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On Discrete Semi-Flows with Universally Measurable Ellis Semigroup

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Abstract

We show that, in the framework of ZFC axioms, any continuous compact discrete semi-flow whose Ellis semigroup consists of universally measurable transformations is tame, and the union of its minimal sets is dense in the minimal attraction center. At the same time, under ZFC, the Ellis semigroups of a broad class of discrete dynamical systems contain transformations that are not universally measurable.

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1. Introduction

We restrict ourselves to semicascades (discrete semi-flows) (Ω, φ) generated by a continuous (not necessarily invertible) transformation φ of a metrizable compact set Ω.

The dynamic and ergodic properties of (Ω, φ) are closely related to the characteristics of associated enveloping semigroups [2, 4, 5, 7, 8, 10]. We mean the Ellis semigroup $E(\Omega, \varphi)$ of mappings $\Omega \to \Omega$ and the Köhler semigroup $K(\Omega, \varphi)$ of transformations of the space $\mathcal{P}(\Omega)$ of Radon probability measures, which is compact and metrizable in the w^*-topology of the space $C^*(\Omega)$. The semigroup $E(\Omega,\varphi)$ is the closure of the shifts $\{\varphi^n,\, n\in\mathbb{N}_0\}$ in the direct product topology on the compact set Ω^Ω. If $Ux = x\circ\varphi$ for $x\in C(\Omega)$, then the semigroup $K(\Omega,\varphi)$ is defined as the closure of the operator family $\{V^n,\, n\in\mathbb{N}_0\}$ with $V=U^*$ in the $\mathrm{W^*O}$-topology of the space $\operatorname{End} C^*(\Omega)$ [7]. The operator V generates the semicascade $(\mathcal{P},V)$ on the compact set $\mathcal{P}=\mathcal{P}(\Omega)$, and one essentially has $K(\Omega,\varphi)\simeq E(\mathcal{P},V)$.

2. Preliminary Notes

There is an important relationship between characteristics of the semicascade (Ω,φ) and the regularity of elements of the semigroup $E=E(\Omega,\varphi)$. For example, (Ω,φ) is weakly almost periodic if and only if $E\subseteq C(\Omega,\Omega)$ [2]. At the same time, there exist compact dynamical systems with $E=\Omega^\Omega$ [4]. We will consider three types of semicascades.

2.1 We define the class $\mathcal{D}_{\mathrm{tm}}$ of *tame* dynamical systems by the relation $\operatorname{card} E\le \mathfrak{c}$ [7, 4]. Let us present a number of well-known properties of such $\mathbb{N}_0$-systems [4, 5].

(A) The action of $E(\Omega,\varphi)$ can naturally be lifted from Ω to $\mathcal{P}(\Omega)$, which permits one to identify the Ellis and Köhler semigroups.

(B) The minimal subsystems are uniquely ergodic.

(C) The topological entropy of minimal subsystems is zero. Either of the following two conditions is necessary and sufficient for $(\Omega,\varphi)\in\mathcal{D}_{\mathrm{tm}}$: (i) $E(\Omega,\varphi)$ consists of Borel transformations [10]; (ii) $E(\Omega,\varphi)$ consists of sequential limits $E_1(\Omega,\varphi)$ of the shifts $\{\varphi^n,\, n\in\mathbb{N}_0\}$.

2.2 Following [4], we say that a dynamical system (Ω,φ) is *injective* if it has property (A). We denote the class of such semicascades by $\mathcal{D}_{\mathrm{in}}$; then $\mathcal{D}_{\mathrm{tm}}\subseteq\mathcal{D}_{\mathrm{in}}$ [7] and the Bernoulli shift on the Cantor space $\Omega=\{0,1\}^{\mathbb{N}_0}$ provides an example of an injective semicascade that is not tame [5, Exercise 7.5]. Injectivity is equivalent to equicontinuity for minimal distal dynamical systems [4, 5], and hence the semicascade

$$\left(\mathbb{T}^2,\varphi\right),\quad \varphi(\omega_1,\omega_2)=(\omega_1+\theta,\ \omega_2+\omega_1)\ (\mathrm{mod}\,1),\quad \theta\in\mathbb{R}\setminus\mathbb{Q}, \tag{1}$$

(see [3]) is not injective.

2.3 Let us introduce the class $\mathcal{D}_{\mathrm{um}}$ of semicascades with Ellis semigroup consisting of transformations of minimal regularity, namely, of $\mathcal{P}$-universally measurable transformations. Clearly, $\mathcal{D}_{\text{tm}} \subseteq \mathcal{D}_{\text{um}}$. Our main result (Theorem 3.2) establishes that the proposition $\mathcal{D}_{\text{tm}} = \mathcal{D}_{\text{um}}$ is consistent under the ZFC axioms (the Zermelo–Fraenkel axioms plus the axiom of choice). In the same sense, all properties of tame semicascades can be extended to the class $\mathcal{D}_{\text{um}}$, and one has the strict inclusion $\mathcal{D}_{\text{um}} \subset \mathcal{D}_{\text{in}}$. *In what follows, the word "consistently" means being consistent under the* ZFC *axioms*. The following result in [9] proves to be crucial: the set of universally measurable subsets of metrizable compact set is consistently at most continual.

Thus, the following alternative holds for the Ellis semigroup of an $\mathbb{N}_0$-dynamical system: *either* $E(\Omega,\varphi) = E_1(\Omega,\varphi)$, *or* $E(\Omega,\varphi)$ *consistently contains transformations nonmeasurable with respect to some Radon measures*. The latter possibility is realized, say, for the classical Bernoulli shift, because one has $\operatorname{card} E(\Omega,\varphi) = 2^{\mathfrak{c}}$ in this case [5]. It follows from Theorem 3.2 and properties (B) and (C) in **2.1** that the Ellis semigroup of a minimal semicascade (Ω,φ) consistently contains a transformation that is not universally measurable, provided that (Ω,φ) is not uniquely ergodic or its topological entropy is not zero.

3. Main Results

Further, let $\sum_b$ and $\sum_u$ be the σ-algebras of Borel subsets of Ω and subsets universally measurable with respect to $\mu \in \mathcal{P}(\Omega)$, respectively; then $\sum_b \subseteq \sum_u$. We say that a mapping $p:\Omega \to \Omega$ is Borel $\left(p \in \prod_b\right)$ or universally measurable $\left(p \in \prod_u\right)$ if $p^{-1} \in \sum_b$ for $e \in \sum_b$ or $p^{-1}e \in \sum_u$ for $e \in \sum_u$, respectively. Let $\sum_u^2$ be the σ-algebra of universally measurable subsets of the metrizable compact set $\Omega \times \Omega$.

Lemma 3.1. $\operatorname{card} \prod_u \le \operatorname{card} \sum_u^2$.

Proof. We generalize the argument in [1, §2.12 (ii)] and [10, pp. 208-209] and identify an arbitrary mapping $p \in \prod_u$ with its graph $G_p = \{(\omega, p\omega),\ \omega \in \Omega\} \subset \Omega \times \Omega$. The transformation $(\omega_1,\omega_2) \to (p\omega_1,\omega_2)$ of the metrizable compact set $\Omega \times \Omega$ is universally measurable, and hence so is the scalar function $g(\omega_1,\omega_2) = \rho(p\omega_1,\omega_2)$ on $\Omega \times \Omega$ for any of the metrics ρ compatible with the original topology on Ω. The graph G_p coincides with $g^{-1}(0)$, and hence $G_p \subset \sum_u^2$ and $\operatorname{card} \prod_u \le \operatorname{card} \sum_u^2$.

Theorem 3.2 (main theorem). *Consistently,* $\mathcal{D}_{\mathrm{um}} = \mathcal{D}_{\mathrm{tm}}$.

Proof. Since $\mathcal{D}_{\mathrm{tm}} \subseteq \mathcal{D}_{\mathrm{um}}$, it suffices to prove that the inclusion $\mathcal{D}_{\mathrm{um}} \subseteq \mathcal{D}_{\mathrm{tm}}$ is consistent with ZFC. By [9], one consistently has $\operatorname{card}\sum_u^2 \le \mathbf{c}$, and hence $\operatorname{card}\prod_u \le \mathbf{c}$ by Lemma 3.1. If $E(\Omega,\varphi) \subseteq \prod_u$, then consistently $\operatorname{card} E(\Omega,\varphi) \le \mathbf{c}$, and the dynamical system (Ω,φ) proves to be tame.

However, it is well known (Hausdorff, 1908) that $\operatorname{card}\sum_u \ge 2^{\aleph_1}$ for $\Omega = [0,1]$, and hence the use of the continuum hypothesis may affect the corresponding logical landscape.

Corollary 3.3. *Consistently,* $E(\Omega,\varphi) \not\subseteq \prod_u$ *for the Bernoulli shift on* $\Omega = \{0,1\}^{\mathbb{N}_0}$.

Indeed, this dynamical system is not tame [5]. Neither is the minimal distal semicascade (1) [4, 5], although the relation $E(\Omega,\varphi) \not\subseteq \prod_u$ for this case can directly be obtained from the argument in [3]. The construction of minimal cascades $(\mathbb{T}^2,\varphi)$ that are not uniquely ergodic (and hence are not tame) is described in [6, §4.2.b and §12.6.b].

Proposition 3.4. *Consistently,* $\mathcal{D}_{\mathrm{um}} \subset \mathcal{D}_{\mathrm{in}}$.

This is a corollary of Theorem 3.2, the inclusion $\mathcal{D}_{\mathrm{tm}} \subseteq \mathcal{D}_{\mathrm{in}}$, and the characteristics of the Bernoulli shift indicated in **2.2**.

If $M^c(\Omega,\varphi)$ is the closure of the union of all minimal sets and $Z(\Omega,\varphi)$ is the minimal attraction center of the semicascade (Ω,φ), which coincides with the closure of the union of supports of φ-ergodic measures $\mu \in \mathcal{P}(\Omega)$, then $M^c(\Omega,\ \varphi) \subseteq Z(\Omega,\ \varphi)$.

Proposition 3.5. *Consistently,* $M^c(\Omega,\varphi) = Z(\Omega,\varphi)$ *for a semicascade* $(\Omega,\varphi) \in \mathcal{D}_{\mathrm{um}}$.

Proof. The semicascade (Ω,φ) is consistently tame by Theorem 3.2, and the desired relation was established for this case in [10, Proposition 4.4].

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