A Weyl Module Stratification of Integrable Representations

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Abstract

We construct a filtration on integrable highest weight module of an affine Lie algebra whose adjoint graded quotient is a direct sum of global Weyl modules. We show that the graded multiplicity of each Weyl module there is given by a corresponding level-restricted Kostka polynomial. This leads to an interpretation of level-restricted Kostka polynomials as the graded dimension of the space of conformal coinvariants. In addition, as an application of the level one case of the main result, we realize global Weyl modules of current algebras of type ADE in terms of Schubert manifolds of thick affine Grassmanian, as predicted by Boris Feigin.

Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Associated to this, we have an untwisted affine Kac-Moody algebra $\widetilde{\mathfrak{g}}$ and a current algebra $\mathfrak{g}[z] := \mathfrak{g} \otimes \mathbb{C}[z] \subset \widetilde{\mathfrak{g}}$. The representation theory of $\widetilde{\mathfrak{g}}$ attracts a lot of attention in 1990s because of its relation to mathematical physics [9, 41, 19, 21] in addition to its own interest [18, 25]. There, they derive numerous interesting equalities and combinatorics. The representation theory of $\mathfrak{g}[z]$ and its variants are studied in detail by many people from 2000s [24, 6, 3, 12, 4, 33] as a fork project on the study of representation theory of $\widetilde{\mathfrak{g}}$.

The representation theory of $\mathfrak{g}[z]$ is essentially the same as $\mathfrak{g}[z, z^{-1}]$, that can be also seen as a part of the representation theory of $\widetilde{\mathfrak{g}}$, and it also affords the natural space of intertwiners of representations of $\widetilde{\mathfrak{g}}$. Moreover, $\mathfrak{g}[z]$ inherits many representations directly from $\mathfrak{g}$. Therefore, the representation theory of current algebra can be seen as a bridge between these of $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$. The goal of this paper is to provide several basic results that support this idea, and to connect them with the combinatorics/equalities from 1990s.

We have an integral weight lattice P of $\mathfrak{g}$ and the integral weight lattice $\widetilde{P}$ of $\widetilde{\mathfrak{g}}$, so that we have a canonical surjection $\widetilde{P} \ni \Lambda \mapsto \overline{\Lambda} \in P$. Let P_+ be the set of

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dominant weights, and let $\widetilde{P}_+^k \subset \widetilde{P}$ be the set of level k dominant weights. For each $\lambda \in P_+$, we have a local Weyl module $W(\lambda, 0)$ and a global Weyl module $W(\lambda)$ defined by Chari-Pressley [6].

Both local Weyl modules and global Weyl modules constitute a basis in K_0 of the category of graded $\mathfrak{g}[z]$-modules. It was recently shown ([28, 29, 4, 30, 32]) that these bases are orthogonal each to other with respect to Ext^* scalar product (see Theorem 2.2) and their characters are related to a specialization of Macdonald polynomials at $t = 0$. It appeared that characters of some natural representations of $\mathfrak{g}[z]$ can be expressed via Weyl module characters with positive coefficients. In particular, the Cauchy identity implies that the projective $\mathfrak{g}[z]$-modules have a filtrations whose adjoint graded space consists of global Weyl modules ([4]).

Let us choose an inclusion of $\mathfrak{g}[z]$ into $\widetilde{\mathfrak{g}}$ contains as the *nonpositive* part of $\widetilde{\mathfrak{g}}$, so the representations of $\widetilde{\mathfrak{g}}$ can be considered as $\mathfrak{g}[z]$-modules. For each $\Lambda \in \widetilde{P}_+^k$, we have a level k integrable highest weight $\widetilde{\mathfrak{g}}$-module $L_k(\Lambda)$. In [7], it is shown that the characters of $L_1(\Lambda)^{\otimes n}$ $(n \geq 1)$ can be expressed via characters of the global Weyl modules with positive coefficients.

Our main result provides an explanation of this phenomena:

Theorem A. *For each $\Lambda \in \widetilde{P}_+^k$, the $\mathfrak{g}[z]$-module $L_k(\lambda)$ admits a filtration by global Weyl modules.*

Let M be a finitely generated graded $\mathfrak{g}[z]$-module stratified by global Weyl modules. We denote the graded multiplicity of $W(\lambda)$ in M by $(M : W(\lambda))_q$. Theorem A implies that we have a well-defined notion of the graded multiplicity $(L_k(\Gamma) : W(\lambda))_q$ that counts the number of occurrences of $W(\lambda)$ in the filtration of $L_k(\Gamma)$ (with grading shifts counted).

Since the representation theory of $\mathfrak{g}[z]$ naturally carries the information of Macdonald polynomials specialized to $t = 0$, Theorem A and a version of the BGG reciprocity implies:

Corollary B. *For each $\lambda \in P_+$ and $\Gamma \in \widetilde{P}_+^k$, we have*

$$\mathsf{gch}\, H_i(\mathfrak{g}[z], \mathfrak{g}; W(-w_0\lambda, 0) \otimes L_k(\Gamma)) = \begin{cases} (L_k(\Gamma) : W(\lambda))_q & (i = 0) \\ 0 & (i \neq 0) \end{cases},$$

where w_0 denote the longest element in the Weyl group of $\mathfrak{g}$. In addition, $(L_k(\Gamma) : W(\lambda))_q$ coincides with the restricted Kostka polynomial of level k defined combinatorially in §5.

Corollary B can be seen as a direct extension of Feigin-Feigin [8]. The same idea as the proof of Theorem B also yield an elementary proof of Teleman's Borel-Weil-Bott theorem [41, Theorem 0 a)] that avoids the use of Laplacian calculations (= Corollary 3.7). From this view point, it is natural to reformulate our results in terms of conformal coinvariants.

Theorem C. *For each $\lambda \in P$ and $\Gamma \in \widetilde{P}_+^k$, the vector space*

$$H_0(\mathfrak{g}[z], \mathfrak{g}; W(\lambda) \otimes L_k(\Gamma))$$

is a free module over $\mathbb{C}[\mathbb{A}^{(\lambda)}]$, where $\mathbb{A}^{(\lambda)}$ is a certain configuration space of $\langle \rho^\vee, \lambda \rangle$-points in $\mathbb{A}^1$. Its specialization to each point $\vec{x} \in \mathbb{A}^{(\lambda)}$ gives the space of the generalized conformal coinvariants (see §3).

We remark that if we specialize to a generic $\vec{x}$, then Theorem C reduces into Teleman's result [41]. In general, the above homology group has subtle cancelations that is observed in [8] when $\mathfrak{g} = \mathfrak{sl}(2)$.

Assume that $\mathfrak{g}$ is of type ADE. Then, the G-invariant Schubert variety of the thick affine Grassmanian [22] is in bijection with P_+ (see e.g. [42]). Our analysis, together with that of Cherednik-Feigin [7] and a result of [27], implies the following realization of global Weyl modules predicted by Boris Feigin:

Theorem D. *Assume that* $\mathfrak{g}$ *is of type* ADE*. For each* $\Omega \in \widetilde{P}^1_+$ *and* $\lambda \in P_+$ *so that* $\lambda \geq \overline{\Omega}$*, we have the following isomorphism of* $\mathfrak{g}[z]$*-modules:*

$$\Gamma_c(\mathbf{Gr}^{\lambda}_G, \mathcal{O}_{\mathbf{Gr}_G}(1)) \xrightarrow{\cong} W(\lambda)^*.$$

The organization of the paper is as follows: In section one, we prepare basic notation and environments. In section two, we exhibit that the Bernstein-Gelfand-Gelfand-Lepowsky resolution gives a projective resolution of an integrable highest weight module. In section three, we define the level-restricted Kostka polynomials and provide its main properties (Theorem A and Theorem B). In addition, we identify our coinvariant space as a natural enhancement of conformal coinvariants (Theorem C). In section four, we derive Feigin's realization of global Weyl modules in terms of a Schubert subscheme of thick affine Grassmanian. In section five, we provide an alternating sum formula (Theorem 5.11) for the polynomials that naturally extends the level-restricted Kostka polynomials (that is implicit in the literature), which plays a crucial role in the comparison with our level-restricted Kostka polynomials (Corollary 3.11). The appendix (by Ryosuke Kodera) contains a bypass of Theorem A in the proof of Theorem D using free field realizations.

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1 Preliminaries

A vector space is always a $\mathbb{C}$-vector space, and a graded vector space refers to a $\mathbb{Z}$-graded vector space whose grading is either bounded from the below or bounded from the above and each of its graded piece is finite-dimensional. For a graded vector space $M = \bigoplus_{i\in\mathbb{Z}} M_i$ or its completion $M^\wedge = \prod_{i\in\mathbb{Z}} M_i$, we define its dual as $M^* := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{C}}(M_i, \mathbb{C})$, where $\mathrm{Hom}_{\mathbb{C}}(M_i, \mathbb{C})$ is understood to have degree $-i$. We define the graded dimension of a graded vector space as

$$\mathsf{gdim}\, M := \sum_{i\in\mathbb{Z}} q^i \dim_{\mathbb{C}} M_i \in \mathbb{Q}[\![q, q^{-1}]\!].$$

For each $n \in \mathbb{Z}$, let us define the grade n-shift of a graded vector space M as: $(M\langle n\rangle)_i := M_{n+i}$ for every $i \in \mathbb{Z}$. For $f(q) \in \mathbb{Q}(q)$, we set $\overline{f(q)} := f(q^{-1})$.

1.1 Algebraic groups and its Lie algebras

Let G be a connected, adjoint semi-simple algebraic group over $\mathbb{C}$, and let B and H be a Borel subgroup and a maximal torus of G so that $H \subset B$. Let

G_{sc} be the simply connected cover of G, and H_{sc} be the preimage of H in G_{sc}. We set U $(= [B, B])$ to be the unipotent radical of B and let U^- be the opposite unipotent subgroup of U with respect to H. We denote the Lie algebra of an algebraic group by the corresponding German letter. We have a (finite) Weyl group $W := N_G(H)/H$. For an algebraic group E, we denote its set of $\mathbb{C}[z]$-valued points by $E[z]$, its set of $\mathbb{C}[\![z]\!]$-valued points by $E[\![z]\!]$, and its set of $\mathbb{C}(z)$-valued points by $E(z)$.

Let $P := \mathrm{Hom}_{gr}(H_{\mathrm{sc}}, \mathbb{C}^\times)$ be the weight lattice of H_{sc}, let $\Delta \subset P$ be the set of roots, let $\Delta^+ \subset \Delta$ be the set of roots belonging to $\mathfrak{b}$, and let $\Pi \subset \Delta^+$ be the set of simple roots. We denote by $\Pi^\vee$ the set of simple coroots of $\mathfrak{g}$. For $\lambda, \mu \in P$, we define $\lambda \geq \mu$ if and only if $\lambda - \mu \in \mathbb{Z}_{\geq 0}\Delta^+$. Let $Q^\vee$ be the dual lattice of P with a natural pairing $\langle \bullet, \bullet \rangle : Q^\vee \times P \to \mathbb{Z}$. Let r be the rank of G and we set $\mathtt{I} := \{1, 2, \ldots, r\}$. We fix bijections $\mathtt{I} \cong \Pi \cong \Pi^\vee$ so that $i \in \mathtt{I}$ corresponds to $\alpha_i \in \Pi$, its coroot $\alpha_i^\vee \in \Pi^\vee$, (non-zero) root vectors E_i, F_i corresponding to $\alpha_i, -\alpha_i$, and a simple reflection $s_i \in W$ corresponding to α_i. We also define a reflection $s_\alpha \in W$ corresponding to $\alpha \in \Delta^+$. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function and let $w_0 \in W$ be the longest element. Let $P_+ := \{\lambda \in P \mid \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}_{\geq 0}, i \in \mathtt{I}\}$. Let $\{\varpi_i\}_{i \in \mathtt{I}} \subset P_+$ denote the dual basis of $\Pi^\vee$. For each $\lambda \in P_+$, we have a finite-dimensional irreducible $\mathfrak{g}$-module $V(\lambda)$ with highest weight λ.

Let $\Delta_{\mathrm{af}} := \Delta \times \mathbb{Z}\delta \cup \mathbb{Z}_{\neq 0}\delta$ be the untwisted affine root system of Δ with its positive part $\Delta^+ \subset \Delta^+_{\mathrm{af}}$. We set $\alpha_0 := -\vartheta + \delta$, $\Pi_{\mathrm{af}} := \Pi \cup \{\alpha_0\}$, and $\mathtt{I}_{\mathrm{af}} := \mathtt{I} \cup \{0\}$, where ϑ is the highest root of Δ^+. We define a normalized inner product $(\bullet, \bullet) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ to be the unique W-invariant inner product such that $(\vartheta, \vartheta) = 2$. We set $W_{\mathrm{af}} := W \ltimes Q^\vee$ and call it the affine Weyl group. It is a reflection group generated by $\{s_i \mid i \in \mathtt{I}_{\mathrm{af}}\}$, where s_0 is the reflection with respect to α_0. This equips W_{af} a length function $\ell : W_{\mathrm{af}} \to \mathbb{Z}_{\geq 0}$ extending that of W. For a subgroup $W' \subset W_{\mathrm{af}}$ generated by a subset of $\{s_i\}_{i \in \mathtt{I}}$, we identify W_{af}/W' with the set of minimal length representatives of right W'-cosets in W_{af}.

1.2 Affine Lie algebras

Let $\widetilde{\mathfrak{g}}$ be the untwisted affine Kac-Moody algebra associated to $\mathfrak{g}$. I.e. we have

$$\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[\xi, \xi^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

where K is central, $[d, X \otimes \xi^m] = mX \otimes \xi^m$ for each $X \in \mathfrak{g}$ and $m \in \mathbb{Z}$, and for each $X, Y \in \mathfrak{g}$ and $f, g \in \mathbb{C}[\xi^{\pm 1}]$ it holds:

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + (X, Y)_{\mathfrak{g}} \cdot K \cdot \mathrm{Res}_{\xi=0} f \frac{\partial g}{\partial \xi},$$

where $(\bullet, \bullet)_{\mathfrak{g}}$ denotes the unique $\mathfrak{g}$-invariant bilinear form such that $(\alpha^\vee, \alpha^\vee)_{\mathfrak{g}} = 2$ for a long simple root α. We set $E_0 := F_\vartheta \otimes \xi$ and $F_0 := E_\vartheta \otimes \xi^{-1}$ (these are root vectors of $\alpha_0, -\alpha_0$, respectively). Then, $\{E_i, F_i\}_{i \in \mathtt{I}_{\mathrm{af}}}$ generates a subalgebra $\widehat{\mathfrak{g}}$ of $\widetilde{\mathfrak{g}}$. We set $\widehat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}K$ and $\widetilde{\mathfrak{h}} := \widehat{\mathfrak{h}} \oplus \mathbb{C}d$. For each $i \in \mathtt{I}_{\mathrm{af}}$, we denote by $\alpha_i^\vee \in \widetilde{\mathfrak{h}}$ the coroot of α_i whose set enhances $\Pi^\vee \subset \mathfrak{h}$. Let $\mathfrak{I} \subset \widetilde{\mathfrak{g}}$ (resp. $\mathfrak{I}^-$) be the subalgebra of $\widetilde{\mathfrak{g}}$ generated by $\widetilde{\mathfrak{h}}$ and $\{E_i\}_{i \in I_{\mathrm{af}}}$ (resp. $\mathfrak{h}$ and $\{F_i\}_{i \in I_{\mathrm{af}}}$).

By introducing $z = \xi^{-1} \in \mathbb{C}[\xi^{\pm 1}]$, we define a Lie subalgebra $\mathfrak{g}[z] \subset \widetilde{\mathfrak{g}}$. We have an involution θ on $\widetilde{\mathfrak{g}}$ that is identity on $\mathfrak{g}$ and swaps z^m with ξ^m for each $m \in \mathbb{Z}$ (and hence K is sent to $-K$ and d is sent to to $-d$).

We set $\widehat{P}$ to be the lattice spanned by a fixed choice of fundamental weights $\Lambda_0, \Lambda_1, \ldots, \Lambda_r \in \widetilde{\mathfrak{h}}^*$ of $\widehat{\mathfrak{g}}$. We define $\widehat{P}_+ := \sum_{i \in I_{\mathrm{af}}} \mathbb{Z}_{\geq 0} \Lambda_i$ and $\widehat{P} := \sum_{i \in I_{\mathrm{af}}} \mathbb{Z}\Lambda_i$. For $k \in \mathbb{Z}_{\geq 0}$, we also set

$$\widehat{P}^k_+ := \{\Lambda \in \widehat{P}_+ \mid \langle K, \Lambda \rangle = k\} \subset \{\Lambda \in \widehat{P} \mid \langle K, \Lambda \rangle = k\} =: \widehat{P}^k.$$

We define $\widetilde{P} := \widehat{P} \oplus \mathbb{Z}\delta$. We have projection maps

$$\widetilde{P} \to \widetilde{P}/\mathbb{Z}\delta \cong \widehat{P} \to \widehat{P}/\mathbb{Z}\Lambda_0 \cong P.$$

We denote the projection of $\Lambda \in \widetilde{P}$ or $\widehat{P}$ to P by $\overline{\Lambda}$. Let us denote the image of $\widehat{P}^k_+$ under this projection by P^k_+. We also identify P with $\widehat{P}^0 \subset \widetilde{P}$.

Let $k \in \mathbb{Z}$. We fix an element $\rho_k \in \widetilde{\mathfrak{h}}^*$ so that $\langle \alpha_i^\vee, \rho_k \rangle = 1$ for each $i \in I$, and $\langle \alpha_0^\vee, \rho_k \rangle = k+1$. For each $w \in W_{\mathrm{af}}$ and $\Lambda \in \widetilde{P}$, we define

$$w \circ_k \Lambda := w(\Lambda + \rho_k) - \rho_k.$$

When $k = 0$, then we simply write $\circ$ instead of $\circ_k$. Note that for $w \in W_{\mathrm{af}}$ and $\lambda \in P$, we have $w \circ (\lambda + k\Lambda_0) = W \circ_k \lambda + k\Lambda_0$.

Every element of P is either $\circ_k$-conjugate to P^k_+ modulo $\mathbb{Z}\delta$ or W_{af} has a non-trivial stabilizer group with respect to the $\circ_k$-action.

Finally, we set $U_k(\widehat{\mathfrak{g}}) := U(\widehat{\mathfrak{g}})/(K-k)U(\widehat{\mathfrak{g}})$ and $U_k(\widetilde{\mathfrak{g}}) := U(\widetilde{\mathfrak{g}})/(K-k)U(\widetilde{\mathfrak{g}})$. We refer $U_k(\widehat{\mathfrak{g}})$-modules and $U_k(\widetilde{\mathfrak{g}})$-modules by the $\widehat{\mathfrak{g}}_k$-modules and $\widetilde{\mathfrak{g}}_k$-modules, respectively.

1.3 Representations of current algebras

We review some results from current algebra representations (cf. [26] §1.2).

Definition 1.1 ($\mathfrak{g}$-integrable module)**.** A $\mathfrak{g}[z]$-module M is said to be $\mathfrak{g}$-integrable if M is finitely generated and it decomposes into a sum of finite-dimensional $\mathfrak{g}$-modules. Let $\mathfrak{g}[z]\text{-}\mathsf{mod}$ be the category of $\mathfrak{g}$-integrable $\mathfrak{g}[z]$-modules. For each $\lambda \in P_+$, let $\mathfrak{g}[z]\text{-}\mathsf{mod}^{\leq \lambda}$ be the full subcategory of $\mathfrak{g}[z]\text{-}\mathsf{mod}$ whose object is isomorphic to a direct sum of $\mathfrak{g}$-modules in $\{V(\mu)\}_{\mu \leq \lambda}$.
A $\mathfrak{g}[z]$-module M is said to be graded if M is a graded vector space and we have $(X \otimes z^n) M_m \subset M_{n+m}$ for each $X \in \mathfrak{g}$ and $n, m \in \mathbb{Z}$. We denote the category of graded $\mathfrak{g}$-integrable $\mathfrak{g}[z]$-modules by $\mathfrak{g}[z]\text{-}\mathsf{gmod}$.

Definition 1.2 (projective modules and global Weyl module)**.** For each $\lambda \in P_+$, we define the non-restricted projective module $P(\lambda)$ as

$$P(\lambda) := U(\mathfrak{g}[z]) \otimes_{U(\mathfrak{g})} V(\lambda).$$

Let $P(\lambda; \mu)$ be the largest $\mathfrak{g}[z]$-module quotient of $P(\lambda)$ so that

$$\mathrm{Hom}_{\mathfrak{g}}(V(\gamma), P(\lambda; \mu)) = \{0\} \quad \text{if} \quad \gamma \not\leq \mu. \tag{1.1}$$

We define the global Weyl module $W(\lambda)$ of $\mathfrak{g}$ to be $P(\lambda; \lambda)$.

For each $\lambda \in P_+$ and $a \in \mathbb{C}$, the $\mathfrak{g}$-module $V(\lambda)$ can be regarded as a $\mathfrak{g}[z]$-module through the evaluation at $z = a$ map $\mathfrak{g}[z] \to \mathfrak{g}$ for each $\lambda \in P_+$. We denote it by $V(\lambda, a)$. We can further regard $V(\lambda, 0)$ as a module in $\mathfrak{g}[z]\text{-}\mathsf{gmod}$ by putting a grading concentrated in a single degree. For a module $M \in \mathfrak{g}[z]\text{-}\mathsf{gmod}$, we have $M\langle n \rangle \in \mathfrak{g}[z]\text{-}\mathsf{gmod}$ for every $n \in \mathbb{Z}$.

Lemma 1.3. *Let $\lambda, \mu \in P_+$. The projective module $P(\lambda)$, its quotient $P(\lambda;\mu)$, and global Weyl modules $W(\lambda)$ can be regarded as graded modules with a simple head $V(\lambda, 0)$ sitting at degree 0.* □

Lemma 1.4. *The module $P(\lambda)$ is the projective cover of $V(\lambda, x)$ as a $\mathfrak{g}$-integrable $\mathfrak{g}[z]$-module for every $x \in \mathbb{C}$;*

In order to establish relation between global and local Weyl modules we set $|\lambda| := \sum_{i \in \mathtt{I}} \langle \alpha_i^\vee, \lambda \rangle$ and

$$\mathbb{C}[\mathbb{A}^{(\lambda)}] := \bigotimes_{i \in \mathtt{I}} \mathbb{C}[X_1^{(i)}, X_2^{(i)}, \ldots, X_{m_i}^{(i)}]^{\mathfrak{S}_{m_i}} \qquad m_i = \langle \alpha_i^\vee, \lambda \rangle,$$

where we understand that $\mathbb{C}[\mathbb{A}^{(\lambda)}]$ is a graded ring by setting $\deg X_j^{(i)} = 1$ for every $i \in \mathtt{I}$ and $j \in \mathbb{Z}$.

Theorem 1.5 (Chari-Fourier-Khandai [2])**.** *The module $W(\lambda)$ admits a free action of $\mathbb{C}[\mathbb{A}^{(\lambda)}]$ induced by the $U(\mathfrak{h}[z])$-action on the $\mathfrak{h}$-weight λ-part of $W(\lambda)$, that commutes with the $\mathfrak{g}[z]$-action and respects the grading of $W(\lambda)$.*

Definition 1.6. For each $x \in \mathbb{A}^{(\lambda)}$, we have a specialization $W(\lambda, x) := W(\lambda) \otimes_{\mathbb{C}[\mathbb{A}^{(\lambda)}]} \mathbb{C}_x$. These modules are called *local Weyl modules.*

Lemma 1.7 (see [2])**.** *If $x \in \mathbb{A}^{(\lambda)}$ is the orbit of $|\lambda|$-distinct points, then we have*

$$W(\lambda, x) \cong \bigotimes_{i=1}^{r} \bigotimes_{j=1}^{\lambda_i} W(\varpi_i, x_{i,j}).$$

Here $(x_{i,1}, \ldots, x_{i,\lambda_i}) \in \mathbb{A}^{\lambda_i}$ corresponds to x (up to $\mathfrak{S}_{\lambda_i}$-action).

Theorem 1.8 (Chari-Loktev [5], Fourier-Littelmann [12], Naoi [37])**.** *The action of $\mathbb{C}[\mathbb{A}^{(\lambda)}]$ on $W(\lambda)$ is free, so $W(\lambda, x) \cong W(\lambda, y)$ as $\mathfrak{g}$-modules for each $x, y \in \mathbb{A}^{(\lambda)}$, in particular $W(\lambda, x)$ is finite-dimensional for any x.*

1.4 Verma and Parabolic Verma modules

We set $\mathfrak{g}[z]_1 := z\mathfrak{g}[z] = \ker(\mathfrak{g}[z] \to \mathfrak{g}) \cong \mathfrak{g} \otimes z\mathbb{C}[z]$ and $\mathfrak{g}[\xi]_1 := \xi\mathfrak{g}[\xi] = \ker(\mathfrak{g}[\xi] \to \mathfrak{g}) \cong \mathfrak{g} \otimes \xi\mathbb{C}[\xi]$ (recall that $\xi = z^{-1}$).

Definition 1.9. We say that a $\widetilde{\mathfrak{g}}$-module M belongs to the parabolic category $\mathcal{O}$ if

1) M graded with respect to the action of $\widetilde{\mathfrak{h}}$ with finite-dimensional graded spaces and bounded from above eigenvalues of $d \in \widetilde{\mathfrak{h}}$;

2) The action of $\mathfrak{g}[\xi]$ is locally finite (each vector generates a finite-dimesional space under this action).

Remark 1.10. A restriction of such a module to $\mathfrak{g}[z]$ is $\mathfrak{g}$-integrable.

Remark 1.11. One can replace the condition 2 in Definition 1.9 with locally finiteness of the action of $\mathfrak{I}$ to obtain the usual category $\mathcal{O}$.

Definition 1.12 (integrable modules)**.** A module from the parabolic category $\mathcal{O}$ is said to be integrable if the actions of E_0 and F_0 are locally finite.

Let $\lambda \in P_+$ and $k \in \mathbb{Z}$. We have the corresponding one-dimensional module $\mathbb{C}_{(\lambda+k\Lambda_0)}$ of $\widetilde{\mathfrak{h}}$, that can be inflated to a module of $\mathfrak{I}$ by the trivial action of $[\mathfrak{I}, \mathfrak{I}]$. We define the Verma module $\mathbf{M}_k(\lambda)$ as

$$\mathbf{M}_k(\lambda) := U(\widetilde{\mathfrak{g}}) \otimes_{U(\mathfrak{I})} \mathbb{C}_{(\lambda+k\Lambda_0)},$$

and its (unique) simple quotient by $L_k(\lambda)$ (see e.g. [25, §1.2]). In case $\lambda \in P_+$, we have a finite-dimensional irreducible $\mathfrak{g}+\widetilde{\mathfrak{h}}$-module $V(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_{k\Lambda_0}$ with $\mathfrak{b}$-fixed vector v_λ of $\widetilde{\mathfrak{h}}$-weight $\lambda + k\Lambda_0$. We inflate $V(\lambda)$ into a $(\mathfrak{g}[\xi] + \widetilde{\mathfrak{h}})$-module by the trivial action of $\mathfrak{g}[\xi]_1$. We define the Parabolic Verma module $M_k(\lambda)$ as

$$M_k(\lambda) := U(\widetilde{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[\xi]+\widetilde{\mathfrak{h}})} (V(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_{k\Lambda_0}).$$

Twisting $\mathbb{C}_{\pm\delta}$ on the RHS gives rise to isomorphic $\widehat{\mathfrak{g}}_k$-modules (up to the shift of the $\mathbb{C}d$-action). We understand this twist as the grading shift of $\widehat{\mathfrak{g}}_k$-modules.

By construction, we have $\widetilde{\mathfrak{g}}$-module surjections

$$\mathbf{M}_k(\lambda) \longrightarrow M_k(\lambda) \longrightarrow L_k(\lambda)$$

for each $\lambda \in P_+$. We understand that their cyclic vectors are degree 0 by convention. The following fact is crucial in what follows:

Proposition 1.13. *Modules $M_k(\lambda)$ and $L_k(\lambda)$ belongs to the parabolic category $\mathcal{O}$. Moreover, $L_k(\lambda)$ is integrable if and only if $\lambda \in P_+^k$.*

Proposition 1.14 (see e.g. Chari-Greenstein [3])**.** *For each $\lambda \in P_+$, we have*

$$M_k(\lambda) \cong U(\mathfrak{g}[z]_1) \otimes_{\mathbb{C}} V(\lambda)$$

as graded $\mathfrak{g}$-modules. Moreover, $M_k(\lambda)$ is isomorphic to a projective module $P(\lambda)$ in $\mathfrak{g}[z]$-mod. □

Lemma 1.15. *For each $\lambda \in P_+$ and a $\widehat{\mathfrak{g}}$-module N from the parabolic category $\mathcal{O}$, we have an isomorphism*

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}}(M_k(\lambda), N) \cong \mathrm{Hom}_{\widehat{\mathfrak{h}}}(\mathbb{C}_\lambda, N^{\mathfrak{g}[\xi]_1}).$$

Proof. The RHS corresponds to the space $\widehat{\mathfrak{g}}$-module morphisms $\psi : \mathbf{M}_k(\lambda) \to N$. Consider the $\widehat{\mathfrak{g}}$-module maps

$$\mathbf{M}_k(w \circ \lambda) \longrightarrow \mathbf{M}_k(\lambda) \qquad w \in W$$

induced by the $U(\mathfrak{g})$-module inclusion of their $\mathfrak{g}[\xi]_1$-fixed parts. By the classification of finite-dimensional highest weight $\mathfrak{g}$-modules, we deduce that $\mathbf{M}_k(w \circ \lambda)^{\mathfrak{g}[\xi]_1}$ has a $\mathfrak{g}$-integrable quotient if and only if $w = e$. Since N belongs to the parabolic category $\mathcal{O}$ and, hence, $\mathfrak{g}$-integrable, ψ must factor through the quotient of $\mathbf{M}_k(\lambda)$ by the images of $\mathbf{M}_k(w \circ \lambda)$ $(w \neq e)$, that is $M_k(\lambda)$. So the assertion follows. □

Theorem 1.16 (see [18], Chapter 7)**.** *Let $\lambda \in P_+^k$. Then, we have*

$$L_k(\lambda) \cong M_k(\lambda)/U(\widehat{\mathfrak{g}})F_0^{k-\langle\vartheta,\lambda\rangle+1}v_\lambda.$$

This isomorphism can be extended to the resolution of $L_k(\lambda)$ by parabolic Verma modules.

Theorem 1.17 (Hackenberger-Kolb [15], Theorem 3.6)**.** *For each $\lambda \in P_+^k$, we have a resolution by parabolic Verma modules*

$$\cdots \xrightarrow{d_3} \bigoplus_{w \in W \backslash W_{\mathrm{af}}, \ell(w)=2} M_k(w \circ_k \lambda) \xrightarrow{d_2} M_k(s_0 \circ_k \lambda) \xrightarrow{d_1} M_k(\lambda) \to L_k(\lambda) \to 0. \tag{1.2}$$

Combaining this statement with Proposition 1.14, we obtain

Proposition 1.18. *By restricting to $\mathfrak{g}[z]$, the resolution* (1.2) *can be seen as a graded projective resolution of $L_k(\lambda)$ whose grading arises from the $(-d)$-action.*

2 The Weyl filtration

We retain the setting of the previous section. In this section, we utilize the following two rather recent results on current algebra representations to prove Theorem 2.15.

For $M \in \mathfrak{g}[z]\text{-}\mathsf{gmod}$ and $\lambda \in P_+$, we set

$$[M : V(\lambda, 0)]_q := \sum_{n \in \mathbb{Z}} q^n \cdot \dim \mathrm{Hom}_{\mathfrak{g}}(V(\lambda), M_n)$$
$$\mathsf{ch}\, M := \sum_{n \in \mathbb{Z}} e^{\mu} \cdot \mathsf{gdim}\, \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_\mu, M_n).$$

In case M is a $\widetilde{\mathfrak{g}}_k$-module, then q represents $e^{-\delta}$ in the standard convention.

Theorem 2.1 (Chari-Ion [4])**.**

1. *The projective module $P(\lambda)$ of $\mathfrak{g}[z]\text{-}\mathsf{mod}^{\le \mu}$ admits a finite graded filtration by $\{W(\mu)\}_{\mu \ge \lambda}$ with suitable grading shifts;*
2. *If we denote by $(P(\lambda) : W(\mu))_q$ the number of $W(\mu)$ appearing in the filtration (in a graded sense), then for each $\lambda, \mu \in P_+$ we have*

$$(P(\lambda) : W(\mu))_q = [W(\mu, 0) : V(\lambda)]_q.$$

In particular, the both sides belongs to $\mathbb{Z}_{\ge 0}[q]$.

Theorem 2.2 (Kleshchev [30] Theorem 7.21 and Lemma 7.23, cf. §10.3)**.**

1. *We have*

$$\mathrm{Ext}^i_{\mathfrak{g}[z]\text{-}\mathsf{mod}}(W(\lambda), W(\mu, 0)^* \langle n \rangle) = \begin{cases} \mathbb{C} & (\lambda = -w_0\mu, i = 0, n = 0) \\ \{0\} & (\textit{otherwise}) \end{cases};$$

2. *A finitely generated $\mathfrak{g}$-integrable $\mathfrak{g}[z]$-module M admits a filtration by global Weyl modules if and only if*

$$\mathrm{Ext}^1_{\mathfrak{g}[z]}(M, W(\lambda, 0)^*) = \{0\} \quad \forall \lambda \in P_+;$$

3. A finite-dimensional graded $\mathfrak{g}[z]$-module M admits a filtration by the dual of local Weyl modules if and only if

$$\mathrm{Ext}^1_{\mathfrak{g}[z]}(W(\lambda), M) = \{0\} \qquad \forall \lambda \in P_+.$$

Remark 2.3. For each $M, N \in \mathfrak{g}[z]\text{-}\mathsf{mod}$, we have

$$\mathrm{Ext}^i_{\mathfrak{g}[z]\text{-}\mathsf{mod}}(M, N) \xrightarrow{\cong} \mathrm{Ext}^i_{\mathfrak{g}[z]}(M, N) \qquad i \in \mathbb{Z}$$

since the category of finite-dimensional $\mathfrak{g}$-modules is completely reducible (cf. [31, §3.1] or [11, Remark 0.2]).

2.1 The Demazure property

Note that we can consider the current algebra representation as the modules of $(\mathfrak{g}[z] + \widetilde{\mathfrak{h}})$ with $K = 0$ and the action of $(-d)$ as the grading operator.

Definition 2.4. Let V be a representation of a subalgebra of $\widetilde{\mathfrak{g}}$ contating $\widetilde{\mathfrak{h}}$ and E_0. We say that V satisfies the Demazure property if for each $n > 0$ and $\gamma \in P$ such that $\langle \alpha_0^\vee, \gamma \rangle = -n$, the action of $(E_0)^{n-1}$ on the corresponding weight space $V^\gamma = \mathbb{C}_\gamma \otimes \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_\gamma, V) \subset V$ has trivial kernel.

Remark 2.5. **1)** This definition is motivated by the fact that for the simply-laced $\mathfrak{g}$, local Weyl modules coincide with the level one Demazure modules, and this property can be deduced from the structure of its Demazure crystal.

2) The assumption on the Lie algebra of Definition 2.4 is automatic for $(\mathfrak{g}[z] + \widetilde{\mathfrak{h}})$, $(\mathfrak{g}[\xi] + \widetilde{\mathfrak{h}})$, and $\mathfrak{I}$.

Let us introduce some notation. For each $i \in I_{\mathrm{af}}$, we have a minimal parabolic subalgebra $\mathfrak{I} \subset \mathfrak{I}(i) = \mathbb{C}F_i \oplus \mathfrak{I}$. We denote by $\mathfrak{sl}(2, i)$ the Lie subalgebra of $\widetilde{\mathfrak{g}}$ generated by E_i and F_i, and $\mathfrak{b}(i)$ the Lie subalgebra of $\widetilde{\mathfrak{g}}$ generated by E_i and $\widetilde{\mathfrak{h}}$. A string $\mathfrak{b}(i)$-module is a finite-dimensional $\mathfrak{b}(i)$-module that is $\widetilde{\mathfrak{h}}$-semisimple and have unique simple submodule and unique simple quotient (that is automatically isomorphic to an irreducible $\mathfrak{sl}(2, i)$-module up to a $\widetilde{\mathfrak{h}}$-weight twist).

Theorem 2.6 ([26] Theorem 4.12 (2))**.** *For each $\lambda \in P_+$, there exists a $\mathfrak{I}(0)$-module $W(\lambda)_{s_0}$ so that*

$$W(\lambda) \subset W(\lambda)_{s_0}.$$

The $\mathfrak{I}(0)$-module $W(\lambda)_{s_0}$ is completely reducible as a $\mathfrak{sl}(2, 0)$-module. For each $m \in \mathbb{Z}_{\geq 0}$, let $W(\lambda)^m_{s_0}$ denote the sum of irreducible $\mathfrak{sl}(2, 0)$-submodules of dimension m. We have $W(\lambda)_{s_0} = \bigoplus_{m \geq 0} W(\lambda)^m_{s_0}$.

We define the canonical filtration of $W(\lambda)_{s_0}$ as:

$$F_n W(\lambda)_{s_0} := \sum_{m \geq n} W(\lambda)^m_{s_0}.$$

This defines a decreasing separable filtration of $W(\lambda)_{s_0}$ so that $F_0 W(\lambda)_{s_0} = W(\lambda)_{s_0}$.

Theorem 2.7. *For each $\lambda \in P_+$ and $m \geq 0$, the module*

$$(F_m W(\lambda)_{s_0} \cap W(\lambda))/(F_{m+1} W(\lambda)_{s_0} \cap W(\lambda))$$

constructed from the inclusion in Theorem 2.6 *is* $\mathfrak{b}(0)$*-stable, and it is the direct sum of irreducible* $(\mathfrak{sl}(2,0) + \widetilde{\mathfrak{h}})$*-modules of dimension m, and one-dimensional* $\widetilde{\mathfrak{h}}$*-modules* $\mathbb{C}_\mu$ *so that* $\langle \alpha_0^\vee, \mu \rangle = m - 1$.

Proof. The assertion follows by [26, Lemma 4.4 and Corollary 4.8] □

Theorem 2.8. *Keep the setting of Theorem 2.7 and Theorem 1.8. Then, the* $\mathbb{C}[\mathbb{A}^{(\lambda)}]$*-action on* $W(\lambda)$ *naturally induces* $\mathfrak{b}(0)$*-endomorphisms of* $W(\lambda)_{s_0}$*. In addition, this* $\mathbb{C}[\mathbb{A}^{(\lambda)}]$*-action on* $W(\lambda)_{s_0}$ *is free.*

Proof. The first part of the assertion follows by Theorem 1.8 since $\mathbb{C}[\mathbb{A}^{(\lambda)}]$ is also isomorphic to the $\mathfrak{h}$-weight $s_\vartheta \lambda$-part of $W(\lambda)$ (by the $\mathfrak{g}$-invariance) and $W(\lambda)_{s_0}$ is cyclically generated by the $\mathfrak{h}$-weight $s_\vartheta \lambda$-part inside $W(\lambda)$ by [26, Proof of Theorem 5.1]. The freeness assertion is [26, Theorem 5.1]. □

Lemma 2.9. *Global and local Weyl modules satisfy the Demazure property.*

Proof. We first check the Demazure property for the global Weyl module $W(\lambda)$. Suppose $v \in W(\lambda)$ is a non-zero vector of ($\widetilde{\mathfrak{h}}$-)weight γ with $\langle \alpha_0^\vee, \gamma \rangle = -n < 0$. Then $v \in F_m W(\lambda)_{s_0} \cap W(\lambda)$ with some $m > n$. As $\langle \alpha_0^\vee, \gamma \rangle < 0$, the only possibility provided by Theorem 2.7 is that v belongs to $F_m W(\lambda)_{s_0} \cap W(\lambda)$ alltogether with the irreducible $(\mathfrak{sl}(2,0) + \widetilde{\mathfrak{h}})$-modules of dimension $m > n$. In particular, we have even stronger statement $(E_0)^n v \neq 0$.

We consider the case of local Weyl modules. Suppose that $u \in W(\lambda, 0)$ is a non-zero vector of weight γ with $\langle \alpha_0^\vee, \gamma \rangle = -n < 0$. Then u is the image of some $v \in F_m W(\lambda)_{s_0} \cap W(\lambda)$ with $m > n$ (as we discussed in the above) under the natural projection of $W(\lambda)$ onto $W(\lambda, 0)$. As we know that $(E_0)^{n-1} v \neq 0$ in $W(\lambda)$, it remains to check that the projection of $(E_0)^{n-1} v$ is not zero in $W(\lambda, 0)$.

By Definition 1.6 the module $W(\lambda, 0)$ is isomorphic to a quotient of $W(\lambda)$ by the action of the augmentation ideal $I \subset \mathbb{C}[\mathbb{A}^{(\lambda)}]$. As $(E_0)^{n-1} v \in F_m W(\lambda)_{s_0} \cap W(\lambda)$, it is enough to show that $(E_0)^{n-1} v \notin I \cdot (F_m W(\lambda)_{s_0} \cap W(\lambda))$ in view of Theorem 2.8.

We prove this assertion by finding a contradiction. So suppose $(E_0)^{n-1} v = P \cdot v'$, where $P \in I$ and $v' \in F_m W(\lambda)_{s_0} \cap W(\lambda)$ is a vector of weight $\gamma - (n-1)\alpha_0$. Here we have $\langle \alpha_0^\vee, \gamma - (n-1)\alpha_0 \rangle = n-2 < m-1$, the above consideration shows that this weight appears only in the sum of irreducible $(\mathfrak{sl}(2,0) + \widetilde{\mathfrak{h}})$-modules inside $F_m W(\lambda)_{s_0} \cap W(\lambda)$. Therefore, v' belongs to a $(\mathfrak{sl}(2,0) + \widetilde{\mathfrak{h}})$-module inside $F_m W(\lambda)_{s_0} \cap W(\lambda)$.

Since the action of $\mathbb{C}[\mathbb{A}^{(\lambda)}]$ on $W(\lambda)_{s_0}$ preserves the action of $\mathfrak{sl}(2,0)$, so is $P \in I$. As this $\mathbb{C}[\mathbb{A}^{(\lambda)}]$-action is free, we have $Pv \in F_m W(\lambda)_{s_0} \cap W(\lambda)$ for $v \in W(\lambda)_{s_0}$ if and only if $v \in F_m W(\lambda)_{s_0}$. This implies that $(F_0)^{n-1} v'$ is a non-zero element of $F_m W(\lambda)_{s_0} \cap W(\lambda)$ and $P \cdot (F_0)^{n-1} v'$ is proportional to v, hence we conclude $u = 0$. This is a contradiction, and hence we conclude that the projection of $(E_0)^{n-1} v$ is not zero in $W(\lambda, 0)$ as required. □

2.2 Ext's in the Parabolic Category $\mathcal{O}$

Proposition 2.10. *Let $\lambda \in P_+^k$. For each $w, v \in W_{\mathrm{af}}$ so that $\overline{w \circ_k \lambda}, \overline{v \circ_k \lambda} \in P_+$, we have $\overline{w \circ_k \lambda} \le \overline{v \circ_k \lambda}$ if $w \le v$.*

Proof. We replace λ with $\lambda' := \frac{1}{k+1}\lambda \in \mathfrak{h}$. The weight λ' belongs to the fundamental alcove (denoted by A_v^+ for $v = 0$ in [34]). We have $\overline{w \circ_k \lambda} \in A = wA_0^+$, $\frac{1}{k+1}\overline{v \circ_k \lambda} \in A' = vA_0^+$. By [34, Lemma 3.6], we have $d(A, A_0^+) = \ell(w)$, and $d(A', A_0^+) = \ell(v)$. By the above identification of $d(\bullet, A_0^+)$ and $\ell(\bullet)$, we deduce $\ell(wt_\gamma) = \ell(w) + \ell(t_\gamma)$ and $\ell(vt_\gamma) = \ell(v) + \ell(t_\gamma)$ for some large dominant coroot γ. Now the subward property of the Bruhat order yields $wt_\gamma < vt_\gamma$, that is equivalent to $A \preceq A'$ (inside the dominant chamber with respect to W) by the equivalence of assertions after [40, Claim 4.14]. Since $\overline{w \circ_k \lambda} - \overline{v \circ_k \lambda} \in Q$, we conclude the result. □

Theorem 2.11 (Kac-Kazhdan [17] Theorem 2, cf. Fiebig [10] §3.2)**.** *Let $\lambda, \mu \in P_+$. Then, we have*

$$\bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}^i_{\widehat{\mathfrak{g}}_k}(L_k(\lambda), L_k(\mu)) \neq \{0\}$$

only if λ and μ belongs to the same $\circ_k$-orbit of W_{af}. □

For a $\widehat{\mathfrak{g}}_k$-module M, let us denote $M^\#$ the module obtained from M by taking the action of the opposition of $\mathfrak{g}$ on the dual space to M (with ξ being fixed). Then, the $\widehat{\mathfrak{g}}_k$-module $L_k(\lambda)$ viewed as a $\mathfrak{g}[z]$-module is sent to the $\widehat{\mathfrak{g}}_k$-module $L_k(\lambda)$, with its $\mathfrak{g}[z]$-module structure given through $X \otimes z^m \mapsto X \otimes \xi^m$ for each $X \in \mathfrak{g}$ and $m \in \mathbb{Z}_{\ge 0}$. The same procedure makes $M_k(\mu)^\#$ into an injective envelope of $V(\mu)$ (as $M_k(\mu)$ is a projective cover of $V(\mu)$).

Proposition 2.12 (Shapiro's Lemma)**.** *Let V be a graded $\mathfrak{g}[\xi]$-module (or a graded $\mathfrak{I}$-module) and let M be a $\widehat{\mathfrak{g}}_k$-module. Then, we have*

$$\mathrm{Ext}^i_{\widehat{\mathfrak{g}}_k}(U_k(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[\xi]+\mathbb{C}d)} V, M) \cong \mathrm{Ext}^i_{\mathfrak{g}[\xi]+\mathbb{C}d}(V, M) \qquad i \in \mathbb{Z},$$

(*or the isomorphism obtained by replacing* $(\mathfrak{g}[\xi] + \mathbb{C}d)$ *with* $\mathfrak{I}$).

Proof. It is straight-forward to see that $U(\widehat{\mathfrak{g}})_k$ is a free $U(\mathfrak{g}[\xi])$-algebra and also a free $U(\mathfrak{I})/(K-k)$-algebra (by the PBW theorem). Hence, Shapiro's lemma imply the results. □

Corollary 2.13. *Let $\lambda, \mu \in P_+$ so that $\lambda \not> \mu$. Then, we have*

$$\mathrm{Ext}^1_{\widehat{\mathfrak{g}}_k}(M_k(\lambda), L_k(\mu)) = \{0\}.$$

In particular, $M_k(\lambda)$ is projective in the parabolic category $\mathcal{O}$ when $\lambda \in P_+^k$.

Proof. By Proposition 2.12, the assertion is equivalent to

$$\mathrm{Ext}^1_{\mathfrak{g}[\xi]}(V(\lambda), L_k(\mu)) = \{0\}.$$

Since $\#$ exchanges an injective resolution of $L_k(\mu)$ (viewed as $\mathfrak{g}[\xi]$-module) and a projective resolution of $L_k(\mu)$ viewed as $\mathfrak{g}[z]$-modules, we deduce that

$$\mathrm{Ext}^1_{\mathfrak{g}[\xi]}(V(\lambda), L_k(\mu)) \cong \mathrm{Ext}^1_{\mathfrak{g}[z]}(L_k(\mu), V(\lambda)). \tag{2.1}$$

Since $M_k(\mu)$ is the projective cover of $L_k(\mu)$ as $\mathfrak{g}[z]$-modules, the non-trivial extension in the RHS of (2.1) occurs in the highest weights in $M_k(\mu)$. Thanks to Proposition 2.10, we need $\mu < \lambda$ to obtain a non-trivial extension. This proves the first assertion. In view of Theorem 2.11 and the fact that P_+^k is contained in the fundamental domain of the $\circ_k$-action on (the real part of) $\mathfrak{h}^*$, we deduce the latter assertion again by Proposition 2.10. □

Lemma 2.14. *Suppose that $\lambda \in P_+^k$. Let V be a finite-dimensional string $\mathfrak{b}(0)$ module containg weights $(\lambda + k\Lambda_0)$ and $(s_0 \circ_k \lambda + k\Lambda_0)$. We inflate to regard it as a $\mathfrak{I}$-module, where other graded elements act by zero. Then, the maximal $\mathfrak{g}$-integrable quotient of $U_k(\widetilde{\mathfrak{g}}) \otimes_{U(\mathfrak{I})} V$ contains the non-trivial extension of $M_k(\lambda)$ by $M_k(s_0 \circ_k \lambda)$ as its subquotient.*

Proof. Note that as $\lambda \in P_k^+$ we have $\lambda - s_0 \circ_k \lambda = m\alpha_0$ for some $m > 0$.

By $\mathbf{v}$ denote a highest weight vector of V, that is, a non-zero vector of the head of V (it is unique up to a scalar). We denote by μ the $\widetilde{\mathfrak{h}}$-weight of $\mathbf{v}$.

We introduce an increasing filtration on V defined by

$$F^m = \langle \mathbf{v}, E_0\mathbf{v}, \ldots, E_0^m\mathbf{v} \rangle \qquad \text{for} \qquad m \in \mathbb{Z}_{\geq 0}.$$

Then each adjoint graded factor is spanned by a single vector with trivial action of E_0.

Thanks to the exactness of the induction, this filtration produce a filtration on the maximal $\mathfrak{g}$-integrable quotient of $U_k(\widetilde{\mathfrak{g}}) \otimes_{U(\mathfrak{I})} V$ with its adjoint graded quotients isomorphic to $M_k(\mu - m\alpha_0)$ for $m = 0, 1, \ldots$.

We denote by Ω the Casimir element of $\widetilde{\mathfrak{g}}$ (see [18, Chapter 2]). It belongs to the center of a suitable completion of $U_k(\widetilde{\mathfrak{g}})$ and acts on each highest weight module, particularly on finite successive extensions of $\{M_k(\lambda - m\alpha_0)\}_{m \in \mathbb{Z}_{\geq 0}}$.

For each $\mu \in \widetilde{P}$, the action of Ω on each $M_k(\mu - m\alpha_0)$ is by a scalar, depending on m as a degree two polynomial on m (see [18, Chapters 2,7]). Also action of Ω on $M_k(\mu)$ is invariant with respect to the $\circ_k$-action of the affine Weyl group on P that arises from highest weights (see [18], Chapter 7). In particular, the scaling factor of Ω is the same for $M_k(\lambda)$ and $M_k(s_0 \circ_k \lambda)$ and differs for other factors. We denote the scaling factor of Ω on $M_k(\lambda)$ by c.

As Ω belongs to the center and our module is a finite successive extension of the modules in which Ω acts by scalars, the generalized c-eigenspace M of Ω in $U_k(\widetilde{\mathfrak{g}}) \otimes_{U(\mathfrak{I})} V$ is a direct summand. Moreover, by the above eigenvalue analysis, it admits a filtration whose adjoint graded quotients are $M_k(\lambda)$ and $M_k(s_0 \circ_k \lambda)$. Therefore, M is a trivial or non-trivial extension between $M_k(\lambda)$ and $M_k(s_0 \circ_k \lambda)$.

Note that the $\widetilde{\mathfrak{h}}$-weight $(s_0 \circ_k \lambda + k\Lambda_0)$-parts of the both of $M_k(\lambda)$ and $M_k(s_0 \circ_k \lambda)$ consists of highest weights (see Theorem 1.16). Hence, in case M is a trivial extension, the $\widetilde{\mathfrak{h}}$-weight $(s_0 \circ_k \lambda + k\Lambda_0)$-part of both modules can not get to a non-trivial vector of weight λ. But in our case it can be done by the action of E_0 in V itself, and hence M cannot be a trivial extension as required. □

2.3 The Main Theorem

Theorem 2.15. *For each $\lambda \in P_+^k$, the $\mathfrak{g}[z]$-module $L_k(\lambda)$ admits a filtration by global Weyl modules.*

Remark 2.16. In fact, our proof of Theorem 2.15 carries over to the case of the twisted affinization of $\mathfrak{g}$ by a straight-forward modification.

Proof of Theorem 2.15. We check the condition in Theorem 2.2. Applying # to the BGGL-resolution of $L_k(\lambda)$, we deduce:

$$0 \to L_k(\lambda)^{\#} \to M_k(\lambda)^{\#} \xrightarrow{d_1^{\#}} M_k(s_0 \circ_k \lambda)^{\#} \xrightarrow{d_2^{\#}} \bigoplus_{\mu \in W_{\mathrm{af}} \circ \lambda} (M_k(\mu)^{\#})^{\oplus m^2(\mu)} \to \cdots . \tag{2.2}$$

Note that the opposition of $\mathfrak{g}$ sends $W(\lambda)$ (the projective cover of $V(\lambda)$ in $\mathfrak{g}[z]\text{-}\mathsf{mod}^{\le \lambda}$) to the projective cover of $V(-w_0\lambda)$ in $\mathfrak{g}[z]\text{-}\mathsf{mod}^{\le -w_0\lambda}$, that is $W(-w_0\lambda)$. Hence, it sends $W(\mu, 0)$ to $W(-w_0\mu, 0)$. By chasing the image of the maps obtained by applying $\mathrm{Hom}_{\mathfrak{g}[z]}(W(\mu, 0), \bullet)$ to (2.2), we deduce

$$\mathrm{Ext}^1_{\mathfrak{g}[z]}(L_k(\lambda), W(-w_0\mu, 0)^*) \cong \mathrm{Ext}^1_{\mathfrak{g}[z]}(W(\mu, 0), L_k(\lambda)^{\#}). \tag{2.3}$$

Here we want to show the vanishing of the LHS of (2.3) for every $\mu \in P_+$.

By Proposition 2.12, we have

$$\mathrm{Ext}^1_{\mathfrak{g}[z]}(W(\mu, 0), L_k(\lambda)^{\#}) \cong \mathrm{Ext}^1_{\widehat{\mathfrak{g}}_k}(U_k(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[\xi])} W(\mu, 0), L_k(\lambda)). \tag{2.4}$$

By using the $\mathfrak{g}[\xi]$-module filtration on $W(\mu, 0)$ and the exactness of the induction, we deduce that $U(\widehat{\mathfrak{g}}_k) \otimes_{U(\mathfrak{g}[\xi])} W(\mu, 0)$ is a finite successive extensions of parabolic Verma modules of level k. In view of Proposition 2.12, we have

$$\begin{aligned} \mathrm{Ext}^1_{\mathfrak{g}[\xi] \oplus \mathbb{C}d}(V(\gamma, 0), L_k(\lambda)) &\cong \mathrm{Ext}^1_{\widehat{\mathfrak{g}}_k}(M_k(\gamma), L_k(\lambda)) \\ \mathrm{Ext}^1_{\mathfrak{I}}(\mathbb{C}_\gamma, L_k(\lambda)) &\cong \mathrm{Ext}^1_{\widehat{\mathfrak{g}}_k}(\mathbf{M}_k(\gamma), L_k(\lambda)). \end{aligned}$$

By Theorem 1.17, and the genuine BGG-resolution, we deduce

$$\begin{aligned} \mathrm{Ext}^1_{\widehat{\mathfrak{g}}_k}(M_k(\gamma), L_k(\lambda)) \neq \{0\} \quad &\Rightarrow \quad \gamma = s_0 \circ_k \lambda, \\ \mathrm{Ext}^1_{\widehat{\mathfrak{g}}_k}(\mathbf{M}_k(\gamma), L_k(\lambda)) \neq \{0\} \quad &\Rightarrow \quad \gamma \in \{s_i \circ_k \lambda\}_{i \in \mathtt{I}_{\mathrm{af}}}, \end{aligned} \tag{2.5}$$

and these extensions are at most one-dimensional. It follows that

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathfrak{g}[\xi] \oplus \mathbb{C}d}(V(\gamma, 0), L_k(\lambda)) & = & \mathrm{Ext}^1_{\widehat{\mathfrak{g}}_k}(M_k(\gamma), L_k(\lambda)) \\ \big\downarrow & & \\ \mathrm{Ext}^1_{\mathfrak{I}}(\mathbb{C}_\gamma, L_k(\lambda)) & = & \mathrm{Ext}^1_{\widehat{\mathfrak{g}}_k}(\mathbf{M}_k(\gamma), L_k(\lambda)) \end{array}$$

and the extension is at most one-dimensional.

As $\lambda \in P_+^k$, the classical weights $\overline{\lambda}$ and $\overline{s_0 \circ_k \lambda}$ are both dominant.

We have the following map

$$\begin{array}{ccc} W(\mu, 0) & \xrightarrow{E_0^m} & W(\mu, 0) \\ \big\uparrow & & \big\uparrow \\ W(\mu, 0)^{\overline{\lambda}} & \xrightarrow[E_0^m]{} & W(\mu, 0)^{\overline{s_0 \circ_k \lambda}} \end{array} \qquad m := -\langle \vartheta^\vee, \lambda \rangle + k \ge 0,$$

where $W(\mu, 0)^\gamma$ for $\gamma \in P$ denotes the $\mathfrak{h}$-weight γ-part of $W(\mu, 0)$.

By Lemma 2.9, the map E_0^m in the bottom line is injective. Therefore, Lemma 2.14 implies that the extension of $M_k(s_0 \circ_k \lambda)$ by $M_k(\lambda)$ is already attained in the subquotient of $U_k(\widetilde{\mathfrak{g}}) \otimes_{U(\mathfrak{I})} W(\mu, 0)$.

This forces the RHS of (2.4) to be zero for $\gamma = s_0 \circ_k \lambda$. In view of (2.5), we deduce

$$\mathrm{Ext}^1_{\widetilde{\mathfrak{g}}_k}(U_k(\widetilde{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[\xi]+\widetilde{\mathfrak{h}})} W(\mu, 0), L_k(\lambda)) = \{0\} \quad \forall \mu \in P_+,$$

that implies the result. □

Corollary 2.17. *Let $\lambda \in P_+^k$. The projective resolution of $L_k(\lambda)$ borrowed from Theorem 1.17 respects the filtration by global Weyl modules.*

Proof. Applying Theorem 2.2 to the long exact sequence obtained by applying $\mathrm{Ext}^{\bullet}_{\mathfrak{g}[z]}(\bullet, W(\mu, 0)^*)$ to the short exact sequence

$$0 \to \ker_0 \to M_k(\lambda) \to L_k(\lambda) \to 0,$$

we deduce $\ker_0$ admits a filtration by global Weyl modules.

We have $\ker d_i \cong \mathrm{Im}\, d_{i+1}$ for each $i \geq 0$, and it is finitely generated by examing the next term. Hence, we apply the same argument by replacing $M_k(\lambda)$ with $\bigoplus_{w \in W \backslash W_{\mathrm{af}}, \ell(w)=i} M_k(w \circ_k \lambda)$ and $L_k(\lambda)$ with $\ker d_i$ to deduce the assertion inductively. □

3 The level-restricted Kostka polynomials

We work in the setting of the previous section. Fix a positive integer k in the sequel. Let M be a $\mathfrak{g}$-integrable graded $\mathfrak{g}[z]$-module. Then, we define its i-th relative homology group as:

$$H_i(\mathfrak{g}[z], \mathfrak{g}; M) := \mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, \mathbb{R}^{-i}\mathrm{Hom}_{\mathfrak{g}[z]_1}(M, \mathbb{C})).$$

This is a graded vector space.

Remark 3.1. Thanks to Proposition 1.14, the projective resolution of M is $\mathfrak{g}$-semisimple. In particular, $\mathbb{R}^{-i}\mathrm{Hom}_{\mathfrak{g}[z]_1}(M, \mathbb{C})$ is semi-simple as $\mathfrak{g}$-modules and hence our definition of the relative homology group coincides with these in [1, Chapter I] (see also Remark 2.3).

Lemma 3.2. *Let $k \in \mathbb{Z}_{>0}$ and let $\lambda, \mu \in P_+$. We have*

$$\mathsf{gdim}\, H_0(\mathfrak{g}[z], \mathfrak{g}; W(\mu, 0) \otimes_{\mathbb{C}} M_k(\lambda)) = [W(\mu, 0)^* : V(\lambda)]_q.$$

Proof. By unwinding the definition and applying Proposition 1.14, we have

$$H_0(\mathfrak{g}[z], \mathfrak{g}; W(\mu, 0) \otimes_{\mathbb{C}} M_k(\lambda)) = \mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, \mathrm{Hom}_{\mathfrak{g}[z]_1}(W(\mu, 0) \otimes_{\mathbb{C}} M_k(\lambda), \mathbb{C}))$$
$$\cong \mathrm{Hom}_{\mathfrak{g}[z]}(M_k(\lambda), W(\mu, 0)^*) = \mathrm{Hom}_{\mathfrak{g}[z]}(P(\lambda), W(\mu, 0)^*).$$

Therefore, the assertion holds. □

Definition 3.3 (Feigin and Feigin [8])**.** Let $\lambda \in P_+^k$ and $\mu \in P_+$. We define the level-restricted Kostka polynomial $P^{(k)}_{\mu,\lambda}(q) \in \mathbb{Z}[q, q^{-1}]$ by

$$P^{(k)}_{\mu,\lambda}(q^{-1}) := \mathsf{gdim}\, H_0(\mathfrak{g}[z], \mathfrak{g}; W(-w_0\mu, 0) \otimes_{\mathbb{C}} L_k(\lambda)).$$

Remark 3.4. **1)** The original definition of the level-restricted Kostka polynomial is due to Schilling-Warnaar [39]. We provide a comparison with a straightforward generalization of their definition with ours in Corollary 3.11. **2)** Since $P(\lambda) \cong \varprojlim_{k\to\infty} L_k(\lambda)$ as graded $\mathfrak{g}[z]$-modules, it holds that

$$\lim_{k\to\infty} P^{(k)}_{\mu,\lambda}(q) = P_{\mu,\lambda}(q).$$

3) If $\mathfrak{g}$ is of type A, then $P_{\mu,\lambda}(q)$ coincides with the Kostka polynomial if we take "transpose" of μ.

Theorem 3.5. *Let $\lambda \in P^k_+$ and $\mu \in P_+$. We have*

$$H_i(\mathfrak{g}[z], \mathfrak{g}; W(\mu, 0) \otimes_{\mathbb{C}} L_k(\lambda)) = \{0\} \quad i \neq 0.$$

Proof. Taking account into Remark 3.1, we have

$$\begin{aligned} H_i(\mathfrak{g}[z], \mathfrak{g}; W(\mu, 0) \otimes_{\mathbb{C}} L_k(\lambda)) &= \mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, \mathbb{R}^{-i}\mathrm{Hom}_{\mathfrak{g}[z]_1}(W(\mu,0) \otimes_{\mathbb{C}} L_k(\lambda), \mathbb{C})) \\ &\cong \mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, \mathbb{R}^{-i}\mathrm{Hom}_{\mathfrak{g}[z]_1}(L_k(\lambda), W(\mu,0)^*)) \\ &\cong \mathbb{R}^{-i}\mathrm{Hom}_{\mathfrak{g}[z]\text{-}\mathsf{mod}}(L_k(\lambda), W(\mu,0)^*) \\ &\cong \mathrm{Ext}^{-i}_{\mathfrak{g}[z]\text{-}\mathsf{mod}}(L_k(\lambda), W(\mu,0)^*). \end{aligned}$$

By Theorem 2.15 and Theorem 2.2 2), we deduce that

$$\mathrm{Ext}^{-i}_{\mathfrak{g}[z]\text{-}\mathsf{mod}}(L_k(\lambda), W(\mu,0)^*) = \{0\} \quad \text{for each} \quad -i \neq 0$$

as required. □

Corollary 3.6. *For each $\lambda \in P^k_+$ and $\mu \in P_+$, we have*

$$P^{(k)}_{\mu,\lambda}(q) = (L_k(\lambda) : W(\mu))_q.$$

Proof. By Theorem 2.15, we can repeatedly apply (the Ext^1-part of) Theorem 2.2 to short exact sequences that respects the filtration by the global Weyl modules $L_k(\lambda)$. This yields the additivity of the Hom-part, namely

$$\mathsf{gdim}\,\mathrm{Hom}_{\mathfrak{g}[z]}(L_k(\lambda), W(\mu,0)^*) = \sum_{\gamma \in P_+} \overline{(L_k(\lambda) : W(\gamma))_q} \cdot \mathsf{gdim}\,\mathrm{Hom}_{\mathfrak{g}[z]}(W(\gamma), W(\mu,0)^*).$$

Applying the Hom-part of Theorem 2.2, we conclude the result. □

The following result is the Teleman's Borel-Weil-Bott theorem [41], that we supply an elementary proof that essentially depends only on Proposition 1.14 and Theorem 1.17 (and does not depend on any results in §1.3).

Corollary 3.7 (Teleman [41])**.** *For each $\lambda, \mu \in P^k_+$ and $w \in W \backslash W_{\mathrm{af}}$, we have*

$$\mathsf{gdim}\, H_i(\mathfrak{g}[z], \mathfrak{g}; V(\overline{w \circ_k \mu}, 0)^* \otimes_{\mathbb{C}} L_k(\lambda)) = \begin{cases} q^{\langle d, w \circ_k \lambda \rangle} & (\lambda = \mu, i = -\ell(w)) \\ 0 & (\textit{otherwise}) \end{cases}.$$

Proof. Note that we have $\overline{w \circ_k \mu} \in P_+$. By unwinding the definition as in the proof of Theorem 3.5, we have

$$H_i(\mathfrak{g}[z], \mathfrak{g}; V(\overline{w \circ_k \mu}, 0)^* \otimes_{\mathbb{C}} L_k(\lambda)) \cong \mathrm{Ext}^{-i}_{\mathfrak{g}[z]}(L_k(\lambda), V(\overline{w \circ_k \mu})).$$

As the BGGL resolution is minimal as a projective resolution, we conclude the result. $\square$

For each $\mu \in P_+$, let us regard $W(\mu, 0)$ as a $\mathfrak{g}[\xi]$-module through the involution $X \otimes z^n \mapsto X \otimes \xi^n$ ($X \in \mathfrak{g}, n \geq 0$). We define

$$\mathbf{W}_k(\mu) := U_k(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[\xi])} W(\mu, 0)$$

and call it the generalized Weyl module of $\mathfrak{g}$ with highest weight μ and level k. We fix a sequence of distinct points $\vec{a} = (a_1, \ldots, a_m) \in \mathbb{C}^m$ and weights $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_m) \in (P_+)^m$ so that $\mu = \mu_1 + \mu_2 + \cdots + \mu_m$. Then, we define the space of generalized conformal coinvariants as:

$$CC(\vec{a}, \vec{\mu}, \lambda) := \frac{L_k(\lambda) \otimes \bigotimes_{i=1}^m \mathbf{W}_k(\mu_i)}{U(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\vec{x}})\,(L_k(\lambda) \otimes \bigotimes_{i=1}^m \mathbf{W}_k(\mu_i))},$$

where $\mathcal{O}_{\vec{x}} = \mathbb{C}[x, \frac{1}{x-a_1}, \ldots, \frac{1}{x-a_m}]$ and an element $X \otimes f \in \mathfrak{g} \otimes \mathcal{O}_{\vec{x}}$ acts on $\mathbf{W}_k(\mu_i)$ ($1 \leq i \leq m$) through its Laurent expansion along $z = x - a_i$, and acts on $L_k(\lambda)$ through its Laurent expansion along $z = \infty$ (cf. Teleman [41, §3.6]).

Theorem 3.8. *Let $\lambda \in P_+^k$ and let $\mu \in P_+$. The space*

$$H_0(\mathfrak{g}[z], \mathfrak{g}; W(\mu) \otimes_{\mathbb{C}} L_k(\lambda))$$

is a free $\mathbb{C}[\mathbb{A}^{(\mu)}]$-module, and the specialization to $\vec{a} \in \mathbb{A}^{(\mu)}$ corresponding to a distinct points $a_1, a_2, \ldots, a_m$ with their multiplicities $\mu_1, \mu_2, \ldots, \mu_m \in P_+$ (i.e. when the multiplicity of a_k with respect to the i-th set of unordered points is $\langle \alpha_i^\vee, \mu_k \rangle$) yields an isomorphism of vector spaces

$$\mathbb{C}_{\vec{a}} \otimes_{\mathbb{C}[\mathbb{A}^{(\mu)}]} H_0(\mathfrak{g}[z], \mathfrak{g}; W(\mu) \otimes_{\mathbb{C}} L_k(\lambda)) \cong CC(\vec{a}, \vec{\mu}, \lambda).$$

Proof. By Theorem 1.8, the module $H_0(\mathfrak{g}[z], \mathfrak{g}; W(\mu) \otimes_{\mathbb{C}} L_k(\lambda))$ admits a decreasing separable filtration whose adjoint graded quotient is the direct sum of quotients of $H_0(\mathfrak{g}[z], \mathfrak{g}; W(\mu, 0) \otimes_{\mathbb{C}} L_k(\lambda))$ with grading shifts. By Theorem 3.5, we deduce that the adjoint graded quotient of $H_0(\mathfrak{g}[z], \mathfrak{g}; W(\mu) \otimes_{\mathbb{C}} L_k(\lambda))$ is the direct sum of $H_0(\mathfrak{g}[z], \mathfrak{g}; W(\mu, 0) \otimes_{\mathbb{C}} L_k(\lambda))$ (instead of its proper quotient) with grading shifts. Our homology group commutes with inverse limit by the degree-wise Mittag-Leffler condition thanks to [41, Theorem 0] and Theorem 3.5. Therefore, we conclude that the free $\mathbb{C}[\mathbb{A}^{(\mu)}]$-action on $W(\mu)$ lifts to $H_0(\mathfrak{g}[z], \mathfrak{g}; W(\mu) \otimes_{\mathbb{C}} L_k(\lambda))$. In addition, we have

$$H_i(\mathfrak{g}[z], \mathfrak{g}; W(\mu) \otimes_{\mathbb{C}} L_k(\lambda)) = \{0\} \quad i \neq 0.$$

By the semi-continuity theorem applied to $0 \in \mathbb{A}^{(\mu)}$ by regarding $W(\mu) \otimes_{\mathbb{C}} L_k(\lambda)$ as a finitely generated $U(\mathfrak{g}[z]) \otimes \mathbb{C}[\mathbb{A}^{(\mu)}]$-module (through the above construction), we deduce that the natural map

$$\mathbb{C}_{\vec{a}} \otimes_{\mathbb{C}[\mathbb{A}^{(\mu)}]} H_0(\mathfrak{g}[z], \mathfrak{g}; W(\mu) \otimes_{\mathbb{C}} L_k(\lambda)) \to H_0(\mathfrak{g}[z], \mathfrak{g}; \mathbb{C}_{\vec{a}} \otimes_{\mathbb{C}[\mathbb{A}^{(\mu)}]} W(\mu) \otimes_{\mathbb{C}} L_k(\lambda))$$

is an isomorphism. The RHS is further isomorphic to

$$H_0(\mathfrak{g}[z], \mathfrak{g}; \bigotimes_{i=1}^{m} W(\mu_i, a_i) \otimes_{\mathbb{C}} L_k(\lambda)),$$

where $\{a_1, \ldots, a_m\} \subset \mathbb{C}$ denotes the set of distinct points in the configuration $\vec{a}$, and μ_i is the sum of fundamental weights that is supported on a_i. This last vector space is precisely $CC(\vec{a}, \vec{\mu}, \lambda)$ in view of [41, Corollary 3.6.6]. $\square$

Theorem 3.9 (Weyl-Kac character formula [18], see also [31] Theorem 2.2.1)**.** *For each $\lambda \in P_+^k$, we have the following equality of characters:*

$$\mathsf{ch}\, L_k(\lambda) = \sum_{w \in W \backslash W_{\mathrm{af}}} (-1)^{\ell(w)} \mathsf{ch}\, M_k(\overline{w \circ_k \lambda}) \otimes_{\mathbb{C}} \mathbb{C}_{\langle d, w \circ_k \lambda \rangle \delta}.$$

Corollary 3.10. *For each $\lambda \in P_+^k$ and $\mu \in P_+$, we have*

$$P^{(k)}_{\mu,\lambda}(q) = \sum_{w \in W \backslash W_{\mathrm{af}}} (-1)^{\ell(w)} q^{-\langle d, w \circ_k \lambda \rangle} P_{\mu, \overline{w \circ_k \lambda}}(q).$$

Proof. Combine Theorem 3.9, Corollary 3.6, and Lemma 3.2, taking into account into the fact that $\{\mathsf{ch}\, W(\lambda)\}_{\lambda \in P^+}$ forms a basis in the space of characters. $\square$

Corollary 3.11. *For each $\lambda \in P_+^k$ and $\mu \in P_+$, we have $P^{(k)}_{\mu,\lambda}(q) = X^{(k)}_{\mu,\lambda}(q)$, where $X^{(k)}_{\mu,\lambda}(q)$ is defined in section 5.*

Proof. Compare them using Corollary 3.10 and Theorem 5.11 since we have $P_{\lambda,\mu}(q) = X_{\lambda,\mu}(q)$ by Lemma 3.2 and Theorem 5.10. $\square$

4 The Feigin realization of global Weyl modules

We retain the setting of the previous section. We assume that $\mathfrak{g}$ is of type ADE in addition.

Theorem 4.1. *Let $\varpi \in P_+^1$ and let $\mu \in P_+$. Then, we have*

$$(L_1(\varpi) : W(\mu))_q = \begin{cases} q^{-\langle d, w(\varpi + \Lambda_0) \rangle} & (\mu = \overline{w(\varpi + \Lambda_0)}, w \in W_{\mathrm{af}}) \\ 0 & (\textit{otherwise}) \end{cases}.$$

Proof. By Cherednik-Feigin [7, (1.25)], the character of $L_1(\varpi)$ is the multiplicity-free sum of these of $W(\mu)$ up to grading shifts (note that we used $Q = Q^\vee \subset W_{\mathrm{af}}$ here). Since the graded characters of $\{W(\mu)\}_{\mu \in P_+}$ are linearly independent, we deduce the $q = 1$ case of the assertion.

Let us normalize $L_1(\varpi)$ so that its highest weight vector has d-degree 0. For $w \in W_{\mathrm{af}}$ we set $a_w := \langle d, w(\varpi + \Lambda_0) \rangle$. Then, we have

Claim A. *Assume that $w \in W_{\mathrm{af}}$. Let μ be the $\mathfrak{h}$-weight of $L_1(\varpi)$ that appears in the $(-d)$-degree $> a_w$-part. Then, we have $\mu < \overline{w(\varpi + \Lambda_0)}$.*

Proof. Easy consequence of [18, Proposition 11.3] $\square$

We return to the proof of Theorem 4.1. In view of the $q = 1$ case, Claim A forces the degree shift of $W(w(\varpi + \Lambda_0))$ appearing in $L_1(\varpi)$ to be exactly a_w as required. □

Let us define the thick affine Grassmanian as

$$\mathbf{Gr}_G := G(\!(\xi)\!)/G[z],$$

that can be presented as an infinite type scheme [20, 27]. The scheme $\mathbf{Gr}_G$ carries a canonical $G_{\mathrm{sc}}(\!(\xi)\!)$-equivariant line bundle $\mathcal{O}(1)$ that we call the determinant line bundle (see e.g. Kashiwara [22]).

By our adjointness assumption on G, the set of connected components of the scheme $\mathbf{Gr}_G$ is in bijection with the set of level one fundamental weights P_+^1 (see Zhu [42, §0.2.5] but beware that our affine Grassmanian is "thick"). For each $\varpi \in P_+^1$, we denote by ${}^{\varpi}\mathbf{Gr}_G$ the corresponding component.

Each $\lambda \in P_+$ defines a cocharacter of H by our assumptions of G. Hence, it defines a point of $H(\!(\xi)\!)$, and hence a point $[\xi^\lambda] \in \mathbf{Gr}_G$. From this, we define the Schubert variety $\mathbf{Gr}_G^\lambda$ as the $G[\![\xi]\!]$-orbit through $[\xi^\lambda]$. We have $\mathbf{Gr}_G^\lambda \subset \overline{\mathbf{Gr}_G^\mu}$ if and only if $\lambda \geq \mu$ (see e.g. Kashiwara-Tanisaki [25, §1.2]).

Theorem 4.2 ([27] Theorem 2.13 and Theorem 2.14)**.** *Let $\varpi \in P_+^1$ and let $\lambda \in P_+$ so that $\varpi \leq \lambda$, we have*

$$\Gamma(\mathbf{Gr}_G^\lambda, \mathcal{O}(1))^* \cong U(\mathfrak{g}[z])v_\lambda \subset L_1(\varpi),$$

where v_λ is an extremal weight vector of $L_1(\varpi)$ of $\mathfrak{h}$-weight λ. Moreover, the natural restriction map

$$\Gamma({}^{\varpi}\mathbf{Gr}_G, \mathcal{O}(1)) \to \Gamma(\mathbf{Gr}_G^\lambda, \mathcal{O}(1))$$

is surjective. □

Theorem 4.3. *Let $\varpi \in P_+^1$ and let $\lambda \in P_+$ so that $\varpi \leq \lambda$. We have an isomorphism*

$$W(\lambda)^* \cong \ker \left(\Gamma(\mathbf{Gr}_G^\lambda, \mathcal{O}(1)) \longrightarrow \bigoplus_{\mu > \lambda} \Gamma(\mathbf{Gr}_G^\mu, \mathcal{O}(1)) \right)$$

as $\mathfrak{g}[z]$-modules. (Here the maps in the RHS is the restriction maps.)

Proof. We borrow the setting in the proof of Theorem 4.1. By Claim A, the extremal weight vector of $L_k(\varpi)$ with its $\widetilde{\mathfrak{h}}$-weight $w(\varpi + \Lambda_0)$ is contained in the head of $W(\lambda)$ in the $\mathfrak{g}[z]$-module stratification of $L_1(\varpi)$ by global Weyl modules. Hence, Theorem 4.2 implies that the graded $\mathfrak{g}[z]$-module

$$\Gamma_c(\mathbf{Gr}_G^\lambda, \mathcal{O}(1))^* := \ker \left(\Gamma(\mathbf{Gr}_G^\lambda, \mathcal{O}(1)) \longrightarrow \bigoplus_{\mu > \lambda} \Gamma(\mathbf{Gr}_G^\mu, \mathcal{O}(1)) \right)^*$$

admits a surjection to $W(\lambda)$, that we denote by κ_λ. Thus, we have

$$\sum_{\lambda \in \varpi + Q_+ \cap P_+} q^{-\langle d, t_\lambda(\varpi + \Lambda_0) \rangle} \mathsf{ch}\, W(\lambda) \leq \sum_{\lambda \in \varpi + Q_+ \cap P_+} \mathsf{ch}\, \Gamma_c(\mathbf{Gr}_G^\lambda, \mathcal{O}(1))^* = \mathsf{ch}\, L_1(\varpi), \tag{4.1}$$

where $\leq$ means we have the inequality for every coefficients of the monomial in $P \times \mathbb{Z}\delta$. By construction, the inequality in (4.1) is a genuine inequality if and only if κ_λ fails to be surjective for some $\lambda \in (\varpi + Q_+ \cap P_+)$. By Theorem 4.1, the most RHS of (4.1) is equal to the most LHS of (4.1). Therefore, we deduce that every κ_λ induces an isomorphism

$$\Gamma_c(\mathbf{Gr}^\lambda_G, \mathcal{O}(1))^* = \ker \left(\Gamma(\mathbf{Gr}^\lambda_G, \mathcal{O}(1)) \longrightarrow \bigoplus_{\mu > \lambda} \Gamma(\mathbf{Gr}^\mu_G, \mathcal{O}(1)) \right)^* \xrightarrow{\cong} W(\lambda)$$

as required. □

5 A combinatorial definition of level restricted Kostka polynomials

This section exhibits a collection of folklore statements that are (most likely) originally due to Okado [35]. Thus, we do not claim the novelty of the materials here. However, we provide their proofs that maybe of independent interest.

We employ the setting of §1.2. We have quantum algebras $U_q(\widetilde{\mathfrak{g}})$ and $U_q(\widehat{\mathfrak{g}})$ associated to $\widetilde{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}$ so that $U_q(\widehat{\mathfrak{g}}) \subset U_q(\widetilde{\mathfrak{g}})$ (cf. [16, §3.1]).

Definition 5.1 (Crystals, cf. [16] §4.2)**.** An affine crystal

$$\mathbb{B} = (B, \mathrm{wt}, \{\varepsilon_i, \psi_i, \tilde{e}_i, \tilde{f}_i\}_{i \in \mathtt{I}_{\mathrm{af}}})$$

consists of the following data:

1. B is a set, $\mathrm{wt} : B \to \widetilde{P}$, $\varepsilon_i : B \to \mathbb{Z}_{\geq 0}$ $(i \in \mathtt{I}_{\mathrm{af}})$, and $\psi_i : B \to \mathbb{Z}_{\geq 0}$ $(i \in \mathtt{I}_{\mathrm{af}})$ are maps;
2. For $b \in B$ and $i \in \mathtt{I}_{\mathrm{af}}$, we have $\langle \alpha_i^\vee, \mathrm{wt}\, b \rangle = -\varepsilon_i(b) + \psi_i(b)$;
3. For $b \in B$ and $i \in \mathtt{I}_{\mathrm{af}}$, we have maps $\tilde{e}_i : B \to B \sqcup \{\emptyset\}$ and $\tilde{f}_i : B \to B \sqcup \{\emptyset\}$ with the following properties:
   - We have $\widetilde{e}_i(b) \neq \emptyset$ if and only if $\varepsilon_i(b) > 0$. We have $\widetilde{f}_i(b) \neq \emptyset$ if and only if $\psi_i(b) > 0$;
   - If $\varepsilon_i(b) > 0$, then $\mathrm{wt}\, \widetilde{e}_i(b) = \mathrm{wt}\, b + \alpha_i$. If $\psi_i(b) > 0$, then $\mathrm{wt}\, \widetilde{f}_i(b) = \mathrm{wt}\, b - \alpha_i$;
   - If $\varepsilon_i(b) > 0$, then $\widetilde{f}_i(\widetilde{e}_i(b)) = b$. If $\psi_i(b) > 0$, then $\widetilde{e}_i(\widetilde{f}_i(b)) = b$.

A classical crystal $\mathbb{B} = (B, \mathrm{wt}, \{\varepsilon_i, \psi_i, \tilde{e}_i, \tilde{f}_i\}_{i \in I_{\mathrm{af}}})$ is the data obtained from the definition of an affine crystal by replacing $\widetilde{P}$ with $\widehat{P}$. We refer crystal as either affine, classical, or finite crystal, and refer B as its underlying set. By abuse of notation, we may abbreviate $b \in B$ by $b \in \mathbb{B}$. A morphism of a crystal is a map of underlying set that intertwines $\mathrm{wt}, \{\varepsilon_i, \psi_i, \tilde{e}_i, \tilde{f}_i\}_{i \in \mathtt{I}_{\mathrm{af}}}$ (that is usually referred to as a strict morphism in the literature).
A highest weight element (resp. finite highest weight element) in a crystal $\mathbb{B}$ is an element b so that $\varepsilon_i(b) = 0$ for every $i \in \mathtt{I}_{\mathrm{af}}$ (resp. every $i \in \mathtt{I}$). A crystal is connected if and only if each two elements are connected by finitely many sequences of $\tilde{f}_i$ and $\tilde{e}_i$. For an affine crystal $\mathbb{B}$ and $i \in \mathtt{I}_{\mathrm{af}}$, an i-string

is a connected subgraph $B' \subset B$ that is closed under the action of $\{\tilde{e}_i, \tilde{f}_i\}$ and satisfies $\langle \alpha_i^\vee, \mathrm{wt}\, b\rangle = \varepsilon_i(b) - \psi_i(b)$. For $b \in B$ and $i \in \mathtt{I}_{\mathrm{af}}$, we denote the i-string that contains b as $S_i(b)$.

Definition 5.2 (Tensor product of crystals, cf. [16] §4.4)**.** Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be crystals with underlying sets B_1, B_2. We define the tensor product crystal $\mathbb{B}_1 \otimes \mathbb{B}_2$ with its underlying set $B_1 \times B_2$ (to which we refer its element (b_1, b_2) as $b_1 \otimes b_2$) by defining:

- For each $b_1 \in B_1$ and $b_2 \in B_2$, we have $\mathrm{wt}\,(b_1 \otimes b_2) := \mathrm{wt}\, b_1 + \mathrm{wt}\, b_2$;
- For each $b_1 \in B_1$, $b_2 \in B_2$, and $i \in \mathtt{I}_{\mathrm{af}}$, we define

$$\tilde{e}_i(b_1 \otimes b_2) := \begin{cases} \tilde{e}_i b_1 \otimes b_2 & (\psi_i(b_1) \geq \varepsilon_i(b_2)) \\ b_1 \otimes \tilde{e}_i b_2 & (\psi_i(b_1) < \varepsilon_i(b_2)) \end{cases}, \text{ and } \tilde{f}_i(b_1 \otimes b_2) := \begin{cases} \tilde{f}_i b_1 \otimes b_2 & (\psi_i(b_1) > \varepsilon_i(b_2)) \\ b_1 \otimes \tilde{f}_i b_2 & (\psi_i(b_1) \leq \varepsilon_i(b_2)) \end{cases},$$

where we understand that $\emptyset \otimes b_2 = b_1 \otimes \emptyset = \emptyset$.

The functions ε_i and ψ_i ($i \in \mathtt{I}_{\mathrm{af}}$) are uniquely determined by the above.

For a crystal $\mathbb{B}$, we define its character as:

$$\mathsf{ch}\, \mathbb{B} := \sum_{b \in \mathbb{B}} e^{\mathrm{wt}\,(b)}.$$

Theorem 5.3 (Kashiwara [20], cf. [16] §5.1)**.** *For each $\Lambda \in \widetilde{P}_+$ so that $\langle K, \Lambda\rangle = k \in \mathbb{Z}_{>0}$, we have an affine crystal $\mathbb{B}(\Lambda)$ that parametrizes a basis of $L_k(\overline{\Lambda})$ so that $\mathsf{ch}\, \mathbb{B}(\Lambda) = \mathsf{ch}\, L_k(\overline{\Lambda})$ up to $e^{m\delta}$-twist for some $m \in \mathbb{Z}$. Each $\mathbb{B}(\Lambda)$ contains a unique element b_Λ so that $\mathrm{wt}\, b_\Lambda = \Lambda$ and $\widetilde{e}_i b_\Lambda = \emptyset$ for every $i \in \mathtt{I}_{\mathrm{af}}$.*

Theorem 5.4 (Kashiwara [23] §3)**.** *For each $\Lambda \in \widetilde{P}_+$ and $w \in W_{\mathrm{af}}$, we have a subset $\mathbb{B}(\Lambda)_w \subset \mathbb{B}(\Lambda)$ that is stable under the action of $\widetilde{e}_i$ for every $i \in \mathtt{I}_{\mathrm{af}}$. For each $i \in \mathtt{I}_{\mathrm{af}}$ so that $s_i w > w$, we have*

$$\mathbb{B}(\Lambda)_{s_i w} = \bigcup_{m \geq 0} \widetilde{f}_i^m \mathbb{B}(\Lambda)_w.$$

For each $b \in \mathbb{B}(\Lambda)_w$, the set

$$\{\widetilde{e}_i^n \widetilde{f}_i^m b\}_{n,m \geq 0} \cap \mathbb{B}(\Lambda)_w$$

is either singleton or isomorphic to an i-string.

For each $\Lambda \in \widetilde{P}$ and $i \in \mathtt{I}_{\mathrm{af}}$, we define

$$\chi(e^\Lambda) := \frac{\sum_{w \in W_{\mathrm{af}}} (-1)^{\ell(w)} e^{w \circ \Lambda}}{\prod_{\alpha \in \Delta_{\mathrm{af}}^-} (1 - e^\alpha)^{\mathrm{mult}\,\alpha}} \quad \text{and,} \quad D_i(e^\Lambda) := \frac{e^\Lambda - e^{s_i \circ \Lambda}}{(1 - e^{-\alpha_i})}$$

where $\mathrm{mult}\,\alpha$ denote the dimension of the α-root space of $\widetilde{\mathfrak{g}}$. We define

$$\mathbb{Z}[\widetilde{P}]^\wedge := \varprojlim_n \mathbb{Z}[\widetilde{P}]/(e^\beta \mid \beta \in \widetilde{P}, \langle d, \beta\rangle < -n)$$

The original form of the Weyl-Kac character formula asserts (see Theorem 3.9) that

$$\chi(e^\Lambda) = \mathsf{ch}\, L_k(\Lambda) \qquad \Lambda \in \widetilde{P}_+. \tag{5.1}$$

In addition, χ naturally extend to a linear operator

$$\mathbb{Z}[\widetilde{P}] \to \mathbb{Z}[\widetilde{P}]^{\wedge},$$

while D_i ($i \in \mathtt{I}_{\mathrm{af}}$) define linear operators on $\mathbb{Z}[\widetilde{P}]$. By Kumar [31, Theorem 8.2.9 and §8.3], there exists an infinite sequence

$$\mathbf{i} = (i_1, i_2, \ldots) \in \mathtt{I}_{\mathrm{af}}^{\infty}$$

so that

$$\chi = \lim_{k \to \infty} D_{i_k} \circ \cdots D_{i_2} \circ D_{i_1} \quad \text{on} \quad \mathbb{Z}[\widetilde{P}]. \tag{5.2}$$

Definition 5.5 (Restricted paths)**.** Let $\mathbb{B}$ be a classical crystal. For each $\Gamma \in \widehat{P}_+$, we define the set of restricted paths as:

$$\mathcal{P}(\mathbb{B}, \Gamma) := \{ b \in \mathbb{B} \mid \varepsilon_i(b) \le \langle \alpha_i^{\vee}, \Gamma \rangle \quad \text{for each} \quad i \in \mathtt{I}_{\mathrm{af}} \}.$$

Similarly, for each $\gamma \in P_+$, we define the set of finitely restricted paths as:

$$\mathcal{P}_0(\mathbb{B}, \gamma) := \{ b \in \mathbb{B} \mid \varepsilon_i(b) \le \langle \alpha_i^{\vee}, \gamma \rangle \quad \text{for each} \quad i \in \mathtt{I} \}.$$

Theorem 5.6 (Kashiwara, Naito-Sagaki)**.** *We have a classical crystal $\mathbb{B}(\varpi_i)$ corresponding to a (finite-dimensional) level zero fundamental representation $\mathbb{W}(\varpi_i)$ so that $\mathsf{ch}\, \mathbb{B}(\varpi_i) = \mathsf{ch}\, \mathbb{W}(\varpi_i)$ for each $i \in \mathtt{I}$.*
For each $\lambda = \sum_{i=1}^{r} m_i \varpi_i \in P_+$, we define the tensor product (classical) crystal $\mathbb{B}_{\mathrm{loc}}(\lambda)$ as

$$\mathbb{B}_{\mathrm{loc}}(\lambda) := \mathbb{B}(\varpi_1)^{\otimes m_1} \otimes \cdots \otimes \mathbb{B}(\varpi_r)^{\otimes m_r}.$$

Then, $\mathbb{B}_{\mathrm{loc}}(\lambda)$ is a connected crystal and is equipped with a function $D : \mathbb{B}_{\mathrm{loc}}(\lambda) \to \mathbb{Z}$ with the following conditions:

1. *D is preserved by the action of $\tilde{e}_i$ and $\tilde{f}_i$ for each $i \in \mathtt{I}$;*
2. *$D(b_0) = 0$, where $b_0 \in \mathbb{B}_{\mathrm{loc}}(\lambda)$ is the unique element so that $\mathrm{wt}\, b_0 = \lambda$;*
3. *we have $D(\tilde{e}_0 b) = D(b) - 1$ for each $b \in \mathbb{B}_{\mathrm{loc}}(\lambda)$ so that $\varepsilon_0(b) \ge 2$.*

Proof. The definition of $\mathbb{B}(\varpi_i)$ is due to Kashiwara [24, Theorem 5.17]. The character comparison follows from the works of Naito-Sagaki [36] and Chari-Ion [4] (cf. [26, Theorem 1.6]). The function D is studied in [36, §3] under the name of degree function and its relation to the energy statistic is in [33, Theorem 4.5]. The first two properties of D is in [36, Lemma 3.2.1], while the third property of D also follow from [36, (3.2.1)] in view of [36, Lemma 2.2.11]. □

Theorem 5.7 (Kashiwara [19, 21], see Hong-Kang [16] §10)**.** *Let $\mu \in P_+$. We have an isomorphism*

$$\mathbb{B}(k\Lambda_0) \otimes \mathbb{B}_{\mathrm{loc}}(\mu) \cong \bigoplus_{b \in \mathcal{P}(\mathbb{B}_{\mathrm{loc}}(\mu), k\Lambda_0)} \mathbb{B}(\mathrm{wt}\, b + k\Lambda_0 - D(b)\delta)$$

of affine crystals.

Proof. Let $\mathbb{W}$ be the tensor product representation of $U_q(\widehat{\mathfrak{g}})$ corresponding to $\mathbb{B}_{\mathrm{loc}}(\mu)$ borrowed from Theorem 5.6. Being a finite-dimensional integrable representation of the quantum group of $\widehat{\mathfrak{g}}$, we can apply the argument in [16, §10.4]. By calculating using the lower global basis, we have

$$\sum_{b \in \mathcal{P}(\mathbb{B}_{\mathrm{loc}}(\mu), k\Lambda_0)} e^{\mathrm{wt}\, b} = \mathsf{ch}\, \{v \in \mathbb{W} \mid \mathsf{E}_0^{k+1} v = 0, \mathsf{E}_i v = 0, \quad i \in \mathtt{I}\},$$

where $\mathsf{E}_i \in U_q(\widehat{\mathfrak{g}})$ ($i \in \mathtt{I}_{\mathrm{af}}$) is e_i in [16, Definition 3.1.1]. Applying [16, Theorem 10.4.3, Theorem 10.4.4], we deduce a classical crystal morphism

$$\Psi : \bigoplus_{b \in \mathcal{P}(\mathbb{B}_{\mathrm{loc}}(\mu), k\Lambda_0)} \mathbb{B}(\mathrm{wt}\, b + k\Lambda_0) \longrightarrow \mathbb{B}(k\Lambda_0) \otimes \mathbb{B}_{\mathrm{loc}}(\mu).$$

Since each direct summand of the LHS is a connected crystal (Theorem 5.3) and the maps are distinct, it follows that Ψ is an embedding of crystals. Therefore, the set $\mathbb{B}(k\Lambda_0) \otimes \mathbb{B}_{\mathrm{loc}}(\mu) \backslash \mathrm{Im}\, \Psi$ is a classical crystal. As all the highest weight elements of $\mathbb{B}(k\Lambda_0) \otimes \mathbb{B}_{\mathrm{loc}}(\mu)$ are contained in $\mathrm{Im}\, \Psi$, it does not contain a highest weight element. A successive application of $\{\widetilde{e}_i\}_{i \in \mathtt{I}_{\mathrm{af}}}$ sends an arbitrary element of $\mathbb{B}(k\Lambda_0) \otimes \mathbb{B}_{\mathrm{loc}}(\mu)$ to $b_{k\Lambda_0} \otimes \mathbb{B}_{\mathrm{loc}}(\mu)$. The set $b_{k\Lambda_0} \otimes \mathbb{B}_{\mathrm{loc}}(\mu)$ is stable under the action of $\{\widetilde{e}_i\}_{i \in \mathtt{I}_{\mathrm{af}}}$. Therefore, it suffices to prove that applying sufficiently many $\{\widetilde{e}_i\}_{i \in \mathtt{I}_{\mathrm{af}}}$ annihilates every element of $b_{k\Lambda_0} \otimes \mathbb{B}_{\mathrm{loc}}(\mu)$ in order to prove $\mathbb{B}(k\Lambda_0) \otimes \mathbb{B}_{\mathrm{loc}}(\mu) = \mathrm{Im}\, \Psi$.

By the definition of the tensor product action and Theorem 5.6, we deduce that the value of D decreases (by one) when we apply $\widetilde{e}_0$ to $b_{k\Lambda_0} \otimes \mathbb{B}_{\mathrm{loc}}(\mu)$. Since $b_{k\Lambda_0} \otimes \mathbb{B}_{\mathrm{loc}}(\mu)$ is a finite set, it follows that applying $\widetilde{e}_0$ (and other $\{\widetilde{e}_i\}_{i \in \mathtt{I}}$) sufficiently many times annihilates the whole of $b_{k\Lambda_0} \otimes \mathbb{B}_{\mathrm{loc}}(\mu)$. Therefore, we deduce that Ψ is a bijection.

By the same argument, we conclude that $-D(\bullet)\delta$ equips $b_{k\Lambda_0} \otimes \mathbb{B}_{\mathrm{loc}}(\mu)$ with $\widetilde{P}$-valued weights. This makes Ψ into an isomorphism of affine crystals as required. □

Remark 5.8. Theorem 5.6 only states that we can equip $\mathbb{B}(k\Lambda_0) \otimes \mathbb{B}_{\mathrm{loc}}(\mu)$ a structure of affine highest weight crystals so that $b_{k\Lambda} \otimes \mathbb{B}_{\mathrm{loc}}(\mu)$ contains all the highest weight vectors and the D-function gives the affine weights of the tensor product. In particular, we have

$$\sum_{b \in \mathcal{P}(\mathbb{B}_{\mathrm{loc}}(\mu), k\Lambda_0)} \mathsf{ch}\, \mathbb{B}(\mathrm{wt}\, b + k\Lambda_0 - D(b)\delta) \neq \sum_{b' \otimes b \in \mathbb{B}(k\Lambda_0) \otimes \mathbb{B}_{\mathrm{loc}}(\mu)} q^{D(b)\delta} e^{\mathrm{wt}\,(b') + \mathrm{wt}\,(b)}$$

in general. Note that $\mathrm{wt}\,(b) \in P$ (as $\mathbb{B}_{\mathrm{loc}}(\mu)$ is a classical crystal), while $\mathrm{wt}\,(b') - k\Lambda_0 \in P \times \mathbb{Z}\delta$ (as $\mathbb{B}(\mathrm{wt}\, b + k\Lambda_0 - D(b)\delta)$ is an affine crystal).

Using Theorem 5.6, we define

$$\mathsf{gch}\, \mathbb{B}_{\mathrm{loc}}(\lambda) := \sum_{b \in \mathbb{B}_{\mathrm{loc}}(\lambda)} q^{D(b)} e^{\mathrm{wt}\,(b)} \in \mathbb{Z}[\widetilde{P}^0]$$

for each $\lambda \in P_+$.

Definition 5.9 (Restricted Kostka polynomials)**.** For each $\mu, \lambda \in P_+$, we define

$$X_{\mu,\lambda}(q) := \sum_{b \in \mathcal{P}_0(\mathbb{B}_{\mathrm{loc}}(\mu), \lambda)} q^{-D(b)}.$$

For each $\mu \in P_+$ and $\lambda \in P_+^k$, we define

$$X_{\mu,\lambda}^{(k)}(q) := \sum_{b \in \mathcal{P}(\mathbb{B}_{\mathrm{loc}}(\mu), k\Lambda_0), \mathrm{wt}\,(b)=\lambda} q^{-D(b)}.$$

Theorem 5.10 (Lenart-Naito-Sagaki-Schilling-Shimozono [33]). *For each $\mu, \lambda \in P_+$, we have an equality*

$$P_\mu(q,0) = \sum_{b \in \mathbb{B}_{\mathrm{loc}}(\mu)} q^{-D(b)} e^{\mathrm{wt}\,(b)} = \sum_{\lambda \in P_+} X_{\mu,\lambda}(q) \cdot \mathsf{ch}\, V(\lambda) = \overline{\mathsf{gch}\, \mathbb{B}_{\mathrm{loc}}(\mu)},$$

where $P_\mu(q,t)$ is the Macdonald polynomial. □

The following result is a straight-forward extension of a result due to Okado [35] (which in turn uses [14]) for type A (see also Schilling-Shimozono [38, §3.6]).

Theorem 5.11. *Let $\mu \in P_+$ and $\lambda \in P_+^k$. We have*

$$X_{\mu,\lambda}^{(k)}(q) = \sum_{w \in W_{\mathrm{af}}, \overline{w \circ \lambda} \in P_+} (-1)^{\ell(w)} q^{\langle d, \lambda - w \circ_k \lambda \rangle} X_{\mu, \overline{w \circ_k \lambda}}(q).$$

Proof. We set $\mathbb{B}_{\mathrm{loc}}(\mu; \lambda) := \{ b \in \mathbb{B}_{\mathrm{loc}}(\mu) \mid \mathrm{wt}\,(b) = \lambda \}$ for each $\lambda \in P$. In view of (5.1) and the $\mathbb{Z}$-linearity of χ, we deduce

$$\begin{aligned} \chi(e^{k\Lambda_0} \cdot \mathsf{gch}\, \mathbb{B}_{\mathrm{loc}}(\mu)) &= \sum_{b \in \mathbb{B}_{\mathrm{loc}}(\mu)} \chi(e^{\mathrm{wt}\,(b) - D(b)\delta + k\Lambda_0}) \\ &= \sum_{w \in W_{\mathrm{af}}, \overline{w \circ_k \lambda} \in P_+^k} \sum_{b \in \mathbb{B}_{\mathrm{loc}}(\mu;\lambda)} \chi(e^{\lambda + k\Lambda_0 - D(b)\delta}) \\ &= \sum_{w \in W_{\mathrm{af}}, \lambda_+ = \overline{w \circ_k \lambda} \in P_+^k} \sum_{b \in \mathbb{B}_{\mathrm{loc}}(\mu;\lambda)} (-1)^{\ell(w)} q^{\langle d, \lambda - \lambda_+ \rangle} \chi(e^{\lambda_+ + k\Lambda_0 - D(b)\delta}) \qquad (5.3) \\ &= \sum_{\lambda \in P_+} \sum_{w \in W \backslash W_{\mathrm{af}}, \lambda_+ = \overline{w \circ_k \lambda} \in P_+^k} (-1)^{\ell(w)} q^{\langle d, \lambda - \lambda_+ \rangle} X_{\mu,\lambda}(q^{-1}) \chi(e^{\lambda_+ + k\Lambda_0}). \qquad (5.4) \end{aligned}$$

Here we used the fact that the reflection by $\circ_k$ is compatible with the function D by Theorem 5.6 1) and 3) in order to derive the third equality.

The tensor product crystal $\mathbb{B}(k\Lambda_0) \otimes \mathbb{B}_{\mathrm{loc}}(\mu)$ is a classical crystal generated by its highest weight elements.

The set $b_{k\Lambda_0} \otimes \mathbb{B}_{\mathrm{loc}}(\mu)$ decomposes into the disjoint union

$$b_{k\Lambda_0} \otimes \mathbb{B}_{\mathrm{loc}}(\mu) = \bigsqcup_{t \ge 1} b_{k\Lambda_0} \otimes \mathbb{B}_{\mathrm{loc}}(\mu)^t \subset \bigsqcup_{t \ge 1} \mathbb{B}(\Lambda^t) \qquad \Lambda^t \in \widetilde{P}_+^k \qquad (5.5)$$

that respects the tensor product decomposition in Theorem 5.7. Here we warn that we equip $\mathbb{B}(\Lambda^t)$ a structure of affine crystals. Since $b_{k\Lambda_0} \otimes \mathbb{B}_{\mathrm{loc}}(\mu)^t$ is stable under the action of $\{\widetilde{e}_i\}_{i \in \mathtt{I}_{\mathrm{af}}}$, the embedding $b_{k\Lambda_0} \otimes \mathbb{B}_{\mathrm{loc}}(\mu)^t \subset \mathbb{B}(\Lambda^t)$ is stable under the action of $\{\widetilde{e}_i\}_{i \in \mathtt{I}}$.

For each $w \in W$ and $i \in \mathtt{I}_{\mathrm{af}}$, Theorem 5.4 implies that $\mathbb{B}(k\Lambda_0)_w$ is a disjoint union of i-strings or a highest weight elements in i-strings. Here $\mathbb{B}_{\mathrm{loc}}(\mu)$ is stable under the action of $\{\widetilde{e}_i, \widetilde{f}_i\}$.

By a rank one calculation, we deduce that $\mathbb{B}(k\Lambda_0)_w \otimes \mathbb{B}_{\mathrm{loc}}(\mu)$ is a disjoint union of i-strings or a highest weight elements in i-strings. Therefore, the set

$$\mathbb{B}(\Lambda^t)_w := \mathbb{B}(k\Lambda_0)_w \otimes \mathbb{B}_{\mathrm{loc}}(\mu) \cap \mathbb{B}(\Lambda^t) \subset \mathbb{B}(\Lambda^t)$$

is a disjoint union of i-strings or a highest weight elements in i-strings. In addition, we have

$$\mathbb{B}(\Lambda^t)_{s_i w} = \bigcup_{m \geq 0} \widetilde{f}_i^m \mathbb{B}(\Lambda^t)_w$$

by Theorem 5.4 and rank one calculation.

In particular, we have

$$\sum_{b \in \mathbb{B}(\Lambda^t)_{s_i w}} e^{\mathrm{wt}\, b} = \sum_{b \in \mathbb{B}(\Lambda^t)_w} D_i(e^{\mathrm{wt}\, b})$$

for each $i \in \mathtt{I}_{\mathrm{af}}$ so that $s_i w > w$ (here we again warn that the weight here is affine weight).

Applying (5.2), we deduce that

$$\begin{aligned}
\chi(e^{k\Lambda_0} \cdot \mathsf{gch}\, \mathbb{B}_{\mathrm{loc}}(\mu)) &= \sum_{b' \in \mathbb{B}_{\mathrm{loc}}(\mu)} q^{D(b')} \chi(e^{k\Lambda_0 + \mathrm{wt}\, b'}) \\
&= \sum_{t \geq 1} \sum_{b' \in \mathbb{B}_{\mathrm{loc}}(\mu)^t} q^{D(b')} \chi(e^{k\Lambda_0 + \mathrm{wt}\, b'}) \\
&= \lim_{w \to \infty} \sum_{t \geq 1} \sum_{b \in \mathbb{B}(\Lambda^t)_w} \chi(e^{k\Lambda_0 + \mathrm{wt}\, b}) \\
&= \sum_{t \geq 1} \chi(\Lambda^t) = \sum_{b \in \mathcal{P}(\mathbb{B}_{\mathrm{loc}}(\mu), k\Lambda_0)} q^{D(b)} \chi(e^{k\Lambda_0 + \mathrm{wt}\, b}), \qquad (5.6)
\end{aligned}$$

where $w \to \infty$ means that we take a limit $\lim_{k \to \infty} s_{i_k} \cdots s_{i_2} s_{i_1}$. Therefore, equating (5.4) and (5.6) implies the result as required (with $q \mapsto q^{-1}$). $\square$

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Appendix A A free field realization proof of Theorem 4.1 by Ryosuke Kodera[1]

For general notation, we refer to §1 and the beginning of §2 in the main body. Let $\mathfrak{g}$ be a simple Lie algebra of type ADE. The goal of this appendix is to provide a proof of the following result (without using Theorem 2.15) based on the free field realizations.

Theorem A.1 (= Theorem 4.1)**.** *Let $\varpi \in P_+^1$. The $\widetilde{\mathfrak{g}}$-module $L_1(\varpi)$, viewed as a $\mathfrak{g}[z]$-module, admits a filtration by $\{W(\mu)\}_{\mu \in P_+}$. Moreover, we have*

$$(L_1(\varpi) : W(\mu))_q = \begin{cases} q^{-\langle d, w(\varpi+\Lambda_0)\rangle} & (\mu = \overline{w(\varpi+\Lambda_0)}, w \in W_{\mathrm{af}}) \\ 0 & (\mathit{otherwise}) \end{cases}$$

for each $\mu \in P_+$.

Thanks to [7, (1.25)], it suffices to show:

Proposition A.2. *There exists a filtration of $L_1(\varpi)$ whose adjoint graded yields the inequality*

$$\mathsf{ch}\ L_1(\varpi) \leq \sum_{\lambda \in P_+ \cap \overline{W_{\mathrm{af}}(\varpi+\Lambda_0)}} q^{\frac{1}{2}((\lambda,\lambda)-(\varpi,\varpi))} \mathsf{ch}\ W(\lambda).$$

The rest of this appendix is devoted to the proof of Proposition A.2.

We recall the Frenkel-Kac construction of level one integrable representations of $\widehat{\mathfrak{g}}$. Let ϖ be an element of P_+^1 and $L_1(\varpi)$ be the integrable highest weight $\widehat{\mathfrak{g}}$-module with highest weight $\varpi + \Lambda_0$.

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Lemma A.3. *We have $W_{\mathrm{af}}(\varpi+\Lambda_0) = \{t_\gamma(\varpi+\Lambda_0) \mid \gamma \in Q\}$. In particular, extremal weights in $L_1(\varpi)$ are parametrized by Q.*

Proof. The equality follows by [18, Lemma 12.6]. $\square$

Remark A.4. In this case we also have $\max(\varpi+\Lambda_0) = W_{\mathrm{af}}(\varpi+\Lambda_0)$. Here $\max(\varpi+\Lambda_0)$ denotes the set of the maximal weights of $L_1(\varpi)$.

Lemma A.5. *We have $P_+ \cap \overline{W_{\mathrm{af}}(\varpi+\Lambda_0)} = P_+ \cap (\varpi + Q_+)$.*

Proof. We have $\overline{W_{\mathrm{af}}(\varpi+\Lambda_0)} = \varpi + Q$ by Lemma A.3. It is well known that for any $\lambda \in P_+$ there exists unique $\varpi' \in P_+^1 \simeq P/Q$ such that $\lambda \geq \varpi'$. Thus we have $P_+ \cap (\varpi+Q) = P_+ \cap (\varpi+Q_+)$. $\square$

We use the symbol $X(k) = X \otimes \xi^k = X \otimes z^{-k} \in \widehat{\mathfrak{g}}$ for an element $X \in \mathfrak{g}$. Define a Lie subalgebra $\widetilde{\mathfrak{s}}$ of $\widehat{\mathfrak{g}}$ to be $\widetilde{\mathfrak{s}} = \mathfrak{h}[\xi,\xi^{-1}] \oplus \mathbb{C}K$. Then $\widetilde{\mathfrak{s}}$ is a direct sum of the Heisenberg Lie algebra $\widehat{\mathfrak{s}} = \bigoplus_{k\neq 0}(\mathfrak{h}\otimes\xi^k) \oplus \mathbb{C}K$ and an abelian Lie algebra $\mathfrak{h}\otimes 1$. Put $\widehat{\mathfrak{s}}_{\geq 0} = \bigoplus_{k>0}(\mathfrak{h}\otimes\xi^k)\oplus\mathbb{C}K$ and let $\mathbb{C}_1$ be the one-dimensional representation of $\widehat{\mathfrak{s}}_{\geq 0}$ via $\mathfrak{h}\otimes\xi^k \mapsto 0$ $(k>0)$ and $K \mapsto \mathrm{id}$. Let F be the Fock representation of the Heisenberg Lie algebra $\widehat{\mathfrak{s}}$ defined by the induction

$$F = U(\widehat{\mathfrak{s}}) \otimes_{U(\widehat{\mathfrak{s}}_{\geq 0})} \mathbb{C}_1.$$

We denote by $|0\rangle$ the element $1\otimes 1 \in F$. Consider $\mathbb{C}[Q]$ the group algebra of Q. It has a $\mathbb{C}$-basis e^γ $(\gamma\in Q)$ and the multiplication is given by $e^\beta e^\gamma = e^{\beta+\gamma}$. We denote by $e^\varpi\mathbb{C}[Q]$ a $\mathbb{C}$-vector space which has a $\mathbb{C}$-basis $e^{\varpi+\gamma}$ $(\gamma\in Q)$. An action of $\mathfrak{h}\otimes 1$ on $e^\varpi\mathbb{C}[Q]$ is given by $h(0)e^{\varpi+\gamma} = \langle h, \varpi+\gamma\rangle e^{\varpi+\gamma}$ for $h\in\mathfrak{h}$. Then $F\otimes e^\varpi\mathbb{C}[Q]$ is naturally a module of $\widetilde{\mathfrak{s}} = \widehat{\mathfrak{s}} \oplus (\mathfrak{h}\otimes 1)$. We define a $\mathbb{Z}$-grading on $F\otimes e^\varpi\mathbb{C}[Q]$ by

$$\deg(\mathfrak{h}\otimes\xi^{-k}) = k \ (k\geq 1) \text{ and } \deg e^{\varpi+\gamma} = (\varpi,\gamma) + \frac{1}{2}(\gamma,\gamma).$$

Thus $F\otimes e^\varpi\mathbb{C}[Q]$ is extended to a module of $\widetilde{\mathfrak{s}}\oplus\mathbb{C}d$ so that $(-d)$ counts the degree.

Remark A.6. The degree of $e^{\varpi+\gamma}$ is determined so that the $\widetilde{\mathfrak{h}}$-weight of $|0\rangle\otimes e^{\varpi+\gamma}$ is $t_\gamma(\varpi+\Lambda_0)$.

We take a certain 2-cocycle $\varepsilon\colon Q\times Q\to\{\pm 1\}$ as in [13, 2.3]. For an element $\gamma\in Q$, we define an operator $\widetilde{T}_\gamma$ on $e^\varpi\mathbb{C}[Q]$ by $\widetilde{T}_\gamma e^{\varpi+\beta} = \varepsilon(\gamma,\beta)e^{\varpi+\beta+\gamma}$ $(\beta\in Q)$. By [13, Proposition 2.2], we can choose root vectors $E_\alpha\in\mathfrak{g}_\alpha$ for $\alpha\in\Delta$ satisfying certain relations, e.g., $[E_\alpha, E_\beta] = \varepsilon(\beta,\alpha)E_{\alpha+\beta}$ if $\alpha+\beta\in\Delta$.

Theorem A.7 (Frenkel-Kac [13])**.**

1. *The restriction of the level one representation $L_1(\varpi)$ of $\widehat{\mathfrak{g}}$ to $\widetilde{\mathfrak{s}}\oplus\mathbb{C}d$ is isomorphic to $F\otimes e^\varpi\mathbb{C}[Q]$.*
2. *The $(\widetilde{\mathfrak{s}}\oplus\mathbb{C}d)$-module $F\otimes e^\varpi\mathbb{C}[Q]$ is extended to $\widehat{\mathfrak{g}}$ by*

$$\sum_{k\in\mathbb{Z}} E_\alpha(k)u^{-k} \mapsto \exp\left(\sum_{k>0}\frac{\alpha^\vee(-k)}{k}u^k\right)(\widetilde{T}_\alpha u^{1+\alpha^\vee})\exp\left(-\sum_{k>0}\frac{\alpha^\vee(k)}{k}u^{-k}\right)$$

and it is isomorphic to $L_1(\varpi)$ as a $\widehat{\mathfrak{g}}$-module.

Proof. The assertion is proved by [13] for the case $\varpi = 0$.

A proof of (i) for a general ϖ is similar. We give a sketch. Let $v'_{\varpi+\gamma}$ be an extremal weight vector in $L_1(\varpi)$ of weight $t_\gamma(\varpi+\Lambda_0)$. Then $U(\widetilde{\mathfrak{s}}\oplus\mathbb{C}d)v'_{\varpi+\gamma}$ is isomorphic to $F\otimes e^{\varpi+\gamma}$ as a module of $\widetilde{\mathfrak{s}}\oplus\mathbb{C}d$. Hence we have an injection $F\otimes e^\varpi\mathbb{C}[Q]\to L_1(\varpi)$. By comparing their characters, we see that they are isomorphic.

A proof of (ii) is same as [13]. $\square$

For each $\gamma \in Q$, we put $v_{\varpi+\gamma} = \widetilde{T}_\gamma(|0\rangle \otimes e^{\varpi}) = |0\rangle \otimes e^{\varpi+\gamma} \in F \otimes e^{\varpi}\mathbb{C}[Q]$. (We note that $\varepsilon(\gamma, 0) = 1$.) We regard $v_{\varpi+\gamma}$ as an extremal weight vector in $L_1(\varpi)$ via the Frenkel-Kac construction. The $\widetilde{\mathfrak{h}}$-weight of $v_{\varpi+\gamma}$ is

$$\varpi + \gamma - \left((\varpi, \gamma) + \frac{1}{2}(\gamma, \gamma)\right)\delta + \Lambda_0$$

by construction. Hence the following lemma follows.

Lemma A.8. *The vector $v_{\varpi+\gamma}$ is an extremal weight vector of weight $t_\gamma(\varpi + \Lambda_0)$. Moreover, $\mathfrak{n}_+ v_{\varpi+\gamma} = 0$ if and only if $\varpi + \gamma$ is dominant.*

The following lemma follows from Lemma A.3.

Lemma A.9. *Any extremal weight vector in $L_1(\varpi)$ is of the form $v_{\varpi+\gamma}$ $(\gamma \in Q)$ up to scalar.*

We use the Frenkel-Kac construction to prove that $L_1(\varpi)$ has a filtration whose successive quotients are quotients of global Weyl modules. We set

$$\mathrm{gr}_\lambda L_1(\varpi) := U(\mathfrak{g}[z])v_\lambda / \sum_{\substack{\mu \in P_+ \cap (\varpi + Q_+),\\ \mu > \lambda}} U(\mathfrak{g}[z])v_\mu \qquad \lambda \in P_+ \cap (\varpi + Q_+).$$

Proposition A.10. *Let λ be an element of $P_+ \cap (\varpi + Q_+)$. The image $\bar{v}_\lambda$ of v_λ in $\mathrm{gr}_\lambda L_1(\varpi)$ satisfies*

$$h(0)\bar{v}_\lambda = \langle h, \lambda\rangle \bar{v}_\lambda \ (h \in \mathfrak{h}) \quad \text{and} \quad \mathfrak{n}_+[z]\bar{v}_\lambda = 0.$$

Hence we have a surjective morphism of degree $\frac{1}{2}((\lambda, \lambda) - (\varpi, \varpi))$ from the global Weyl module $W(\lambda)$ to $\mathrm{gr}_\lambda L_1(\varpi)$.

Proof. The relation $h(0)\bar{v}_\lambda = \langle h, \lambda\rangle \bar{v}_\lambda$ follows since the $\mathfrak{h}$-weight of v_λ is λ by construction.

Let $\alpha \in \Delta_+$. Then for $k \geq 0$, we have

$$E_\alpha(-k)v_\lambda \in F \otimes e^{\lambda+\alpha} = U(\xi^{-1}\mathfrak{h}[\xi^{-1}])(|0\rangle \otimes e^{\lambda+\alpha})$$

by Theorem A.7. Here $|0\rangle \otimes e^{\lambda+\alpha} = v_{\lambda+\alpha}$ is an extremal weight vector in $L_1(\varpi)$. The $\mathfrak{g}$-submodule $U(\mathfrak{g})v_{\lambda+\alpha}$ is finite-dimensional and simple. Let $\mu \in P_+$ be the highest weight of this module. Then $U(\mathfrak{g})v_{\lambda+\alpha}$ contains v_μ as its highest weight vector by Lemma A.8 and A.9. Hence we see that $v_{\lambda+\alpha} \in U(\mathfrak{n}_-)v_\mu$ and $\lambda < \lambda + \alpha \leq \mu$. This implies that

$$E_\alpha(-k)v_\lambda \in U(\xi^{-1}\mathfrak{h}[\xi^{-1}])U(\mathfrak{n}_-)v_\mu.$$

and completes the proof. ☐

Proof of Proposition A.2. The filtration is constructed as above. The inequality

$$\mathsf{ch}\ L_1(\varpi) \leq \sum_{\lambda \in P_+ \cap (\varpi + Q_+)} q^{\frac{1}{2}((\lambda,\lambda)-(\varpi,\varpi))}\mathsf{ch}\ W(\lambda)$$

follows from Proposition A.10. The summation in the right-hand side is over $P_+ \cap \overline{W_{\mathrm{af}}(\varpi + \Lambda_0)}$ by Lemma A.5. ☐

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