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Tests based on characterizations, and their efficiencies: a survey

Ya. Yu. Nikitin

Abstract. A survey of goodness-of-fit and symmetry tests based on the characterization properties of distributions is presented. This approach became popular in recent years. In most cases the test statistics are functionals of U-empirical processes. The limiting distributions and large deviations of new statistics under the null hypothesis are described. Their local Bahadur efficiency for various parametric alternatives is calculated and compared with each other as well as with diverse previously known tests. We also describe new directions of possible research in this domain.

1. Introduction

This survey is dedicated to the statistical tests based on characterizations. This is a relatively new idea which manifests growing popularity in the context of goodness-of-fit and symmetry testing. The idea to build goodness-of-fit tests using the characterizations of distributions belongs to Yu. V. Linnik [47]. At the end of this wide-ranging paper he wrote: *"... one can raise the issue of the construction of goodness-of-fit tests for testing composite hypotheses based on the equal distribution of the two relevant statistics $g_1(x_1, \dots, x_r)$ and $g_2(x_1, \dots, x_r)$, and on the reduction of such question to the homogeneity tests."* This sentence was the guiding star which showed the researchers the right direction in the new and unexplored domain.

Currently, in the world literature there exist hundreds of various characterizations of probability distributions, see, e.g., [37], [26], [40], and [38]. Many characterizations according to Linnik's idea imply the corresponding statistical tests. Such tests are attractive because they reflect some *intrinsic and hidden properties* of probability distributions connected with the given

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characterization, and therefore can be more efficient or more robust than others.

Moreover, one should keep in mind that any hypothesis has to be tested with several possible criteria. The point of the matter is that with absolute confidence we can only reject it, while each new test which fails to reject the null-hypothesis gradually brings the statistician closer to the perception that this hypothesis is true. We find it pertinent to quote here the famous assertion by Einstein [23]: *"No amount of experimentation can ever prove me right; a single experiment can prove me wrong."* Hence, we are interested in building new statistical tests based on novel ideas, specifically using the characterizations.

But the theory of such tests is intricate, and the study of their asymptotic properties including limiting behavior, and especially their asymptotic efficiency began only after 1990. Before that there existed few exceptions like the paper [97], of which later Mudholkar and Tian [71] wrote: *"Vašicek (1976) was the first to recognize that the characterization results can be logical starting points for developing goodness-of-fit tests."*

Probably these authors were unfamiliar with the seminal paper by Linnik cited above who was surely the first to propose the idea under discussion. In the abstract of the paper [29] published in 1993, Hashimoto and Shirahata proposed one of the first tests of fit based on characterizations and wrote: *"However, since no test statistics based on characterizations are known, our test will be worth considered."* This citation shows that in the beginning of 1990s the tests based on characterizations were unusual and sparse. But since that time the state of affairs changed significantly. Numerous new tests based on characterizations were build, and their study *gradually acquired the traits of a theory.* We want to trace an outline of this theory and its main achievements within the last 25 years.

We begin by general constructions explaining the structure of tests used in this domain. Next we pass to concrete problems like testing of exponentiality, normality or symmetry, and describe the main developments of the last period of time. We are *mainly interested in the asymptotic efficiency of our tests* though the results of power simulation are also possible and interesting. At the end of the paper we pose some problems and trace new directions of research. In most cases, the proofs are omitted, otherwise this survey would exceed the size of the paper in a journal.

2. U-statistics and U-empirical distributions

Let $X_1, X_2, \dots, X_n$ be i.i.d. observations with continuous df F. We begin by testing the *composite goodness-of-fit* problem

$$H_0 : F \in \mathfrak{F},$$

where $\mathfrak{F}$ is some family of df's, against the alternative

$$H_1 : F \notin \mathfrak{F}.$$

Typical examples are testing exponentiality, normality or symmetry of a sample.

Next exposition will be based on U-statistics and their variants. Currently U-statistics play an important role in statistics and probability. They appeared in the middle of 1940s in problems of unbiased estimation, but after the crucial paper of Hoeffding [32] it became clear that the numerous valuable statistics are just U-statistics (or von Mises functionals having very similar asymptotic theory). The most complete exposition of this theory can be found in the monographs [39] and [42].

We consider U-statistics of the form

$$U_n = \binom{n}{m}^{-1} \sum_{1 \leqslant i_1 < \ldots < i_m \leqslant n} \Psi(X_{i_1}, \ldots, X_{i_m}), \qquad n \geqslant m,$$

where $X_1, X_2, \ldots$ is a sequence of i.i.d. rv's with common df F, while the kernel $\Psi : \mathbb{R}^m \to \mathbb{R}^1$ is a measurable symmetric function of $m \geq 1$ variables. The number m is called the *degree* of the kernel. We assume that the kernel Ψ is integrable on R^m and denote

$$\theta(F) := \int \ldots \int_{\mathbb{R}^m} \Psi(x_1, \ldots, x_m) dF(x_1) \ldots dF(x_m).$$

In the sequel we need the notations

$$\psi(x) := \mathbb{E}_F\{\Psi(X_1, \ldots, X_m) | X_1 = x\}, \quad \Delta^2 := \mathbb{E}_F \psi^2(X_1) - (\theta(F))^2.$$

The function ψ is called the one-dimensional *projection* of the kernel Ψ and plays an important role in asymptotic theory. If $\Delta^2 > 0$ that specifies the so-called non-degenerate case, the limiting distribution of U-statistics is normal as discovered by Hoeffding [32]. He proved that if $\mathbb{E}_F \Psi^2(X_1, \ldots, X_m) < \infty$ and $\Delta^2 > 0$, then as $n \to \infty$ one has convergence in distribution

$$\frac{\sqrt{n}}{m\Delta} (U_n - \theta(F)) \xrightarrow{\mathrm{d}} N(0,1). \tag{1}$$

Consider, in conformity with Linnik, the characterization of some probability law by the identical distribution of two statistics $g_1(X_1, \ldots, X_r)$ and $g_2(X_1, \ldots, X_s)$. The examples of such characterizations will be given further.

We can build two *U-empirical df's*

$$L_n^1(t) = \binom{n}{r}^{-1} \sum_{1 \leq i_1 < \ldots < i_r \leq n} \mathbf{1}\{g_1(X_{i_1}, \ldots, X_{i_r}) < t\}, \quad t \in \mathbb{R}^1,$$

$$L_n^2(t) = \binom{n}{s}^{-1} \sum_{1 \leq i_1 < \ldots < i_s \leq n} \mathbf{1}\{g_2(X_{i_1}, \ldots, X_{i_s}) < t\}, \quad t \in \mathbb{R}^1.$$

The theory of U-empirical df's was developed in 1980s (see, e.g., [31], [33], [39]) and is similar to the theory of usual empirical df's. By the Glivenko–Cantelli theorem for U-empirical df's we have (wp 1) as $n \to \infty$:

$$L_n^1(t) \rightrightarrows L^1(t) := P(g_1(X_1, \ldots, X_r) < t),$$

$$L_n^2(t) \rightrightarrows L^2(t) := P(g_2(X_1, \ldots, X_s) < t).$$

Under H_0 for large n we have $L_n^1(t) \approx L_n^2(t)$ so that we can use this closeness for goodness-of-fit testing. Over much of this survey we consider two types of statistics: the integral one

$$I_n = \int_{\mathbb{R}^1} \left(L_n^1(t) - L_n^2(t)\right) dF_n(t),$$

where $F_n(t)$ is the usual empirical df, and of Kolmogorov type, namely

$$D_n = \sup_{t \in \mathbb{R}^1} \mid L_n^1(t) - L_n^2(t) \mid .$$

Such statistics can have rather different behavior depending on the type of characterization and underlying distribution, accordingly the statistical tests based on them can have distinct limiting properties, power and efficiency.

3. Outline of Bahadur theory

Suppose that we want compare two sequences of statistics I_n and D_n by their asymptotic efficiency. Among many types of efficiencies (see [75, Ch.1]) we select the Bahadur efficiency because, unlike Pitman efficiency, it can be calculated for statistics with non-normal limiting distribution. This is the primary reason to use it in the present context as the Kolmogorov type statistics have non-normal limiting distributions. Hodges–Lehmann efficiency has other drawbacks, in particular, it does not discriminate two-sided tests, see, e.g., [75, Ch.1]. In this section we shortly describe main points of Bahadur theory, see the complete exposition in [12] and [13].

Let $s = (X_1, X_2, \ldots)$ be a sequence of i.i.d. rv's with the distribution $P_\theta, \theta \in \Theta$, on $(\mathcal{X}, \mathcal{A})$. We are testing the null-hypothesis

$$H_0 : \theta \in \Theta_0 \subset \Theta \subset \mathbb{R}^1$$

against the alternative

$$H_1 : \theta \in \Theta_1 = \Theta \setminus \Theta_0.$$

For this problem we use the sequence of test statistics $T_n(s) = T_n(X_1, \ldots, X_n)$. The Bahadur approach prescribes one to fix the power of concurrent tests and to compare the exponential rates of decrease of their sizes for the increasing number of observations and fixed alternative. This exponential rate for a sequence of statistics $\{T_n\}$ is usually proportional to some non-random function $c_T(\theta)$ depending on the alternative parameter $\theta \in \Theta_1$ which is called the *exact slope* of the sequence $\{T_n\}$. The Bahadur asymptotic relative efficiency (ARE) $e^B_{V,T}(\theta)$ of two sequences of statistics $\{V_n\}$ and $\{T_n\}$ is defined by means of the formula

$$e^B_{V,T}(\theta) = c_V(\theta) \,/\, c_T(\theta)\,.$$

The exact slope can be found by the fundamental theorem of Bahadur [12].

Theorem 1. *Suppose that the following two conditions hold:*

a) $T_n \xrightarrow{P_\theta} b(\theta), \quad \theta > 0,$
where $-\infty < b(\theta) < \infty$, *and* $\xrightarrow{P_\theta}$ *denotes convergence in probability under* P_θ.

b) $\lim_{n\to\infty} n^{-1} \ln \mathbb{P}_{H_0}\,(T_n \geq t) = -h(t)$
for any t *in an open interval* I, *on which* h *is continuous and* $\{b(\theta), \theta > 0\} \subset I$.

Then $c_T(\theta) \;=\; 2\; h(b(\theta))$.

Often the exact Bahadur ARE is uncomputable for any alternative depending on θ, but it is possible to calculate the local Bahadur ARE as $\theta \in \Theta_1$ approaches the null-hypothesis. Then one speaks about the *local* Bahadur efficiency and *local* Bahadur exact slopes [75].

Let $K(\theta, \theta_0) \equiv K(P_\theta, P_{\theta_0})$ be the Kullback–Leibler distance between P_θ and P_{θ_0}, see, e.g., [13] or [98]. Put for any $\theta \in \Theta_1$

$$K(\theta, \Theta_0) := \inf\{K(\theta, \theta_0) : \theta_0 \in \Theta_0\}.$$

The Bahadur–Raghavachari inequality (the analog of Cramér–Rao inequality in testing), see [12], [75], states that for any $\theta \in \Theta_1$ one has

$$c_T(\theta) \leq 2K(\theta, \Theta_0).$$

Hence we may define the (absolute) *local Bahadur efficiency* of the sequence $\{T_n\}$ by the formula

$$eff_T = \lim_{\theta \to \partial\Theta_0} c_T(\theta)/2K(\theta, \Theta_0).$$

Only in exceptional cases

$$c_T(\theta) = 2K(\theta, \Theta_0), \quad \forall \theta \in \Theta_1.$$

Therefore one can be interested in those $F = \mathbb{L}(X_1)$ for which

$$eff_T = \lim_{\theta \to \partial\Theta_0} c_T(\theta)/2K(\theta, \Theta_0) = 1.$$

We call this property the *local optimality* in Bahadur sense. An interesting question is to describe those alternatives for which the considered tests are locally optimal in Bahadur sense. The idea ascends to Bahadur [13] but was developed by the author, see [74], [75, Ch.6], and subsequent papers, e.g., [80]. However, we leave this direction apart as it requires considerable space to enounce the obtained results.

The first condition of Theorem 1 is a variant of the Law of Large Numbers and its verification is easy. On the contrary, the second condition of this theorem describes the rough (logarithmic) large deviation asymptotics of test statistics under the null-hypothesis and is non-trivial. To verify it, we often use the following theorem on large deviations of U-statistics by Nikitin and Ponikarov [81].

Theorem 2. *Let V_n be a sequence of U-statistics with centered, bounded and non-degenerate kernel Ψ. Then*

$$\lim_{n\to\infty} n^{-1} \ln \mathbb{P}\{V_n \geq a\} = -\sum_{j=2}^{\infty} b_j a^j, \tag{2}$$

where the series with numerical coefficients b_j converges for sufficiently small $a > 0$, and $b_2 = (2m^2\Delta^2)^{-1}$, where Δ^2 is the variance of the projection of the kernel Ψ.

Large deviations for the supremum of the family of non-degenerate U-statistics $\sup_{t\in T} U_n(t)$, where $U_n(t)$ for each $t \in T$ is a U-statistic with the non-degenerate kernel $\Xi(X, Y; t)$ which corresponds to Kolmogorov type statistics, were studied in [78]. The result is similar to (2) but slightly more involved.

4. Desu's characterization and corresponding tests of exponentiality

One of the most simple characterizations of exponential distribution belongs to Desu [21].

Theorem 3. *Let X and Y be non-negative i.i.d. rv's with df differentiable at zero. Then $X \stackrel{d}{=} 2\min(X, Y)$ if and only if X and Y are exponentially distributed.*

Using this characterization we will show how to build and analyze the corresponding tests of exponentiality.

Let $X_1, \ldots, X_n$ be i.i.d. observations with non-degenerate df F, and let F_n be the corresponding empirical df. We are testing the composite hypothesis

$H_0 : F(x)$ *is the df of exponential law with the density* $f(x) = \lambda e^{-\lambda x}, x \geq 0,$

where $\lambda > 0$ is some unknown parameter, against the alternative H_1 under which the hypothesis H_0 is wrong.

In this case we need the U-statistical empirical df H_n which is defined as

$$H_n(t) = \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} \mathbf{1}\{2\min(X_i, X_j) < t\}, \ t \ge 0.$$

We will study the two statistics

$$I_n = \int_0^\infty (F_n(t) - H_n(t)) dF_n(t),$$

and

$$D_n = \sup_{t \ge 0} |F_n(t) - H_n(t)|.$$

Clearly their distribution under the null-hypothesis does not depend on λ.

The statistic I_n is asymptotically equivalent to the U-statistic of degree 3 with the centered kernel

$$\Psi(X, Y, Z) = \frac{1}{2} - \frac{1}{3}[\mathbf{1}\{2\min(X, Y) < Z\} - \\ - \mathbf{1}\{2\min(Y, Z) < X\} - \mathbf{1}\{2\min(X, Z) < Y\}].$$

The projection of this kernel is

$$E[\Psi(X, Y, Z)|Z = t] := \psi(s) = \frac{1}{3}e^{-s} - \frac{1}{18} - \frac{4}{9}e^{-3s},$$

and the variance of the projection is $\Delta^2 := E\psi^2(Z) = 11/3780 \approx 0.003$.

By Hoeffding's theorem (see [32]) we get the following result.

Theorem 4. *Under the hypothesis H_0 one has convergence in distribution*

$$\sqrt{n} I_n \xrightarrow{d} \mathcal{N}(0, 9\Delta^2), \quad \textit{as} \ \ n \to \infty.$$

As to the large deviations, in our case we get for $a > 0$

$$\lim_{n \to \infty} n^{-1} \ln \mathbb{P}(I_n > a) = -f_I(a),$$

where the function f_I is continuous for sufficiently small $a > 0$, and, moreover,

$$f_I(a) = \frac{210}{11} a^2 (1 + o(1)), \quad \text{as} \ \ a \to 0.$$

By way of an example, let us calculate the local Bahadur efficiency of I_n for the Weibull alternative. This means that the alternative df of observations is

$$F(x, \theta) = 1 - \exp(-x^{1+\theta}), \quad x \ge 0, \ \theta \ge 0.$$

We find after some simple calculations that, as $\theta \to 0$,

$$c_I(\theta) \sim b_I(\theta)^2 / (9\Delta^2) \sim 1.147\theta^2.$$

The Kullback–Leibler distance $K(\theta)$ between H_0 and H_1 satisfies

$$K(\theta) \sim \pi^2 \theta^2 / 12, \quad \theta \to 0.$$

The *local Bahadur ARE* of our test is consequently equal to

$$eff(I) := \lim_{\theta \to 0} \frac{c_I(\theta)}{2K(\theta)} \approx 0.697.$$

Consider now the Kolmogorov-type statistic D_n. The difference $F_n(t) - H_n(t)$ is a *family* of U-statistics with the kernels depending on $t \geq 0$:

$$\Xi(X, Y; t) = \frac{1}{2}(\mathbf{1}\{X < t\} + \mathbf{1}\{Y < t\}) - \mathbf{1}\{2 \min(X, Y) < t\}). \tag{3}$$

The limiting distribution of the sequence D_n is unknown. Critical values for the statistics D_n can be found via simulation.

In our case the family $\{\Xi(X, Y; t), t \geq 0\}$ from (3) is centered, bounded, non-degenerate and hence satisfies all the conditions of Theorem 2.4 from [78] on large deviations of U-empirical Kolmogorov statistics. Therefore, as $a > 0$, by [78],

$$\lim_{n \to \infty} n^{-1} \ln P(D_n > a) = -f_D(a),$$

where

$$f_D(a) = 2a^2(1 + o(1)), \text{ as } a \to 0.$$

Consider again the Weibull alternative. Arguments similar to the case of integral statistic, see [78], show that the local Bahadur efficiency of the sequence D_n is equal to 0.158. We see that this efficiency is low and considerably smaller than in the integral case. It is a rule that Kolmogorov–Smirnov type statistics are less efficient than the integral ones. There exist some exceptions but they are rare.

5. Tests of exponentiality based on characterizations

There are numerous characterizations of exponential law, probably more than of any other probability law, see, e.g., [4], [10], [14], and [26]. We consider only few typical examples where the tests of fit are build and studied.

5.1. Lack of memory property and corresponding tests. First, we mention the celebrated "lack of memory" property which consists in that only the exponential distribution satisfies the functional equation in df's

$$1 - F(x + y) - (1 - F(x))(1 - F(y)) = 0 \quad \forall\, x, y \geq 0.$$

Replacing F by empirical df F_n, one obtains some empirical field, and the functionals of it can be used as test statistics for exponentiality, see as examples of many papers in this direction [1], [41], and [30].

The "lack of memory" property can be simplified. Denote, following Angus [7], by $\mathbf{D}_1$ the class of right-continuous df's F with $F(0-) = 0$ and

$$\lim_{h \to 0} \frac{F(h) - F(0)}{h} = l \in [0, \infty].$$

Let $\bar{F}(x) = 1 - F(x)$. Angus used the following statement that belongs to Arnold and Gupta: the functional equation

$$\bar{F}(2x) = \bar{F}^2(x) \quad \forall\, x \geq 0$$

characterizes the exponential distribution in the class of such distributions in $\mathbf{D}_1$ which are not concentrated at 0. He introduced a Kolmogorov type test based on this characterization and studied its properties in [7]. Later its local Bahadur efficiency against standard alternatives was calculated in [76] and [78]. It turned out to be rather low.

5.2. Characterizations based on order statistics. Another example is given by Riedel–Rossberg characterization in terms of order statistics, see [90]. Denote, as usually, by $X_{k,n}$ the k-th order statistic in the sample of size n, $1 \leq k \leq n$. Then the following characterization holds.

Theorem 5. *Two statistics $X_{2,3} - X_{1,3}$ and $\min(X_1, X_2)$ are identically distributed if and only if the sample X_1, X_2, X_3 consists of exponential rv's.*

The construction of tests based on this characterization and their asymptotic analysis is performed similarly to the case of Desu characterization (see [99]).

Next consider the Ahsanullah's characterization. Suppose that the df F belongs to the class of df's $\mathbb{F}_1$, where the failure rate function $f(t)/(1-F(t))$ is monotone for $t \geq 0$. Ahsanullah [2] proved some characterizations of exponentiality within the class $\mathbb{F}_1$. We consider here only one of his characterizations.

Theorem 6. *Let X and Y be non-negative i.i.d. rv's from the class $\mathbb{F}_1$. Then $|X - Y| \stackrel{d}{=} 2\min(X, Y)$ if and only if X and Y are exponentially distributed.*

The corresponding tests were build and analyzed by Nikitin and Volkova in [104].

5.3. Characterization of Arnold and Villaseñor. Recently Arnold and Villaseñor [9] expressed in the form of a conjecture the following characterization of exponentiality:

Let $X_1, X_2, \dots$ be non-negative i.i.d. rv's with density f having derivatives of all orders around zero. Then for any $k \geq 2$

$$\max(X_1, X_2, \dots, X_k) \stackrel{d}{=} \sum_{i=1}^{k} \frac{X_i}{i}$$

if and only if f is exponential.

Arnold and Villaseñor were able to prove this conjecture only for $k = 2$. Later Yanev and Chakraborty [105] proved that it is true for $k = 3$, and later in [106] proved it for arbitrary k (see also [61]). Tests of exponentiality based on these characterizations and their efficiencies were studied in [35] and in [100].

Other tests based on characterizations of exponential distribution in terms of order statistics were built and studied in [84] and [58]. One can mention also the characterization of exponential law by the same distribution of X and $|X-Y|$ where X, Y are i.i.d. rv's having absolute continuous distribution (see [88]). Some steps towards using it for testing were made in [78].

5.4. Table of efficiencies. Now we present a table of local Bahadur efficiencies of the majority of tests of exponentiality described above. We will compare them with well-known classic scale-free tests of exponentiality based on Greenwood statistic R_n, Moran statistic M_n, and Gini statistic G_n. We recall that

$$R_n = 2 - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\overline{X}}\right)^2, \; M_n = \frac{1}{n}\sum_{i=1}^{n}\ln\left(\frac{X_i}{\overline{X}}\right) + \mathbf{C}, \; G_n = \frac{\sum_{i,j=1}^{n}|X_i - X_j|}{2n(n-1)\overline{X}},$$

where $\mathbf{C}$ denotes the Euler constant. We consider also the famous Lilliefors [43] statistic which has the form

$$Li_n = \sup_{x\geq 0}\left|1 - F_n(x) - e^{-x/\overline{X}}\right|,$$

and belongs to Kolmogorov type statistics with estimated parameters. On efficiencies of these statistics, see [83], [82], [94].

We consider the following standard alternatives against exponentiality:

i) Weibull alternative with the density

$$(1+\theta)x^{\theta}\exp(-x^{1+\theta}), \theta \geq 0, x \geq 0;$$

ii) Makeham alternative with the density

$$(1+\theta(1-e^{-x}))\exp(-x-\theta(e^{-x}-1+x)), +quad\theta \geq 0, \;\; x \geq 0;$$

iii) linear failure rate alternative with the density

$$(1+\theta x)e^{-x-\frac{1}{2}\theta x^2}, \quad \theta \geq 0, \;\; x \geq 0.$$

Now let us compare the values of local Bahadur efficiency for various statistics. All of them are collected in Table 1 below and were calculated according to the approach developed above for the tests based on Desu characterization. The superscripts *Ross* and *Ahs* denote the statistics based on Riedel–Rossberg's or Ahsanullah's characterization.

We see that our tests based on characterizations are competitive with respect to other tests of exponentiality, all the more given that the alternatives were taken almost at random. However, the Gini test reaffirms its high reputation.

Table 1. Local efficiencies of tests for exponentiality.

Statistic	Alternative Weibull	Alternative Makeham	Alternative linear failure rate
Integral type statistics			
I_n^{Ross}	0.650	0.450	0.119
I_n^{Ahs}	0.795	0.692	0.257
Gini	0.876	1	0.750
Moran	0.943	0.694	0.388
Greenwood	0.608	0.750	1
Kolmogorov type statistics			
D_n^{Ross}	0.320	0.207	0.047
D_n^{Ahs}	0.450	0.470	0.187
Angus	0.158	0.187	0.073
Lilliefors	0.538	0.607	0.356

6. Tests of normality

Characterizations of normality are also numerous and mathematically content-rich. They have been described in [5], [37], [53], and [19], apart from many articles. We discuss here only few papers based on selected characterizations.

6.1. Polya characterization. One of first characterizations in the history of statistics belongs to Polya [87].

Theorem 7. *Let X and Y be i.i.d. centered rv's. Then $X \stackrel{d}{=} (X+Y)/\sqrt{2}$ if and only if X and Y have the normal distribution with some positive variance.*

The integral test of normality based on this property was proposed by Muliere and Nikitin [72]. Their statistic is asymptotically normal with the variance $9\delta^2$, where

$$\delta^2 = \frac{13}{108} - \frac{4}{9\pi}(\arctan\sqrt{\frac{3}{5}} + \frac{1}{2}\arctan\frac{1}{\sqrt{7}}) \approx 1.571236 \cdot 10^{-3} > 0.$$

The expression for the variance shows the non-trivial character of the calculations. The efficiency of this test is very high and equals 0.967 for shift and skew (see [11]) alternatives.

We can generalize these findings by considering a general characterization, which is a particular case of [37, Theorem 13.7.2].

Theorem 8. *Let X and Y be centered i.i.d. rv's, and let a and b are such constants that $0 < a, b < 1$, $a^2 + b^2 = 1$. Then $X \stackrel{d}{=} aX + bY$ if and only if $X, Y \in N(0, \sigma^2)$.*

We can rebuild our statistics using Theorem 8, and the result should depend on a. The theory of integral statistic in this generalized setting is developed in [50]. In particular, the local efficiency of integral test for shift alternative equals to

$$eff^*(a) = \Big(a - 1 + \sqrt{1-a^2}\Big)^2 \Big/ \Omega(a),$$

where

$$\Omega(a) = \Big(\frac{7}{3}\pi - 4\arctan\sqrt{\frac{1+a^2}{3-a^2}} - 4\arctan\sqrt{\frac{2-a^2}{2+a^2}} - 4\arctan\sqrt{\frac{1-a^2}{3+a^2}} - 4\arctan\sqrt{\frac{a^2}{4-a^2}} + 4\arctan\sqrt{\frac{a^2(1-a^2)}{a^4-a^2+4}}\Big).$$

The maximum of $eff^*(a)$ is 1 but is attained for $a = 0$ and $a = 1$, where the test is inconsistent. The worst case (quite unexpectedly) is just the Polya case for $a = \frac{\sqrt{2}}{2}$ with the efficiency 0.966. We recommend $a = \frac{24}{25}$, and $b = \frac{7}{25}$. Then we have $a^2 + b^2 = 1$, and the efficiency is 0.990, this is a very high value.

The Kolmogorov type test based on this characterization was studied in [51]. The results are similar but the efficiencies are considerably lower.

6.2. Characterization by Shepp property. In 1964 Shepp [91] proved that if X and Y are i.i.d., $X, Y \in N(0, \tau^2)$, then the rv

$$k(X,Y) := 2XY/\sqrt{X^2+Y^2} \in \mathbb{N}(0,\tau^2) \quad \text{again.}$$

This statement is usually called the *Shepp property.*

Later Galambos and Simonelli [25] proved that the Shepp property characterizes the normal law in some class $\mathfrak{F}_0$ which consists of such df's F which satisfy $0 < F(0) < 1$ and for which $F(x) - F(-x)$ is changing regularly in zero with the exponent 1. They proved the following result

Theorem 9. *Let X and Y be i.i.d. rv's with common df F from the class $\mathfrak{F}_0$. Then the equality in distribution $X \stackrel{d}{=} k(X,Y)$ holds if and only if $X \in \mathbb{N}(0,\tau^2)$ for some variance $\tau^2 > 0$.*

Nikitin and Volkova [102] constructed tests of normality based on this characterization and found the efficiencies of corresponding tests. It turned out that for shift and skew alternatives the efficiencies coincide and are equal in case of integral and supremum tests to

$$eff_I = \frac{3}{\pi} = 0.955, \quad eff_D = \frac{2}{\pi} = 0.637.$$

7. Tests of fit for other distributions

The reader has probably noticed that the majority of characterizations used above for testing exponentiality and normality was formulated in terms of equal distribution of some simple statistics. There arises the question if such characterizations exists for other probability laws and if it is possible to build goodness-of-fit tests based on them. The answer is positive, but the set of corresponding characterizations is more sparse, the calculations are more involved, and therefore the whole subject is underdeveloped.

7.1. Puri–Rubin characterization. We begin by the characterization of the power law. We are testing the composite hypothesis

$$H_0 : F \text{ is the df of the power law so that } F(x) = x^{\mu}, x \in [0,1], \mu > 0,$$

against general alternatives. We use the characterization which is given in the paper by Puri and Rubin [88].

Theorem 10. *Let X and Y be i.i.d. non-negative rv's with df F. Then the equality*

$$X \stackrel{d}{=} \min\left(\frac{X}{Y}, \frac{Y}{X}\right)$$

holds if and only if X and Y have the power distribution.

The tests for the power law based on this characterization were build and studied by Nikitin and Volkova [103]. The efficiencies of integral test are between 0.71 and 0.97, the efficiencies of the Kolmogorov test are between 0.47 and 0.63 depending on the alternative under consideration.

7.2. Some other laws. The power law is closely related to the Pareto law, so Obradovic, Jovanovic, and Miloševic, see [86], were able to use almost the same characterization (by replacing min by max) when testing for Pareto law. Volkova [101] introduced and studied some tests of fit for the Pareto distribution based on another characterization.

Goodness-of-fit test for the Cauchy law was build and studied by Litvinova [49]. She used the characterization of Ramachandran and Rao [89]. Its simplified variant is the following.

Theorem 11. *Let X and Y be i.i.d. rv's. Then X and $\frac{1}{3}X - \frac{2}{3}Y$ are identically distributed if and only if X and Y have the Cauchy df with arbitrary scale factor.*

Litvinova in [49] explored the integral test; its local efficiency under the shift alternative turned out to be 0.665.

Some tests of uniformity based on characterizations were developed in [22], [29], [64]. In [57] there are interesting efficiency calculations for such tests.

We finish this section by briefly mentioning numerous results on testing goodness-of-fit based on characteristic properties of entropy and Kullback–Leibler information, see [8], [97], [22], [71], [28], [85], etc. However, there is almost nothing known on efficiencies of new tests, these tests are mainly compared on the basis of simulated power.

8. Testing of symmetry

Testing of *symmetry* based on characterizations has been much less explored than goodness-of-fit testing. Consider the classical hypothesis

$$H_0 : 1 - F(x) - F(-x) = 0, \quad x \in \mathbb{R}^1, \tag{4}$$

against the alternative H_1 under which the equality (4) is violated at least in one point. The first step in construction of such tests was done in the crucial paper by Baringhaus and Henze [16].

Suppose that X and Y are i.i.d. rv's with continuous df F. Baringhaus and Henze proved that the equidistribution of the rv's $|X|$ and $|\max(X,Y)|$ is valid iff F is symmetric with respect to zero, that is, (4) holds. They also proposed suitable Kolmogorov-type and omega-square type tests of symmetry. Some efficiency calculations for Kolmogorov type test were later performed in [77] (see also [78]). Integral test of symmetry was next proposed and studied by Litvinova [48].

Another characterization of symmetry with respect to 0 belongs to Ahsanullah and was published in [3].

Theorem 12. *Suppose that $X_1, \dots, X_k, k \geq 2$, are i.i.d. rv's with absolutely continuous df $F(x)$. Denote $X_{1,k} = \min(X_1, \dots, X_k)$ and $X_{k,k} = \max(X_1, \dots, X_k)$. Then*

$$|X_{1,k}| \stackrel{d}{=} |X_{k,k}|$$

if and only if F is symmetric about zero, i.e.,

$$1 - F(x) - F(-x) = 0, \quad x \in \mathbb{R}^1.$$

Subsequently we refer to this result as *Ahsanullah's characterization of order k.*

Nikitin and Ahsanullah [79] published a paper on tests of symmetry based on these characterizations and their efficiencies. It was found that corresponding tests of symmetry for $k = 2$ and $k = 3$ are asymptotically equivalent to the test of Litvinova and to the Kolmogorov-type test of Baringhaus and Henze. In case of location alternative they are competitive and manifest rather high local Bahadur efficiency in comparison to many other tests of symmetry. At the same time, higher values of k, $k > 3$, lead us to different tests with lower values of efficiencies in the case of common alternatives. It would be interesting to calculate the efficiencies of such tests for more realistic alternatives, for instance, for *skew* alternatives (see [11]). First steps in this direction were undertaken in the recent paper [18].

Similar research based on a certain modification of Ahsanullah's characterization was done recently by Obradovic and Miloševic [59]. The authors of [59] were able to build a corresponding integral test and a Kolmogorov-type test based on their theorem and studied its efficiency.

9. Directions of further research and perspectives

9.1. Tests based on characterizations of stable laws. Only three stable laws have explicit densities: normal, Cauchy, and Lévy one-sided density given by the formula

$$l(x) = \frac{1}{\sqrt{2\pi x^3}} \exp\left(-\frac{1}{2x}\right), \quad x \geq 0.$$

The tests for normal and Cauchy law based on characterizations were described above. One of the simplest characterizations of the Lévy law obtained by Ahsanullah and Nevzorov [6] runs as follows.

Theorem 13. *Let X, Y, and Z be i.i.d. rv's. Then the equality in distribution*

$$X \stackrel{d}{=} \frac{Y+Z}{4}$$

holds if and only if X, Y, and Z have the one-sided Lévy distribution with arbitrary scale factor.

The tests based on this characterization are unknown. Nothing is known about testing for general stable distributions using similar characterizations.

9.2. Tests based on characterizations by independence. The characterizations of distributions can be formulated not only in terms of the equidistribution of statistics as in majority of examples given above but also in terms of their independence. Consider, as an example, the well-known classical result obtained independently by Kac [36] and Bernstein [17] long ago.

Theorem 14. *If X and Y are independent rv's, then $X+Y$ and $X-Y$ are independent if and only if X and Y are normal.*

As far as we know, this approach is unexplored. Further development of the plot led finally to the famous Skitovich–Darmois theorem (see [19], [37]) which is also suitable for the construction of tests. We can construct corresponding U-empirical distributions and test statistics which are more difficult for analysis. Nobody has studied the corresponding goodness-of-fit tests.

Another option consists in the well-known result that the independence of $\bar{x}$ and s^2 implies normality, which was first proved by Geary in [27]. The same is true for higher central moments. These characterizations were used in a number of papers, see [52], [46], [70], and a preprint by Thulin [95] with power study via simulations. However, there are no calculations of efficiency and analytic comparison with other tests of normality.

First steps in the calculations of efficiency for tests based on characterizations by independence were done recently by Miloševic and Obradovic [62]. For instance, they used the following characterization of the exponential law from [24].

Theorem 15. *If X and Y are independent i.i.d. rv's with an absolutely continuous distribution, and* $\min\{X, Y\}$ *and* $|X-Y|$ *are independent, then both X and Y have exponential distribution with distribution function* $F(x) = 1 - e^{-\lambda x},\ \ x > 0,\ \lambda > 0.$

In [62] there are also related results concerning other distributions.

9.3. Use of empirical integral transforms. For certain characterizations one can build the test statistics based not on U-empirical distributions but on *empirical transforms, e.g., on empirical characteristic functions or empirical Laplace transforms.*

Let $f_n(t) = n^{-1}\sum_{k=1}^{n}\exp(itX_k)$ be the empirical characteristic function of the sample $X_1, \dots, X_n$. Then it is clear that using the Polya characterization ($X \sim (X+Y)/\sqrt{2}$) we have

$$f_n(t) - f_n^2(t/\sqrt{2}) \approx 0.$$

Hence the statistics for testing normality of the sample can be

$$Z_n = \sup_t |f_n(t) - f_n^2(t/\sqrt{2})|$$

or

$$W_n = \int_{-\infty}^{\infty} |f_n(t) - f_n^2(t/\sqrt{2})|^2 Q(t)dt,$$

where Q is some appropriate weight function.

The asymptotic properties and efficiencies of empirical integral transforms are unknown. However, the technique of asymptotic analysis of similar statistics was substantially developed in recent years, see, e.g., the papers by Meintanis and Jimenez-Gamero [55], [54], [34], [56], etc.

The use of empirical Laplace transform with interesting calculation of efficiencies for testing of exponentiality is presented in [60].

9.4. Characterizations based on records. There are many characterizations of distributions based on *record statistics,* see, e.g., [74], [20], [15], [92], and many others. Only few of them have been used for construction of goodness-of-fit tests, mainly in the works of Morris and Szynal (see, e.g., [68],

[69], [66], [93]), where they essentially used the characterizations based on moments of record values. However, nothing is known about the efficiencies of such tests.

9.5. Characterizations based on moments. Some characterizations of distributions are based on their moments or on moments of corresponding order statistics, see, e.g., [44], [45], [63], [65], [96]. They can be used for the construction of goodness-of-fit tests, but their efficiencies are unexplored.

9.6. Multivariate generalizations. It seems that little or nothing is known about *multivariate goodness-of-fit tests and multivariate symmetry tests.* One of the few exceptions is the recent paper [62].

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Department of Mathematics and Mechanics, Saint-Petersburg State University, Universitetskaia nab. 7/9, Saint-Petersburg 199034, Russia; National Research University - Higher School of Economics, Souza Pechatnikov 16, Saint-Petersburg 190008, Russia

E-mail address: y.nikitin@spbu.ru