

On hyperbolic attractors and repellers of endomorphisms

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Abstract. It is well known that topological classification of dynamical systems with hyperbolic dynamics is significantly defined by dynamics on non-wandering set. F. Przytycki generalized axiom A for smooth endomorphisms that was previously introduced by S. Smale for diffeomorphisms and proved spectral decomposition theorem which claims that nonwandering set of an A -endomorphism is a union of a finite number basic sets. In present paper the criterion for a basic sets of an A -endomorphism to be an attractor is given. Moreover, dynamics on basic sets of codimension one is studied. It is shown, that if an attractor is a topological submanifold of codimension one of type $(n - 1, 1)$, then it is smoothly embedded in ambient manifold and restriction of the endomorphism to this basic set is an expanding endomorphism. If a basic set of type $(n, 0)$ is a topological submanifold of codimension one, then it is a repeller and restriction of the endomorphism to this basic set is also an expanding endomorphism.

Keywords: endomorphism, axiom A , basic set, attractor, repeller.

1 Introduction

It is well known that axiom A introduced by S. Smale along with strong transversality condition are necessary and sufficient for structural stability of a dynamical system (either smooth flow or diffeomorphism) given on a smooth manifold. There are several classification results for such systems based on Smale's theorem on spectral decomposition which states that the

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nonwandering set of a structurally stable system can be uniquely decomposed into finitely many closed invariant basic sets each of which contains a transitive trajectory (see [7, 9, 2, 10] for information and references).

As for noninvertible discrete dynamical systems (endomorphisms), there are only few classes of systems satisfying axiom A with a well studied structure of basic sets. Among them there are endomorphisms of the circle and the interval [1, 22], endomorphisms of the Riemann sphere that occur in holomorphic dynamics [4, 12], expanding endomorphisms of closed manifolds [20]. In the present paper we study dynamics of endomorphisms on basic sets of codimension one which are topological submanifolds of ambient manifold.

Let M^n be a smooth closed manifold. By C^r -endomorphism we mean a C^r -smooth, $r \geq 1$, surjective map $f: M^n \rightarrow M^n$. If an endomorphism f has a C^r -smooth inverse, then f is called a C^r -diffeomorphism.

Let $f: M^n \rightarrow M^n$ be an C^r -endomorphism. Let us define \hat{M} the subset of Tikhonov product $\widetilde{M} = \prod_{i=-\infty}^{+\infty} M^n$ as $\hat{M} = \left\{ \{x_i\}_{i \in \mathbb{Z}} \in \widetilde{M} \mid f(x_i) = x_{i+1} \right\}$.

For any $x \in M^n$ set $\hat{x} = \left\{ \{x_i\}_{i \in \mathbb{Z}} \in \hat{M} \mid x_0 = x \right\}$.

If f is a diffeomorphism, then for any point $x \in M^n$ the set \hat{x} consists of exactly one point. If f is not one-to-one, then it is not true in general. Let use symbol \bar{x} for a particular element of \hat{x} . For an f -invariant set Λ (i. e. $f(\Lambda) = \Lambda$) let us introduce $\hat{\Lambda} \subset \hat{M}$ as $\hat{\Lambda} = \left\{ \{x_i\}_{i \in \mathbb{Z}} \in \hat{M} \mid x_i \in \Lambda, \forall i \in \mathbb{Z} \right\}$. For $\bar{x} \in \hat{M}$ let $\hat{f}(\bar{x})$ be a shift of \bar{x} (i.e. $\hat{f}(\{x_i\}_{i \in \mathbb{Z}}) = \{x_{i+1}\}_{i \in \mathbb{Z}}$; $\hat{f}^{-1}(\{x_i\}_{i \in \mathbb{Z}}) = \{x_{i-1}\}_{i \in \mathbb{Z}}$). Let $\Lambda \subset M^n$ be closed f -invariant set. The following definition of hyperbolic set given by F. Przytycki [17] generalizes Smale's definition for diffeomorphisms [19].

Definition 1. *Let f be an endomorphism of a manifold M^n . An invariant set Λ is called hyperbolic if there exist constants $C > 0$, $0 < \lambda < 1$ such that for every $x \in \Lambda$ and every $\bar{x} \in \hat{x} \cap \hat{\Lambda}$ ($\bar{x} = \{x_i\}_{i \in \mathbb{Z}}$) there exists a continuous splitting of the tangent subbundle $\bigcup_{i \in \mathbb{Z}} T_{x_i} M^n$ into the direct sum*

$$\bigcup_{i \in \mathbb{Z}} T_{x_i} M^n = \bigcup_{i \in \mathbb{Z}} E_{x_i, \hat{f}^i(\bar{x})}^s \oplus E_{x_i, \hat{f}^i(\bar{x})}^u \text{ such that:}$$

1. $Df \left(E_{x_i, \hat{f}^i(\bar{x})}^s \right) = E_{x_{i+1}, \hat{f}^{i+1}(\bar{x})}^s$, $Df \left(E_{x_i, \hat{f}^i(\bar{x})}^u \right) = E_{x_{i+1}, \hat{f}^{i+1}(\bar{x})}^u$, where $E_{x_i, \hat{f}^i(\bar{x})}^s, E_{x_i, \hat{f}^i(\bar{x})}^u \subset T_{x_i} M^n$;
2. $\|Df^k(v)\| \leq C\lambda^k \|v\|$, for all $k \geq 0, i \in \mathbb{Z}, v \in E_{x_i, \hat{f}^i(\bar{x})}^s$;
3. $\|Df^k(v)\| \geq (1/C)\lambda^{-k} \|v\|$, for all $k \geq 0, i \in \mathbb{Z}, v \in E_{x_i, \hat{f}^i(\bar{x})}^u$.

Remark 1. One can show that the stable subspace $E_{x_i, \hat{f}^i(\bar{x})}^s$ in the tangent space $T_{x_i}M^n$ at point x_i is independent of the choice of $\bar{x} \in \hat{x}$ (see, for example, [17]). Also note, that $E_{x_0, \bar{x}}^s = E_{x_0, \hat{f}^0(\bar{x})}^s$ and $E_{x_0, \bar{x}}^u = E_{x_0, \hat{f}^0(\bar{x})}^u$.

For a smooth map $f: M^n \rightarrow M^n$, a point $x \in M^n$ is called *regular* if the rank of the map $Df(x): T_x M^n \rightarrow T_{f(x)} M^n$ is exactly n . Otherwise the point x is called *singular*.

Next definition is a generalization of Smale's axiom A that was given in the paper [17].

Definition 2. An endomorphism $f: M^n \rightarrow M^n$ satisfies axiom A if the following conditions hold:

1. the nonwandering set Ω_f is hyperbolic and does not contain singular points of f ;
2. the set of periodic points Per_f of f is dense in nonwandering set Ω_f .

For A-endomorphisms there exists a spectral decomposition theorem proved in [17] that generalizes Smale's result for diffeomorphisms [19].

Proposition 1. Let f be an A-endomorphism. Then its nonwandering set Ω_f can be uniquely decomposed into a finite union of closed f -invariant sets (called basic sets) $\Omega_f = \bigcup_{i=1}^l \Lambda_i$ such that the restriction of f to every basic set Λ_i is topologically transitive.

Definition 3. A basic set Λ of an A-endomorphism f is called attractor if it has a closed neighborhood $U \supset \Lambda$ such that $f(U) \subset \text{Int } U$ and $\bigcap_{n=0}^{\infty} f^n(U) = \Lambda$.

Definition 4. A basic set Λ of an A-endomorphism f is called repeller if it has an open neighborhood U such that $cl(U) \subset f(U)$ and $\bigcap_{n=0}^{\infty} f^{-n}(U) = \Lambda$.

The following definition belongs to M. Shub [20].

Definition 5. A C^r -endomorphisms $f: M^n \rightarrow M^n$ is called expanding if there exist constants $C > 0$ and $\lambda > 1$ such that $\|Df^n(v)\| \geq C\lambda^n \|v\|$ for all $v \in TM^n$, $n = 0, 1, 2, \dots$

Nevertheless, is possible to define an expanding endomorphism not only for smooth manifolds but for arbitrary metric space as well. The following definition was given in [3].

Definition 6. A continuous map $f: X \rightarrow X$ of a metric space X is called *expanding* if there exist constants $\varepsilon > 0$ and $\mu > 1$ such that for all $x, y \in X$, $x \neq y$, $\rho(x, y) < \varepsilon$ the following inequality holds $\rho(f(x), f(y)) > \mu\rho(x, y)$.

Note that in the case when X is a C^1 -smooth, compact manifold and f is a C^1 -smooth map it follows from [3] that conditions of definition 5 imply conditions of definition 6.

It follows from definition 5 that an ambient manifold M^n of an expanding endomorphism f is hyperbolic set. Moreover, it was shown in [20] that if M^n is compact, then periodic points of expanding endomorphism f are dense in M^n . Thus, any expanding endomorphism of a compact manifold is A -endomorphism and its nonwandering set coincides with the ambient manifold. It was also shown in this paper that an ambient manifold of an expanding endomorphism has an Euler characteristics equal to zero and that its universal covering is diffeomorphic to \mathbb{R}^n . Moreover if a compact manifold M^n is locally flat, then it admits an expanding endomorphism [11],.

It was shown in [20] that if ambient manifold M^n is diffeomorphic to the n -torus \mathbb{T}^n , then expanding endomorphism f is topologically conjugated to the algebraic expanding endomorphism.

Definition 7. An endomorphism $f: M^n \rightarrow M^n$ is called *Anosov endomorphism* if the ambient manifold M^n is hyperbolic set.

It follows from definition 7 that expanding endomorphism is an Anosov endomorphism.

Other examples of endomorphisms such that their unique basic set coincides with the ambient manifold are provided by Anosov algebraic endomorphisms of n -torus induced by matrix $A_{n \times n}$ with eigenvalues which are inside and outside of the unit circle and with no eigenvalues on the unit circle.

It is well known that an arbitrary Anosov diffeomorphism of n -torus is conjugate with the algebraic hyperbolic automorphism [24, 15, 14]. However there is no such result for an Anosov endomorphism that are not expanding or diffeomorphisms [17, 13]. Moreover it was shown in paper [23] that the set of endomorphisms of the n -torus that are not conjugated to any algebraic endomorphism is the residual¹ subset in the set of all Anosov endomorphisms on the torus. Thus, the question whether an ambient manifold of an arbitrary Anosov endomorphism is nonwandering is still open.

As it follows from [6, 20] Anosov diffeomorphisms and expanding endomorphism are structurally stable. However, F. Przytycki [17] and R. Mane

¹The set X is called residual, if it is an intersection of countably many sets with dense interiors.

with Ch. Pugh [13] independently proved that Anosov endomorphisms that are not expanding and diffeomorphisms are not structurally stable.

For a basic set Λ of an A -endomorphism $f: M^n \rightarrow M^n$ the pair of integers $(\dim E_{x_0, \bar{x}}^u, \dim E_{x_0, \bar{x}}^s)$ is called *the type of basic set*. It follows from [17] that this definition is correct since $\dim E_{x_0, \bar{x}}^u, \dim E_{x_0, \bar{x}}^s$ do not depend on the point $\bar{x} \in \hat{\Lambda}$.

In the case when f is A -diffeomorphism the topological structure of basic sets of codimension one is well studied. It follows from R. V. Plykin that any basic set of codimension one is necessarily either attractor or repeller. In the case when $n = 2$ it is locally homeomorphic to the product of the Cantor set and the interval. If $f: M^3 \rightarrow M^3$ is an A -diffeomorphism of closed 3-manifold and Λ is a 2-dimensional basic set of the type $(2, 1)$ $((1, 2))$ that coincides with the union of unstable (stable) manifolds of its points, then it is called *expanding attractor (contracting repeller)*, and it follows from [16, 8], that it is locally homeomorphic to the product of \mathbb{R}^2 and the Cantor set. If Λ doesn't coincides with the union of unstable (stable) manifolds of its points, then Λ is homeomorphic to the 2-torus \mathbb{T}^2 and restriction of f_Λ is topologically conjugated to the Anosov algebraic diffeomorphism.

In the present paper we prove a criterion for a basic set Λ to be an attractor of an endomorphism $f: M^n \rightarrow M^n$. Moreover, we study the dynamics of restriction of an A -endomorphisms $f: M^n \rightarrow M^n$ to the basic set Λ in the case when it is submanifold of codimension one of the manifold M^n . The existence of a smooth structure on such basic set is also considered.

Theorem 1. *A basic set Λ of an A -endomorphism $f: M^n \rightarrow M^n$ is an attractor if and only if there exists $\varepsilon > 0$ such that for every $x \in \Lambda$ and $\bar{x} \in \hat{\Lambda} \cap \hat{\Lambda}$ one has $W_{x, \bar{x}, \varepsilon}^u \subset \Lambda$.²*

Theorem 2. *Let basic set Λ be a codimension one topological submanifold of M^n . If Λ is of type $(n, 0)$, then:*

- 1) Λ is a repeller;
- 2) the restriction of f to Λ is an expanding endomorphism.³

Remark 2. *If in theorem 2 A -endomorphism $f: M^n \rightarrow M^n$ is an A -diffeomorphism, then the statement of the theorem is true only in case $n = 1$. In case $n > 1$, basic set of a diffeomorphism of type $(n, 0)$ is a periodic source, thus it cannot be a submanifold of codimension one.*

Remark 3. *Basic set Λ from theorem 2 is not necessarily smoothly embedded in ambient manifold M^n . As an example we can consider an endomorphism*

² $W_{x, \bar{x}, \varepsilon}^u$ is introduced in section 2 (definition 8).

³In sense of definition 6 in metric on M^n induced by Riemannian metric $\langle \cdot, \cdot \rangle_\Lambda$ defined in proposition 2.

of Riemann sphere induced by the map $z \rightarrow z^2 + c$. If parameter c is small enough but does not equal to zero, then there is a basic set which is a repeller homeomorphic to the circle but it is not smooth at any point (see, for example, [4, 12]).

Remark 4. There exist repellers of type $(n, 0)$ that are not submanifolds. It is well known that endomorphisms of the Riemann sphere that are considered in holomorphic dynamics can have one-dimensional fractal repellers [4, 12].

Theorem 3. Let a basic set Λ be a codimension one topological submanifold of M^n . If Λ is an attractor of an A -endomorphism $f: M^n \rightarrow M^n$ of type $(n - 1, 1)$, then:

- 1) Λ is smooth;
- 2) the restriction f to Λ is an expanding endomorphism.⁴

Remark 5. There exist attractors of type $(n - 1, 1)$ that are not submanifolds. Let us consider an endomorphism of 3-torus $f: \mathbb{T}^3 \rightarrow \mathbb{T}^3$ obtained as the direct product of an expanding endomorphism of the circle and a DA-diffeomorphism of the 2-torus with one-dimensional attractor (see, for example, [19]). Then nonwandering set of f contains an attractor Λ of type $(2, 1)$ locally homeomorphic to the product of the Cantor set and the 2-dimensional disk.

Remark 6. Examples of basic sets on n -manifolds defined in theorems 2 and 3 can be easily constructed. It is sufficient to consider a direct product of an expanding endomorphism of $(n - 1)$ -torus and a Morse-Smale diffeomorphism of the circle.

2 Auxiliary information

Let $\langle \cdot, \cdot \rangle$ be a smooth Riemannian metric on manifold TM^n . And ρ be a metric on M^n induced by $\langle \cdot, \cdot \rangle$. Let Λ be an invariant hyperbolic set of an endomorphism f . Then it is possible to introduce local stable and unstable manifolds for points from Λ , as it was done for diffeomorphisms. However, there is a significant difference from the case of diffeomorphisms. For endomorphism local unstable manifold of a point $x \in \Lambda$ depend on $\bar{x} \in \hat{x} \cap \hat{\Lambda}$.

Definition 8. Let Λ be hyperbolic invariant set of endomorphism $f: M^n \rightarrow M^n$ and $x \in \Lambda$, $\bar{x} \in \hat{x} \cap \hat{\Lambda}$. The set

$$W_{x,\varepsilon}^s = \{y \in M^n \mid \rho(f^n(x), f^n(y)) < \varepsilon, n = 0, 1, 2, \dots\}$$

⁴In sense of definition 5.

is called local stable manifold of point x , and the set

$$W_{x,\bar{x},\varepsilon}^u = \{y \in M^n \mid \exists \bar{y} \in \hat{y}, \rho(x_n, y_n) < \varepsilon, n = 0, -1, -2, \dots\}$$

is called local unstable manifold of a point x .

The structure of hyperbolic sets of endomorphisms was studied in detail in [17]. We will mention here some results important for the present paper.

Proposition 2. *For any hyperbolic set Λ of endomorphism f there exists a smooth Riemannian metric $\langle \cdot, \cdot \rangle_\Lambda$ on TM^n equivalent to $\langle \cdot, \cdot \rangle$ and a real number λ , $0 < \lambda < 1$ such that for any $x \in \Lambda$ and $\bar{x} \in \hat{\Lambda} \cap \hat{\Lambda}$ following inequalities hold*

$$\|Df_{x_i}(v)\|_\Lambda \leq \lambda \|v\|_\Lambda \quad \text{where } v \in E_{x_i, \hat{f}^i(\bar{x})}^s,$$

$$\|Df_{x_i}(v)\|_\Lambda \geq (1/\lambda) \|v\|_\Lambda \quad \text{where } v \in E_{x_i, \hat{f}^i(\bar{x})}^u,$$

$i \in \mathbb{Z}$.

Proposition 3. *Let Λ be a hyperbolic set of endomorphism f then:*

1. *there exists $\varepsilon > 0$ such that for any $\bar{x} \in \hat{x} \cap \hat{\Lambda}$ the local stable $W_{x,\varepsilon}^s$ and the local unstable $W_{x,\bar{x},\varepsilon}^u$ manifolds are smoothly embedded disks of topological dimension $\dim E_{x_0,\bar{x}}^s$ and $\dim E_{x_0,\bar{x}}^u$ tangent to $E_{x_0,\bar{x}}^s$ and $E_{x_0,\bar{x}}^u$ at point x ;*
2. *$W_{x,\varepsilon}^s$ and $W_{x,\bar{x},\varepsilon}^u$ depend continuously in C^1 topology on point x and \bar{x} , respectively;*
3. *there exists $\mu > 1$ such that in metric ρ on M^n induced by Riemannian metric $\langle \cdot, \cdot \rangle_\Lambda$*
 - (a) *for any points $y, z \in W_{x,\varepsilon}^s$ inequalities $\rho(f^{n+1}(y), f^{n+1}(z)) \leq (1/\mu)\rho(f^n(y), f^n(z))$, $n = 0, 1, 2, \dots$ hold,*
 - (b) *for any points $y, z \in W_{x,\bar{x},\varepsilon}^u$ and $\bar{y} \in \hat{y}$, $\bar{z} \in \hat{z}$ satisfying inequalities from the definition of $W_{x,\bar{x},\varepsilon}^u$, inequalities $\rho(y_{-n-1}, z_{-n-1}) \leq (1/\mu)\rho(y_{-n}, z_{-n})$, $n = 0, 1, 2, \dots$ hold.*

Proposition 4. *Let $f: M^n \rightarrow M^n$ be an A -endomorphism and $\Omega = \bigcup_{j=1}^l \Lambda_j$*

its spectral decomposition. Then:

1. *For any point $x \in M^n$ there exists unique basic set Λ_{j_1} ($j_1 = \overline{1, l}$) such that $f^k(x) \rightarrow \Lambda_{j_1}$ as $k \rightarrow +\infty$. Moreover, there exists a point $y \in \Lambda_{j_1}$ such that $\rho(f^k(x), f^k(y)) \rightarrow 0$ as $k \rightarrow +\infty$.*

2. For any $\bar{x} \in \hat{M}$ there exists a unique basic set Λ_{j_2} ($j_1 = \overline{1, l}$) such that $x_i \rightarrow \Lambda_{j_2}$ as $i \rightarrow -\infty$. Moreover, there exists $\bar{y} \in \hat{\Lambda}_{j_2}$ such that $\rho(x_i, y_i) \rightarrow 0$ as $i \rightarrow -\infty$.

For $\delta > 0$ and for $x \in M^n$ let $B_\delta(x)$ be an open ball of radius δ at point x (i.e. $B_\delta(x) = \{y \in M^n \mid \rho(x, y) < \delta\}$) and $\bar{B}_\delta(x)$ be a closed ball of radius δ at point x (i.e. $\bar{B}_\delta(x) = \{y \in M^n \mid \rho(x, y) \leq \delta\}$).

Next statement is a corollary of compactness. We present its prove here for the sake of completeness.

Proposition 5. *Let K be a compact set of some metric space X and U be an open neighborhood of K . Then there exists $\delta > 0$ such that for any point $x \in K$ the inclusion $B_\delta(x) \subset U$ holds.*

Proof. Suppose that the statement of the lemma is not true. Then for any $\delta > 0$ there exists a point $x \in K$ such that $B_\delta(x) \setminus U \neq \emptyset$. Let us consider the sequence of positive numbers $\{\delta_i\}_{i=1}^\infty$ such that $\delta_i \rightarrow 0$ as $i \rightarrow \infty$ and a sequence of points $\{x_i\}_{i=1}^\infty$ such that $B_{\delta_i}(x_i) \setminus U \neq \emptyset$.

Since K is compact, without loss of generality we can assume that the sequence $\{x_i\}_{i=1}^\infty$ is convergent to some point $x_0 \in K$. Since U is open, there exists $\delta_0 > 0$ such that $B_{\delta_0}(x_0) \subset U$. Since sequences $\{x_i\}_{i=1}^\infty$ and $\{\delta_i\}_{i=1}^\infty$ are convergent, there exists an integer $k \in \mathbb{N}$ such that inequalities $\rho(x_0, x_k) < \delta_0/2$ and $\delta_k < \delta_0/2$ hold.

Let us consider an arbitrary point $y \in B_{\delta_k}(x_k)$. Then $\rho(x_0, y) \leq \rho(x_0, x_k) + \rho(x_k, y) < \delta_0/2 + \delta_k < \delta_0$ and $y \in B_{\delta_0}(x_0) \subset U$. It contradicts the assumption that $B_{\delta_k}(x_k) \setminus U \neq \emptyset$. \square

3 The criterion for the existence of the attractor

Let Λ be a basic set of f . Hereinafter we assume that TM^n is equipped with a Riemannian metric $\langle \cdot, \cdot \rangle_\Lambda$ defined in proposition 2 and that ρ is a metric on M^n induced by $\langle \cdot, \cdot \rangle_\Lambda$.

Theorem 1 follows from lemmas 1, 2 and 3.

Lemma 1. *Let f be a A -endomorphism satisfying to the axiom A , Λ be a basic set which is an attractor. Then there exists $\varepsilon > 0$ such that for any point $x \in \Lambda$ and for any $\bar{x} \in \hat{x} \cap \hat{\Lambda}$ the inclusion $W_{x, \bar{x}, \varepsilon}^u \subset \Lambda$ holds.*

Proof. Let Λ be of type $(k, n - k)$. In case $k = 0$, lemma is trivial, since $W_{x, \bar{x}, \varepsilon}^u$ coincides with a point x . That is why, we consider only the case $k \geq 1$. Suppose that Λ does not coincide with the manifold M^n (otherwise

the statement of the lemma is obvious). Let U be a neighborhood from the definition of an attractor. Since Ω does not contain singular points (see item 1 of definition 2) tangent map $Df_x: T_x M^n \rightarrow T_x M^n$ is nondegenerate at any point $x \in \Lambda$. So, it follows from inverse function theorem (see, for example, [5], p. 499) that the restriction of f to a sufficiently small neighborhood of the point x is a local diffeomorphism. Therefore, for any $k \geq 0$, the set $f^k(U)$ contains an open neighborhood of the set Λ . Choose $\varepsilon > 0$, which satisfies the conclusions of the proposition 3. Suppose that the statement of the lemma is not true, then there is a point $x \in \Lambda$ and $\bar{x} \in \hat{x} \cap \hat{\Lambda}$ such that $W_{x, \bar{x}, \varepsilon}^u \not\subset \Lambda$. It follows from proposition 3 that $W_{x, \bar{x}, \varepsilon}^u$ is homeomorphic to k -disk. Therefore, there exists $y \in (U \setminus \Lambda) \cap W_{x, \bar{x}, \varepsilon}^u$. By the definition of attractor $f^l(y) \rightarrow \Lambda$ for $l \rightarrow +\infty$. By definition of an attractor $\bigcap_{j=0}^{+\infty} f^j(U) = \Lambda$, therefore there exists $m \in \mathbb{N}$ such that $y \notin f^m(U)$. It follows from item 3 of proposition 3 that for the point y there exists $\bar{y} \in \hat{y}$ such that $\rho(y_l, \Lambda) \rightarrow 0$ as $l \rightarrow -\infty$. This fact along with the fact that $f^m(U)$ contains an open neighborhood of Λ imply an existence of a number $t \in \mathbb{N}$ such that $y_{-t} \in f^m(U)$ and $f^t(y_{-t}) = y$. Since $f^t(f^m(U)) \subset f^m(U)$, it follows that $y = f^t(y_{-t}) \in f^m(U)$, which contradicts the choice of the number m . \square

Lemma 2. *Let Λ be basic set of an A -endomorphism of the type $(n, 0)$. If there exists $\varepsilon_1 > 0$ such that $W_{x, \bar{x}, \varepsilon_1}^u \subset \Lambda$ for any point $x \in \Lambda$ and for some $\bar{x} \in \hat{x} \cap \hat{\Lambda}$, then Λ coincides with the ambient manifold M^n .*

Proof. Put $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, where ε_2 satisfies the conclusions of assertion 3. According to the spectral decomposition theorem (see proposition 1) Λ is a closed set. Since Λ is of type $(n, 0)$, then by proposition 3 local unstable manifold $W_{x, \bar{x}, \varepsilon}^u$ contains an open n -dimensional disk. Therefore, Λ is an open set. Thus, Λ is simultaneously open and closed set, hence it coincides with M^n . \square

Lemma 3. *Let Λ be a basic set of an A -endomorphism of the type $(k, n - k)$, $0 \leq k \leq n - 1$. If there exists $\varepsilon_1 > 0$ such that $W_{x, \bar{x}, \varepsilon_1}^u \subset \Lambda$ for any point $x \in \Lambda$ and some $\bar{x} \in \hat{x} \cap \hat{\Lambda}$, then Λ is an attractor.*

Proof. Put $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, where ε_2 satisfies conclusions of proposition 3. Let us show that there exists $\delta > 0$ such that for any point $x \in \Lambda$ and for any η satisfying inequalities $0 < \eta \leq \delta$:

1. the intersection of $\bar{B}_\eta(x) \cap W_\varepsilon^s$ consists of one connected component,
2. the intersection $\partial \bar{B}_\eta(x) \cap W_{x, \varepsilon}^s$ is homeomorphic to $(n - k - 1)$ -dimensional sphere.

Assume the contrary. Since $W_{x,\varepsilon}^s$ is a smoothly embedded $(n - k)$ -dimensional disk, for any point $x \in \Lambda$ there exists $\delta(x) > 0$ such that for any η satisfying inequality $0 < \eta \leq \delta(x)$ properties 1, 2 holds.

Consider a sequence $\{\delta_i\}_{i=1}^{+\infty}$ such that $\delta_i > 0$ and $\delta_i \rightarrow 0$ as $i \rightarrow \infty$. The contrary assumption implies that for any δ_i there exists at least one point $x_i \in \Lambda$, such that δ_i does not satisfy at least one of the properties 1, 2 at the point x_i . Since Λ is compact, without loss of generality we can assume that the sequence $\{x_i\}_{i=1}^{+\infty}$ is convergent to some point $x_0 \in \Lambda$. For a point x_0 there exists $\delta_0 > 0$ satisfying conditions 1, 2. By continuous dependence $W_{x,\varepsilon}^s$ on a point $x \in \Lambda$ in C^1 topology there exists a neighborhood V of the point x_0 such that $\delta_0/2$ satisfies to the properties 1, 2 for any point $x \in V \cap \Lambda$. It contradicts the fact that $V \cap \Lambda$ contains points of the sequence $\{x_i\}_{i=1}^{+\infty}$ such that $\delta_i < \delta_0/2$.

Let δ satisfies the properties 1, 2 for any point $x \in \Lambda$. Put $U = \bigcup_{x \in \Lambda} (\bar{B}_\delta(x) \cap W_{x,\varepsilon}^s)$.

Let us show that U is a closed neighborhood of the attractor. According to the choice of δ for any point $x \in \Lambda$ the set $B_\delta(x) \cap W_{x,\varepsilon}^s$ is homeomorphic to the closed $(n - k)$ -dimensional disk. Consider an arbitrary point $x \in \Lambda$ and $\bar{x} \in \hat{\Lambda}$ such that the inclusion $W_{x,\bar{x},\varepsilon}^u \subset \Lambda$ holds. Since the unstable manifold $W_{x,\bar{x},\varepsilon}^u$ is a smoothly embedded open k -dimensional disk, and the stable manifold $W_{x,\varepsilon}^s$ depends on x continuously in C^1 topology the set $\bigcup_{x \in W_{x,\bar{x},\varepsilon}^u} (\bar{B}_\delta(x) \cap W_{x,\varepsilon}^s) \subset U$ is homeomorphic to the direct product of the open k -dimensional disk and $(n - k)$ -dimensional closed disk. Thus U contains an open neighborhood of the set Λ .

Let us show that U is a closed set. Consider an arbitrary point $y \in \text{cl}(U)$. There exists a sequence of points $\{y_i\}_{i=1}^{+\infty}$, $y_i \in U$ converging to y_0 . By construction, for any element of the sequence $\{y_i\}_{i=1}^{+\infty}$ one can find a point $x_i \in \Lambda$ such that $y \in W_{x_i,\varepsilon}^s$ and $\rho(x_i, y_i) \leq \delta$. Since Λ is compact, without loss of generality the sequence $\{x_i\}_{i=1}^{+\infty}$ can be considered as convergent to some point $x_0 \in \Lambda$. Since the metric ρ is continuous (as a map from the direct product $M^n \times M^n$ to \mathbb{R}) and the sequence of pairs (x_i, y_i) is convergent in $M^n \times M^n$, inequality $\rho(x_0, y) \leq \delta$ holds as $\rho(x_i, y_i) \leq \delta$ for any $i \in \mathbb{N}$. By continuous dependence of $W_{x,\varepsilon}^s$ on a point $x \in \Lambda$ in C^1 topology the point y belongs to $W_{x_0,\varepsilon}^s$. Thus, $y \in \bar{B}_\delta(x_0) \cap W_{x_0,\varepsilon}^s$ and hence $x \in U$.

Let us show that $f(U) \subset \text{int}(U)$. Consider an arbitrary point $y \in U$. By construction, there exists a point $x \in \Lambda$ such that $x \in \bar{B}_\delta(x_0) \cap W_{x,\varepsilon}^s$. By item 3 of proposition 3 inequality $\rho(f(y), f(x)) \leq \delta/\mu$ holds for some $\mu > 1$. Thus, $f(y)$ belongs to the interior of $(n - k)$ -dimensional closed disk $\bar{B}_\delta(f(x)) \cap W_{f(x),\varepsilon}^s$. By continuous dependence of local stable manifolds on

the point in C^1 topology the inclusion $y \in \text{int } U$ holds.

Equality $\bigcap_{n=0}^{\infty} f^n(U) = \Lambda$ follows from the fact that for any $y \in U$ one has $y \in W_{x,\varepsilon}^s$ for some $x \in \Lambda$ and from item 3 of proposition 3.

Thus, U is the desired neighborhood from the definition of an attractor. \square

4 The structure of attractors of type $(N - 1, 1)$ and repellers of type $(N, 0)$ that are topological manifolds of codimension one

4.1 The proof of the theorem 2

Lemma 4. *If Λ is a compact hyperbolic set of type $(n, 0)$ of an endomorphism f and $\varepsilon > 0$ satisfies the conclusions of the proposition 3, then there exists $\delta > 0$ such that for any $x \in \Lambda$, $\bar{x} \in \hat{\Lambda}$ the inclusion $B_\delta(x) \subset (W_{x,\bar{x},\varepsilon}^u)$ holds.*

Proof. Since f is continuous and Λ is compact, the set $\hat{\Lambda}$ is a closed subset in \tilde{M} . Since M^n is compact, the set \tilde{M} is also compact. Thus, $\hat{\Lambda}$ is compact as a closed subset of a compact space.

Suppose that the statement of the lemma is not true, then there exists a sequence $\{\bar{x}^i\}_{i=1}^{\infty}$ of points in $\hat{\Lambda}$ and a sequence of numbers $\{\delta_i\}_{i=1}^{\infty}$ such that $B_{\delta_i}(x) \setminus W_{x_0^i, \bar{x}^i, \varepsilon}^u \neq \emptyset$ and $\delta_i \rightarrow 0$ as $i \rightarrow \infty$. Without loss of generality, we assume that the original sequence $\{\bar{x}^i\}_{i=0}^{\infty}$ is convergent to point $\bar{x}^0 \in \hat{\Lambda}$. By item 1 of proposition 3 local unstable manifold $W_{x_0^0, \bar{x}^0, \varepsilon}^u$ contains an open disk $B_{\delta_0}(x_0^0)$. By item 2 of the same proposition there exists an integer N such that for any $i > N$ one has $B_{\delta_0/2}(x) \subset W_{x_0^i, \bar{x}^i, \varepsilon}^u$. It is contradicts the choice of the sequences $\{\bar{x}^i\}_{i=0}^{\infty}$ and $\{\delta_i\}_{i=0}^{\infty}$. \square

Lemma 5. *Let $K \subset M^n$ be compact and any point $x \in K$ is regular with respect to f . Then there exist $\varepsilon > 0$ and an open neighborhood V of K such that if $x', x'' \in V$ and $\rho(x', x'') < \varepsilon$, then $f(x') \neq f(x'')$.*

Proof. Since for any point $x \in K$ the tangent map $Df_x: T_x M^n \rightarrow T_{f(x)} M^n$ is nondegenerate, it follows from the inverse function theorem that there exists a neighborhood $U(x)$ of a point x such that the restriction $f|_{U(x)}: U(x) \rightarrow f(U(x))$ is a diffeomorphism. Let us consider the open cover $U = \bigcup_{x \in K} U(x)$ of the set K with such neighborhoods. Since M^n is normal space there exists

an open neighborhood V of the set K such that $U \supset \text{cl}(V) \supset V \supset K$ (see for example [21]).

As the manifold M^n is compact, the set $\text{cl}(V)$ is also compact. By Lebesgue's lemma⁵ there exists a real number $\lambda > 0$ such that for any pair of points $x', x'' \in \text{cl}(V)$ such that $\rho(x', x'') < \lambda$, there exists an element of the cover $U(x)$ that contains x' and x'' . If one set $\varepsilon = \lambda$, then V will be the desired neighborhood. \square

Lemma 6. *Let a basic set Λ of an A -endomorphism f be a topological submanifold of codimension one. Then there exists a neighborhood U of Λ such that $f^{-1}(\Lambda) \cap U = \Lambda$.*

Proof. Suppose the contrary, then there exists a sequence $\{x_i\}_{i=0}^{+\infty}$, $x_i \in M^n \setminus \Lambda$, such that $\rho(x_i, \Lambda) \rightarrow 0$ as $i \rightarrow +\infty$ and $f(x_i) \in \Lambda$ for any $i \in \mathbb{N}$. By compactness of M^n the sequence $\{x_i\}_{i=0}^{+\infty}$ can be considered as convergent to some point $x_0 \in \Lambda$. By inverse function theorem there exists a neighborhood V_1 of the point x_0 such that the restriction of f to V_1 is a local diffeomorphism. Since Λ is a topological $(n-1)$ -dimensional submanifold, there exists a neighborhood V_2 of the point x_0 such that $V_2 \cap \Lambda$ is homeomorphic to $(n-1)$ -dimensional disk. Let V be an open disk such that $x \in V \subset V_1 \cap V_2$. The set $V \cap \Lambda$ is also homeomorphic to an open $(n-1)$ -dimensional disk. Since Λ is invariant, $f(V \cap \Lambda) \subset f(V) \cap \Lambda$. Since $x_i \rightarrow x_0$ as $i \rightarrow +\infty$, and $f(x_i) \in \Lambda$, by continuity f inclusion $f(x_i) \in f(V) \cap \Lambda$ holds for sufficiently big numbers i . Thus, the restriction of f to V is not injective map. It contradicts the fact that $f|_V$ is local diffeomorphism. \square

Lemma 7. *Let a basic set Λ of an A -endomorphism f be a topological submanifold of codimension one. If Λ is of type $(n, 0)$, then there exists a neighborhood Q of Λ such that f is expanding on Q in terms of definition 6.⁶*

Proof. Let ε_1 satisfy conclusions of proposition 3. Let neighborhood $\Pi \supset \Lambda$ and constant $\varepsilon_2 > 0$ satisfy the conclusion of lemma 5. Since Λ is a basic set of an A -endomorphism, there exists $x \in \Lambda$ such that $\bigcup_{i=0}^{\infty} f^i(x)$ is dense in Λ .

By proposition 5 there exists $\varepsilon_3 > 0$ and $\bar{x} \in \hat{x} \cap \hat{\Lambda}$ such that the inclusion $\bigcup_{i \in \mathbb{Z}} W_{x_i, \hat{f}^i(\bar{x}), \varepsilon_3}^u \subset \Pi$ holds.

⁵Let X be a compact metric space, and $\{U\}$ be its open cover. Then there exists a real number $\lambda > 0$ (called a Lebesgue's number) such that any subset of X with a diameter less than λ lies entirely in some element of the cover $\{U\}$ (see for example [21]). This Lemma also holds for any compact subset of metric space and its open covering.

⁶ $f: X \rightarrow X$ is called expanding on $A \subset X$, if there exist $\varepsilon > 0$ and $\mu > 1$ such that for any $x, y \in A$, $x \neq y$ one has $\rho(f(x), f(y)) > \mu\rho(x, y)$.

Set $\varepsilon = \min\{\varepsilon_1, \frac{\varepsilon_2}{2}, \varepsilon_3\}$ and $V = \bigcup_{i \in \mathbb{Z}} \left[W_{x_i, \hat{f}^i(\bar{x}), \varepsilon}^u \cap f^{-1} \left(W_{x_{i+1}, \hat{f}^{i+1}(\bar{x}), \varepsilon_1}^u \right) \right]$.

Let us check that the set V is a neighborhood of the basic set Λ . Since $\bigcup_{i=0}^{\infty} f^i(x)$ is dense in Λ , it is sufficient to show that there exist $\delta > 0$ such that for any x_i one has $B_\delta(x_i) \subset W_{x_i, \hat{f}^i(\bar{x}), \varepsilon}^u \cap f^{-1} \left(W_{x_{i+1}, \hat{f}^{i+1}(\bar{x}), \varepsilon_1}^u \right)$. By lemma 4 there exist $\delta_1 > 0$ such that $B_{\delta_1}(x_i) \subset W_{x_i, \hat{f}^i(\bar{x}), \varepsilon}^u$ for any $i \in \mathbb{Z}$. Since $\varepsilon \leq \varepsilon_1$ one has $B_{\delta_1}(x_{i+1}) \subset W_{x_{i+1}, \hat{f}^{i+1}(\bar{x}), \varepsilon_1}^u$ for any $i \in \mathbb{Z}$. Show that there exist $\delta_2 > 0$ such that $f(B_{\delta_2}(x_i)) \subset W_{x_{i+1}, \hat{f}^{i+1}(\bar{x}), \varepsilon_1}^u$. Suppose the contrary. Choose some sequence $\{\eta_j\}_{j=1}^{+\infty}$ such that $\eta_j \rightarrow 0$ as $j \rightarrow +\infty$. Then for any $j \in \mathbb{N}$ there exist points x_{i_j} and y_j such that $\rho(x_{i_j}, y_j) < \eta_j$ and $f(y_j) \notin W_{x_{i+1}, \hat{f}^{i+1}(\bar{x}), \varepsilon_1}^u$. Since $B_{\delta_1}(x_{i+1}) \subset W_{x_{i+1}, \hat{f}^{i+1}(\bar{x}), \varepsilon_1}^u$, then $\rho(f(y_j), f(x_{i_j})) = \rho(f(y_j), x_{i_j+1}) > \delta_1$. Since M^n is compact without loss of generality one can consider the sequences x_{i_j} and y_j to be convergent to some point x_0 . Since $\rho(f(x_0), f(x_0)) = 0$, one has a contradiction with continuity of f . Then one can set $\delta = \min\{\delta_1, \delta_2\}$.

Since M^n is normal and Λ is closed, there exists an open set Q such that $V \supset \text{cl}(Q) \supset Q \supset \Lambda$. Let us show that Q is a desired neighborhood.

It follows from Lebesgue's lemma and compactness of $\text{cl}(Q)$ that there exists $\lambda > 0$ such that for any pairs of points $y, z \in Q$ such that $\rho(y, z) < \lambda$ there exists x_i such that $y, z \in W_{x_i, \hat{f}^i(\bar{x}), \varepsilon}^u$.

By definition of Q one has $f \left(W_{x_i, \hat{f}^i(\bar{x}), \varepsilon}^u \right) \subset W_{x_{i+1}, \hat{f}^{i+1}(\bar{x}), \varepsilon_1}^u$ and $f(y), f(z) \in W_{x_{i+1}, \hat{f}^{i+1}(\bar{x}), \varepsilon_1}^u$. Set $y^+ = f(y), z^+ = f(z)$. By item 3 of proposition 3 there exist $\bar{y}^+ \in \hat{y}^+$ and $\bar{z}^+ \in \hat{z}^+$ such that $\rho(y_i^+, x_{i+1}) < \varepsilon$, $\rho(z_i^+, x_{i+1}) < \varepsilon$ for $i \leq 0$.

Show that $y = y_{-1}^+$ and $z = z_{-1}^+$. Let $y' = y_{-1}^+$ and $z' = z_{-1}^+$. Then one has $\rho(x_i, y') < \varepsilon$ and $\rho(x_i, z') < \varepsilon$. Let us show that these inequalities cannot hold for any preimages of the points y^+ and z^+ different from the points y and z . By the choice of ε we have $\rho(x_i, y) < \frac{\varepsilon_2}{2}$, $\rho(x_i, y') < \frac{\varepsilon_2}{2}$ and by the triangle inequality $\rho(y, y') \leq \rho(x_i, y) + \rho(x_i, y') < \varepsilon_2$. Then by Lemma 5 the equality $y = y'$ holds. Similarly the equality $z = z'$ holds.

Hence for any points $y, z \in Q$ such that $\rho(y, z) < \lambda$ the inequality $\rho(f(y), f(z)) > \mu \rho(y, z)$ holds for some $\mu > 1$. Thus the restriction of f to Q is expanding endomorphism in the sense of definition 6. \square

Proof of the theorem 2. The fact that $f|_\Lambda$ is an expanding endomorphism follows from the lemma 7. Show that the Λ has a neighborhood U defined in definition 4.

Consider $x \in \Lambda$ such that $\bigcup_{i=0}^{\infty} f^i(x)$ is dense in Λ . Choose ε satisfying the conclusions of proposition 3. Fix any $\bar{x} \in \hat{x} \cap \hat{\Lambda}$. By lemma 4 the set $V_1 = \bigcup_{i \in \mathbb{Z}} W_{x_i, \hat{f}^i(\bar{x}), \varepsilon}^u$ is an open cover of the set Λ . Let V_2 be a neighborhood of Λ satisfying the conclusion of lemma 6. Since M^n is normal, and basic sets $\Lambda_1, \dots, \Lambda_l$ are closed, then there exist disjoint open neighborhoods Q_1, \dots, Q_l of basic sets. Set $V_3 = Q_s$, where Q_s is a neighborhood of Λ .

Since Λ is compact, it follows from proposition 5 that there exists $\delta_1 > 0$, $\delta_2 > 0$ and $\delta_3 > 0$ such that $B_{\delta_1}(x_i) \subset V_1$, $B_{\delta_2}(x_i) \subset V_2$ and $B_{\delta_3}(x_i) \subset V_3$ for any $i \in \mathbb{Z}$. Set $\delta = \min\{\delta_1/\mu, \delta_2, \delta_3\}$, where $\mu > 1$ satisfies the conclusion of item 3 of proposition 3. Set $U = \bigcup_{i \in \mathbb{Z}} B_{\delta}(x_i)$. By construction, U is also an open cover of the set Λ and $U \subset V_1 \cap V_2$. Let us show that U is the desired neighborhood of the repeller Λ , i.e. that the following conditions are satisfied:

1) $f(U) \supset \text{cl}(U)$ and

2) $\bigcap_{k=0}^{+\infty} f^{-k}(U) = \Lambda$.

1) Set $W = \bigcup_{i \in \mathbb{Z}} B_{\mu\delta}(x_i)$, then $\text{cl}(U) \subset W \subset V_1$. Show that inclusion $W \subset f(U)$ holds. Indeed, by definition of W for any point $y \in W$ there exists $i \in \mathbb{Z}$ such that $\rho(y, x_i) < \mu\delta$, and hence $y \in W_{x_i, \hat{f}^i(\bar{x}), \varepsilon}^u$. By item 3 of proposition 3 there exists a point $y' \in f^{-1}(y)$ such that $\rho(y', x_{i-1}) \leq (1/\mu)\rho(y, x_i) < (1/\mu)(\mu\delta) = \delta$. Thus, $y' \in B_{\delta}(x_{i-1}) \subset U$ and $y = f(y') \in f(U)$.

2) Assume the contrary, i.e. $\bigcap_{k=0}^{+\infty} f^{-k}(U) \neq \Lambda$. Then there exists a point $x \in \bigcap_{k=0}^{\infty} f^{-k}(U) \setminus \Lambda$. By definition of the point x for any $k \geq 0$ one has $f^k(x) \in U$. By proposition 4 for any point $x \in M^n$ the sequence $\{f^i(x)\}_{i=0}^{+\infty}$ converges to some basic set Λ_j . Since all points from $U \setminus \Lambda$ are wandering and $\text{cl}U \cap \Lambda_j = \emptyset$ for any basic set $\Lambda_j \neq \Lambda$, one has $\rho(f^i(x), \Lambda) \rightarrow 0$ as $i \rightarrow +\infty$. By proposition 4 there exists $y \in \Lambda$ such that $\rho(f^i(x), f^i(y)) \rightarrow 0$ as $i \rightarrow +\infty$ and, by lemma 6 $f^i(x) \neq f^i(y)$. It contradicts the results of lemma 7. \square

4.2 The proof of the theorem 3

Proof of the theorem 3. Let us show that Λ is a smooth submanifold of the manifold M^n . Since Λ is a topological submanifold, for a sufficiently small neighborhood $U(x)$ of any point $x \in \Lambda$ the set $\Lambda \cap U(x)$ is homeomorphic to

$(n - 1)$ -dimensional disk. Choose $\varepsilon > 0$ such that it satisfies the conclusions of the proposition 3. Since the basic set Λ is attractor of type $(n - 1, 1)$, by lemma 1 any point $x \in \Lambda$ has a neighborhood (in Λ) coinciding with a local unstable manifold $W_{x,\bar{x},\varepsilon}^u$. As $W_{x,\bar{x},\varepsilon}^u$ is a smoothly embedded $(n - 1)$ -dimensional disk, then the basic set Λ is a smooth submanifold.

Let us show that the restriction of f_Λ is an expanding map. As the fiber $E_{x_0,\bar{x}}^u$ is tangent to $W_{x,\bar{x},\varepsilon}^u$ it does not depend on $\bar{x} \in \hat{x} \cap \hat{\Lambda}$. Hence, $T\Lambda = \bigcup_{x \in \Lambda} E_{x_0,\bar{x}}^u$. Then it follows from the hyperbolicity of the set Λ and the definition 5 that the restriction $f|_\Lambda$ is an expanding map. \square

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