TRIANGULATED ENDOFUNCTORS OF THE DERIVED CATEGORY OF COHERENT SHEAVES WHICH DO NOT ADMIT DG LIFTINGS

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Abstract. In [RV], Rizzardo and Van den Bergh constructed an example of a triangulated functor between the derived categories of coherent sheaves on smooth projective varieties over a field k of characteristic 0 which is not of the Fourier-Mukai type. The purpose of this note is to show that if $\operatorname{char} k = p$ then there are very simple examples of such functors. Namely, for a smooth projective Y over $\mathbb{Z}_p$ with the special fiber $i : X \hookrightarrow Y$, we consider the functor $Li^* \circ i_* : D^b(X) \to D^b(X)$ from the derived categories of coherent sheaves on X to itself. We show that if Y is a flag variety which is not isomorphic to $\mathbb{P}^1$ then $Li^* \circ i_*$ is not of the Fourier-Mukai type. Note that by a theorem of Toen ([T], Theorem 8.15) the latter assertion is equivalent to saying that $Li^* \circ i_*$ does not admit a lifting to a $\mathbb{F}_p$-linear DG quasi-functor $D^b_{dg}(X) \to D^b_{dg}(X)$, where $D^b_{dg}(X)$ is a (unique) DG enhancement of $D^b(X)$. However, essentially by definition, $Li^* \circ i_*$ lifts to a $\mathbb{Z}_p$-linear DG quasi-functor.

Given smooth proper schemes X_1, X_2 over a field k and an object $E \in D^b(X_1 \times X_2)$ of the bounded derived category of coherent sheaves on $X_1 \times X_2$ define a triangulated functor

$$\Phi_E : D^b(X_1) \to D^b(X_2) \tag{0.1}$$

sending a bounded complex M of coherent sheaves on X_1 to $Rp_{2*}(E \overset{L}{\otimes} p_1^* M)$, where $p_i : X_1 \times X_2 \to X_i$ are the projections. Recall that a triangulated functor $D^b(X_1) \to D^b(X_2)$ is said to be of the Fourier-Mukai type if it is isomorphic to Φ_E for some E.

Let Y be a smooth projective scheme over $\operatorname{Spec} \mathbb{Z}_p$ and let X be its special fiber, $i : X \hookrightarrow Y$ the closed embedding. Consider the triangulated functor $G : D^b(X) \to D^b(X)$ given by the formula

$$G = Li^* \circ i_*$$

We shall see that in general G is not of the Fourier-Mukai type.

Theorem 1. *Let Z a smooth projective scheme over $\operatorname{Spec} \mathbb{Z}_p$, $Y = Z \times Z$, $X = Y \times_{\operatorname{Spec} \mathbb{Z}_p} \operatorname{Spec} \mathbb{F}_p$. Assume that*

1. *The Frobenius morphism $Fr : \overline{Z} \to \overline{Z}$, where $\overline{Z} = Z \times \operatorname{Spec} \mathbb{F}_p$, does not lift modulo p^2.*
2. *$H^1(X, T_X) = 0$, where T_X is the tangent sheaf on X.*

Then $G = Li^* \circ i_* : D^b(X) \to D^b(X)$ is not of the Fourier-Mukai type.

For example, let GL_n be the general linear group over $\operatorname{Spec} \mathbb{Z}_p$, $B \subset GL_n$ a Borel subgroup. Then, by Theorem 6 from [BTLM], for any $n > 2$, the flag variety $Z = GL_n/B$ satisfies the first assumption of the Theorem *i.e.*, the Frobenius $Fr : \overline{Z} \to \overline{Z}$ does not lift on $Z \times \operatorname{Spec} \mathbb{Z}/p^2\mathbb{Z}$. By ([KLT], Theorem 2), we have that $H^1(\overline{Z}, T_{\overline{Z}}) =$

$H^1(\overline{Z}, \mathcal{O}_{\overline{Z}}) = 0$. It follows that $H^1(X, T_X) = 0$. Hence, by the Theorem, for $n > 2$, $G : D^b(X) \to D^b(X)$ is not of the Fourier-Mukai type.

Proof. Assume the contrary and let $E \in D^b(X \times X)$ be the Fourier-Mukai kernel. By definition, for every $M \in D^b(X)$ we have a functorial isomorphism

$$G(M) \xrightarrow{\sim} Rp_{2*}(E \overset{L}{\otimes} p_1^* M). \tag{0.2}$$

By the projection formula ([H], Chapter II, Prop. 5.6) we have that

$$i_* \circ Li^* \circ i_*(M) \xrightarrow{\sim} i_*(M) \overset{L}{\otimes} i_*(\mathcal{O}_X) \xrightarrow{\sim} i_*(M) \otimes (\mathcal{O}_Y \xrightarrow{p} \mathcal{O}_Y) \xrightarrow{\sim} i_*(M) \oplus i_*(M)[1]$$

In particular, if M is a coherent sheaf then $\underline{H}^i(G(M)) \simeq M$ for $i = 0, -1$ and $\underline{H}^i(G(M)) = 0$ otherwise. Applying this observation and formula (0.2) to skyscraper sheaves, $M = \delta_x$, $x \in X(\overline{\mathbb{F}}_p)$, we conclude that the coherent sheaves $\underline{H}^i(E)$ are set theoretically supported on the diagonal $\Delta_X \subset X \times X$. Applying the same formulas to $M = \mathcal{O}_X$ we see that $p_{2*}(\underline{H}^i(E)) = \mathcal{O}_X$ for $i = 0, -1$ and $p_{2*}(\underline{H}^i(E)) = 0$ otherwise. In fact, every coherent sheaf F on $X \times X$ which is set theoretically supported on the diagonal and such that $p_{2*}F = \mathcal{O}_X$ is isomorphic to $\mathcal{O}_{\Delta_X}$. It follows that $\underline{H}^0(E) = \underline{H}^{-1}(E) = \mathcal{O}_{\Delta_X}$. In the other words, E fits into an exact triangle in $D^b(X \times X)$

$$\mathcal{O}_{\Delta_X}[1] \xrightarrow{\alpha} E \longrightarrow \mathcal{O}_{\Delta_X} \xrightarrow{\beta} \mathcal{O}_{\Delta_X}[2] \tag{0.3}$$

for some $\beta \in Ext^2_{\mathcal{O}_{X\times X}}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X})$. We wish to show that the second assumption in the Theorem implies that $\beta = 0$, while the first one implies that $\beta \neq 0$. For every $M \in D^b(X)$, (0.3) gives rise to an exact triangle

$$M[1] \xrightarrow{\alpha_M} G(M) \longrightarrow M \xrightarrow{\beta_M} M[2] \tag{0.4}$$

Our main tool is the following result.

Lemma 0.1. *For a coherent sheaf M the following conditions are equivalent.*

1. *$\beta_M = 0$.*
2. *$G(M) \xrightarrow{\sim} M \oplus M[1]$.*
3. *There exists a morphism $\lambda : G(M) \to M[1]$ such that $\lambda \circ \alpha_M$ is an isomorphism.*
4. *M admits a lift modulo p^2 i.e., there is a coherent sheaf $\tilde{M}$ on Y flat over $\mathbb{Z}/p^2\mathbb{Z}$ such that $i^*\tilde{M} \simeq M$.*

Proof. The equivalence of (1), (2) and (3) is immediate. Let us check that (3) is equivalent to (4). A morphism $\lambda : G(M) \to M[1]$ gives rise by adjunction a morphism $\gamma : i_*M \to i_*M[1]$. Note that $\tilde{M} = \operatorname{cone}\gamma[1]$ is a coherent sheaf on Y which is an extension of i_*M by itself. It suffices to prove that $\lambda \circ \alpha_M : M[1] \to M[1]$ is an isomorphism if and only if $\tilde{M}$ is flat over $\mathbb{Z}/p^2\mathbb{Z}$. Indeed, from the exact triangle

$$Li^*i_*M \to Li^*(\tilde{M}) \to Li^*i_*M \to Li^*i_*M[1]$$

we get a long exact sequence of the cohomology sheaves

$$0 \to M \to L_1 i^*(\tilde{M}) \to M \xrightarrow{\lambda\circ\alpha_M} M \to i^*(\tilde{M}) \to M \to 0$$

Thus $\lambda \circ \alpha_M$ is an isomorphism if and only if in the exact sequence

$$0 \to i_*M \to \tilde{M} \to i_*M \to 0$$

the image of second map is the kernel of the multiplication by p on $\tilde{M}$ and also the image of this map. The latter is equivalent to flatness of $\tilde{M}$ over $\mathbb{Z}/p^2\mathbb{Z}$. □

We have a spectral sequence converging to $Ext^*_{\mathcal{O}_{X\times X}}(\mathcal{O}_{\Delta_X},\mathcal{O}_{\Delta_X})$ whose second page is $H^*(X,\mathcal{E}xt^*_{\mathcal{O}_{X\times X}}(\mathcal{O}_{\Delta_X},\mathcal{O}_{\Delta_X}))$. In particular, we have a homomorphism

$$Ext^2_{\mathcal{O}_{X\times X}}(\mathcal{O}_{\Delta_X},\mathcal{O}_{\Delta_X})\to H^0(X,\mathcal{E}xt^2_{\mathcal{O}_{X\times X}}(\mathcal{O}_{\Delta_X},\mathcal{O}_{\Delta_X}))\xrightarrow{\sim} H^0(X,\wedge^2 T_X).$$

Let us check that the image μ of β under this map is 0. To do this we apply the Lemma to skyscraper sheaves δ_x, where x runs over closed points of X. On the one hand, the evaluation of the bivector field μ at x is equal to the class of β_{δ_x} in $Ext^2_{\mathcal{O}_X}(\delta_x,\delta_x)\xrightarrow{\sim}\wedge^2 T_{x,X}$. On the other hand, by the Lemma, $\beta_{\delta_x}=0$ since δ_x is liftable modulo p^2. Next, the assumption that $H^1(X,T_X)=0$ implies that β lies in the image of the map

$$v: H^2(X,\mathcal{O}_X)\xrightarrow{\sim} H^2(X,\mathcal{E}xt^0_{\mathcal{O}_{X\times X}}(\mathcal{O}_{\Delta_X},\mathcal{O}_{\Delta_X}))\to Ext^2_{\mathcal{O}_{X\times X}}(\mathcal{O}_{\Delta_X},\mathcal{O}_{\Delta_X}). \tag{0.5}$$

The map (0.5) has a left inverse $u: Ext^2_{\mathcal{O}_{X\times X}}(\mathcal{O}_{\Delta_X},\mathcal{O}_{\Delta_X})\to H^2(X,\mathcal{O}_X)$ which takes β to $\beta_{\mathcal{O}_X}$. But, by the Lemma, the later class is equal to 0 since $\mathcal{O}_X$ is liftable modulo p^2. It follows that β is 0.

On the other hand, let $\Gamma\subset X=\overline{Z}\times\overline{Z}$ be the graph of the Frobenius morphism $Fr:\overline{Z}\to\overline{Z}$ and $\mathcal{O}_\Gamma$ the structure sheaf of Γ viewed as a coherent sheaf on X. Then, by our first assumption, the sheaf $\mathcal{O}_\Gamma$ is not liftable modulo p^2. Hence, by the Lemma, $\beta_{\mathcal{O}_\Gamma}$ is not 0. This contradiction completes the proof. □

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