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Conditional probabilities and van Lambalgen's theorem revisited

Bruno Bauwens · Alexander Shen · Hayato Takahashi

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Abstract The definition of conditional probability in the case of continuous distributions (for almost all conditions) was an important step in the development of mathematical theory of probabilities. Can we define this notion in algorithmic probability theory for individual random conditions? Can we define randomness with respect to the conditional probability distributions? Can van Lambalgen's theorem (relating randomness of a pair and its elements) be generalized to conditional probabilities? We discuss the developments in this direction. We present almost no new results trying to put known results into perspective and explain their proofs in a more intuitive way. We assume that the reader is familiar with basic notions of measure theory and algorithmic randomness (see, e.g., [8] or [7] for a short introduction).

Keywords Conditional probability; Algorithmic randomness; Van Lambalgen's theorem.

1 Conditional probability

Let P be a computable distribution on the product of two copies of Cantor space $\Omega_1 \times \Omega_2$, and let P_1 be its marginal distribution (= the projection of P onto Ω_1).

Bruno Bauwens
National Research University Higher School of Economics (HSE), Faculty of Computer Science, Kochnovskiy Proezd 3, Moscow, 125319, Russia, E-mail: bbauwens@hse.ru.

Alexander Shen
Laboratoire d'Informatique, de Robotique et de Microélectronique de Montpellier, CNRS, Université de Montpellier, E-mail: alexander.shen@lirmm.fr. Supported by ANR-15-CE40-0016-01 RaCAF grant.

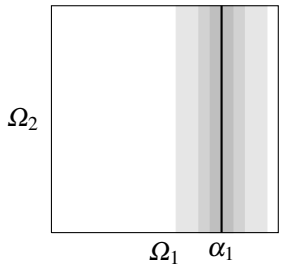
Hayato Takahashi
1-1 Yanagido, Gifu City 501-1193, Japan. Organization for Promotion of Higher Education and Student Support, Gifu University. E-mail: hayato.takahashi@ieee.org. Supported by JSPS KAKENHI grant number 24540153.

The paper records the discussion of conditional randomness results by the participants of the Heidelberg Focus Semester on algorithmic randomness, 2015 (Laurent Bienvenu, Mathieu Hoyrup, Rupert Hölzl, Wolfgang Merkle, Jason Rute and others).

Consider some $\alpha_1 \in \Omega_1$. We want to define the conditional distribution on Ω_2 with the condition "the first coordinate is equal to α_1". For that we consider a prefix a_1 of α_1 and the conditional distribution on Ω_2 with the condition "the first coordinate starts with a_1". (For this we need that $P_1(a_1)$, the probability of the interval $[a_1]$ of all extensions of a_1, is positive.) In this way we get a family of measures P_{a_1} on Ω_2:

$$P_{a_1}(a_2) = \frac{P(a_1, a_2)}{P_1(a_1)}$$

Here the numerator is the P-measure of the product $[a_1] \times [a_2]$, and the denominator is the P-measure of $[a_1] \times \Omega_2$. Then, for a given a_2, we consider the limit of probability $P_{a_1}(a_2)$ as the length of prefix a_1 (of α_1) tends to infinity.



Theorem 1 ([9]) *If α_1 is Martin-Löf random with respect to P_1, then this limit is well defined and determines a distribution on Ω_2.*

Proof The proof goes as follows: for a fixed a_2 the function $m\colon a_1 \mapsto P_{a_1}(a_2)$ is a computable martingale on Ω_1 with respect to P_1, being a ratio of some measure and P_1, and an effective version of the martingale convergence theorem can be applied. Let us now elaborate the details.

We assume for now that $P_1(a_1) \neq 0$ for all a_1, so the ratio is well defined. This assumption is not always true, but at least $P_1(a_1) \neq 0$ for strings that are prefixes of random sequences. We return to the general case later. Under this assumption, the function m is well defined (for every fixed a_2) and indeed is a P_1-martingale; this means that

$$m(x) = \frac{P_1(x0)}{P_1(x)} m(x0) + \frac{P_1(x1)}{P_1(x)} m(x1)$$

and is true because $P_1(x)m(x)$ is a measure.

The classical martingale convergence theorem says that (with P_1-probability 1 over α_1) the values $m(\alpha_1 \restriction n)$ have finite limit as $n \to \infty$ (where $\alpha_1 \restriction n$ is the n-bit prefix of α_1). We need the effective version of this statement saying that this convergence happens for every P_1-random α_1.

For the proof we use the effective version of the martingale convergence theorem. To see why this theorem can be effectivized, let us recall its proof. First we show that every martingale is bounded on almost all sequences (the martingale m is always bounded by 1, but this argument will be used later for a different martingale). For that, we note that for every $c > 0$ and random α_1, the probability of the event "the

martingale exceeds c at some prefix of α_1" is at most $1/c$, and sequences where the martingale is unbounded belong to this event for every c.

To prove the convergence, it remains to show that for every pair of positive rationals $u < v$ the set of sequences where the martingale m is infinitely often less than u and infinitely often greater than v, is a null set. Indeed, the set of sequences where there are at least N changes across (u,v) has small measure due to a 'buy low–sell high" argument. Formally, we apply the argument above to another martingale m' that follows m's bets, starting when m becomes less than u and doing this until m becomes greater than v; then m' waits (keeping the capital unchanged) for the next time when m becomes less than u, then starts following m again, etc.

An effective version of this argument says that if P_1 and m are computable, then the P_1-null set of sequences that have no limit is an *effectively* P_1-null set, and therefore it does not contain P_1-random sequences. Indeed, the set of prefixes where the martingale is large, as well as the set of sequences where at least N low–high oscillations happen, are effectively open, and have small measure (as the classical proof shows). Therefore, the convergence happens for all P_1-random α_1.

To finish the proof, we need to consider the case when some $P_1(a_1)$ are zeros. Still we may consider the set of prefixes where P_1 does not vanish and more than N oscillations happen. This set is effectively open and has small measure as the same martingale argument shows.

Now we know that the limit exists (and is finite) for every P_1-random sequence α_1. To see that we indeed get a measure on Ω_2, it remains to check finite additivity which is obvious: a limit of a sum is the sum of limits. The measure of Ω_2 is 1, so we get a probability distribution. □

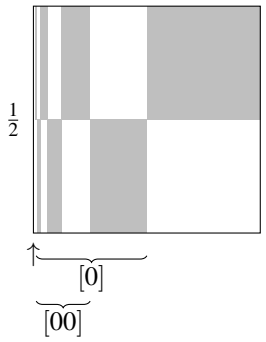
The conditional distribution can be denoted by $P(\cdot\,|\,\alpha_1)$. Of course, if we start with a product distribution $P = P_1 \times P_2$, the conditional distribution is the same (equals P_2) for all α_1.

Remark 1 The definition of a conditional distribution in classical probability theory is usually given using the Radon–Nikodym theorem; the Lebesgue differentiation theorem can be used to show that the conditional distribution defined in this way coincides almost everywhere with the limit we considered.

2 Non-computable conditional probability

Let us note first that the limit in the definition of conditional probability may not exist for some conditions (though these conditions form a null set, as we have seen). This is shown by the following example. In this section we identify Ω_2 with the real interval $[0,1]$; the dyadic rationals have two representations as sequences, but this does not matter much.

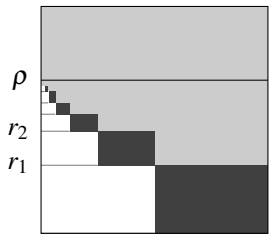
Example 1 Consider the following distribution on $\Omega_1 \times [0,1]$. In our distribution the grey areas have double density compared with the uniform distribution on the square, while the white areas have zero density.



It is easy to show that for the leftmost point (shown by an arrow) the limit that defines the conditional distribution does not exist. Indeed, the conditional probabilities of the two halves $[0,1/2]$ and $[1/2,1]$ oscillate between $1/3$ and $2/3$ depending on the length of a_1, as one may easily check.

Example 2 below shows that the conditional probability for a computable distribution on pairs might exist but at the same time might be non-computable. An example of this type was first constructed in [1] and was more complicated.[1]

Example 2 Let $r_1, r_2, \ldots$ be an increasing computable sequence of rational numbers whose limit ρ is non-computable. Consider the following distribution:



Vertical lines are drawn at right endpoints of the intervals $[0]$, $[00]$,...; in the grey zone the density is the same as for the uniform distribution; in the black zone the density is twice bigger, and in the white zone the density is zero. Since the widths of the black and white stripes on every horizontal line are the same, the total amount of mass does not change; we just move all the mass horizontally from the white part to the black part.

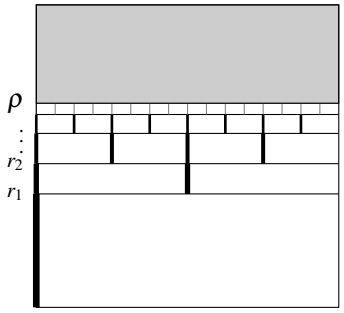
Note that this distribution (on the square) is computable even though ρ is not computable. Indeed, if we are interested in the mass of some rectangle R (the product of two dyadic intervals), the mass transfers in small rectangles (thinner than R) do not matter, and we may look only on finitely many r_i (they can be computed).

[1] In fact, the example in [1] has an additional property that our example does not have: the set of α_1 for which $P(\cdot|\alpha_1)$ is not computable with oracle α_1, has positive P_1-measure. The distribution constructed in Example 3 (see below) also has this property, see also [2, Corollary 1]. On the other hand, the example in [1] has a conditional distribution $P(\cdot|\alpha_1)$ that is continuous in α_1, unlike Example 3. A simplified construction of a continuous (in α_1) distribution can be found in [2, Theorem 5].

It is easy to see that the limit distribution (at the leftmost point) is the uniform distribution on $[\rho,1]$, and it is not computable, since the density $1/(1-\rho)$ is not computable.

In this example the conditional distribution is non-computable only at one point. However, the example can be easily changed so that the conditional distribution is the same non-computable distribution for all conditions α_1 except for sequences that contain finitely many ones. Here is the corresponding construction.

Example 3 Consider again the increasing computable sequence $r_1, r_2, \ldots$ of rational numbers that converges to a non-computable real ρ.



In the grey area above the horizontal line with coordinate ρ we keep the same density as in the uniform distribution. Below ρ all the mass is concentrated on black vertical segments. For example, the mass r_1 is concentrated on the segment $\{000\ldots\}\times[0,r_1]$, and is distributed uniformly there (so the mass transfer happens only in the horizontal direction). The mass r_2-r_1 is then split evenly between two vertical segments shown (at horizontal coordinates $0000\ldots$ and $1000\ldots$), etc. One can say that each vertical segment "horizontally grabs" all the mass of the white rectangle on the right of it (so this white rectangle has zero density except for its left side, a sequence with finitely many ones).

As before, it is easy to see that the resulting distribution is computable: to find the mass of a dyadic rectangle of width 2^{-n}, it is enough to take into account only $r_1,\ldots,r_n$ and use the uniform distribution above r_n.

It is also easy to see that for every α_1 with infinitely many ones (in other words, for vertical lines that do not contain black segments), the conditional probability $P(\cdot|\alpha_1)$ is uniformly distributed on $[\rho,1]$.

The conditional probability exists also for α_1 with finitely many ones, i.e., for the case when $\alpha_1 = x000\ldots$ and in this case it is even computable: if the non-zero prefix of α_1 consists of k bits, the density of the conditional distribution is zero below r_k, the density is constant on $[r_k,r_{k+1}]$, twice smaller on $[r_{k+1},r_{k+2}]$, again twice smaller on $[r_{k+2},r_{k+3}]$, etc., and is zero above ρ. The density is computable due to fast convergence of the series $\sum(r_{k+1}-r_k)2^{-k}$.

We will reuse this example in Section 5.

3 Van Lambalgen's theorem

Now we consider the relation between randomness of a pair and its components. The basic result goes back to Michiel van Lambalgen (see [6, Theorem 5.10], where this result is stated in an implicit way). It considers the case of the product P of two computable distributions P_1 on Ω_1 and P_2 on Ω_2, and says that the pair (α_1, α_2) is Martin-Löf random with respect to P if and only if two conditions are satisfied:

- α_1 is random with respect to the distribution P_1;
- α_2 is random with respect to the distribution P_2 with oracle α_1.

(See, for example, [8, chapter 5] for the proof.) Note that these conditions are not symmetric; of course, one can exchange the coordinates and conclude that α_1 is also random with respect to P_1 with oracle α_2.

It is natural to look for some version of van Lambalgen's theorem that can be applied to non-product distributions P. Informally speaking, such a version should say that (α_1, α_2) is P-random if and only if

- α_1 is P_1-random (where P_1 is the projection of P);
- α_2 is random with respect to the conditional probability distribution $P(\cdot|\alpha_1)$ with oracle α_1.

But some precautions are needed. The problem is that Martin-Löf randomness is usually defined for computable distributions, while the conditional distribution may not be computable. Moreover, it may not be computable even with oracle α_1 for P_1-random α_1. Indeed, in Example 3 the conditional distribution was $[\rho, 1]$ for every irrational condition α_1, and among them there are P_1-random conditions that do not compute ρ. To prove their existence, let us note that random reals with respect to the uniform Lebesgue measure are all P_1-random and some of them do not compute ρ. Indeed, if the uniform Lebesgue measure of conditions α_1 that compute ρ were positive, then the same would be true for some fixed oracle machine due to countable additivity. Then the Lebesgue density theorem says that in this case there exists some interval where most of the oracles compute ρ, so ρ can be computed without oracle by majority voting — but ρ is not computable. (The last argument is known as de Leeuw – Moore – Shannon – Shapiro theorem [5].)

Still we can make several observations.

4 Image randomness and beyond

If (α_1, α_2) is (Martin-Löf) random with respect to P, then α_1 is (Martin-Löf) random with respect to the marginal distribution P_1. This is obvious; every cover of α_1 with small P_1-measure gives a cover of (α_1, α_2) with the same P-measure.

This result can be considered also as a special case of the image randomness theorem (see the section about image randomness in [8]) applied to the projection mapping. Moreover, the reverse direction of the image randomness theorem ("no randomness from nothing") guarantees that *every P_1-random α_1 is a first component of some P-random pair (α_1, α_2).*

So we know that for every P_1-random α_1 there exists at least one α_2 that makes the pair (α_1, α_2) P-random. It is natural to expect that most α_2 should have this property. Indeed this is the case, as the following result from [9] shows.

Theorem 2 *Assume that α_1 is P_1-random. Then the set of α_2 such that (α_1, α_2) is P-random, has probability 1 according to the conditional probability distribution $P(\cdot|\alpha_1)$.*

Proof Consider a universal Martin-Löf test on the product space. Let U_n be the effectively open set of measure at most 2^{-2n} provided by this test.

First, let us consider the case where P is a product $P_1 \times P_2$ of two computable distributions. In this case the proof goes as follows:

- Consider the set V_n of all α_1 such that the α_1-section of U_n has P_2-measure greater than 2^{-n}.
- Note that V_n is an effectively open set of measure at most 2^{-n}.
- Conclude that a random α_1 does not belong to V_n for all sufficiently large n. Indeed, if a point α_1 is covered by infinitely many sets V_n, it is covered by all sets $V'_n = \bigcup_{k>n} V_k$, so α_1 is not random. (This is often called the *Solovay randomness criterion*.)
- Now we know that for every random α_1 and for sufficiently large values of n the α_1-section of U_n has measure at most 2^{-n}. If (α_1, α_2) is not random, then this pair is contained in all U_n, so α_2 is covered by the α_1-section of all U_n.
- Therefore, the set of α_2 such that (α_1, α_2) is not random, is a null set.

Moreover, for every α_2 that is P_2-random with oracle α_1 the pair (α_1, α_2) is random, since the open cover for α_2 provided by the α_1-section of U_n is α_1-enumerable. (This is how one direction of van Lambalgen's theorem is proven.)

For the general case of a non-product distribution P we should be more careful since the conditional probability is defined as a limit, so we cannot use it directly to construct effectively open sets. Instead of V_n, we consider all *n-heavy* intervals $I \subset \Omega_1$, i.e., all intervals I such that U_n occupies more than 2^{-n}-fraction in $I \times \Omega_2$ measured according to P. In other words, an interval I is n-heavy if $P(U_n \cap (I \times \Omega_2)) > 2^{-n} P_1(I)$. This is an enumerable family of intervals since P and P_1 are computable.

Lemma 1 *The P_1-measure of the union of all n-heavy intervals is at most 2^{-n}.*

Proof To prove that the union of all n-heavy intervals has measure at most 2^{-n}, it is enough to prove this for an arbitrary finite union of n-heavy intervals. Without loss of generality we may assume that the intervals in this union are disjoint (consider only maximal intervals). For every n-heavy interval I the fraction of U_n in the stripe $I \times \Omega$ exceeds 2^{-n}, so the total P_1-measure of disjoint n-heavy intervals cannot exceed 2^{-n}, otherwise U_n would be too big (its measure would be greater than 2^{-2n}). □

In other words, the function

$$I \mapsto \text{fraction of } U_n \text{ in } I \times \Omega_2$$

is a (lower semicomputable) martingale with initial value 2^{-2n}. So due to the martingale inequality the union of intervals where the martingale exceeds 2^{-n} is at most 2^{-n}.

Lemma 2 *If α_1 is not covered by any n-heavy interval, then the α_1-section of U_n has measure at most 2^{-n} according to the conditional probability distribution with condition α_1 (assuming this probability is defined for condition α_1).*

Proof If the conditional measure of the α_1-section of U_n exceeds 2^{-n}, then there exists a finite set of disjoint vertical intervals $J_1, \ldots, J_k$ that have total conditional measure more than 2^{-n} and all belong to the α_1-section of U_n. Since U_n is open, a compactness argument shows that for sufficiently small intervals I containing α_1 we have

$$I \times J_1, \ldots, I \times J_k \subset U_n.$$

By assumption, the conditional measure of $J_1 \cup \ldots \cup J_n$ exceeds 2^{-n}, and the conditional probability is defined as the limit of conditional probabilities with condition I when intervals I containing α_1 decrease. So for all sufficiently small I the conditional measure of $J_1 \cup \ldots \cup J_k$ with condition I exceeds 2^{-n}, but this means that I is n-heavy and α_1 is covered by I, contrary to our assumption. □

Now consider (for some fixed random α_1) all "bad" α_2, i.e., all α_2 such that (α_1, α_2) is non-random. Being non-random, these pairs belong to all U_n, so every bad α_2 is inside α_1-sections of U_n for all n. Since α_1 is random, it is not covered by n-heavy intervals I for all sufficiently large n. By Lemma 2, for those n the α_1-section of U_n covers α_2 and is a set of conditional measure at most 2^{-n}. So the α_1-conditional measure of the set of all bad α_2 is equal to 0. □

In fact, we have proven the following result (see Theorem 3 below) from [10,11] (one direction of van Lambalgen's theorem extended to non-product distributions). It uses the notion of *blind* (*Hippocratic*) randomness, a version of Martin-Löf definition of randomness extended to noncomputable distributions. In this version (studied by Kjos-Hansen [4]) uniformly effectively open tests are considered and the random sequence is required to pass all of them (if the distribution is non-computable, there may be no universal test). It is opposed to *uniform* randomness where the test is effectively open with respect to the distribution (see [3] for the details); uniform randomness implies blind randomness (since more tests are allowed in the uniform version), so we may require uniform randomness in Theorem 3.

Theorem 3 *If α_1 is P_1-random and α_2 is blind (Hippocratic) random with respect to the conditional probability distribution $P(\cdot|\alpha_1)$ with oracle α_1, then the pair (α_1, α_2) is P-random.*

Proof (of Theorem 3) Indeed, the construction above gives a cover for bad α_2 that is *effectively* open relative to oracle α_1 (namely, the α_1-section of U_n for large n). □

5 A counterexample

The following counterexample from [2] shows that the statement of Theorem 3 cannot be reversed.

Theorem 4 *There exists a computable distribution P on $\Omega_1 \times \Omega_2$ such that the conditional distribution $P(\cdot|\alpha_1)$ is defined for all α_1, and a P-random pair (α_1, α_2) such that α_2 is not blind random with respect to the conditional distribution $P(\cdot|\alpha_1)$ on Ω_2.*

This result shows that the implication stated by Theorem 3 cannot be reversed, and even a bit more: the reverse implication would state that α_2 is blind random with oracle α_1, but in our example α_2 is not blind random even without oracle.

Proof We use the distribution from Example 3. The second component α_2 of the pair is now a lower semicomputable *random* real ρ that is the limit of a computable increasing sequence of dyadic rationals r_i (e.g., Chaitin's Ω-number). We start with the following observation: for this α_2 and for arbitrary α_1 *the pair (α_1, α_2) is random if and only if this pair is random with respect to the uniform distribution*. Indeed, non-randomness of a point means that we can effectively cover the point by an enumerable set of rectangles with arbitrarily small total measure. Now we have two distributions (the uniform one and the distribution from Example 3); let us show that the difference between them does not matter for covering points with second coordinate $\alpha_2 = \rho$. If we have some enumerable set of rectangles that covers (α_1, α_2), we can safely discard parts of the rectangles that are below some r_i, since this does not change anything for (α_1, α_2). In this way we may ensure that the P-measure of these rectangles equals their uniform measure (for a thin rectangle we need to discard more of it), so a P-test can be transformed into a uniform test and vice versa if we are interested only in points with second coordinate $\alpha_2 = \rho$.

Now we can find a random point (α_1, α_2) with second coordinate $\alpha_2 = \rho$ (according to the classical van Lambalgen's theorem it is enough to choose a sequence α_1 that is random with respect to the uniform distribution with oracle ρ). Since α_1 is random and thus not dyadic rational, the conditional probability $P(\cdot|\alpha_1)$ is uniformly distributed on $[\rho, 1]$. It remains to show that the *lower semicomputable real ρ is not blind random with respect to the uniform distribution on $[\rho, 1]$*. Indeed, for every rational $\varepsilon > 0$ the interval $(0, \rho + \varepsilon)$ is effectively open, since it can be represented as the union of the intervals $(0, r_i + \varepsilon)$, and its measure with respect to the uniform distribution on $[\rho, 1]$ is proportional to ε (so it is small for small ε). □

6 The case of computable conditional distribution

Still van Lambalgen's result can be generalized to non-product distributions with an additional computability assumption. As before, we consider a computable distribution P on $\Omega_1 \times \Omega_2$ and its projection P_1 on Ω_1 (the marginal distribution). The following result was proven in [9, 10]:

Theorem 5 *If a pair (α_1, α_2) is P-random, and for this α_1 the conditional distribution $P(\cdot|\alpha_1)$ is computable with oracle α_1, then α_2 is Martin-Löf random with oracle α_1 with respect to this conditional distribution.*

Before proving this theorem, let us make several remarks about its statement:

- If (α_1, α_2) is P-random, then α_1 is P_1-random, and by Theorem 1, $P(\cdot|\alpha_1)$ is well defined.
- Under our assumption Martin-Löf randomness with respect to this conditional distribution and with oracle α_1 is well defined, because this distribution is computable with oracle α_1.
- We assume the computability of the conditional distribution *only for one given condition*: $P(\cdot|\beta)$ is computable with oracle β for $\beta = \alpha_1$, but for other β it is not required (or guaranteed) even if β is P_1-random.

Recalling Theorem 3 we get a randomness criterion for pairs:

Corollary 1 *For all α_1 such that $P(\cdot|\alpha_1)$ is defined and computable with oracle α_1, the following statements are equivalent:*

- *(α_1, α_2) is Martin-Löf random with respect to distribution P,*
- *α_1 is Martin-Löf random with respect to P_1 and α_2 is Martin-Löf random with respect to $P(\cdot|\alpha_1)$ with oracle α_1.*

Proof (of Theorem 5) Let us first recall the proof for the case of a product distribution $P_1 \times P_2$. Assume that α_2 is *not* random with oracle α_1. Then there is a set $Z \subset \Omega_2$ of arbitrarily small P_2-measure that covers α_2 and is effectively open with oracle α_1. The latter statement means that Z is a section of some effectively open set of pairs $U \subset \Omega_1 \times \Omega_2$ obtained by fixing the first coordinate equal to α_1. This set U covers (α_1, α_2) by construction. The problem is that only the α_1-section of U is guaranteed to be small while other sections may be large, and we need a bound for the total measure of U to show the non-randomness of (α_1, α_2).

The solution is that we "trim" U making all its sections small. Enumerating the rectangles in U, we look at the P_2-size of all sections. When some section attempts to become too big, we prevent this and stop increasing that section. In this way we miss nothing in the α_1-section of U since it was small in the first place.

This argument works for the case of product distributions. How can we do similar things in the general case of arbitrary computable distributions on $\Omega_1 \times \Omega_2$? Again we start with a set Z of small conditional measure containing α_2 and represent Z as the α_1-section of some effectively open $U \subset \Omega_1 \times \Omega_2$. But trimming U is now not so easy; we need to trim enough to obtain a set of small P-measure, but we should keep the α_1-section untouched. To understand the problem better, let us first consider two simple approaches that do not work.

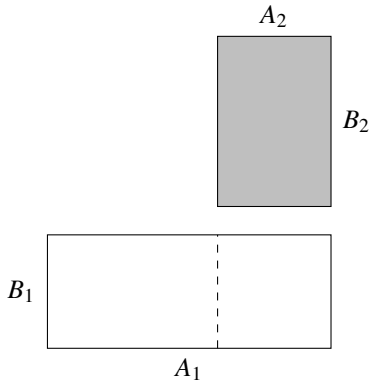
First non-working approach. The problem for the general case is that we have no "etalon" distribution on sections that can be used for trimming. It is natural to use the conditional distribution $P(\cdot|\alpha_1)$, and by assumption, there is an algorithm Γ that computes it using α_1 as oracle. Given some rectangle, we can split this rectangle horizontally (i.e., fix more and more bits of α_1) and use Γ to compute the conditional distribution with more and more precision for all the parts, letting through only the rectangles where the vertical side is guaranteed to have small (Γ-computed) measure for the values of α_1 that belong to the horizontal side.

The problem with this approach is that the conditional distribution is computable given α_1 *only on the α_1-section* but not elsewhere. So the algorithm Γ may be

unrelated to the distribution P outside this section. In this case small values produced by Γ do not guarantee anything about the measure of the rectangles that are let through, and we are in trouble.

Second non-working approach. Instead of using the tentative conditional probability provided by Γ, we may use the actual conditional probability when the condition is an interval; unlike the limit probability, it is computable. Imagine that we have some rectangle $A \times B$. Then we can compute the conditional probability of B with condition "the first coordinate is in A", i.e., the ratio $P(A \times B)/P_1(A)$, and let the rectangle through if this ratio is small (we assume that there are no earlier rectangles in the same vertical stripe). This guarantees that the P-measure of the rectangle is small; if we have several allowed rectangles with disjoint horizontal footprints, and for each of them this conditional probability is at most ε, then the P-measure of their union is also at most ε, since for each of them the P-measure is bounded by ε times the horizontal size of the rectangle, and the sum of horizontal sizes is at most 1.

What is the problem with this approach? (There should be a problem, since in this argument we do not use Γ, but we have to use it, as the counterexample above shows.) The problem is that the argument does not work if the horizontal footprints are not disjoint, as the following example shows.



For example, imagine that the set we want to trim contains some rectangle $A_1 \times B_1$. We compute the conditional probability $P(B_1|A_1) = P(A_1 \times B_1)/P_1(A_1)$. (Note that $P_1(A_1) = P(A_1 \times \Omega_2)$, so this conditional probability is the density of the rectangle in the vertical A_1-stripe, measured according to P.) We find that this conditional probability is slightly less than the threshold ε, so we let this rectangle $(A_1 \times B_1)$ through. Then we discover another rectangle $A_2 \times B_2$ where A_2 is a part of A_1, but B_2 is outside B_1 (as shown in the picture), and again find that $P(B_1 \cup B_2|A_2)$ is slightly less than ε. But if we let the second rectangle through, the average vertical measure of the resulting union may exceed ε. This could happen, for example, if all the mass in $A_1 \times B_1$ is concentrated outside $A_2 \times B_1$; then the conditional measure of B_1 exceeds ε outside A_2 and is zero inside A_2, thus leaving space for additional measure from B_2.

So the second approach also does not work. How can we deal with this problem?

Main idea: We combine the two approaches and always check (before adding something) that the actual conditional probability (with the interval as the condition)

is close to the tentative conditional probability computed by Γ. The latter will remain almost the same for smaller intervals (a valid computation remains valid when more information about the oracle is available), so the errors related to the change will be bounded.

Details. First we need to introduce some terminology and notation. We consider *basic* (=clopen) sets in $\Omega_1 \times \Omega_2$, i.e., finite unions of products of intervals. Every effectively open set is a union of an increasing computable sequence of basic sets. A basic set is a *rectangle* if it is the product of two clopen sets in Ω (not necessarily intervals). By a *vertical stripe* we mean a rectangle $S = I \times \Omega_2$, where I is some interval in Ω_1 (i.e., I consists of all extensions of some finite string). A basic set U is *stable in the stripe* $S = I \times \Omega_2$ if $U \cap S = I \times V$ for some $V \subset \Omega_2$. This means that all the vertical sections of U inside S are the same; we denote these sections by $U|S$.

The *horizontal size* $h(S)$ of a stripe $S = I \times \Omega$ is defined as $P_1(I)$ (and is equal to the P-measure of this stripe). If a set U is stable in the stripe S, its *vertical size in* S is defined as $P(U \cap S)/P(S)$, i.e., the conditional probability of $U|S$ with condition I. We denote the vertical size by $v(U|S)$. Note that the vertical size can increase or decrease if we replace S by a smaller stripe S' in S (and if it increases, say, for the left half of S, then it decreases for the right half); so "average vertical size" would be a better name for $v(U|S)$.

We want to trim an effectively open set U that is the union of an increasing computable sequence of basic sets

$$U_1 \subset U_2 \subset U_3 \subset \dots$$

Let us explain first which parts of U_1 will be let through. We divide Ω_1 into two stripes, then divide each stripe into two halves, and so on. We use the algorithm Γ to get the approximations for the tentative conditional probabilities for all stripes. Let us agree, for example, that for stripes S of level n (with footprints of length 2^{-n}) we always make n steps of the Γ-computation, using the first n bits of the oracle α_1 (i.e., the bits that are fixed for a stripe S) and produce some lower and upper bounds $\underline{P}^S(V)$ and $\overline{P}^S(V)$ for the tentative conditional probability of all intervals $V \subset \Omega_2$. Note that for a given V the interval $[\underline{P}^S(V), \overline{P}^S(V)]$ can only decrease as S becomes smaller. We know that these intervals should converge to $P(V|\alpha_1)$ as the length of the intervals S containing α_1 approaches zero; for other points (not α_1) the convergence is not guaranteed.

As soon as a stripe becomes small enough to make U_1 stable in this stripe, we compute the lower and upper bounds $\underline{v}(U_1|S)$ and $\overline{v}(U_1|S)$ for the vertical size $v(U_1|S)$ such that the difference between the lower and upper bounds is at most 2^{-n} for stripes of level n.[2] Unlike for $\underline{P}$ and $\overline{P}$, the interval $[\underline{v}(U_1|S), \overline{v}(U_1|S)]$ does not necessarily decrease as S becomes smaller: for a smaller stripe we may get an interval that is not a subset of the interval for a bigger stripe. Still the endpoints of these

[2] Since P is computable, we can compute $v(U_1|S)$ for each U_1 and S with arbitrary precision. The only exception is the case when $P(S) = 0$; to avoid this problem, let us agree that we start processing stripe S only after we discover that $P(S) > 0$. In this way we lose all stripes with $P(S) = 0$ but this does not matter since these stripes do not contain random pairs (α_1, α_2). (Recall that our goal was to prove that (α_1, α_2) is not random contrary to the assumption.)

intervals converge to the conditional probability of the α_1-section of U_1 if decreasing stripes around α_1 are used as S (recall that conditional probability is well defined for every P_1-random point in Ω_1).

If for some stripe S all the four numbers $\underline{P}^S(U_1|S)$, $\overline{P}^S(U_1|S)$, $\underline{v}(U_1|S)$, and $\overline{v}(U_1|S)$ are close to each other, more precisely, if all four can be covered by some interval of size δ_1 (where δ_1 is a small number, see below), and at the same time the upper bound $\overline{v}(U_1|S)$ is less than the threshold ε selected for trimming, we say that S is U_1-*good* and let U_1 through inside S. Note that smaller stripes may be U_1-good or not, but this does not matter at this stage, since U_1 is already let through inside S. If S is not U_1-good, we continue the process and consider two halves of S, checking whether they are good or not (making $n+1$ steps of the computation, using precision $2^{-(n+1)}$), etc.

In this way we get a trimmed version $\hat{U}_1 \subset U_1$. Formally speaking, $\hat{U}_1$ is the disjoint union of $U_1 \cap S$ for all maximal U_1-good stripes S. (We can take the union over all U_1-good stripes, but only maximal ones really matter.) The set $\hat{U}_1$ may not be a basic set, but it is effectively open. Before going further, let us prove some properties of this construction:

Lemma 3

(**a**) *Assume that some pair* (β_1, β_2) *is covered by* U_1, *the conditional probability* $P(\cdot|\beta_1)$ *is well defined and is computed by* Γ *with oracle* β_1, *and the* β_1*-section of* U_1 *has conditional measure* (*with condition* β_1) *less than* ε. *Then* (β_1, β_2) *is covered by* $\hat{U}_1$.

(**b**) *The* P*-measure of the trimmed set* $\hat{U}_1$ *is at most* ε.

Proof To prove (a), look at the smaller and smaller stripes that contain β_1. Starting from some point, U_1 is stable in these stripes, and the vertical size and tentative probabilities converge to some number smaller than ε. So they finally get into a δ_1-interval with both endpoints below ε. Therefore all small enough stripes containing β_1 are U_1-good, and the β_1-section of U_1 is not trimmed.

Proof of (b): In every U_1-good stripe S the vertical size $v(U_1|S)$ is less than ε, so the measure of U_1 inside this stripe is at most $\varepsilon h(S)$. Since all the maximal U_1-good stripes are disjoint, the sum of their horizontal sizes is bounded by 1. □

Now we switch to the next set U_2 (we should decide which part of it should remain after trimming). We consider U_2 only inside maximal U_1-good stripes selected at the first stage.[3] Let S be one of them. We start dividing S into smaller stripes; at some point they are small enough to make both U_1 and U_2 stable. Then we start checking if they are both U_1-good (according to the definition above) and U_2-*good*. The latter means that they satisfy the similar requirement for U_2 with some smaller error tolerance δ_2 (the four numbers for U_2 are in some δ_2-interval and the upper bound for the vertical size of U_2 in the stripe is less than ε). If a stripe S' inside S turns out to be both U_1-good and U_2-good, then the set U_2 is let through inside S',

[3] Note that a U_1-good stripe may have empty intersection with U_1, so this does not prevent us from adding some stripes that intersect U_2 but not U_1.

otherwise we continue the process and consider the halves of S'. So finally we have (inside S) the set

$$(S\cap U_1)\cup\bigcup_{S'}(S'\cap U_2)$$

where the union is taken over maximal stripes $S'\subset S$ that are both U_1- and U_2-good. Doing this for all maximal U_1-good stripes S, we get the trimmed version $\hat{U}_2$ of U_2. By construction $\hat{U}_1\subset\hat{U}_2\subset U_2$.

The key part of the proof is the following bound for the size of $\hat{U}_2$ inside a maximal U_1-good stripe S:

Lemma 4 *The P-measure of $\hat{U}_2\cap S$ is bounded by $(\varepsilon+2\delta_1)h(S)$.*

Then, summing up the inequalities provided by Lemma 4 for all maximal U_1-good stripes S (they are disjoint), we see that the total measure of $\hat{U}_2$ is bounded by $\varepsilon+2\delta_1$. (Note that $\hat{U}_2$ is contained in the union of maximal U_1-good stripes.)

Proof (of Lemma 4) The measure in question can be rewritten as

$$P(S\cap U_1)+\sum_{S'}P(S'\cap(U_2\setminus U_1))$$

(we separate points added during the first and second stages). This sum can be rewritten as

$$h(S)v(U_1|S)+\sum_{S'}h(S')[v((U_2\setminus U_1)|S')]$$

or

$$h(S)v(U_1|S)+\sum_{S'}h(S')v(U_2|S')-\sum_{S'}h(S')v(U_1|S').$$

Imagine for the moment that in the last term the condition is S, not S'. Then we could combine the first and last term and get

$$\left(h(S)-\sum_{S'}h(S')\right)v(U_1|S)+\sum_{S'}h(S')v(U_2|S'). \qquad (*)$$

The factors $v(U_1|S)$ and $v(U_2|S')$ are bounded by ε (for all S' that are U_1-U_2-good), and the sum of horizontal sizes is just $h(S)$, so the statement of the lemma would be true even without $2\delta_1$-term. This term compensates for the error that we made. Indeed, the difference between $v(U_1|S)$ and $v(U_1|S')$ is bounded by $2\delta_1$, and the sum of all $h(S')$ is at most $h(S)$. To see why the difference is bounded, note that the interval between the lower and upper approximations $\underline{P}$, $\overline{P}$ only decreases when we switch from S to S', and both sizes $v(\cdot|S)$ and $v(\cdot|S')$ are in a δ_1-neighborhood of every point in the smaller interval (that corresponds to S'). □

Let us look again at this argument and introduce some notation that will be useful for the next stages. The second term in $(*)$ is the size of U_2 inside U_1-U_2-good stripes S' (that are in a given maximal U_1-good stripe S), while the first term is an approximate (the difference is bounded by $2\delta_1 h(S)$) bound for the size of U_1 inside S but outside maximal U_1-U_2-good stripes $S'\subset S$.

We can sum up these bounds for all maximal U_1-good stripes S. Let G_1 be the union of these stripes, and let G_2 be the union of maximal U_1-U_2-good stripes (note that $G_2 \subset G_1$). Then $\hat{U}_2$ is empty outside G_1, coincides with U_1 inside $G_1 \setminus G_2$, and coincides with U_2 inside G_2. The bounds for the size of $\hat{U}_2$ in the last two cases are $\varepsilon P(G_1 \setminus G_2) + 2\delta_1 h(G_2)$ and $\varepsilon P(G_2)$ respectively.

We have proved a statement similar to Lemma 3 (b), for the second step of the construction. Now we extend the statement (a) of that lemma in the same way. Assume that (1) (β_1, β_2) is covered by U_2; (2) the conditional probability $P(\cdot|\beta_1)$ with condition β_1 is well defined and the β_1-conditional size of the β_1-section of U_2 is computed by Γ with oracle β_1; (3) this size is less than ε. Then we claim that (β_1, β_2) is covered by the trimmed set $\hat{U}_2$. Indeed, consider smaller and smaller stripes containing β_1. Starting from some point, both sets U_1 and U_2 are stable in those stripes, all the approximations converge (both for U_1 and U_2), and the limits are less than ε. So at some stage the stripes become both U_1- and U_2-good, and at this moment (β_1, β_2) is covered (unless this happened earlier).

Now we consider the next set U_3. The same construction is used: consider the maximal U_1-U_2-good stripes (they are considered during the second stage). For each of them we look for stripes inside that are both U_2-good and U_3-*good* (the latter means that the set U_3 is stable, all four parameters are δ_3-close for a suitable δ_3, see below, and the upper bound for the vertical size is less than ε). Then we do the same thing as before, but not for U_1-U_2-good stripes inside some U_1-good one, but for U_2-U_3-good stripes inside some maximal U_1-U_2-good one. The same approach is used for U_4, U_5, etc.

In this way we get the set G_3 that is the union of maximal U_2-U_3-good sub-stripes inside maximal U_1-U_2-good stripes considered earlier. Then $U_3 \cap G_3$ is added to $\hat{U}_2$ to obtain $\hat{U}_3$. In the next stage we define G_4 and let $\hat{U}_4 = (U_4 \cap G_4) \cup \hat{U}_3$, etc. The same reasoning as in the proof of Lemma 4 gives us the following bounds:

Lemma 5

- $P(U_{i-1} \cap (G_{i-1} \setminus G_i)) \le \varepsilon P(G_{i-1} \setminus G_i) + 2\delta_{i-1} P(G_i)$;
- $P(U_i \cap G_i) \le \varepsilon P(G_i)$.

What have we achieved? We explained how to trim the set U_k for each k and get $\hat{U}_k \subset U_k$. The union $\hat{U} = \bigcup_k \hat{U}_k$ is the trimmed version of the effectively open set U we started with. In other words, $\hat{U}$ coincides with U_{i-1} inside $G_{i-1} \setminus G_i$ and coincides with U in $\cap_i G_i$. What are the properties of this $\hat{U}$?

- The trimming procedure is effective: the set $\hat{U}$ is effectively open uniformly in U. This is guaranteed by the construction.
- The P-measure of $\hat{U}$ can be made arbitrarily small. Indeed, for each k the measure of $\hat{U}_k$ is bounded by $\varepsilon + 2\sum_i \delta_i$ (sum up the first bounds from Lemma 5 for $i = 2, \ldots, k$ and the second bound for $i = k$). Then we note that one can choose computable δ_i such that $\sum \delta_i < \varepsilon$; doing this, we conclude that $P(\hat{U}) \le 3\varepsilon$, since $P(G_i) \le 1$ for all i.
- Assume that the conditional probability is well defined for some condition β_1 and is computed by Γ with oracle β_1. Assume also that the β_1-section of U has

conditional measure less than ε and contains some β_2. Then $(\beta_1,\beta_2)\in\hat{U}$. Indeed, (β_1,β_2) belongs to some U_i and (under the conditions mentioned) belongs to $\hat{U}_i$ as explained above.

Then the proof of Theorem 5 ends in the same way as for the standard van Lambalgen's theorem: if (α_1,α_2) is P-random, the first coordinate α_1 is P_1-random, the conditional probability $P(\cdot|\alpha_1)$ is well defined, and our assumption says that it is computed by Γ with oracle α_1. If α_2 is not (Martin-Löf) random with oracle α_1 with respect to the conditional probability, we consider a Martin-Löf test with oracle α_1 for the distribution $P(\cdot|\alpha_1)$ rejecting α_2, represent its elements as α_1-sections of a sequence of uniformly effectively open subsets of $\Omega_1\times\Omega_2$ with small α_1-sections, and trim each of these sets as described above. This gives us a Martin-Löf test with respect to distribution P that rejects (α_1,α_2), which contradicts the assumption. □

7 A quantitative version for uniformly computable conditional probabilities

In the previous section we started with a computable probability distribution P on $\Omega_1\times\Omega_2$, and then defined a conditional distribution on Ω_2. However, in many cases the natural order could be different: we first generate a sequence ω randomly according to some distribution P_1 on Ω_1, and then generate ω' randomly according to some distribution P^ω on Ω_2 that depends on ω. If the dependence of P^ω on ω is computable (and therefore continuous), then we get some computable distribution P on $\Omega_1\times\Omega_2$. It is easy to check that for P the conditional probabilities indeed coincide with P^ω, so we can apply the results from the preceding section. But in this special case the argument could be easier, and a stronger quantitative version of Theorem 5 could be obtained (as shown by Vovk and Vyugin in [12, Theorem 1, page 261], though in somehow obscure notation).

To state this quantitative version, we need to use the notion of randomness deficiency. More precisely, we use the notion of expectation-bounded randomness deficiency (see [3] for the details). In other words, to define the randomness deficiency for a computable distribution P_1 on Ω_1 we consider the maximal (up to $O(1)$-factor) lower semicomputable function t on Ω_1 with non-negative real values (including $+\infty$) such that

$$\int_{\Omega_1} t(\omega)\,dP_1(\omega)\le 1.$$

One can prove (see [3]) that such a function exists. We denote this maximal function by $\mathbf{t}_{P_1}(\omega)$; the value $\mathbf{t}_{P_1}(\omega)$ is finite for P_1-random ω and infinite otherwise. Then we switch to logarithmic scale and define deficiency as $\mathbf{d}_{P_1}(\omega)=\log\mathbf{t}_{P_1}(\omega)$.

In a similar way one can define randomness deficiency for pairs with respect to a computable distribution P on pairs: it is the logarithm of the maximal lower semicomputable function $t(\omega,\omega')$ on $\Omega_1\times\Omega_2$ such that

$$\iint_{\Omega_1\times\Omega_2} t(\omega,\omega')\,dP(\omega,\omega')=\int_\omega\int_{\omega'} t(\omega,\omega')\,dP^\omega(\omega')\,dP_1(\omega)\le 1.$$

We denote this maximal function by $\mathbf{t}_P(\omega,\omega')$ and its logarithm by $\mathbf{d}_P(\omega,\omega')$.

We need one more variant of randomness deficiency which is a bit more complicated. We want to measure the randomness deficiency of ω' with respect to the distribution P^ω given some additional information as oracle. This additional information is ω itself and some integer (its role will be explained later). We can use the general definition of uniform deficiency (as a function of a sequence and a distribution, see [3]), but let us give an equivalent definition for this special case. A lower semicomputable function $t(\omega', \omega, k)$ of three arguments (ω' and ω are sequences, k is an integer) is called a test if

$$\int_{\omega'} t(\omega', \omega, k)\, dP^\omega(\omega') \le 1$$

for every ω and k. There exists a maximal test, as usual: we may trim all the lower semicomputable functions making them tests, and then take their sum with coefficients $1/2^n$ (other computably converging series can also be used). Trimming is easy since ω is an argument and P^ω is computable given ω (uniformly for all ω, according to our assumption). We denote the maximal test by $\mathbf{t}_{P^\omega}(\omega' | \omega, k)$ and its logarithm by $\mathbf{d}_{P^\omega}(\omega' | \omega, k)$.

Now we can state the Vovk–Vyugin result:

Theorem 6

$$\mathbf{d}_P(\omega, \omega') = \mathbf{d}_{P_1}(\omega) + \mathbf{d}_{P^\omega}(\omega' | \omega, \mathbf{d}_{P_1}(\omega)) + O(1).$$

In this statement we assume that the value of $\mathbf{d}_{P_1}(\omega)$ in the condition is rounded to an integer; the exact nature of rounding does not matter since it may change the deficiency only by $O(1)$.

Again, before proving this theorem, let us make some remarks:

- This result has high precision (it holds up to an $O(1)$ additive term); if we were satisfied with logarithmic precision, we could omit $\mathbf{d}_{P_1}(\omega)$ in the condition. Indeed, the standard argument shows that adding a condition d could increase the deficiency at most by $O(\log d)$ and decrease it at most by $O(1)$.
- It is easy to see that $\mathbf{d}_{P^\omega}(\omega' | \omega, d)$ is finite if and only if ω' is random with respect to distribution P^ω with oracle ω: as we have mentioned, adding the condition d changes the deficiency at most by $O(\log d)$, so [in]finite values remain [in]finite. So we get a qualitative version: (ω, ω') is P-random if and only if ω is P_1-random and ω' is P^ω-random with oracle ω. (This statement generalizes van Lambalgen's theorem and is a special case of Theorem 5 from the previous section.)
- A special case of this statement, when P^ω does not depend on ω and is always equal to some computable distribution P_2, gives a quantitative version of van Lambalgen's theorem for the product distribution $P_1 \times P_2$.
- One can consider a finite version of this theorem. If x is a constructive object, e.g., a string or a pair of strings, and A is a finite set containing x (so A is also a constructive object), we may define the *randomness deficiency of x as an element of A* in the following way:

$$d(x|A) = \log |A| - K(x|A),$$

where $K(x|A)$ is the conditional prefix complexity of x given A. It is easy to check that $d(x|A)$ is positive (up to $O(1)$-error) and that it can also be defined as a logarithm of maximal lower semicomputable function $t(x,A)$ of two arguments (x is an object, A is a finite set) such that

$$\sum_{x\in A} t(x,A) \le 1$$

for each finite set A. Note that we do not define $d(x|A)$ for $x \notin A$; one may also agree that $d(x|A) = +\infty$ in this case. We can also define randomness deficiency with an additional condition as

$$d(x|A;y) = \log|A| - K(x|A,y).$$

Then we can state the following equality for the deficiency of a pair (with $O(1)$-precision):

$$d((x,y)|A\times B) = d(x|A;B) + d(y|B;x,A,d(x|A;B)).$$

It is just the Levin–Gacs formula for the complexity of pairs in disguise. Indeed, this statement can be rewritten (with $O(1)$-precision) as

$$\begin{aligned}\log|A\times B| - K(x,y|A,B) = \\ = \log|A| - K(x|A,B) + \log|B| - K(y|B,x,A,\log|A| - K(x|A,B)).\end{aligned}$$

The logarithms cancel each other and we have (with $O(1)$-precision)

$$K(x,y|A,B) = K(x|A,B) + K(y|B,x,A,\log|A| - K(x|A,B)).$$

which is just the Levin–Gacs theorem about prefix complexity of a pair (note that $\log|A|$ in the condition does not matter since A is there anyway).
The proof below can also be adapted to the finite case.[4]

Proof (of Theorem 6) We need to prove two inequalities. In each case, we construct some test and compare it with the maximal one.

We start with the $\ge$-direction, proving that $d(\omega,\omega')$ is large enough. The function

$$T(\omega,\omega') = \mathbf{t}_{P_1}(\omega)\cdot \mathbf{t}_{P^\omega}(\omega'|\omega,\mathbf{d}_{P_1}(\omega))$$

(where $\mathbf{d}_{P_1}(\omega)$ in the argument is rounded) has integral at most 1 with respect to distribution P. Indeed,

$$\begin{aligned}\int_\omega\int_{\omega'} \mathbf{t}_{P_1}(\omega)\cdot \mathbf{t}_{P^\omega}(\omega'|\omega,\mathbf{d}_{P_1}(\omega))\,dP^\omega(\omega')\,dP_1(\omega) = \\ = \int_\omega \mathbf{t}_{P_1}(\omega)\int_{\omega'} \mathbf{t}_{P^\omega}(\omega'|\omega,\mathbf{d}_{P_1}(\omega))\,dP^\omega(\omega')\,dP_1(\omega) \le \int_\omega \mathbf{t}_{P_1}(\omega)\,dP_1(\omega) \le 1\end{aligned}$$

[4] It would be interesting to derive the statement of the infinite theorem using the formula for expectation-bounded deficiency in terms of prefix complexity and the formula for the complexity of pairs, but it is not clear how (and if) this can be done.

(first we use that $\mathbf{t}_{P^\omega}$ is a test, and then we use that $\mathbf{t}_{P_1}$ is a test). One would like to say that this test T is bounded by the maximal test $\mathbf{t}_P$, but the problem is that the function T is not guaranteed to be a test: it may not be lower semicomputable, since it uses $\mathbf{d}_{P_1}(\omega)$ as a condition. To avoid this problem, we consider a bigger function

$$T'(\omega,\omega') = \sum_{k<\mathbf{d}_{P_1}(\omega)} 2^k \mathbf{t}_{P^\omega}(\omega'|\omega,k).$$

It is indeed bigger (up to $O(1)$-factor) since the last term in the sum coincides with T up to $O(1)$-factor. This function is lower semicomputable since the property $k<\mathbf{d}_{P_1}(\omega)$ is effectively open in the natural sense, and $\mathbf{t}_{P^\omega}$ is lower semicomputable. The integral is bounded not only for T but also for T':

$$\iint \left[\sum_{k<\mathbf{d}_{P_1}(\omega)} 2^k \mathbf{t}_{P^\omega}(\omega'|\omega,k)\right] dP^\omega(\omega')\, dP_1(\omega) =$$

$$= \int_\omega \left[\sum_{k<\mathbf{d}_{P_1}(\omega)} 2^k \int_{\omega'} \mathbf{t}_{P^\omega}(\omega'|\omega,k)\, dP^\omega(\omega')\right] dP_1(\omega) \le \int_\omega \left[\sum_{k<\mathbf{d}_{P_1}(\omega)} 2^k\right] dP_1(\omega) \le$$

$$\le O(1)\cdot \int_\omega 2^{\mathbf{d}_{P_1}(\omega)}\, dP_1(\omega) = O(1)\cdot \int_\omega \mathbf{t}_{P_1}(\omega)\, dP_1(\omega) = O(1).$$

Here we use that the sum of different powers of 2 coincides with its biggest term up to an $O(1)$-factor. So we have found a function T' that is lower semicomputable and is a test, so T' and therefore T are bounded by the maximal P-test, which gives the required inequality.

Now we have to prove the reversed ($\le$) inequality. For that we consider the maximal P-test $\mathbf{t}_P(\omega,\omega')$ and the maximal P_1-test $\mathbf{t}_{P_1}(\omega)$. The function

$$\omega \mapsto \int_{\omega'} \mathbf{t}_P(\omega,\omega')\, dP^\omega(\omega')$$

is lower semicomputable and its integral with respect to distribution P_1 is at most 1, therefore this function is bounded by $O(1)\cdot \mathbf{t}_{P_1}(\omega)$. So the ratio

$$t(\omega',\omega) = \mathbf{t}_P(\omega,\omega')/\mathbf{t}_{P_1}(\omega)$$

has bounded integral over ω' (with respect to P^ω) for each ω (and the bound does not depend on ω). If t were a test, we could compare t with the maximal test $\mathbf{t}_{P^\omega}(\omega'|\omega)$ and get the desired inequality. But again the function t may not be lower semicomputable, since it has a lower semicomputable function in the denominator.

This is why we need an additional argument d for the test function. Namely, we consider

$$t(\omega'|\omega,d) = [\mathbf{t}_P(\omega,\omega')/2^{d+c}],$$

where the square brackets denote trimming that make this function a test (the integral over ω' for each ω and d should be bounded by 1, and the trimming should not change $t(\omega'|\omega,d)$ if for this pair (ω,d) the integral was already bounded by 1). The constant c should be chosen in such a way that for $d=\mathbf{d}_{P_1}(\omega)$ trimming is not needed; this is possible due to the argument above.

It remains to compare this test with the maximal one and note that for this test the required inequality is true by construction. □

There is another generalization of van Lambalgen's theorem. In the previous results we considered only computable distributions. However, one can define a *uniform randomness test* as a lower semicomputable function $t(\omega, P)$ of two arguments (where ω is a sequence, and P is a probability distribution on Cantor space) such that

$$\int_{\Omega} t(\omega, P)\, dP(\omega) \leq 1$$

for every P. Note that we should first define the notion of semicomputability for functions whose arguments are distributions; this can be done in a natural way (even for arguments in an arbitrary constructive metric space, see [3] for the details). There exists a universal uniform test $\mathbf{t}(\omega, P)$ and its logarithm $\mathbf{d}(\omega, P) = \log \mathbf{t}(\omega, P)$ is called *uniform randomness deficiency*. For computable P the value of $\mathbf{d}(\omega, P)$ coincides with $\mathbf{d}_P(\omega)$ up to an additive constant that depends on P. Also we can generalize this definition by allowing additional conditions (strings, oracles or even points in constructive metric spaces); we list these conditions after a semicolon, so $\mathbf{d}(\omega | P; \alpha, k)$ is the uniform randomness deficiency of ω with respect to distribution P with additional conditions α and k.

One could prove that

$$\mathbf{d}((\omega_1, \omega_2) | P_1 \times P_2) = \mathbf{d}(\omega_1 | P_1; P_2) + \mathbf{d}(\omega_2 | P_2; P_1, \omega_1, \mathbf{d}(\omega_1 | P_1; P_2)) + O(1);$$

it would be interesting to combine this generalization with the Vovk–Vyugin result (where P_2 is not fixed, but depends on ω_1). One possibility is to assume that P_1 is a computable function of some parameter p (a point in a constructive metric space). For example, we may let $P_1 = p$, so P_1 itself may be used as such a parameter. Then we assume that $P_2^{\omega}(\cdot)$ is a computable function of p and ω, so the distribution on pairs also becomes a computable function of p. In this case we get the equality

$$\mathbf{d}((\omega_1, \omega_2) | P_1; p) = \mathbf{d}(\omega_1 | P_1; p) + \mathbf{d}(\omega_2 | P_2^{\omega_1}; p, \omega_1, \mathbf{d}(\omega_1 | P_1; p)) + O(1),$$

where the $O(1)$-constant does not depend on p.

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