilon \geq 0$ , and  $e \in \mathbb{R}^d$

$$\begin{aligned} \int |\varphi(x + te) - \varphi(x)|^{1+\varepsilon} d\mu &\leq t^{1+\varepsilon} \|\langle x, e \rangle\|_{L^p(\nu)}^{1+\varepsilon} \cdot \sup_{0 \leq s \leq t} \|e^{\beta se}\|_{L^q(\mu)}, \\ \int (\varphi(x + te) - \varphi(x) - t\partial_e\varphi(x)) d\mu &\leq t \|\langle x, e \rangle\|_{L^p(\nu)} \cdot \sup_{0 \leq s \leq t} \|e^{\beta se} - 1\|_{L^q(\mu)}. \end{aligned}$$

*Proof*

One has  $\varphi(x + te) - \varphi(x) = \int_0^t \partial_e\varphi(x + se) ds$ . Hence,

$$\begin{aligned} &\int |\varphi(x + te) - \varphi(x)|^{1+\varepsilon} d\mu \\ &\leq t^\varepsilon \int \int_0^t |\partial_e\varphi|^{1+\varepsilon}(x + se) ds d\mu \\ &= t^\varepsilon \int_0^t \left[ \int |\partial_e\varphi|^{1+\varepsilon} e^{\beta se} d\mu \right] ds \leq t^{1+\varepsilon} \|\partial_e\varphi\|_{L^p(\mu)}^{1+\varepsilon} \cdot \sup_{0 \leq s \leq t} \|e^{\beta se}\|_{L^q(\mu)} \\ &= t^{1+\varepsilon} \|\langle x, e \rangle\|_{L^p(\nu)}^{1+\varepsilon} \cdot \sup_{0 \leq s \leq t} \|e^{\beta se}\|_{L^q(\mu)}. \end{aligned}$$

Applying the same arguments one gets

$$\begin{aligned} \int (\varphi(x + te) - \varphi(x) - t\partial_e\varphi(x)) d\mu &= \int \int_0^t (\partial_e\varphi(x + se) - \partial_e\varphi(x)) ds d\mu \\ &= \int \left[ \int_0^t (e^{\beta se} - 1) ds \right] \partial_e\varphi(x) d\mu \\ &\leq t^{\frac{1}{p}} \|\partial_e\varphi\|_{L^p(\mu)} \left[ \int \int_0^t |e^{\beta se} - 1|^q ds d\mu \right]^{\frac{1}{q}}. \end{aligned}$$

The desired estimate follows from the change of variables formula and trivial uniform bounds.  $\square$

In addition, we will apply the following elementary lemma.

## LEMMA 3.2

Assume that a sequence  $\{T_n\}$  of measurable mappings  $T_n: \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$  converges to a mapping  $T$  in the following sense: for every  $e_i$ ,  $\lim_n \langle T_n, e_i \rangle = \langle T, e_i \rangle$  in measure with respect to  $\mu$ . Then the measures  $\{\mu \circ T_n^{-1}\}$  converge weakly to  $\mu \circ T^{-1}$ .

## 3.2. Existence theorem

We consider a couple of Borel probability measures  $\mu$  and  $\nu$  on  $\mathbb{R}^\infty$ , where  $\mathbb{R}^\infty$  is the space of all real sequences:  $\mathbb{R}^\infty = \prod_{i=1}^\infty \mathbb{R}_i$ . We deal with the standard coordinate system  $x = (x_1, x_2, \dots, x_n, \dots)$  and the standard basis vectors  $e_i = (\delta_{ij})$ . The projection on the first  $n$  coordinates will be denoted by  $P_n$ :  $P_n(x) = (x_1, \dots, x_n)$ . We use notation  $\|x\|$ ,  $\langle x, y \rangle$  for the Hilbert space norm and inner product, respectively:  $\|x\| = \sum_{i=1}^\infty x_i^2$ ,  $\langle x, y \rangle = \sum_{i=1}^\infty x_i y_i$ . We use the notation  $\mathbb{E}_\mu^n$  for the conditional expectation with respect to  $\mu$  and the  $\sigma$ -algebra generated by  $x_1, \dots, x_n$ . For any product measure  $P = \prod_{i=1}^\infty p_i(x_i) dx_i$  its projection  $P_n = P \circ P_n^{-1}$  has the form  $\prod_{i=1}^n p_i(x_i) dx_i$ , and the projection  $(f \cdot P) \circ P_n^{-1} = f_n \cdot P_n$  of the measure  $f \cdot P$  satisfies  $f_n = \mathbb{E}_P^n f$ . Everywhere below we agree that every cylindrical function  $f = f(x_1, \dots, x_n)$  can be extended to  $\mathbb{R}^\infty$  by the formula  $x \rightarrow f_n(P_n x)$ .

It will be assumed throughout the article that the shifts of  $\mu$  along any vector  $v = te_i$  are absolutely continuous with respect to  $\mu$ :

$$\frac{d\mu_v}{d\mu} = e^{\beta v}.$$

In Section 3, moreover, the following assumption holds.

## ASSUMPTION A

For every basic vector  $e = e_i$  there exist  $p \geq 1$ ,  $q \geq 1$ , satisfying  $\frac{1}{p} + \frac{1}{q} = 1$ , and  $\varepsilon > 0$  such that

$$\int |\langle x, e \rangle|^{(1+\varepsilon)p} d\nu < \infty$$

and

$$p(t) = \sup_{0 \leq s \leq t} \int |e^{\beta s e} - 1|^q d\mu$$

satisfies  $\lim_{t \rightarrow 0} p(t) = 0$ .

Let  $\mu_n = \mu \circ P_n^{-1}(x)$ ,  $\nu_n = \nu \circ P_n^{-1}(y)$  be the projections of  $\mu, \nu$ . For every  $v = te_i$  let us set

$$\frac{d(\mu_n)_v}{d\mu_n} = e^{\beta_v^{(n)}}.$$

It is easy to check that the projections of  $\mu, \nu$  satisfy Assumption A.

LEMMA 3.3

For every  $n \in \mathbb{N}$  and every  $e = e_i$  one has

$$\int |\langle P_n(x), e \rangle|^p d\nu_n \leq \int |\langle x, e \rangle|^p d\nu, \quad \int |e^{\beta_e^{(n)}} - 1|^q d\mu_n \leq \int |e^{\beta_e} - 1|^q d\mu.$$

*Proof*

The first estimate is trivial. To prove the second one, let us note that  $e^{\beta_v^{(n)}} = \mathbb{E}_\mu^n e^{\beta_v}$ . The claim follows from the Jensen inequality and convexity of the function  $t \rightarrow |t - 1|^q$ .  $\square$

We denote by  $\pi_n$  the optimal transportation plan for the couple  $(\mu_n, \nu_n)$ . Let  $\varphi_n(x)$  and  $\psi_n(y)$  solve the dual Kantorovich problem. Let us recall that  $\nabla\varphi_n$  ( $\nabla\psi_n$ ) is the optimal transportation mapping sending  $\mu_n$  to  $\nu_n$  ( $\nu_n$  to  $\mu_n$ ). One has

$$\varphi_n(x) + \psi_n(y) \geq \langle P_n x, P_n y \rangle$$

for every  $x, y$ . The equality is attained on the support of  $\pi_n$ . In particular,

$$\varphi_n(x) + \psi_n(\nabla\varphi_n(x)) = \langle P_n x, \nabla\varphi_n(x) \rangle.$$

It is easy to check that  $\{\pi_n\}$  is a tight sequence. By the Prokhorov theorem, one can extract a weakly convergent subsequence  $\pi_{n_k} \rightarrow \pi$ . Note that  $\pi_n$  is *not* the projection of  $\pi$ .

The main result of the section is the following theorem.

THEOREM 3.4

Assume that Assumption A is fulfilled and, in addition, that

$$F_n(x, y, 0, 0) = \varphi_n(x) + \psi_n(y) - \langle P_n x, P_n y \rangle \rightarrow 0$$

in measure with respect to  $\pi$ . Then there exists a mapping  $T: \mathbb{R}^\infty \mapsto \mathbb{R}^\infty$  such that

$$T(x) = y$$

for  $\pi$ -almost all  $(x, y)$ .

In what follows we will pass several times to subsequences and use for the new subsequences the same index  $n$  again, with the agreement that  $n$  takes values in another infinite set  $\mathbb{N}' \subset \mathbb{N}$ . Let us fix unit vectors  $e_i, e_j$  for some  $i, j \in \mathbb{N}$ , and consider the sequence of nonnegative functions

$$F_n(x, y, t, s) = \varphi_n(x + te_i) + \psi_n(y + se_j) - \langle P_n(x + te_i), P_n(y + se_j) \rangle$$

with  $n > i, n > j$ .

LEMMA 3.5

There exists an  $L^{1+\varepsilon}(\pi)$ -weakly convergent subsequence

$$\varphi_{n_k}(x + te_i) - \varphi_{n_k}(x) \rightarrow U(x).$$

The following relation holds for the limiting function  $U(x)$ :

$$\left| \int U(x) d\mu - t \int \langle y, e_i \rangle d\nu \right| \leq Ctp(t).$$

*Proof*

Taking into account that  $\int F_n(x, y, 0, 0) d\pi_n = 0$ , one obtains

$$\int F_n(x, y, t, 0) d\pi_n = \int F_n(x, y, t, 0) d\pi_n - \int F_n(x, y, 0, 0) d\pi_n \geq 0.$$

Note that the right-hand side equals

$$\int (F_n(x, y, t, 0) - F_n(x, y, 0, 0)) d\pi_n = \int [\varphi_n(x + te_i) - \varphi_n(x) - t\langle y, e_i \rangle] d\pi_n.$$

Taking into account that the projection of  $\pi_n$  onto  $X$  coincides with  $\mu_n$  and  $\varphi_n$  depends on the first  $n$  coordinates, one finally obtains that for  $n > i$  the latter is equal to

$$\begin{aligned} & \int [\varphi_n(x + te_i) - \varphi_n(x)] d\mu - t \int \langle y, e_i \rangle d\nu \\ &= \int [\varphi_n(x + te_i) - \varphi_n(x) - t\partial_{e_i}\varphi_n(x)] d\mu. \end{aligned}$$

It follows from Lemmas 3.1 and 3.3 and Assumption A that

$$(1) \quad \left| \int F_n(x, y, t, 0) d\pi_n \right| \leq Ctp(t).$$

Since  $\varphi_n$  depends on a finite number of coordinates (at most  $n$ ), one has

$$\int |\varphi_n(x + te_i) - \varphi_n(x)|^{1+\varepsilon} d\mu = \int |\varphi_n(x + te_i) - \varphi_n(x)|^{1+\varepsilon} d\mu_n.$$

Hence, by Lemma 3.1

$$U_n(x) = \varphi_n(x + te_i) - \varphi_n(x) \in L^{1+\varepsilon}(\mu)$$

and, moreover,  $\sup_n \|U_n\|_{L^{1+\varepsilon}(\mu)} < \infty$ . Thus, there exists a function  $U \in L^{1+\varepsilon}(\mu)$  such that for some subsequence  $n_k$

$$\varphi_{n_k}(x + te_i) - \varphi_{n_k}(x) \rightarrow U(x)$$

weakly in  $L^{1+\varepsilon}(\mu)$ . Passing to the limit we obtain from (1) that

$$\left| \int U(x) d\mu - t \int \langle y, e_i \rangle d\nu \right| \leq Ctp(t). \quad \square$$

#### LEMMA 3.6

Assume that  $F_n(x, y, 0, 0) \rightarrow 0$  in measure with respect to  $\pi$ . Then

$$U(x) - t\langle y, e_i \rangle \geq 0$$

for  $\pi$ -almost all  $(x, y)$ .

*Proof*

Note that

$$\begin{aligned} & [\varphi_n(x + te_i) - \varphi_n(x) - t\langle y, e_i \rangle] + F_n(x, y, 0, 0) \\ &= \varphi_n(x + te_i) + \psi_n(y) - \langle P_n y, P_n(x + te_i) \rangle \end{aligned}$$

is a nonnegative function for every  $n$ . Since  $F_n(x, y, 0, 0) \rightarrow 0$  in measure, there exists a subsequence (denoted again by  $F_n$ ) which converges to zero  $\pi$ -almost everywhere. Since  $f_n = \varphi_n(x + te_i) - \varphi_n(x) - t\langle y, e_i \rangle$  converges to  $f = U(x) - t\langle y, e_i \rangle$  weakly in  $L^{1+\varepsilon}(\pi)$ , one can assume (passing again to a subsequence) that  $\frac{1}{N} \sum_{n=1}^N f_n \rightarrow f$   $\pi$ -almost everywhere. Since  $f_n + F_n \geq 0$ , this implies that  $f \geq 0$   $\pi$ -almost everywhere.  $\square$

**PROPOSITION 3.7**

Assume that there exists a sequence of continuous functions

$$f_n(x_1, \dots, x_n), g_n(y_1, \dots, y_n) \in L^1(\pi_n)$$

such that  $G_n = f_n(x) + g_n(y) - \sum_{i=1}^n x_i y_i$  has the following properties:

- (1)  $G_n \geq 0$ ,
- (2)  $G_n \leq G_m, \forall n \leq m, x, y \in \mathbb{R}^m$ ,
- (3)  $\sup_n \int G_n d\pi_n < \infty$ .

Then  $F_n(x, y, 0, 0) \rightarrow 0$  in  $L^1(\pi)$ .

*Proof*

We start with the identity  $\int F_n(x, y, 0, 0) d\pi_n = 0$  and rewrite it as

$$(2) \quad 0 = \int (\varphi_n - f_n) d\mu + \int (\psi_n - g_n) d\nu + \int \left( f_n(x) + g_n(y) - \sum_{i=1}^n x_i y_i \right) d\pi_n.$$

Since  $\varphi_n, \psi_n$  are defined up to a constant, one can assume that  $\int (\psi_n - g_n) d\nu = 0$ . Thus,  $-\int (\varphi_n - f_n) d\mu = \int (f_n(x) + g_n(y) - \sum_{i=1}^n x_i y_i) d\pi_n$ . It follows from (1) and (3) that the right-hand side is a bounded sequence of nonnegative numbers. Passing to a subsequence we may assume that the right-hand side has a limit. It follows from the weak convergence  $\pi_n \rightarrow \pi$  and the monotonicity property (2) that for every  $k$

$$\begin{aligned} \underline{\lim}_n \int \left( f_n(x) + g_n(y) - \sum_{i=1}^n x_i y_i \right) d\pi_n &\geq \underline{\lim}_n \int \left( f_k(x) + g_k(y) - \sum_{i=1}^k x_i y_i \right) d\pi_n \\ &= \int \left( f_k(x) + g_k(y) - \sum_{i=1}^k x_i y_i \right) d\pi. \end{aligned}$$

Hence,

$$\underline{\lim}_n \int \left( f_n(x) + g_n(y) - \sum_{i=1}^n x_i y_i \right) d\pi_n \geq \lim_k \int \left( f_k(x) + g_k(y) - \sum_{i=1}^k x_i y_i \right) d\pi,$$

where the limit on the right-hand side exists, because the sequence is monotone. Hence, we get from (2)

$$0 \geq \lim_n \int (\varphi_n - f_n) d\mu + \lim_n \int \left( f_n(x) + g_n(y) - \sum_{i=1}^n x_i y_i \right) d\pi.$$

Taking into account that  $\int g_n d\pi = \int g_n d\nu = \int \psi_n d\nu = \int \psi_n d\pi$ , we obtain

$$\begin{aligned} 0 &\geq \lim_n \int (\varphi_n - f_n)(x) d\mu + \lim_n \int \left( f_n(x) + g_n(y) - \sum_{i=1}^n x_i y_i \right) d\pi \\ &= \lim_n \left( \int \left( \varphi_n(x) + \psi_n(y) - \sum_{i=1}^n x_i y_i \right) d\pi \right) \geq 0. \end{aligned}$$

The proof is complete.  $\square$

Finally, we obtain a sufficient condition for the existence of an optimal mapping in the infinite-dimensional case.

#### *Proof of Theorem 3.4*

Let us fix  $e_i$ , and choose a sequence of numbers  $t_n \rightarrow 0$ . We get from Lemmas 3.5 and 3.6 that there exist  $\pi$ -almost everywhere nonnegative functions  $U_{t_n}(x) - t_n \langle y, e_i \rangle$  with  $\int (U_{t_n}(x) - t_n \langle y, e_i \rangle) d\pi = o(t_n)$ . Hence,  $\lim_{t_n \rightarrow 0} \int \left( \frac{U_{t_n}(x)}{t_n} - \langle y, e_i \rangle \right) d\pi = 0$ . Taking into account that  $\frac{U_{t_n}(x)}{t_n} - \langle y, e_i \rangle \geq 0$  for  $\pi$ -almost all  $(x, y)$ , we conclude that  $\frac{U_{t_n}(x)}{t_n}$  converges  $\mu$ -almost everywhere and in  $L^1(\mu)$  to a function  $u_i(x)$  satisfying  $u_i(x) - \langle y, e_i \rangle \geq 0$   $\pi$ -almost everywhere and  $\int (u_i(x) - \langle y, e_i \rangle) d\pi = 0$ . Clearly,  $u(x) = \langle y, e_i \rangle$  for  $\pi$ -almost all  $(x, y)$ . Repeating these arguments for every  $i \in \mathbb{N}$ , we get the claim.  $\square$

#### **4. Application: Quasiproduct case**

The main result of this section is a generalization of the optimal transport existence theorem for Gaussian measures. Recall that by results from [12] and [15] for the standard Gaussian measure  $\gamma = \prod_{i=1}^{\infty} \gamma_i(dx_i)$ ,  $\gamma_i \sim \mathcal{N}(0, 1)$ , the existence of the optimal transportation mapping pushing forward  $f \cdot \gamma$  onto  $g \cdot \gamma$  is established, for instance, under the assumption that  $\int f \log f d\gamma < \infty$ ,  $\int g \log g d\gamma < \infty$ . We give in this section a generalization of this result for a wide class of quasiproduct measures.

Let us consider two product reference measures

$$P = \prod_{i=1}^{\infty} p_i(x_i) dx_i, \quad Q = \prod_{i=1}^{\infty} q_i(x_i) dx_i,$$

and fix the diagonal infinite transportation mapping

$$T(x) = (T_1(x_1), \dots, T_n(x_n), \dots),$$

where  $T_i(x_i)$  pushes forward  $p_i(x_i) dx_i$  onto  $q_i(x_i) dx_i$ . Clearly,  $T$  takes  $P$  onto  $Q$ . The inverse mapping  $S = T^{-1}$  has the same diagonal structure:

$$S(x) = (S_1(x_1), \dots, S_n(x_n), \dots).$$

**THEOREM 4.1**

Let  $\mu = f \cdot P$  and  $\nu = g \cdot Q$  be probability measures satisfying Assumption A of the previous section. Assume, in addition, that

- (1) there exists  $K > 0$  such that every  $q_i$  is  $K$ -uniformly log-concave;
- (2) there exists  $M > 0$  such that

$$S'_i(x_i) \leq M$$

for all  $i, x_i$ ;

- (3) assume that either (a) or (b) holds for some constants  $C > c > 0$ :
  - (a)  $g \log^2 g \in L^1(Q), \frac{1}{f} \in L^1(P), f \leq C$ ,
  - (b)  $f \log f \in L^1(P), c \leq g \leq C$ .

Then there exists a transportation mapping  $T$  pushing forward  $\mu$  onto  $\nu$  which is a  $\mu$ -almost everywhere limit of finite-dimensional optimal transportation mappings  $T_n$ .

**REMARK 4.2**

It follows from Caffarelli's contraction theorem (see Section 2) that assumption (2) is satisfied if  $(-\log p_i(x_i))'' \geq C_0$ ,  $(-\log q_i(x_i))'' \leq C_1$  for some  $C_0, C_1 > 0$  and every  $i$ . Of course, there exist many other examples where this assumption is satisfied.

*Proof*

Consider the finite-dimensional projections  $\mu_n = f_n \cdot P_n$ ,  $\nu_n = g_n \cdot Q_n$ , where  $P_n = \prod_{i=1}^n p_i(x_i) dx_i$ ,  $Q_n = \prod_{i=1}^n q_i(x_i) dx_i$ . Here  $f_n$  and  $g_n$  are the conditional expectations of  $f, g$  with respect to  $P, Q$  and the  $\sigma$ -algebra  $\mathcal{F}_n$ , generated by the first  $n$  coordinates. Recall that  $\nabla\varphi_n$  is the optimal transportation of  $\mu_n$  to  $\nu_n$ . Let

$$u_i(x_i), \quad v_i(y_i) = u_i^*$$

be the one-dimensional convex potentials associated to the mappings  $T_i, S_i$ , respectively:

$$T_i = u'_i, \quad S_i = v'_i.$$

Note that  $\tilde{T}_n = (T_1, \dots, T_n)$  pushes forward  $P_n$  onto  $Q_n$  and  $\nabla\varphi_n$  pushes forward  $\frac{f_n}{g_n(\nabla\varphi_n)} \cdot P_n$  onto  $Q_n$ .

According to Proposition 2.3 one has the following estimate:

$$(3) \quad \frac{K}{2} \int |\tilde{T}_n - \nabla\varphi_n|^2 dP_n \leq \int \log\left(\frac{g_n(\nabla\varphi_n)}{f_n}\right) dP_n.$$

To see that the right-hand side is finite, let us estimate

$$\begin{aligned} \int \log\left(\frac{g_n(\nabla\varphi_n)}{f_n}\right) dP_n &\leq \int \log \frac{1}{f_n} dP_n + \frac{1}{2} \int \log^2 g_n(\nabla\varphi_n) f_n dP_n + \frac{1}{2} \int \frac{dP_n}{f_n} \\ &= \int \log \frac{1}{f_n} dP_n + \frac{1}{2} \int g_n \log^2 g_n dQ_n + \frac{1}{2} \int \frac{dP_n}{f_n}. \end{aligned}$$

Applying assumption (3.a) of the theorem and the Jensen inequality, one can easily get that the right-hand side is uniformly bounded.

We complete the proof by applying Theorem 3.4 and Proposition 3.7. For the application of Proposition 3.7 set

$$f_n = \sum_{i=1}^n u_i(x_i), \quad g_n = \sum_{i=1}^n v_i(y_i).$$

We need to estimate  $\sum_{i=1}^n \int (u_i(x_i) + v_i(y_i) - x_i y_i) d\pi_n$ . Taking into account that  $\pi_n$  is supported on the graph of  $\nabla\varphi_n$  and the relation  $u_i(x_i) + v_i(T_i(x)) = x_i T_i(x)$ , we obtain that the latter equals

$$\begin{aligned} &\int (u_i(x_i) + v_i(\partial_{x_i}\varphi_n) - x_i \partial_{x_i}\varphi_n(x)) d\mu_n \\ &= \int [v_i(\partial_{x_i}\varphi_n(x)) - v_i(T_i(x)) - x_i(\partial_{x_i}\varphi_n(x) - T_i(x))] d\mu_n \\ &= \int [v_i(\partial_{x_i}\varphi_n(x)) - v_i(T_i(x)) - v'_i(T_i(x))(\partial_{x_i}\varphi_n(x) - T_i(x))] d\mu_n \\ &\leq M \int (\partial_{x_i}\varphi_n(x) - T_i)^2 d\mu_n. \end{aligned}$$

Here we use the uniform bound  $v'_i = S'_i \leq M$ . Finally, using the uniform bound  $f \leq C$  and the Jensen inequality we obtain that

$$\sum_{i=1}^n \int (u_i(x_i) + v_i(y_i) - x_i y_i) d\pi_n \leq MC \int |\nabla\varphi_n - \tilde{T}_n|^2 dP_n.$$

We have already shown that the right-hand side is bounded. The result now follows from Proposition 3.7.

The proof follows along the same lines under assumption (3.b), but we use another corollary of Proposition 2.3:

$$\frac{K}{2} \int |\tilde{T}_n - \nabla\varphi_n|^2 \frac{f_n}{g_n(\nabla\varphi_n)} dP_n \leq \int \log\left(\frac{f_n}{g_n(\nabla\varphi_n)}\right) \frac{f_n}{g_n(\nabla\varphi_n)} dP_n.$$

The details are left to the reader.  $\square$

## 5. Symmetric transportation problem and ergodic decomposition of optimal transportation plans

### 5.1. Symmetric transportation problem

In this section we discuss the mass transportation of symmetric (mainly exchangeable) measures, where the word ‘‘symmetric’’ means ‘‘invariant under the action

of a group  $\Gamma$ ." Recall that a probability measure is exchangeable if it is invariant with respect to any permutation of a finite number of coordinates. Before we consider  $\mathbb{R}^\infty$ , let us make some remarks on the finite-dimensional case.

Consider the group  $S_d$  of all permutations of  $\{1, \dots, d\}$  acting on  $\mathbb{R}^d$  as follows:

$$L_\sigma(x) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(d)}), \quad \sigma \in S_d.$$

Let  $\Gamma \subset S_d$  be any subgroup with the property that for every couple  $i, j$  there exists  $\sigma \in \Gamma$  such that  $\sigma(i) = j$ .

Assume that the source and target measures are both invariant with respect to  $\Gamma$ . Under the additional assumption that the cost function  $c$  is  $\Gamma$ -invariant (e.g.,  $c = |x - y|^2$ ), one can easily check that the Kantorovich potential  $\varphi$  is  $\Gamma$ -invariant as well:  $\varphi = \varphi \circ L_\sigma$  for any  $\sigma \in \Gamma$  (see [21], [26]). Consequently, the optimal transportation  $T = \nabla\varphi$  has the following commutation property:

$$T = L_\sigma^*(T \circ L_\sigma) = L_\sigma^{-1} \circ T \circ L_\sigma.$$

Equivalently,

$$L_\sigma \circ T = T \circ L_\sigma.$$

The optimal transportation plan  $\pi(dx, dy)$  is also  $\Gamma$ -invariant under the following extension of the action of  $\Gamma$  to  $\mathbb{R}^d \times \mathbb{R}^d$ :

$$L_\sigma(x, y) = (L_\sigma x, L_\sigma y).$$

Now let  $\sigma(i) = j$ . One has

$$\begin{aligned} \int x_i y_j d\pi &= \int \langle e_i, x \rangle \langle e_j, y \rangle d\pi = \int \langle L_\sigma e_i, L_\sigma x \rangle \langle L_\sigma e_i, L_\sigma y \rangle d\pi \\ &= \int \langle e_j, L_\sigma x \rangle \langle e_j, L_\sigma y \rangle d\pi = \int x_j y_j d\pi. \end{aligned}$$

Consequently,

$$(4) \quad W_2^2(\mu, \nu) = \int \|x - y\|^2 d\pi = \sum_{i=1}^d \int (x_i - y_i)^2 d\pi = d \int (x_i - y_i)^2 d\pi, \quad \forall i.$$

**LEMMA 5.1**

*The standard quadratic Kantorovich problem on  $\mathbb{R}^d$  with  $\Gamma$ -invariant marginals is equivalent to the transportation problem for the cost  $|x_1 - y_1|^2$  with the additional constraint that the solution is a  $\Gamma$ -invariant probability measure.*

*Proof*

Let  $\pi$  be the solution to the quadratic Kantorovich problem for the marginals  $\mu, \nu$ , and let  $\tilde{\pi}$  be a measure minimizing the functional  $m \mapsto \int |x_1 - y_1|^2 dm$  among the  $\Gamma$ -invariant measures with the same marginals. By the optimality of  $\pi$

$$\int \|x - y\|^2 d\pi \leq \int \|x - y\|^2 d\tilde{\pi}.$$

Since  $\pi$  and  $\tilde{\pi}$  are both  $\Gamma$ -invariant, (4) implies that  $\int |x_1 - y_1|^2 d\pi \leq \int |x_1 - y_1|^2 d\tilde{\pi}$ . By the optimality of  $\tilde{\pi}$  one gets  $\int |x_1 - y_1|^2 d\pi = \int |x_1 - y_1|^2 d\tilde{\pi}$ , and finally,  $\int \|x - y\|^2 d\pi = \int \|x - y\|^2 d\tilde{\pi}$ . This means that  $\tilde{\pi}$  solves the quadratic Kantorovich problem as well, and vice versa,  $\pi$  solves the Kantorovich problem with symmetric constraints.  $\square$

The conclusion made above helps us to give a variational meaning to the transportation problem in the infinite-dimensional case.

**DEFINITION 5.2 (SYMMETRIC KANTOROVICH PROBLEM)**

Let  $\Gamma$  be a group of linear operators acting on  $\mathbb{R}^\infty$ , and let  $\mu, \nu$  be  $\Gamma$ -invariant probability measures. Assume in addition that the following statements hold.

- For every  $i, j \in \mathbb{N}$  there exists  $g \in \Gamma$  such that

$$g(e_i) = e_j.$$

- The space of probability measures  $\Pi^\Gamma(\mu, \nu)$  on  $\mathbb{R}^\infty \times \mathbb{R}^\infty$  that are invariant with respect to the action  $(x, y) \mapsto (g(x), g(y))$ ,  $g \in \Gamma$ , of  $\Gamma$  and have marginals  $\mu, \nu$  is nonempty and closed in the weak topology.

We say that a measure  $\pi \in \Pi^\Gamma(\mu, \nu)$  is a solution to the  $\Gamma$ -symmetric (quadratic) Kantorovich problem if it minimizes the functional

$$(5) \quad \Pi^\Gamma(\mu, \nu) \ni m \mapsto \int (x_1 - y_1)^2 dm.$$

**DEFINITION 5.3 (SYMMETRIC OPTIMAL TRANSPORTATION)**

Let  $m$  be a solution to the symmetric Kantorovich problem. A measurable mapping  $T: \mathbb{R}^\infty \mapsto \mathbb{R}^\infty$  is called the optimal transportation mapping of  $\mu$  onto  $\nu$  if

$$m(\{(x, T(x))\}) = 1.$$

The standard compactness arguments imply that a solution to the Kantorovich problem (5) exists provided  $\int x_1^2 d\mu < \infty$ ,  $\int y_1^2 d\nu < \infty$ . If, in addition, there exists an optimal transportation mapping  $T$ , then it commutes with any  $g \in \Gamma$ . This means that for  $\mu$ -almost all  $x$  and every  $g \in \Gamma$

$$(6) \quad (T \circ g)(x) = (g \circ T)(x).$$

**EXAMPLE 5.4 (EXCHANGEABLE MEASURES)**

We denote by  $S_\infty$  the group of permutations of  $\mathbb{N}$  which change only a finite number of coordinates. We consider its natural action on  $\mathbb{R}^\infty$  defined by

$$\sigma(x) = (x_{\sigma(i)}), \quad x = (x_i) \in \mathbb{R}^\infty, \sigma \in S_\infty.$$

Consider measures  $\mu$  and  $\nu$  which are invariant with respect to any  $\sigma \in S_\infty$ :

$$\mu = \mu \circ \sigma^{-1}, \quad \nu = \nu \circ \sigma^{-1}.$$

The measures of this type are called *exchangeable*. The basic example is given by the countable power  $m^\infty$  of some Borel measure  $m$  on  $\mathbb{R}$ . The structure of mappings satisfying (6) in the case  $\mu = m^\infty$  is very easy to describe. Consider the function  $T_1(x) = \langle T(x), e_1 \rangle$ , and fix the first coordinate  $x_1$ . Then the function  $F: (x_2, x_3, \dots) \rightarrow T_1(x)$  is invariant with respect to  $S_\infty$  (acting on  $(x_2, x_3, \dots)$ ). Hence,  $F$  is constant according by the Hewitt–Savage zero–one law applied to the measure  $\mu$ . Thus,  $T_1(x) = T_1(x_1)$  depends on  $x_1$  only (up to a set of measure zero). The same arguments applied to other coordinates imply that  $T$  is diagonal:  $(T_1(x_1), T_2(x_2), \dots)$ . Moreover,  $T_i(x) = T_1(x)$  because  $T$  commutes with every permutation of coordinates.

**EXAMPLE 5.5 (OPTIMAL TRANSPORTATION DOES NOT ALWAYS EXIST)**

Let  $\mu_1, \mu_2$  be countable powers of two different one-dimensional measures. By the Kakutani dichotomy theorem they are mutually singular. There is no mass transportation  $T$  of  $\mu = \mu_1$  onto  $\nu = \frac{1}{2}(\mu_1 + \mu_2)$  satisfying (6). Indeed, according to Example 5.4 any  $T$  satisfying (6) must be diagonal; hence, the measure  $\mu \circ T^{-1}$  must be a product measure.

Thus, we see that the optimal transportation does not always exist. This example can be easily generalized to many other linear groups  $\Gamma$  and  $\Gamma$ -invariant measures. It can be easily understood that  $T$  does not exist provided the source measure is ergodic, but the target measure is not.

**5.2. Ergodic decomposition of optimal transportation plans**

The connection between the Kantorovich problem and ergodic decomposition has been established under fairly general assumptions by the second author [25]. A particular case of this result is given in the following theorem.

Let  $\Gamma$  be an amenable group acting by continuous one-to-one mappings on a Polish space  $X$ . Let  $\Pi^\Gamma$  be the set of all Borel probability  $\Gamma$ -invariant measures, and let  $\mu, \nu \in \Pi^\Gamma$ . The set of  $\Gamma$ -invariant transportation plans with marginals  $\mu, \nu$  will be denoted by  $\Pi^\Gamma(\mu, \nu)$ . Assume that the cost function  $c$  is lower semicontinuous and that  $\Pi^\Gamma(\mu, \nu)$  is nonempty and closed in the weak topology.

Let us fix a solution  $\pi$  to the  $\Gamma$ -invariant Kantorovich problem with marginals  $\mu, \nu$ . Denote by  $\Delta(X)$  the set of all  $\Gamma$ -invariant ergodic measures on  $X$ . Assume that we are given ergodic decompositions

$$(7) \quad \mu = \int_{\Delta(X)} \mu^x d\sigma_\mu, \quad \nu = \int_{\Delta(Y)} \nu^y d\sigma_\nu$$

of  $\mu, \nu$ , where  $X = Y$  and  $\sigma_\mu, \sigma_\nu$  are probability measures on  $\Delta(X), \Delta(Y)$ . Similarly, assume that we are given the ergodic decomposition of  $\pi$

$$(8) \quad \pi = \int_{\Delta(X \times Y)} \pi^{x,y} d\delta.$$

(Recall that the  $\Gamma$ -invariance for  $\pi$  means the invariance with respect to the action  $(x, y) \mapsto (g(x), g(y))$ .) We stress that in (7) the integrals are taken not

with respect to variables  $x, y$ , but with respect to variables  $\mu^x, \nu^y$ . (Here,  $x, y$  indicate the spaces where the measures are defined.) The same holds for (8). It is straightforward that  $\delta$ -almost all  $\pi^{x,y}$ 's have ergodic marginals, and taking the projections of both sides of (8), we obtain decompositions (7). Moreover, the following statement holds.

**THEOREM 5.6**

*Under  $\delta$  almost every measure  $\pi^{x,y}$  solves the  $\Gamma$ -symmetric Kantorovich problem with marginals  $\mu^x, \nu^y$ :*

$$K_c^\Gamma(\mu^x, \nu^y) = \inf_{m \in \Pi^\Gamma(\mu^x, \nu^y)} \int c dm = \int c d\pi^{x,y},$$

*and the following representation formula holds:*

$$\inf_{\pi \in \Pi^\Gamma(\mu, \nu)} \int c d\pi = \inf_{\delta \in \Pi(\sigma_\mu, \sigma_\nu)} \int K_c^\Gamma(\mu^x, \nu^y) d\delta.$$

**REMARK 5.7**

In the situation of Theorem 5.6 one can decompose the optimal transportation plan for ergodic marginals  $\mu, \nu$ :  $\pi = \int_{\Delta(X \times Y)} \pi^{x,y} d\delta$ . The ergodicity of the marginals implies immediately that  $\delta$ -almost all  $\pi^{x,y}$ 's have the same marginals  $\mu$  and  $\nu$ . The optimality of  $\pi^{x,y}$  for the cost  $c$  follows from Theorem 5.6. Thus, we get that any solvable symmetric Kantorovich problem with ergodic marginals admits, in particular, an ergodic solution.

Thus, the symmetric transportation problem can be reduced to the following steps.

(Q1) Construct a solution to the symmetric Kantorovich problem for ergodic measures.

(Q2) Given two nonergodic measures  $\mu, \nu$  and the corresponding ergodic decompositions (7), construct a solution to the Kantorovich problem for measures  $\sigma_\mu, \sigma_\nu$  on  $\Delta(X)$  with cost function  $K_c^\Gamma$ .

Consider the application of Theorem 5.6 to several classical groups.

**EXAMPLE 5.8 (EXCHANGEABLE MEASURES REVISITED)**

Consider the invariant transportation problem for exchangeable measures and  $c = (x_1 - y_1)^2$ . The answer to (Q1) is trivial, because ergodic measures are countable powers and the structure of the corresponding solution is trivial. As for (Q2), by the de Finetti theorem the space of ergodic measures is isomorphic to the space  $\mathcal{P}(\mathbb{R})$  of probability measures on  $\mathbb{R}$ . Thus, to resolve an optimal transportation problem for exchangeable measures, we need to study the optimal transportation problem for a couple of measures  $\mu_0, \nu_0$  on  $\mathcal{P}(\mathbb{R})$  arising from the de Finetti decomposition. It is clear that the cost function  $c$  on  $\mathcal{P}(\mathbb{R})$  satisfies

$$c(p_1, p_2) = W_2^2(p_1, p_2),$$

where  $W_2$  is the standard Kantorovich distance on  $\mathbb{R}$ .

## EXAMPLE 5.9 (ROTATIONALLY INVARIANT MEASURES)

Consider the invariant transportation problem for measures invariant with respect to operators of the type  $U \times \text{Id}$ , where  $U$  is a rotation of  $\mathbb{R}^n = \text{Pr}_n(\mathbb{R}^\infty)$  and  $\text{Id}$  is the identical operator on the orthogonal complement to  $\mathbb{R}^n$ . As usual,  $c = (x_1 - y_1)^2$ . This is an example where the optimal transportation problem admits a precise solution. By a well-known result (see [14]), every rotationally invariant measure  $\mu$  on  $\mathbb{R}^\infty$  admits a representation

$$\mu = \int \gamma_t dp_\mu(t),$$

where  $\gamma_t$  is the distribution of the Gaussian independent and identically distributed with zero mean and variance  $t$  and  $p_\mu$  is a measure on  $\mathbb{R}_+$ . The optimal transportation problem is obviously reduced to the one-dimensional optimal transportation between  $p_\mu$  and  $p_\nu$ .

## EXAMPLE 5.10 (STATIONARY MEASURES)

These are the measures which are invariant with respect to the shift

$$T: x = (x_1, x_2, \dots) \mapsto (x_2, x_3, \dots).$$

Note that the powers of  $T$  generate the semigroup  $\{0\} \cup \mathbb{N}$  but not the group. However, it makes no difference for our analysis, since we are still able to consider the corresponding ergodic decompositions. In this case the description of ergodic measures is nontrivial, and we do not know any general sufficient conditions for existence even in the case in which both measures are ergodic. Some sufficient conditions are given in Section 7.

We conclude this section by remarking that the existence of a transportation mapping for a (not necessary optimal) symmetric plan  $\pi$  with ergodic  $X$ -marginal implies the ergodicity of  $\pi$ .

## PROPOSITION 5.11

*Let  $X = Y$  be a Polish space, and let  $\Gamma$  be a group of Borel one-to-one transformations acting on  $X$ . Assume that  $\pi$  and  $\mu$  are  $\Gamma$ -invariant Borel probability measures on  $X \times Y$  and  $X$ , respectively. Assume, in addition, that  $\text{Pr}_X \pi = \mu$ ,  $\mu$  is ergodic and that  $\pi(\{x, T(x)\}) = 1$  for some Borel mapping  $T$ . Then  $\pi$  is ergodic.*

*Proof*

Assuming the contrary, we represent  $\pi$  as a convex combination of two  $\Gamma$ -invariant measures

$$\pi = \lambda\pi_1 + (1 - \lambda)\pi_2,$$

$\pi_1 \neq \pi_2$ ,  $0 < \lambda < 1$ . Clearly, this implies a similar decomposition for the projections  $\mu = \lambda\text{Pr}_X \pi_1 + (1 - \lambda)\text{Pr}_X \pi_2$ . If we show that  $\mu_1, \mu_2$  are  $\Gamma$ -invariant and distinct, we will get a contradiction. The  $\Gamma$ -invariance of both measures follows

immediately from the  $\Gamma$ -invariance of  $\pi_i$ . Let us show that  $\mu_1 \neq \mu_2$ . Assume the contrary, and take a Borel set  $B \subset X \times Y$ . We get that  $\pi_i(B)$  equals  $\mu_i(A)$ , where  $A = \text{Pr}_X(B \cap \text{Graph}(T))$ . (Note that  $A$  is universally measurable as a projection of a Borel set.) Then it follows that the  $\pi_i$ 's coincide because the  $\mu_i$ 's do.  $\square$

## 6. Kantorovich duality

In this section we start to study measures which are invariant under actions of some group. The results of this section will not be used in this article, but they are of independent interest.

Let  $X, Y$  be Polish spaces, and let  $\Gamma$  be a locally compact amenable group with *continuous* actions  $L_\Gamma^X, L_\Gamma^Y$  on  $X, Y$ , respectively. The action  $L_\Gamma$  on the product space  $X \times Y$  is defined as

$$L_g(x, y) = (L_g(x), L_g(y)),$$

where  $L_g$  is an element of  $L_\Gamma$  corresponding to  $g \in \Gamma$ .

Let us define the space  $W_\Gamma \subset C_b(X \times Y)$  as the closure of the linear span of the set

$$\{f - f \circ L_g : f \in C_b(X \times Y), g \in \Gamma\}.$$

It can be checked that the property

$$(9) \quad \int \omega d\pi = 0, \quad \forall \omega \in W_\Gamma,$$

of a probability measure  $\pi \in \mathcal{P}(X \times Y)$  is equivalent to its invariance with respect to  $L_\Gamma$ .

Let  $\mu \in \mathcal{P}(X), \nu \in \mathcal{P}(Y)$  be invariant under the actions  $L_\Gamma^X, L_\Gamma^Y$ , respectively. Then a transport plan  $\pi \in \Pi(\mu, \nu)$  is invariant if and only if the property (9) is satisfied. We denote the set of all invariant transport plans by  $\Pi^\Gamma(\mu, \nu)$ .

The following theorem is a refinement of the duality result, which was proved in [26, Theorem 2.5]. That result only considered  $C_b(X \times Y)$  cost functions. (We warn the reader that the classical duality statement from our Section 2 is formulated in a slightly different but equivalent way: in the notation of this section  $\Phi = \frac{x^2}{2} - \varphi, \Psi = \frac{y^2}{2} - \psi$ .)

### THEOREM 6.1

Let  $c \in C(X \times Y)$  be a nonnegative function such that there exist  $f \in L^1(X, \mu), g \in L^1(Y, \nu)$ , and

$$c(x, y) \leq f(x) + g(y), \quad \forall (x, y) \in X \times Y.$$

Then, in the setting described above,

$$\inf_{\pi \in \Pi^\Gamma} \int c d\pi = \sup_{\Phi + \Psi + \omega \leq c} \int_X \Phi(x) d\mu + \int_Y \Psi(y) d\nu,$$

where  $\Phi \in L^1(X), \Psi \in L^1(Y), \omega \in W_\Gamma$ .

*Proof*

The inequality

$$\inf_{\pi \in \Pi^\Gamma} \int c \, d\pi \geq \sup_{\Phi + \Psi + \omega \leq c} \int \Phi \, d\mu + \int \Psi \, d\nu$$

can be easily obtained:

$$\begin{aligned} \inf_{\pi \in \Pi^\Gamma} \int c \, d\pi &\geq \inf_{\pi \in \Pi^\Gamma} \left( \sup_{\Phi + \Psi + \omega \leq c} \int (\Phi + \Psi + \omega) \, d\pi \right) \\ &= \inf_{\pi \in \Pi^\Gamma} \left( \sup_{\Phi + \Psi + \omega \leq c} \int \Phi \, d\mu + \int \Psi \, d\nu \right) = \sup_{\Phi + \Psi + \omega \leq c} \int \Phi \, d\mu + \int \Psi \, d\nu. \end{aligned}$$

To obtain the opposite inequality we use the following statement from [26, Theorem 2.5]:

$$\inf_{\pi \in \Pi^\Gamma} \int c_b \, d\pi = \sup_{\Phi + \Psi + \omega \leq c_b} \int_X \Phi(x) \, d\mu + \int_Y \Psi(y) \, d\nu,$$

for  $c_b \in C_b(X \times Y)$ ,  $\Phi \in C_b(X)$ ,  $\Psi \in C_b(Y)$ ,  $\omega \in W_\Gamma$ . Let  $c_n(x, y) := \min\{c(x, y), n\}$  for each  $n \in \mathbb{N}$ . The inequality

$$\sup_{\Phi + \Psi + \omega \leq c_n} \int_X \Phi(x) \, d\mu + \int_Y \Psi(y) \, d\nu \leq \sup_{\Phi + \Psi + \omega \leq c} \int_X \Phi(x) \, d\mu + \int_Y \Psi(y) \, d\nu$$

is obvious for any natural  $n$ . Thus, it remains to prove that

$$\lim_{n \rightarrow \infty} \inf_{\pi \in \Pi^\Gamma} \int c_n \, d\pi = \inf_{\pi \in \Pi^\Gamma} \int c \, d\pi.$$

Recall that the functional  $\pi \rightarrow \int c_b \, d\pi$  is weakly continuous for every  $c_b \in C_b(X \times Y)$ . It follows, from the characterization (9) of invariant measures, that  $\Pi^\Gamma(\mu, \nu)$  is a closed subset of  $\Pi(\mu, \nu)$ , which is known to be compact. Thus,  $\Pi^\Gamma(\mu, \nu)$  is compact in the topology of weak convergence. If  $\pi_n$  is the solution for

$$\inf_{\pi \in \Pi^\Gamma} \int c_n \, d\pi,$$

then the sequence  $(\pi_n)$  has to have a subsequence converging to some element  $\pi^* \in \Pi^\Gamma$ . Since for any fixed  $m \in \mathbb{N}$  the inequality  $\lim_{n \rightarrow \infty} \int c_n \, d\pi^* \geq \int c_m \, d\pi^*$  is satisfied and, by the monotone convergence theorem,  $\lim_{m \rightarrow \infty} \int c_m \, d\pi^* = \int c \, d\pi^* \leq \int (f(x) + g(y)) \, d\pi^* < \infty$ , we obtain

$$\lim_{n \rightarrow \infty} \int c_n \, d\pi_n \geq \lim_{m \rightarrow \infty} \int c_m \, d\pi^* = \int c \, d\pi^* \geq \inf_{\pi \in \Pi^\Gamma} \int c \, d\pi.$$

This fact concludes the proof of the theorem.  $\square$

As one can see, the form of the duality theorem is similar to the well-known classic result, but the difference is substantial: dual functionals are related to each other in a more complicated way. Moreover, there is no existence result for the dual problem without any additional assumptions.

It was shown in [26, Theorem 5.7] that, in the case of compact group  $\Gamma$  and under the assumptions of Theorem 6.1,

$$\inf_{\pi \in \Pi^\Gamma} \int c d\pi = \sup_{\Phi + \Psi \leq \bar{c}} \int_X \Phi(x) d\mu + \int_Y \Psi(y) d\nu,$$

where  $\bar{c} := \int_\Gamma (c \circ g) d\chi(g)$  and  $\chi(g)$  is the probability Haar measure. It is clear that if the cost function is  $\Gamma$ -invariant, then the invariant dual problem coincides with the usual one. Moameni [21] proved that, for  $\Gamma = \mathbb{Z}$  and an invariant cost function  $c$ , the corresponding invariant dual problem coincides with the usual one, and moreover, both prime and dual Kantorovich problems have an invariant solution.

## 7. Existence of invariant optimal mapping for stationary measures

Recall that the measures on  $\mathbb{R}^\infty$  which are invariant with respect to the shift

$$\sigma(x_1, x_2, \dots) = (x_2, x_3, \dots)$$

are called *stationary measures*. Unlike exchangeable measures, the projections of stationary measures are in general not invariant with respect to some reasonable family of linear transformations.

As usual we assume that  $\mathbb{R}^\infty$  is approximated by the sequence of finite-dimensional spaces  $\mathbb{R}^n$  in the following sense: we identify  $\mathbb{R}^n$  with the subset

$$P_n(\mathbb{R}^\infty) = \{x = (x_1, x_2, \dots, x_n, 0, 0, \dots)\} \subset \mathbb{R}^\infty.$$

On every finite-dimensional space  $\mathbb{R}^n$  we will apply the following operator of cyclical shift:

$$\sigma_n(x_1, x_2, \dots, x_n) = (x_2, x_3, \dots, x_n, x_1).$$

Let us associate with every stationary measure  $\mu$  the cyclical average of its projections:

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^n (\mu \circ P_n^{-1}) \circ \sigma_n^{-(i-1)}.$$

In addition, let us denote by  $\mathbb{R}_{m,n}$  the orthogonal complement of  $\mathbb{R}^m \subset \mathbb{R}^n$ :

$$\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}_{m,n}, \quad m < n.$$

The marginal measures are always assumed to satisfy the following property.

### ASSUMPTION B

The measures  $\mu, \nu$  are stationary Borel probability measures such that their projections on every  $\mathbb{R}^n$

$$\mu \circ \text{Pr}_n^{-1}, \quad \nu \circ \text{Pr}_n^{-1}$$

have Lebesgue densities and bounded second moments.

We consider the symmetric Monge–Kantorovich problem

$$(10) \quad \int (x_1 - y_1)^2 d\pi \rightarrow \min,$$

where the infimum is taken among all stationary measures  $\Pi^\Gamma(\mu, \nu)$  with marginals  $\mu, \nu$ .

REMARK 7.1

Minimizing  $\int (x_1 - y_1)^2 d\pi$  is equivalent to maximizing  $\int x_1 y_1 d\pi$ , because  $\int x_1^2 d\pi = \int x_1^2 d\mu$  and  $\int y_1^2 d\pi = \int y_1^2 d\nu$  are fixed.

THEOREM 7.2

Let  $\mu$  be a stationary measure which satisfies the following assumptions.

- (1)  $\mu$  is a weak limit of a sequence of  $\sigma_n$ -invariant measures  $\mu_n$  on  $\mathbb{R}^n$ .
- (2) For every  $m < n$  there exists a probability measure  $\mu_{m,n}$  on  $\mathbb{R}_{m,n}$  such that the relative entropy (the Kullback–Leibler distance) between  $\mu_m \times \mu_{m,n}$  and  $\mu_n$  is uniformly bounded in  $n$ :

$$\int \log\left(\frac{d\mu_n}{d(\mu_m \times \mu_{m,n})}\right) d\mu_n < C_m,$$

with  $C_m$  satisfying

$$\lim_m \frac{C_m}{m} = 0.$$

- (3) The cyclical average  $\hat{\mu}_n$  of the  $n$ -dimensional projection  $\mu \circ P_n^{-1}$  has finite second moments and admits a density  $\rho_n$  with respect to  $\mu$  satisfying

$$\sup_n \int \rho_n^{-\varepsilon} d\mu < \infty$$

for some  $\varepsilon > 0$ .

Then there exists a mapping  $T$  with the following properties.

- $T$  pushes forward  $\mu$  onto the standard Gaussian measure on  $\mathbb{R}^\infty$ :

$$\nu = \gamma.$$

- $T$  is a  $\mu$ -almost everywhere limit of finite-dimensional mappings  $T_n : \mathbb{R}^n \mapsto \mathbb{R}^n$  such that every  $T_n$  is a solution to an optimal transportation problem on  $\mathbb{R}^n$ .

*Proof*

We consider the sequence of  $n$ -dimensional optimal transportation mappings  $T_n$  with cost function  $\sum_{i=1}^n (x_i - y_i)^2$  pushing forward  $\mu_n$  onto  $\gamma_n$ . It follows from the  $\sigma_n$ -invariance of  $\mu_n$  and  $\gamma_n$  that the mapping  $T_n$  is cyclically invariant:

$$\langle T_n \circ \sigma_n, e_i \rangle = \langle T_n, e_{i-1} \rangle, \quad \mu_n\text{-a.e.}$$

Fix a couple of numbers  $m, n$  with  $n > m$ . Let  $T_{m,n}$  be the optimal transportation mapping for the cost function  $\sum_{i=m+1}^n (x_i - y_i)^2$  pushing forward  $\mu_{m,n}$

onto the standard Gaussian measure on  $\mathbb{R}_{m,n}$ . We stress that  $T_m$  and  $T_{m,n}$  depend on different collections of coordinates.

We extend  $T_m$  onto  $\mathbb{R}^n$  in the following way:

$$T_m(x) = T_m(P_m x) + T_{m,n}(P_{m,n} x).$$

Clearly,  $T_m$  pushes forward  $\mu_m \times \mu_{m,n}$  onto the standard Gaussian measure on  $\mathbb{R}^n$ . Applying Proposition 2.3 to the couple of mappings  $T_m, T_n$ , we get

$$(11) \quad \frac{1}{2} \int \|T_n - T_m\|^2 d\mu_n \leq \int \log \left( \frac{d\mu_n}{d(\mu_m \times \mu_{m,n})} \right) d\mu_n.$$

This implies that

$$(12) \quad \sum_{i=1}^m \int \langle T_n - T_m, e_i \rangle^2 d\mu_n \leq \int \|T_n - T_m\|^2 d\mu_n \leq 2C_m$$

for every  $m, n, m < n$ .

Let us note that for every  $i$  one can extract a weakly convergent subsequence from a sequence of (signed) measures  $\{\langle T_n, e_i \rangle \cdot \mu_n\}$ . Indeed, for any compact set  $K$

$$\left( \int_{K^c} |\langle T_n, e_i \rangle| d\mu_n \right)^2 \leq \int |\langle T_n, e_i \rangle|^2 d\mu_n \cdot \mu_n(K^c) = \int x_i^2 d\gamma \cdot \mu_n(K^c).$$

Using the tightness of  $\{\mu_n\}$  we get that  $\{|\langle T_n, e_i \rangle| \cdot \mu_n\}$  is a tight sequence. In addition, note that for every continuous  $f$

$$\lim_n \left( \int f |\langle T_n, e_i \rangle| d\mu_n \right)^2 \leq \int x_i^2 d\gamma \cdot \int f^2 d\mu.$$

This implies that any limiting point of  $\{\langle T_n, e_i \rangle \cdot \mu_n\}$  is absolutely continuous with respect to  $\mu$ . Applying the diagonal method and passing to a subsequence, one can assume that convergence takes place for all  $i$ 's simultaneously. Consequently, there exists a subsequence  $\{n_k\}$  and a measurable mapping  $T$  with values in  $\mathbb{R}^\infty$  such that

$$\langle T_{n_k}, e_i \rangle \cdot \mu_{n_k} \rightarrow \langle T, e_i \rangle \cdot \mu$$

weakly in the sense of measures for every  $i$ . It is easy to check that the standard property of  $L^2$ -weak convergence holds also in this case:

$$(13) \quad \int \langle T, e_i \rangle^2 d\mu \leq \underline{\lim}_k \int \langle T_{n_k}, e_i \rangle^2 d\mu_{n_k} = \int x_i^2 d\gamma = 1.$$

Finally, we pass to the limit in (12) and get

$$(14) \quad \sum_{i=1}^m \int \langle T - T_m, e_i \rangle^2 d\mu \leq 2C_m.$$

The claim follows from (13) and the fact that  $\lim_n \int \varphi d\mu_n = \int \varphi d\mu$  for every  $\varphi \in L^2(\mu)$ . Indeed, if  $\varphi$  is bounded and continuous, this follows from the weak convergence  $\langle T_n, e_i \rangle \cdot \mu_n \rightarrow \langle T, e_i \rangle \cdot \mu$ . For arbitrary  $\varphi \in L^2(\mu)$  we find a continuous bounded cylindrical function  $\tilde{\varphi}$  such that  $\|\varphi - \tilde{\varphi}\|_{L^2(\mu)} < \varepsilon$ . One has

$\lim_n \int \varphi d\mu_n = \lim_n \int (\varphi - \tilde{\varphi}) d\mu_n + \int \tilde{\varphi} d\mu$ . The claim follows from the estimate

$$\left( \int |\varphi - \tilde{\varphi}| d\mu_n \right)^2 \leq \int (\varphi - \tilde{\varphi})^2 d\mu \cdot \int \rho_n^2 d\mu \leq \left( \sup_n \int \rho_n^2 d\mu \right) \varepsilon^2.$$

Note that  $T$  commutes with the shift  $\sigma$ :  $\langle T \circ \sigma, e_i \rangle = \langle T, e_{i-1} \rangle$ . Indeed, for every bounded cylindrical  $\varphi$  one has

$$\begin{aligned} \int \varphi \langle T_n, e_{i-1} \rangle d\mu_n &= \int \varphi \langle T_n(\sigma_n), e_i \rangle d\mu_n = \int \varphi(\sigma_n^{-1}) \langle T_n, e_i \rangle d\mu_n \\ &= \int \varphi(\sigma^{-1}) \langle T_n, e_i \rangle d\mu_n. \end{aligned}$$

Here we use that  $\varphi(\sigma_n^{-1}) = \varphi(\sigma^{-1})$  for sufficiently large values of  $n$  and the cyclical invariance of  $T_n$ . Passing to the limit in the  $n_k$ -subsequence one gets

$$\int \varphi \langle T, e_{i-1} \rangle d\mu = \int \varphi(\sigma^{-1}) \langle T, e_i \rangle d\mu = \int \varphi \langle T \circ \sigma, e_i \rangle d\mu.$$

Hence,  $T \circ \sigma = \sigma \circ T$ .

Hence, by the assumptions of the theorem and (14) we get

$$\limsup_m \frac{1}{m} \sum_{i=1}^m \int \langle T - T_m, e_i \rangle^2 d\mu = 0.$$

To prove that  $T$  pushes forward  $\mu$  onto  $\gamma$  it is sufficient to show that  $\langle T_m, e_i \rangle \rightarrow \langle T, e_i \rangle$  in measure (see Lemma 3.2). To this end, let us approximate  $T_1$  by a bounded function  $\xi_1(x_1, \dots, x_k)$  depending on a finite number of coordinates in  $L^2(\mu)$ :  $\int \|T_1 - \xi_1\|^2 d\mu < \varepsilon$ , where  $\varepsilon$  is chosen sufficiently small. Set:  $\xi_i = \xi \circ \sigma^{i-1}$ . Clearly, we get by the shift invariance

$$\frac{1}{m} \int \sum_{i=1}^m (T_i - \xi_i)^2 d\mu = \int (T_1 - \xi_1)^2 d\mu < \varepsilon.$$

Hence,

$$\limsup_m \frac{1}{m} \int \|T_m - \xi\|^2 d\mu \leq \varepsilon, \quad \xi = (\xi_1, \xi_2, \dots).$$

Make the change of variables under the cyclical shift  $\sigma_n$ . One has

$$\langle T_m, e_i \rangle \circ \sigma_m^{-(i-1)} = T_1$$

for all  $1 \leq i \leq m$  and

$$\xi_i \circ \sigma_m^{-(i-1)} = \xi_1$$

as soon as  $i - 1 + k \leq m$ . Hence, for the latter values of  $i$  one has

$$\int \langle \xi - T, e_i \rangle^2 d\mu = \int \langle \xi - T, e_1 \rangle^2 d\mu \circ \sigma_n^i.$$

The number of indices which do not satisfy this property is limited by  $k$ . Clearly, it does not affect the limit of averages. Finally, we obtain

$$\varepsilon \geq \limsup_m \frac{1}{m} \int \sum_{i=1}^n \langle \xi - T_m, e_i \rangle^2 d\mu = \limsup_m \int \langle \xi - T_m, e_1 \rangle^2 d\hat{\mu}_m.$$

Recall that  $\int (T_1 - \xi_1)^2 d\mu \leq \varepsilon$ . Finally,

$$\begin{aligned} \limsup_m \int \langle T - T_m, e_1 \rangle^2 d\hat{\mu}_m &\leq 2 \limsup_m \int \langle \xi - T_m, e_1 \rangle^2 d\hat{\mu}_m \\ &\quad + 2 \limsup_m \int (T_1 - \xi_1)^2 d\hat{\mu}_m \leq 4\varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, one gets  $\int \langle T - T_m, e_1 \rangle^2 d\hat{\mu}_m \rightarrow 0$ . By the Hölder inequality

$$\int \langle T - T_m, e_1 \rangle^{\frac{2}{p}} d\mu \leq \left( \int \langle T - T_m, e_1 \rangle^2 d\hat{\mu}_m \right)^{\frac{1}{p}} \left( \int \rho_m^{-\frac{1}{p-1}} d\mu \right)^{\frac{1}{q}}.$$

Taking  $p = 1 + \frac{1}{\varepsilon}$  we get by the assumption of the theorem that the latter tends to zero. The proof is complete.  $\square$

#### REMARK 7.3

In Theorem 7.2 the Gaussian measure  $\gamma$  can be replaced by any countable power of an uniformly log-concave one-dimensional measure.

In the following proposition we prove that the transportation mapping  $T$  is indeed optimal under additional assumptions.

#### PROPOSITION 7.4

Let the assumptions of Theorem 7.2 hold. Assume in addition that

$$\lim_{n \rightarrow \infty} \frac{1}{n} W_2^2(\hat{\mu}_n, \mu_n) = 0.$$

Then there exists a solution  $\pi$  of problem (10) in the class of stationary measures such that  $\pi\{(x, T(x)), x \in \mathbb{R}^\infty\} = 1$ .

#### Proof

We show that the measure  $\pi = \mu \circ (x, T(x))^{-1}$ , which is the weak limit of measures  $\pi_n$ , is optimal. Recall that  $\pi_n$  minimizes  $m \rightarrow \int \sum_{i=1}^n (x_i - y_i)^2 dm$  and has marginals  $\mu_n, \gamma_n$ ; hence, measure  $\pi$  has marginals  $\mu, \gamma$ . Indeed,

$$\int (x_1 - y_1)^2 d\pi = \lim_n \int (x_1 - y_1)^2 d\pi_n = \lim_n \frac{1}{n} \int \sum_{i=1}^n (x_i - y_i)^2 d\pi_n.$$

If  $\pi$  is not optimal, then there exists a stationary measure  $\pi_0$  with projections  $\mu, \nu$  such that

$$\int (x_1 - y_1)^2 d\pi_0 + \varepsilon < \frac{1}{n} \int \sum_{i=1}^N (x_i - y_i)^2 d\pi_n$$

for some  $\varepsilon > 0$  and all sufficiently big values of  $n$ . Taking into account the stationarity of  $\pi_0$ , we get  $\int x_i y_i d\pi_0 = \int x_j y_j d\pi_0$  for every  $i, j$ . Thus,

$$\int \sum_{i=1}^n (x_i - y_i)^2 d\hat{\pi}_0 + n\varepsilon = \int \sum_{i=1}^n (x_i - y_i)^2 d\pi_0 + n\varepsilon < \int \sum_{i=1}^n (x_i - y_i)^2 d\pi_n,$$

where  $\hat{\pi}_0 = \frac{1}{n} \sum_{i=1}^n (\pi_0 \circ \text{Pr}_n^{-1}) \circ \sigma_n^{-(i-1)}$ . The latter inequality implies that

$$W_2^2(\hat{\mu}_n, \gamma_n) + n\varepsilon \leq W_2^2(\mu_n, \gamma_n).$$

By the triangle inequality

$$\begin{aligned} W_2^2(\hat{\mu}_n, \gamma_n) + n\varepsilon &\leq (W_2(\mu_n, \tilde{\mu}_n) + W_2(\hat{\mu}_n, \gamma_n))^2 \\ &\leq W_2^2(\mu_n, \hat{\mu}_n) + 2W_2(\hat{\mu}_n, \gamma_n)W_2(\mu_n, \hat{\mu}_n) + W_2^2(\hat{\mu}_n, \gamma_n). \end{aligned}$$

Hence,

$$(15) \quad \varepsilon \leq \frac{1}{n} (2W_2(\hat{\mu}_n, \gamma_n)W_2(\mu_n, \hat{\mu}_n) + W_2^2(\hat{\mu}_n, \mu_n)).$$

The quantity  $W_2^2(\hat{\mu}_n, \gamma_n)$  can be trivially estimated by  $2 \sum_{i=1}^n (\int x_i^2 d\hat{\mu}_n + \int y_i^2 d\gamma_n) \leq Cn$ . Then using the assumption of the theorem we get that the right-hand side of (15) tends to zero, which contradicts the positivity of  $\varepsilon$ .  $\square$

We finish this section with a concrete application of Theorem 7.2. We study a transportation of a Gibbs measure  $\mu$  which can be formally written in the form

$$\mu = e^{-H(x)} dx,$$

where the potential  $H$  admits the following heuristic representation:

$$H(x) = \sum_{i=1}^{\infty} V(x_i) + \sum_{i=1}^{\infty} W(x_i, x_{i+1}).$$

Here  $V$  and  $W$  are smooth functions and  $W(x, y)$  is symmetric:  $W(x, y) = W(y, x)$ . The existence of such measures was proved in [1].

Let us specify the assumptions about  $V$  and  $W$ . These are a particular case of [1, Assumptions A1–A3].

(1)

$$W(x, y) = W(y, x).$$

(2) There exist numbers  $J > 0$ ,  $L \geq 1$ ,  $N \geq 2$ ,  $\sigma > 0$ , and  $A, B, C > 0$  such that

$$|W(x, y)| \leq J(1 + |x| + |y|)^{N-1}, \quad |\partial_x W(x, y)| \leq J(1 + |x| + |y|)^{N-1}.$$

(3)

$$|V(x)| \leq C(1 + |x|)^L, \quad |V'(x)| \leq C(1 + |x|)^{L-1}.$$

(4) (Coercivity assumption)

$$V'(x) \cdot x \geq A|x|^{N+\sigma} - B.$$

Let us define the following probability measure on  $E_n$ :

$$\mu_n = \frac{1}{Z_n} \exp\left(-\sum_{i=1}^n (V(x_i) + W(x_i, x_{i+1}))\right),$$

with the convention  $x_{n+1} := x_1$ . Here  $Z_n$  is the normalizing constant.

## PROPOSITION 7.5

The sequence  $\mu_n$  admits a weakly convergent subsequence  $\mu_{n_k} \rightarrow \mu$  satisfying the assumptions of Theorem 7.2.

*Proof*

It was proved in [1, Theorem 3.1] that any sequence of probability measures

$$\tilde{\mu}_n = c_n e^{-H_n} dx_{-n} \cdots dx_n,$$

where  $H_n$  is obtained from  $H$  by fixing a boundary condition  $\tilde{x}$

$$H_n = \sum_{i=1}^n V(x_i) + \sum_{i=1}^{n-1} W(x_i, x_{i+1}) + W(x_n, \tilde{x}_1),$$

which has a weakly convergent subsequence  $\tilde{\mu}_{n_k} \rightarrow \tilde{\mu}$ . In addition (see [1]),  $\mu$  satisfies the following a priori estimate: for every  $\lambda > 0$

$$\sup_{k \in \mathbb{N}} \int \exp(\lambda |x_k|^N) d\tilde{\mu} < \infty.$$

The same estimate holds for  $\tilde{\mu}_n$  uniformly in  $n$ .

Following the reasoning from [1] it is easy to show that the sequence  $\{\mu_n\}$  is tight and satisfies the same a priori estimate. Thus, we can pass to a subsequence  $\{\mu_{n'}\}$  which weakly converges to a measure  $\mu$ . For the sake of simplicity this subsequence will be denoted by  $\{\mu_n\}$  again. The limiting measure  $\mu$  satisfies

$$(16) \quad \sup_{k \in \mathbb{N}} \int \exp(\lambda |x_k|^N) d\mu < \infty;$$

moreover,

$$(17) \quad \sup_n \sup_{k \in \mathbb{N}} \int \exp(\lambda |x_k|^N) d\mu_n < \infty.$$

Let us estimate the relative entropy. We note that  $\mu_n$  and  $\mu_m$  ( $n > m$ ) are related in the following way:

$$\frac{e^Z \mu_n}{\int e^Z d\mu_n} = \mu_m \times \nu_{m,n},$$

where  $Z = -W(x_m, x_1) + W(x_m, x_{m+1}) + W(x_n, x_1)$  and  $\nu_{m,n}$  is a probability measure on  $E_{m,n}$ . Set  $\mu_{m,n} = \nu_{m,n}$ . Then

$$\int \log \left( \frac{d\mu_n}{d(\mu_m \times \mu_{m,n})} \right) d\mu_n = \int \left( Z - \log \int e^Z d\mu_n \right) d\mu_n.$$

The desired bound follows immediately from (17) and the assumptions about  $W$ .

To prove assumption (3) we note that

$$\frac{[e^{W(x_n, x_{n+1}) + W(x_1, x_n)} \cdot \mu] \circ P_n^{-1}}{\int e^{W(x_n, x_1) + W(x_1, x_n)} d\mu} = \frac{e^{W(x_1, x_n)} \cdot \mu_n}{\int e^{W(x_1, x_n)} d\mu_n}.$$

The normalizing constants can be easily estimated with the help of a priori bounds for  $\mu$  and  $\mu_n$ . Applying assumptions on  $W$  one can easily get that

$$Ae^{-B(|x_n|^{N-1}+|x_q|^{N-1})} \leq \frac{d\mu_n}{d\mu \circ P_n^{-1}} \leq Ae^{B(|x_n|^{N-1}+|x_1|^{N-1})},$$

where  $A, B > 0$  do not depend on  $n$ . Hence, assumption (3) follows immediately from (17), the Jensen inequality, and convexity if the function is  $x^{-\varepsilon}$ .  $\square$

#### REMARK 7.6

Finally, let us briefly discuss when the transportation mapping obtained in Proposition 7.5 by Theorem 7.2 solves the corresponding optimal transportation problem. To this end we apply Proposition 7.4.

Following the estimates obtained in Proposition 7.5 and applying the Jensen inequality, one can easily show that the sequence of the entropies

$$\int \log\left(\frac{d\hat{\mu}_n}{d\mu_n}\right) d\hat{\mu}_n$$

is bounded. Then the assumption of Proposition 7.4 holds, for instance, if every  $\mu_n$  satisfies the Talagrand inequality

$$W_2^2(\mu_n, \rho \cdot \mu_n) \leq C \int \rho \log \rho d\mu_n$$

with constant which does not depend on  $n$ . We do not investigate here sufficient conditions for measures  $\mu_n$  to satisfy this inequality; we just mention that this clearly holds in many natural situations (e.g., under the assumption of uniform log-concavity or finiteness of the log-Sobolev constant).

In addition, we emphasize that in many applications the measures do indeed satisfy the Talagrand inequality, but Proposition 7.4 should actually work under much milder assumptions.

## References

- [1] S. Albeverio, Y. G. Kondratiev, M. Röckner, and T. V. Tsikalenko, *A priori estimates for symmetrizing measures and their applications to Gibbs states*, J. Funct. Anal. **171** (2000), 366–400. MR 1745630. DOI 10.1006/jfan.1999.3482.
- [2] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2nd ed., Birkhäuser, Basel, 2008. MR 2401600.
- [3] M. Beiglböck, *Cyclical monotonicity and the ergodic theorem*, Ergodic Theory Dynam. Systems **35** (2015), 710–713. MR 3334900. DOI 10.1017/etds.2013.75.
- [4] V. I. Bogachev and A. V. Kolesnikov, *On the Monge–Ampère equation in infinite dimensions*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **8** (2005), 547–572. MR 2184083. DOI 10.1142/S0219025705002141.

- [5] ———, *The Monge–Kantorovich problem: achievements, connections, and perspectives*, Uspekhi Mat. Nauk **67**, no. 5 (2012), 3–110; English translation in Russian Math. Surveys **67** (2012), 785–890. MR 3058744.
- [6] ———, *Sobolev regularity for the Monge–Ampere equation in the Wiener space*, Kyoto J. Math. **53** (2013), 713–738. MR 3160599.  
DOI 10.1215/21562261-2366078.
- [7] C. Borel, *Convex measures on locally convex spaces*, Arkiv Math. **12** (1974), 239–252. MR 0388475. DOI 10.1007/BF02384761.
- [8] F. Cavalletti, *The Monge problem in Wiener space*, Calc. Var. Partial Differential Equations **45** (2012), 101–124. MR 2957652.  
DOI 10.1007/s00526-011-0452-5.
- [9] G. Contreras, A. O. Lopes, and E. R. Oliveira, “Ergodic transport theory, periodic maximizing probabilities and the twist condition” in *Modeling, Dynamics, Optimization, and Bioeconomics, I*, Springer Proc. Math. Stat. **73**, Springer, Cham, 2014, 183–219. MR 3297616.  
DOI 10.1007/978-3-319-04849-9\_12.
- [10] S. Fang and V. Nolot, *Sobolev estimates for optimal transport maps on Gaussian spaces*, J. Funct. Anal. **266** (2014), 5045–5084. MR 3177330.  
DOI 10.1016/j.jfa.2014.02.017.
- [11] S. Fang and J. Shao, *Optimal transport maps for Monge–Kantorovich problem on loop groups*, J. Funct. Anal. **248** (2007), 225–257. MR 2329689.  
DOI 10.1016/j.jfa.2006.11.016.
- [12] D. Feyel and A. S. Üstünel, *Monge–Kantorovich measure transportation and Monge–Ampère equation on Wiener space*, Probab. Theory Related Fields **128** (2004), 347–385. MR 2036490. DOI 10.1007/s00440-003-0307-x.
- [13] N. Ghoussoub and A. Moameni, *Symmetric Monge–Kantorovich problems and polar decompositions of vector fields*, Geom. Funct. Anal. **24** (2014), 1129–1166. MR 3248482. DOI 10.1007/s00039-014-0287-2.
- [14] O. Kallenberg, *Probabilistic Symmetries and Invariance Principles*, Springer, New York, 2005. MR 2161313.
- [15] A. V. Kolesnikov, *Convexity inequalities and optimal transport of infinite-dimensional measures*, J. Math. Pures Appl. (9) **83** (2004), 1373–1404. MR 2096305. DOI 10.1016/j.matpur.2004.03.005.
- [16] ———, *Mass transportation and contractions*, Proc. MIPT **2** (2010), 90–99.
- [17] ———, *On Sobolev regularity of mass transport and transportation inequalities*, Teor. Veroyatn. Primen. **57**, no. 2 (2012), 296–321; English translation in Theory Probab. Appl. **57** (2013), 243–264. MR 3201654.  
DOI 10.1137/S0040585X97985947.
- [18] A. V. Kolesnikov and M. Röckner, *On continuity equations in infinite dimensions with non-Gaussian reference measure*, J. Funct. Anal. **266** (2014), 4490–4537. MR 3170213. DOI 10.1016/j.jfa.2014.01.010.

- [19] A. O. Lopes and J. K. Mengue, *Duality theorems in ergodic transport*, J. Stat. Phys. **149** (2012), 921–942. MR 2999567. DOI 10.1007/s10955-012-0626-3.
- [20] A. O. Lopes, E. O. Oliveira, and P. Thieullen, “The dual potential, the involution kernel and transport in ergodic optimization” in *Dynamics, Games and Science*, CIM Ser. Math. Sci. **1**, Springer, Cham, 2015, 357–398. DOI 10.1007/978-3-319-16118-1\_20.
- [21] A. Moameni, *Invariance properties of the Monge–Kantorovich mass transport problem*, Discrete Contin. Dyn. Syst. **36** (2016), 2653–2671. MR 3485412. DOI 10.3934/dcds.2016.36.2653.
- [22] L. Rüschemdorf and T. Sei, *On optimal stationary couplings between stationary processes*, Electron. J. Probab. **17** (2012), no. 17. MR 2900458. DOI 10.1214/EJP.v17-1797.
- [23] A. M. Vershik, *The problem of describing central measures on the path spaces of graded graphs*, Funktsional. Anal. i Prilozhen. **48**, no. 4 (2014), 26–46; English translation in Funct. Anal. Appl. **48** (2014), 256–271. MR 3372738. DOI 10.1007/s10688-014-0069-5.
- [24] C. Villani, *Topics in Optimal Transportation*, Grad. Stud. Math. **58**, Amer. Math. Soc., Providence, 2003. MR 1964483. DOI 10.1007/b12016.
- [25] D. A. Zaev, *On ergodic decompositions related to the Kantorovich problem*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **437** (2015), 100–130. MR 3499910.
- [26] ———, *On the Monge–Kantorovich problem with additional linear constraints*, Mat. Zametki **98** (2015), 664–683. MR 3438523. DOI 10.4213/mzm10896.

*Kolesnikov*: Higher School of Economics, Moscow, Russia; Sascha77@mail.ru

*Zaev*: Higher School of Economics, Moscow, Russia; zaev.da@gmail.com