TOPOLOGICAL CLASSIFICATION OF MORSE–SMALE DIFFEOMORPHISMS ON 3-MANIFOLDS

C. BONATTI, V. GRINES, and O. POCHINKA

Abstract

The topological classification of even the simplest Morse–Smale diffeomorphisms on 3*-manifolds does not fit into the concept of singling out a skeleton consisting of stable and unstable manifolds of periodic orbits. The reason for this lies primarily in the possibility of "wild" behavior of separatrices of saddle points. Another difference between Morse–Smale diffeomorphisms in dimension* 3 *and their surface analogues lies in the variety of heteroclinic intersections: a connected component of such an intersection may not be only a point, as in the* 2*-dimensional case, but also a curve, compact or noncompact. The problem of topological classification of Morse–Smale cascades on* 3*-manifolds either without heteroclinic points (gradient-like cascades) or without heteroclinic curves was solved in a series of papers from 2000 to 2016 by C. Bonatti, V. Grines, F. Laudenbach, V. Medvedev, E. Pecou, and O. Pochinka. The present article is devoted to completing the topological classification of the set* $MS(M^3)$ *of orientation-preserving Morse–Smale diffeomorphisms* f *on a smooth closed orientable* 3*-manifold* M^3*. The complete topological invariant for a diffeomorphism* $f \in MS(M^3)$ *is the equivalence class of its scheme* S_f *which contains information on the periodic data and the topology of embedding of* 2*-dimensional invariant manifolds of the saddle periodic points of* f *into the ambient manifold.*

Contents

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1. Introduction and formulation of the results

1.1. Introducing of the invariants

Among dynamical systems the Morse–Smale systems represent the simplest evolution imaginable: every orbit goes to a (periodic) equilibrium point and comes (if it is not an equilibrium point itself) from another periodic point. More precisely, a diffeomorphism is called *Morse–Smale* if it has finitely many periodic points, all of them hyperbolic, the stable and the unstable manifolds of any two periodic points are transverse, and every point in the manifold lies in both an unstable manifold and a stable manifold. The importance of Morse–Smale systems comes from the fact that they are *structurally stable* (see [26], [28]): the dynamics remains unchanged (i.e., conjugated to itself) under small perturbation in the C^1-topology.

The time-1 map of the gradient flow of a generic Morse function is a typical example of a Morse–Smale diffeomorphism (the genericity is needed to get the transversality condition). One could therefore hope that the manifolds and the dynamics are characterized by simple combinatorial information about the periodic orbits and the positions of the invariant manifolds. That is indeed the case for Morse–Smale vector fields on compact surfaces where a complete description and a classification (up to topological equivalence) has been obtained by Peixoto [29] as a formalization of the Leontovich–Mayer scheme for 2-spheres (see [24]).

The problem is substantially more complicated in the case of diffeomorphisms. In particular, Morse–Smale diffeomorphisms (while dynamically as "simple as possible") are not necessarily embedded into a flow, even in a topological flow: the simplest obstruction is the existence of the points of transverse intersection of stable and unstable manifolds of complementary dimensions. These *heteroclinic intersection points* lead to the main difficulty for classification of the Morse–Smale diffeomorphisms on surfaces. Nevertheless, A. Bezdenezhnykh and V. Grines (see [2], [3], [15]) proved that they admit an invariant similar to Peixoto's graph in the case of finitely many heteroclinic orbits (see also [18] and [20] for surveys on topological classification of Morse–Smale diffeomorphisms on surfaces). Indeed, a Morse–Smale diffeomorphism on a compact surface may have infinitely many heteroclinic orbits which cut the stable and unstable manifolds into infinitely many orbits of segments; the relative topological position of these segments in the surface is a topological conjugacy invariant of the diffeomorphisms. These infinitely many segments seem to follow a finite pattern. After a long series of papers, C. Bonatti and R. Langevin in [11] provided a complete finite combinatorical invariant for Morse–Smale diffeomorphisms of surfaces, as well as structurally stable diffeomorphisms with nontrivial basic sets (see also the paper by V. Grines [16] devoted to classification of structurally stable diffeomorphisms with 1-dimensional basic sets).

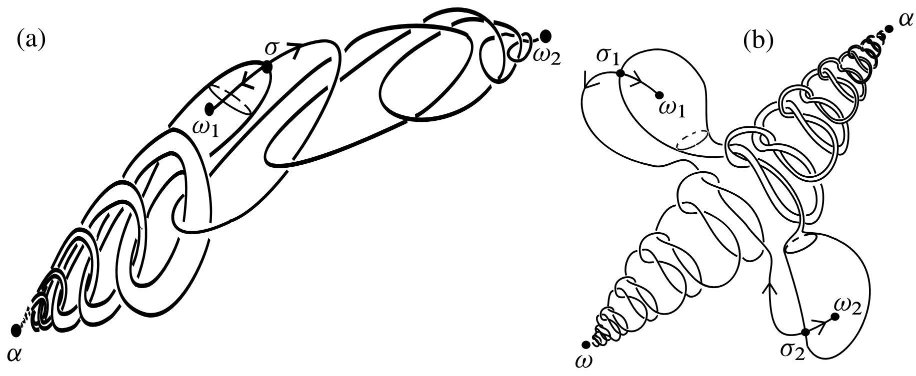

Figure 1. Morse–Smale diffeomorphisms with wild separatrices on the 3-sphere: (a) the separatrices of the saddle point σ form an Artin–Fox arc with wild points at the source α and at the sink ω_2; (b) the separatrices of the saddle points σ_1, σ_2 form a Debruner–Fox frame with wild points at the source α and at the sink ω.

Even if a Morse–Smale diffeomorphism has no heteroclinic intersections in dimension $n > 2$ that does not mean that it can be embedded into a topological flow (see [17] for a characterization of this phenomenon; see also [4]). Thus, the gluing of the dynamics from one periodic point to the next is not necessarily given by a flow, and can be very interesting from the topological viewpoint (see Figure 1).

In view of the complexity of classifying Morse–Smale diffeomorphisms on surfaces and in view of the new topological behaviors allowed in dimensions ≥ 3, a classification of Morse–Smale diffeomorphisms in dimension 3 could appear hopeless. Nevertheless, below we describe a general approach to the classification of the dynamics of Morse–Smale diffeomorphisms in dimension 3.

The most simple among the Morse–Smale diffeomorphisms is the *source-sink* diffeomorphism whose nonwandering set consists of exactly two fixed points: a source and a sink. The source-sink diffeomorphisms have trivial dynamics: all the nonfixed points are wandering, and, under the action of the diffeomorphism, they move from the source to the sink. The ambient space (the manifold M^3 on which the diffeomorphism is given) for such system is the 3-sphere $\mathbb{S}^3$. So the wandering set of such a system is homeomorphic to $\mathbb{S}^2 \times \mathbb{R}$ (3-sphere without two points). Then the result that all source-sink 3-diffeomorphisms are topologically conjugate follows immediately from the fact that the spaces of their wandering orbits are homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ (in the orientation-preserving case).

The next simplest case in dimension 3 is the orientation-preserving Morse–Smale diffeomorphism whose nonwandering set consists of exactly four fixed points: two sinks ω_1, ω_2, one saddle point σ with a 1-dimensional unstable manifold, and a source

α (see Figure 1(a)). Classification of such diffeomorphisms is the aim of [5]. Note that the dynamics still looks like a source-sink dynamics, but the sink has been replaced by the invariant compact segment $\{\omega_1\} \cup W^u_\sigma \cup \{\omega_2\}$, which is the attracting set. This segment can be wildly knotted, as shown in Figure 1(a). The wandering set, being the complement of the attractor-repeller pair, coincides with the punctured basin $W^u_\alpha \setminus \{\alpha\}$. Then this wandering orbit space is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ again. The projection to this space of the punctured stable manifold $W^s_\sigma \setminus \{\sigma\}$ gives the 2-torus which is the boundary of a tubular neighborhood of a knot γ. This means that there is a one-to-one correspondence between the topological conjugacy classes of such Morse–Smale diffeomorphisms f and the knots γ in $\mathbb{S}^2 \times \mathbb{S}^1$ in the homotopy class of the $\mathbb{S}^1$ factor up to homeomorphisms acting trivially on $\mathbb{Z} = H_1(\mathbb{S}^2 \times \mathbb{S}^1)$.

Another simple case consists of the Morse–Smale diffeomorphisms f on a closed 3-manifold M^3 which are the time-1 maps of the gradient X of a generic Morse function. In that case, Smale's result in [34] implies the existence of a closed (connected) surface Σ_f transverse to X dividing M^3 into two components: one repelling component and one attracting component; furthermore, all the sources and all the saddles with 1-dimensional stable manifolds are in the repelling component, whereas all the sinks and all the saddles with 1-dimensional unstable manifolds are in the attracting component. Here the dynamics is analogous to the source-sink, the closure of the 1-dimensional stable (unstable) manifolds playing the role of source (sink).

Then, the surface Σ_f intersects transversally each 2-dimensional stable (resp., unstable) manifold of the saddles and the intersection is exactly one circle. Thus, Σ_f is equipped with a family $\mathcal{C}^s_f$ of disjoint stable circles, a family $\mathcal{C}^u_f$ of disjoint unstable circles, and these two families are transverse. Then $(\Sigma_f, \mathcal{C}^s_f, \mathcal{C}^u_f)$ is a complete invariant of topological conjugacy: the triple $(\Sigma_f, \mathcal{C}^s_f, \mathcal{C}^u_f)$ does not depend, up to homeomorphism, on the choice of the transverse surface Σ_f, and two such diffeomorphisms f, f' are conjugated if and only if $(\Sigma_f, \mathcal{C}^s_f, \mathcal{C}^u_f)$ is homeomorphic to $(\Sigma_{f'}, \mathcal{C}^s_{f'}, \mathcal{C}^u_{f'})$ (see, e.g., [31]).

When one studies an arbitrary Morse–Smale diffeomorphism $f : M^3 \to M^3$ on a closed 3-manifold M^3, one sees that the dynamics looks similar. More precisely, one considers the attractor A_f (resp., repeller R_f) as the closure of all 1-dimensional unstable W^u_p (resp., stable W^s_p) manifolds of saddle points p. That is,

$$A_f = \operatorname{cl}\Big(\bigcup_{\dim W^u_p = 1} W^u_p\Big), \qquad R_f = \operatorname{cl}\Big(\bigcup_{\dim W^s_p = 1} W^s_p\Big).$$

Let

$$V_f = M^3 \setminus (A_f \cup R_f).$$

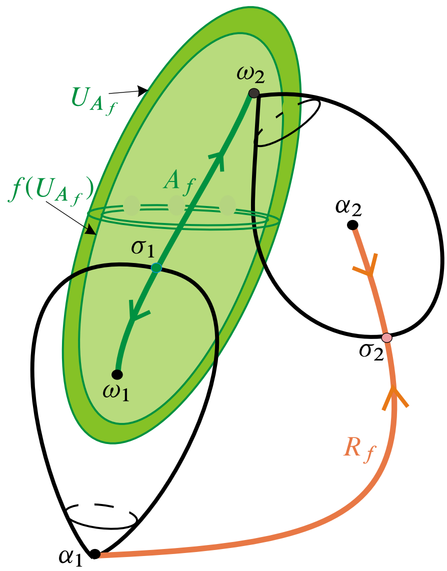

Figure 2. A representation of the Morse–Smale dynamics as a source-sink.

One can check that A_f (R_f) is either a connected 1-dimensional lamination in M^3 or a sink (resp., source) in the exceptional case where there are no saddle points with 1-dimensional unstable (resp., stable) manifolds. These 1-dimensional laminations do not divide the manifold M^3. Thus, the set V_f is a connected 3-manifold. The factor space $\hat{V}_f = V_f/f$ is obtained by taking a trapping neighborhood U_{A_f} in the basin of the attractor A_f and identifying the boundaries of the fundamental domain $U_{A_f} \setminus \operatorname{int} U_{A_f}$ by f (see Figure 2). This means that its orbit space $\hat{V}_f$ is a closed connected manifold. Moreover, in [12] it is proved that $\hat{V}_f$ is a *prime manifold*, that is, a closed orientable 3-manifold which is either homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ or irreducible (any smooth 2-sphere bounds a 3-ball there).

The closed manifold $\hat{V}_f$ is our *first conjugacy invariant*.[1]

We denote by

$$p_f : V_f \to \hat{V}_f$$

the natural projection. Note that p_f is a cyclic cover whose deck transformation group is generated by f. Such a cyclic cover is associated to an epimorphism η_f :

[1]In this approach, the dimension 3 plays a specific role. In particular, in dimensions 1 and 2 there is no such canonical choice for the triple A_f, R_f, V_f where all elements are connected.

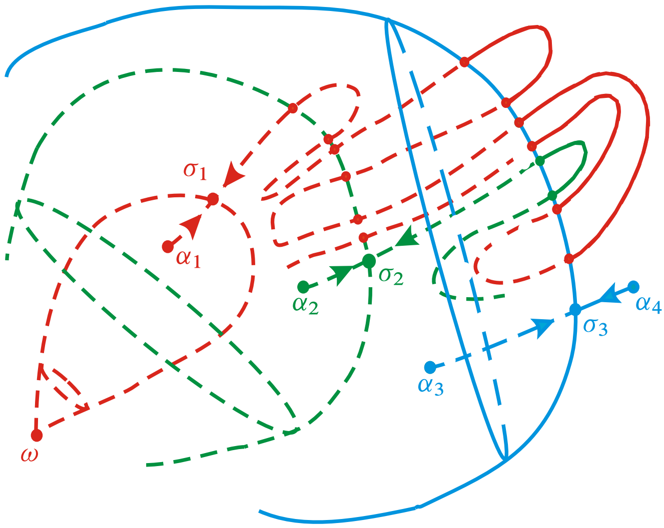

Figure 3. Phase portrait of a Morse–Smale diffeomorphism with heteroclinic points on a 3-manifold.

$\pi_1(\hat{V}_f) \to \mathbb{Z}$ so that $[\hat{c}] \mapsto f^{\eta_f([\hat{c}])}$ is the natural representation of $\pi_1(\hat{V}_f)$ in the deck transformation group.

Then, $\eta_f \in H^1(\hat{V}_f, \mathbb{Z})$ is our *second conjugacy invariant.*

Our next invariants consist of the projection to $\hat{V}_f$ of the 2-dimensional invariant (stable or unstable) manifolds of the periodic saddle. Denote by

$$\Gamma^u_f$$

the intersection with V_f of the 2-dimensional unstable manifolds of the saddle points of f. It is an invariant 2-dimensional lamination with finitely many leaves which is closed in V_f. Each leaf of this lamination is obtained by removing from the unstable manifold its set of intersection points with the 1-dimensional stable manifold; this intersection is at most countable (see Figure 3).

As Γ^u_f is invariant under f, it projects to the quotient in a compact 2-dimensional lamination $\hat{\Gamma}^u_f$ on $\hat{V}_f$. Note that each 2-dimensional unstable manifold is a plane on which f acts as an extension, so that the quotient by f of the punctured unstable manifold is either a punctured torus or a punctured Klein bottle. Thus, the leaves of $\hat{\Gamma}^u_f$ are either tori or Klein bottles punctured at most countable set.

To represent an embedding of a punctured surface, consider a dynamical system whose phase portrait is shown on Figure 3. For this diffeomorphism A_f is the sink ω

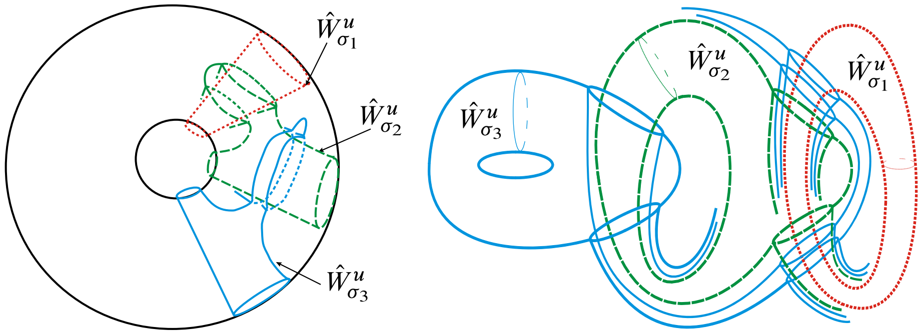

Figure 4. The u-lamination of the diffeomorphism f from Figure 3.

and R_f is the closure of $W^s_{\sigma_1}$, where fixed saddle points $\sigma_1, \sigma_2, \sigma_3$ satisfy the following conditions: $(W^u_{\sigma_1} \setminus \sigma_1) \cap R_f = \emptyset$; $(W^u_{\sigma_2} \setminus \sigma_2) \cap R_f$ consists of a finite number of heteroclinic orbits; and $(W^u_{\sigma_3} \setminus \sigma_3) \cap R_f$ consists of countably many heteroclinic orbits. The quotient $\hat{V}_f$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, and $\hat{W}^u_{\sigma_1} \cup \hat{W}^u_{\sigma_2} \cup \hat{W}^u_{\sigma_3}$ forms a u-lamination (see Figure 4).

For an arbitrary Morse–Smale diffeomorphism $f : M^3 \to M^3$, we get two transversally intersected laminations: the s-lamination $\hat{\Gamma}^s_f$ (as the quotient by f of the intersection with V_f of the 2-dimensional stable manifolds) and the u-lamination $\hat{\Gamma}^u_f$ on $\hat{V}_f$, each leaf of which is either the torus or the Klein bottle with empty, finite, or countable set of punctured points.

The laminations $\hat{\Gamma}^s_f$ and $\hat{\Gamma}^u_f$ are our *last conjugacy invariants*.

1.2. The exact formulation of the results

Let $MS(M^3)$ be a set of orientation-preserving Morse–Smale diffeomorphisms f given on smooth closed orientable 3-manifolds M^3 and $f \in MS(M^3)$. Let Ω_f be the nonwandering set of f. For $q = 0, 1, 2, 3$ denote by Ω_q the set of all periodic points of f with q-dimensional unstable manifolds. Thus

$$\Omega_f = \bigcup_{g=0}^{n} \Omega_q.$$

Let us represent the dynamics of f in a source-sink form in the following way. Let $A_f = W^u_{\Omega_0 \cup \Omega_1}$, $R_f = W^s_{\Omega_2 \cup \Omega_3}$, and $V_f = M^3 \setminus (A_f \cup R_f)$. Then the set $A_f(R_f)$ is a connected attractor (resp., repeller)[2] of f with topological dimension at most 1,

[2] A compact set $A \subset M^n$ is *an attractor of a diffeomorphism* $f : M^n \to M^n$ if there is a neighborhood U of the set A such that $f(U) \subset \operatorname{int} U$ and $A = \bigcap_{n\in\mathbb{N}} f^n(U)$. A set $R \subset M^n$ is called *a repeller* of f if it is an attractor of f^{-1}.

the set V_f is a connected 3-manifold, and $V_f = W^s_{A_f \cap \Omega_f} \setminus A_f = W^u_{R_f \cap \Omega_f} \setminus R_f$. Moreover, a quotient $\hat{V}_f = V_f/f$ is a closed connected orientable 3-manifold on which the natural projection $p_f : V_f \to \hat{V}_f$ induces an epimorphism $\eta_f : \pi_1(\hat{V}_f) \to \mathbb{Z}$, assigning to a homotopy class $[\hat{c}] \in \pi_1(\hat{V}_f)$ of a closed curve $\hat{c} \subset \hat{V}_f$ an integer n such that its lift on V_f joins a point x with the point $f^n(x)$. Let $\hat{\Gamma}^s_f = p_f(W^s_{\Omega_1} \setminus A_f)$ and $\hat{\Gamma}^u_f = p_f(W^u_{\Omega_2} \setminus R_f)$.

Definition 1.1
The collection $S_f = (\hat{V}_f, \eta_f, \hat{\Gamma}^s_f, \hat{\Gamma}^u_f)$ is called the *scheme* of the diffeomorphism $f \in MS(M^3)$.

Definition 1.2
The schemes S_f and $S_{f'}$ of diffeomorphisms $f, f' \in MS(M^3)$ are called *equivalent* if there is a homeomorphism $\hat{\varphi} : \hat{V}_f \to \hat{V}_{f'}$ with the following properties:
(1) $\eta_f = \eta_{f'} \hat{\varphi}_*$;
(2) $\hat{\varphi}(\hat{\Gamma}^s_f) = \hat{\Gamma}^s_{f'}$ and $\hat{\varphi}(\hat{\Gamma}^u_f) = \hat{\Gamma}^u_{f'}$.

THEOREM 1
Two Morse–Smale diffeomorphisms $f, f' \in MS(M^3)$ are topologically conjugate if and only if their schemes are equivalent.

The realization problem has been solved in the general setting in [10], generalizing the solution proposed in [8] for the gradient-like diffeomorphisms.

The structure of the paper is as follows:

- In Section 1, we give a formulation of the classification results for Morse–Smale 3-diffeomorphisms.
- In Section 2, we represent the general properties of Morse–Smale diffeomorphisms and their space of wandering orbits, which are necessary for the topological classification.
- In Section 3, we construct a compatible system of neighborhoods, which is the key point for the construction of the conjugating homeomorphism.
- In Section 4, we construct a conjugating homeomorphism foretelling the construction by an intuitive idea how it has to be.
- In Section 5, we prove some topological lemmas which we used in the classification theorem.

2. General properties of Morse–Smale diffeomorphisms

In this section, we represent the general properties of diffeomorphisms from the class $MS(M^n)$ of orientation-preserving Morse–Smale diffeomorphisms f given on a smooth closed orientable n-manifold M^n, which are necessary for the topological classification. Proofs of the facts below can be found in the following sources: [7], [9], [12], [18], [20], [21], [23], [26], [27], [32], [35], and [36].

2.1. Dynamics

Let $f \in MS(M^n)$. According to the definition of the Morse–Smale diffeomorphism, the nonwandering set Ω_f of the diffeomorphism f consists of a finite number of periodic points ($\Omega_f = Per_f$). The hyperbolic structure of the set Ω_f implies the existence of the invariant manifolds for each periodic point $p \in \Omega_f$ of period m_p: *stable* W^s_p and *unstable* W^u_p, which are defined in topological terms as the following:

$$W^s_p = \{x \in M^n : \lim_{n\to+\infty} d\big(f^{nm_p}(x), p\big) = 0\},$$

$$W^u_p = \{x \in M^n : \lim_{n\to+\infty} d\big(f^{-nm_p}(x), p\big) = 0\},$$

where d is a metric on M^n. Moreover, $\dim W^s_p = n - q_p$ ($\dim W^u_p = q_p$), where q_p is the number of the eigenvalues of Jacobian $(\frac{\partial f^{m_p}}{\partial x})|_p$ with the absolute value greater then 1 (*Morse index*). Further, for any subset $P \subset \Omega_f$, we will denote by W^u_P (W^s_P) a union of the unstable (stable) manifolds of all points from P. A connected component $\ell^s_p(\ell^u_p)$ of the set $W^s_p \setminus p$ ($W^u_p \setminus p$) is called a *stable (unstable) separatrix of the point* p. A number ν_p, which equals $+1$ if the map $f^{m_p}|_{W^u_p}$ preserves orientation and equals -1 if $f^{m_p}|_{W^u_p}$ changes orientation, is called a *type of orientation* of the point p. The triple $(m_p, q_p, \nu_p) = (m_{\mathcal{O}_p}, q_{\mathcal{O}_p}, \nu_{\mathcal{O}_p})$ is called a *periodic data* of the point p or the orbit $\mathcal{O}_p$.

A periodic point p is called a *saddle* if $0 < q_p < n$ and called a *node* otherwise; moreover, p is a *sink (resp., source)* if $q_p = 0$ ($q_p = n$). As a diffeomorphism f preserves orientation, the type orientation of any node point equals $+1$; but, for a saddle point, both values $+1, -1$ are possible. For $q \in \{0, \dots, n\}$, denote by Ω_q the set of all periodic points with the Morse index q and k_f the number of periodic orbits of $f \in MS(M^n)$.

Dynamical properties and topological type of a Morse–Smale diffeomorphism are largely determined by the properties of the embedding and by the mutual disposition of the invariant manifolds of the periodic points. The key role here belongs to the study of asymptotic properties of the invariant manifolds of the saddle periodic points.

Statement 2.1

Let $f \in MS(M^n)$. Then

(1) $M^n = \bigcup_{p\in\Omega_f} W^u_p$;

(2) W^u_p is a smooth submanifold[3] of the manifold M^n which is diffeomorphic to $\mathbb{R}^{q_p}$ for every periodic point $p \in \Omega_f$;

(3) $\mathrm{cl}(\ell^u_p) \setminus (\ell^u_p \cup p) = \bigcup_{r\in\Omega_f : \ell^u_p \cap W^s_r \neq \emptyset} W^u_r$ for every unstable separatrix ℓ^u_p of a periodic point $p \in \Omega_f$, where $\mathrm{cl}(\cdot)$ stands for the closure of $(\cdot)$.

According to item (2) of Statement 2.1, the map $f|_{W^u_{\mathcal{O}_p}} : W^u_{\mathcal{O}_p} \to W^u_{\mathcal{O}_p}$ is a diffeomorphism. Furthermore, the class of topological conjugacy of the diffeomorphism $f^{m_p}|_{W^u_p}$ is completely determined by the Morse index q_p and the orientation type ν_p of the point p. Namely, according to the theorem on the local topological classification of the hyperbolic fixed point of a diffeomorphism (see [27, Theorem 5.5]), the map f^{m_p} is locally conjugated at p to a linear diffeomorphism $a_{q_p,\nu_p} : \mathbb{R}^n \to \mathbb{R}^n$ given by the formula

$$a_{q_p,\nu_p}(x_1,\dots,x_n) = \Big(\nu_p \cdot 2x_1, 2x_2, \dots, 2x_{q_p}, \nu_p \cdot \frac{x_{q_p+1}}{2}, \frac{x_{q_p+2}}{2}, \dots, \frac{x_n}{2}\Big).$$

Let us call $a_{q,\nu} : \mathbb{R}^n \to \mathbb{R}^n$ a *canonical diffeomorphism*. Denote by O the coordinate origin in $\mathbb{R}^n$. Furthermore, denote by $a^u_{q,\nu}$, $a^s_{q,\nu}$ the restrictions of the canonical diffeomorphism to $Ox_1,\dots,x_q$, $Ox_{q+1},\dots,x_n$ and call the diffeomorphism $a^u_{q,\nu}$, $a^s_{q,\nu}$ a *canonical expansion*, or *canonical contraction*, accordingly. According to item (2) of Statement 2.1, $W^u_{\mathcal{O}_p}$ is a smooth submanifold of M^n and, hence, the map $f|_{W^u_{\mathcal{O}_p}} : W^u_{\mathcal{O}_p} \to W^u_{\mathcal{O}_p}$ is a diffeomorphism. Thus we have the following global topological classification of the maps $f|_{W^u_{\mathcal{O}_p}}$.

Statement 2.2
Let $f \in MS(M^n)$. Then for every periodic point $p \in \Omega_f$ the diffeomorphism $f^{m_p}|_{W^u_p} : W^u_p \to W^u_p$ is topologically conjugate to the canonical expansion $a^u_{q_p,\nu_p} : \mathbb{R}^{q_p} \to \mathbb{R}^{q_p}$.

If a periodic point is a saddle, then the embedding of its f-invariant neighborhood is also of importance. We begin with the linear case.

For $q \in \{1,\dots,n-1\}$, $t \in (0,1]$ let

$$\mathcal{N}^t_q = \big\{(x_1,\dots,x_n) \in \mathbb{R}^n : (x_1^2 + \dots + x_q^2)(x_{q+1}^2 + \dots + x_n^2) < t\big\}$$

[3]A subset A of a C^r-manifold X $(r \geq 0)$ is called a C^r*-submanifold* if for some integer $0 \leq k \leq n$ each point of A belongs to a chat (U,ψ) of X such that $\psi(U \cap A) = \mathbb{R}^k$ or $\psi(U \cap A) = \mathbb{R}^k_+$, where $\mathbb{R}^k = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_{k+1} = \dots = x_n = 0\}$ and $\mathbb{R}^k_+ = \{(x_1,\dots,x_n) \in \mathbb{R}^k : x_k \geq 0\}$. Herewith, A becomes C^r-manifolds with chats $\{(U \cap A, \psi|_{U\cap A})\}$. A C^0-submanifold is also called a *topological submanifold*. A classical topological fact says that a subset A of a C^r-manifold X with $r \geq 1$ is a C^r-submanifold if and only if it is an image of a C^r-embedding. That is, there is a C^r-manifold B and a regular C^r-map $g : B \to X$ (the rank of the Jacoby matrix of g at each point equals the dimension of the manifold B) which homeomorphically sends B to the subspace $A = g(B)$ with the topology induced from X. The map g is called a C^r*-embedding*.

and $\mathcal{N}_q^1 = \mathcal{N}_q$. Notice that the set $\mathcal{N}_q^t$ is invariant with respect to the canonical diffeomorphism $a_{q,\nu}$, which has a unique fixed point O that is a saddle point. Its unstable manifold being $W_O^u = Ox_1, \dots, x_q$ and its stable manifold $W_O^s = Ox_{q+1}, \dots, x_n$.

Definition 2.1
Let $f \in MS(M^n)$. We call a neighborhood N_σ of a saddle point $\sigma \in \Omega_f$ *linearizing* if there is a homeomorphism $\mu_\sigma : N_\sigma \to \mathcal{N}_{q_\sigma}$ which conjugates the diffeomorphism $f^{m_\sigma}|_{N_\sigma}$ to the canonical diffeomorphism $a_{q_\sigma,\nu_\sigma}|_{\mathcal{N}_{q_\sigma}}$.

The neighborhood $N_{\mathcal{O}_\sigma} = \bigcup_{k=0}^{m_\sigma-1} f^k(N_\sigma)$ equipped with the map $\mu_{\mathcal{O}_\sigma}$ made up of the homeomorphisms $\mu_\sigma f^{-k} : f^k(N_\sigma) \to \mathcal{N}_{n,q_\sigma}$, $k = 0, \dots, m_\sigma - 1$ is called the *linearizing neighborhood* of the orbit $\mathcal{O}_\sigma$, and the map $\mu_{\mathcal{O}_\sigma}$ is called *linearizing*.

Statement 2.3
Every saddle point (orbit) of the diffeomorphism $f \in MS(M^n)$ has a linearizing neighborhood.

Due to the linear dynamics near saddle points, we have the following fact.

Statement 2.4
Let σ be a saddle point of a diffeomorphism $f \in MS(M^n)$, let $T_\sigma \subset W_\sigma^s$ be a compact neighborhood of the point σ, and let $\xi \in T_\sigma$. Then, for every sequence of points $\{\xi_m\} \subset (M^n \setminus T_\sigma)$ converging to the point ξ, there are a subsequence $\{\xi_{m_j}\}$, a sequence of natural numbers $k_{m_j} \to +\infty$, and a point $\eta \in (W_\sigma^u \setminus \sigma)$ such that the sequence of points $\{f^{k_{m_j}}(\xi_{m_j})\}$ converges to the point η.

Define in the canonical neighborhood $\mathcal{N}_q$ a pair of transversal foliations $\mathcal{F}_q^u$, $\mathcal{F}_q^s$ in the following way:

$$\mathcal{F}_q^u = \bigcup_{(c_{q+1},\dots,c_n)\in Ox_{q+1}\dots x_n} \big\{(x_1, \dots, x_n) \in \mathcal{N}_q : (x_{q+1}, \dots, x_n) = (c_{q+1}, \dots, c_n)\big\},$$

$$\mathcal{F}_q^s = \bigcup_{(c_1,\dots,c_q)\in Ox_1\dots x_q} \big\{(x_1, \dots, x_n) \in \mathcal{N}_q : (x_1, \dots, x_q) = (c_1, \dots, c_q)\big\}.$$

The dimensions of the leaves of $\mathcal{F}_q^u$ and $\mathcal{F}_q^s$ are q and $n-q$ accordingly. Notice that the canonical diffeomorphism $a_{q,\nu}$ sends the leaves of the foliation $\mathcal{F}_q^u$ ($\mathcal{F}_q^s$) to the leaves of the same foliation. According to Statement 2.3, for any saddle point σ of $f \in MS(M^3)$, the foliations $\mathcal{F}_{q_\sigma}^u$, $\mathcal{F}_{q_\sigma}^s$ induce, by means of the linearizing map, f-invariant foliations $F_{\mathcal{O}_\sigma}^u$, $F_{\mathcal{O}_\sigma}^s$ on the linearizing neighborhood $N_{\mathcal{O}_\sigma}$, which are called *linearizing* (see Figure 5).

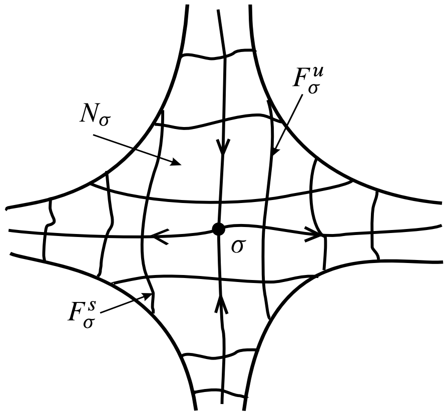

Figure 5. Linearizing foliations in the linearizing neighborhood.

According to item (2) of Statement 2.1, the invariant manifold of a periodic point of a diffeomorphism $f \in MS(M^n)$ is a smooth submanifold of M^n. Nevertheless, its closure can have a complicated topological structure. The nature of this phenomenon is either dynamical or topological. The first case corresponds to a situation when a separatrix of a saddle point takes part in the heteroclinic intersections.

Definition 2.2
If σ_1, σ_2 are distinct periodic saddle points of a diffeomorphism $f \in MS(M^n)$ for which $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \emptyset$, then the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is called a *heteroclinic intersection*.

- If $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) > 0$, then a connected component of the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is called a *heteroclinic manifold*, and if $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) = 1$, then it is called a *heteroclinic curve*;
- If $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) = 0$, then the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is countable; each point of this set is called a *heteroclinic point*, and the orbit of the heteroclinic point is called a *heteroclinic orbit*.

Definition 2.3
A diffeomorphism $f \in MS(M^n)$ is said to be *gradient-like* if from $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \emptyset$ for different points $\sigma_1, \sigma_2 \in \Omega_f$ it follows that $\dim W^u_{\sigma_1} < \dim W^u_{\sigma_2}$.

It follows from the transversality of intersection of invariant manifolds of the periodic point that a diffeomorphism $f \in MS(M^n)$ is gradient-like if and only if it has no heteroclinic points.

According to item (3) of Statement 2.1, the closure of a separatrix of a saddle point which has heteroclinic intersections is not a topological manifold in general, but the closure of a separatrix of a saddle with no heteroclinic intersections is a topologically embedded manifold.[4] The following statement holds.

Statement 2.5
Let $f \in MS(M^n)$, and let σ be a saddle point of f such that its unstable separatrix ℓ^u_σ does not take part in the heteroclinic intersections. Then

$$\mathrm{cl}(\ell^u_\sigma) = \ell^u_\sigma \cup \{\sigma, \omega\},$$

where ω is a sink periodic point of f. Herewith, if $q_\sigma = 1$, then $\mathrm{cl}(\ell^u_\sigma)$ is a topologically embedded arc in M^n; if $q_\sigma \geq 2$, then $\mathrm{cl}(\ell^u_\sigma)$ is a topologically embedded q_σ-sphere in M^n.

According to item (2) of Statement 2.1, $\ell^u_\sigma \cup \sigma$ is a smooth submanifold of the manifold M^n. However, the manifold $\mathrm{cl}(\ell^u_\sigma)$ may be wild at the point ω; in this case the separatrix ℓ^u_σ is called *wild*, and it is called *tame* in the opposite case.

For $n = 2$, according to E. Moise's results in [25], every compact arc, and hence any separatrix with no heteroclinic points, is tamely embedded into M^2. An example of a wild compact arc in $\mathbb{S}^3$ (that has nothing to do with dynamical systems) was constructed by E. Artin and R. Fox in 1948 (see [1]). The first example of a Morse–Smale diffeomorphism with wildly embedded separatrices belongs to D. Pixton [30] (see Figure 6), and it is based on the Artin–Fox arc. The following statement proved in [22] and [13] contains criteria for the tame embedding of separatrices (1- and 2-dimensional) of the saddle points for a diffeomorphism $f \in MS(M^3)$.

Statement 2.6
Let $f \in MS(M^3)$, let ω be a sink, and let ℓ^u_σ be a 1-dimensional (resp., 2-dimensional) separatrix of a saddle σ such that $\ell^u_\sigma \subset W^s_\omega$. The separatrix ℓ^u_σ is tamely embedded into M^3 if and only if there is a smooth 3-ball $B_\omega \subset W^s_\omega$ containing ω in its interior and such that ℓ^u_σ intersects ∂B_ω at exactly one point (resp., one circle).

[4] A C^0-map $g : B \to X$ is called a *topological embedding* to a topological manifold X of a topological manifold B if it homeomorphically sends B to a subspace $g(B)$ with the topology induced from X. The image $A = g(B)$ is called a *topologically embedded manifold*. Notice that a topologically embedded manifold is not a topological submanifold in general. If A is a submanifold, then it is called *tame* or *tamely embedded*, in the opposite case *wild* or *wildly embedded*, and the points of violation of submanifold's condition are called *points of wildness*.

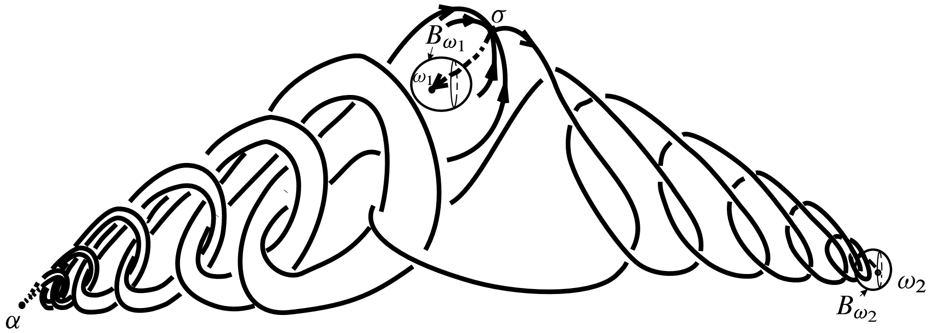

Figure 6. Pixton's example.

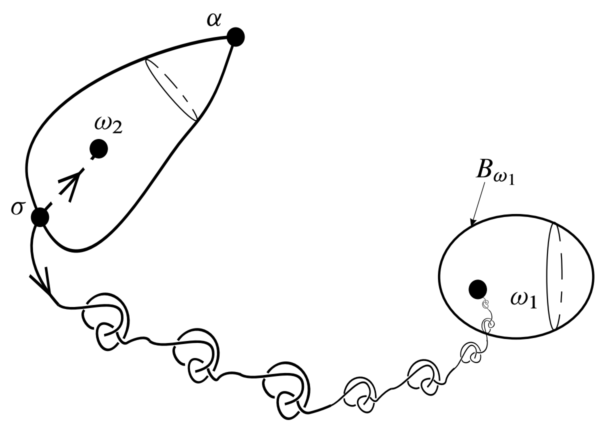

Figure 7. Tame separatrices of the periodic points.

In Figure 6 are represented 3-balls B_{ω_1}, B_{ω_2} around the sinks ω_1, ω_2. By the criteria above, the unstable saddle separatrix going to ω_1 is tame, but the other is wild. Notice that all saddle separatrices in Figure 7 are tame, unlike the Pixton example.

According to S. Smale [35], it is possible to define a partial order in the set of saddle points for a Morse–Smale diffeomorphism f as follows: for different periodic orbits $\mathcal{O}_p \neq \mathcal{O}_q$, one sets

$$\mathcal{O}_p \prec \mathcal{O}_q \qquad \text{if and only if} \qquad W^u_{\mathcal{O}_q} \cap W^s_{\mathcal{O}_p} \neq \emptyset.$$

In that case, it follows from [26, Lemma 1.5] that there is a maximal sequence of distinct periodic orbits $\mathcal{O}_{p_0}, \dots, \mathcal{O}_{p_k}$ satisfying the following conditions: $\mathcal{O}_{p_0} = \mathcal{O}_p$, $\mathcal{O}_{p_k} = \mathcal{O}_q$ and $\mathcal{O}_{p_i} \prec \mathcal{O}_{p_{i+1}}$. In that case, the sequence $\mathcal{O}_{p_0}, \dots, \mathcal{O}_{p_k}$ is said to be a *k-chain connecting $\mathcal{O}_p$ with $\mathcal{O}_q$*. The length of the longest chain connecting $\mathcal{O}_p$, $\mathcal{O}_q$

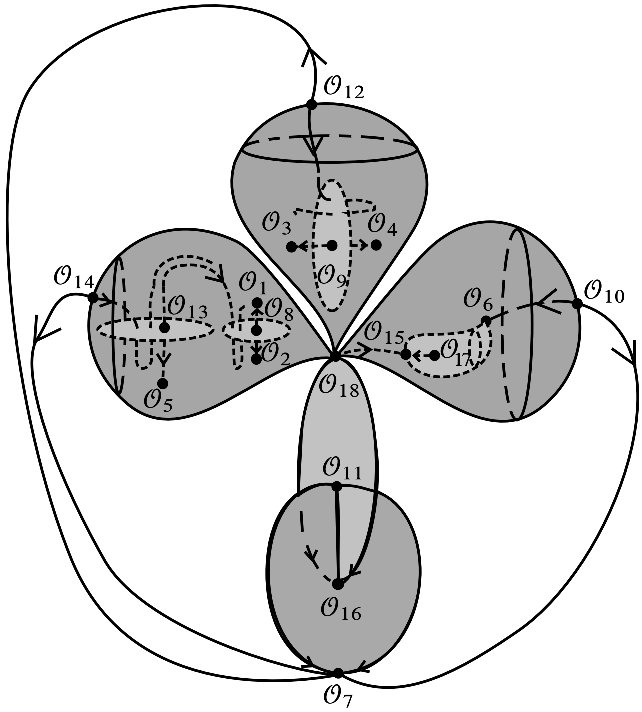

Figure 8. Ordered periodic orbits.

is denoted by

$$\mathrm{beh}(\mathcal{O}_q \mid \mathcal{O}_p)$$

from the word *behavior*. Set $\mathrm{beh}(\mathcal{O}_q \mid \mathcal{O}_p) = 0$ if $W^u_{\mathcal{O}_q} \cap W^s_{\mathcal{O}_p} = \emptyset$. For a subset P of the periodic orbits, let us set $\mathrm{beh}(\mathcal{O}_q \mid P) = \max_{\mathcal{O}_p \in P}\{\mathrm{beh}(\mathcal{O}_q \mid \mathcal{O}_p)\}$.

In Figure 8, ordered periodic orbits for a Morse–Smale 3-diffeomorphism are represented. Here $\mathrm{beh}(\mathcal{O}_{14} \mid \mathcal{O}_8) = 2$, $\mathrm{beh}(\mathcal{O}_{18} \mid \mathcal{O}_1) = 4$, and so on.

Let us divide the set of the saddle orbits of f on two parts Σ_A, Σ_R such that $W^s_{\Sigma_R} \cap W^u_{\Sigma_A} = \emptyset$. Notice that such a division is not unique; moreover, one of the parts may be empty. For arbitrary such decomposition let

$$A = W^u_{\Sigma_A} \cup \Omega_0, \qquad R = W^s_{\Sigma_R} \cup \Omega_n, \qquad V = M^n \setminus (A \cup R).$$

Statement 2.7
Let $f \in MS(M^n)$. Then we have the following:

(1) The set $A(R)$ is an attractor (resp., repeller) of f; moreover, if $\dim A \leq (n-2)$ (resp., $\dim R \leq (n-2)$), then the repeller R (resp., attractor A) is connected and if $\dim(A \cup R) \leq (n-2)$, then the manifold V is connected.

(2) $V = W^s_{A\cap\Omega_f} \setminus A = W^u_{R\cap\Omega_f} \setminus R$.

We call V a *characteristic manifold*. Below we study orbit spaces of some wandering sets and, in particular, the *characteristic space* $\hat{V} = V/f$.

2.2. *Orbit spaces*

In this section, we are interested in the topology of an orbit space for a some diffeomorphism $g : X \to X$ on a manifold X. We use the denotation X/g for the *space of* g*-orbits on* X and $p_{X/g} : X \to X/g$ for the natural projection. Let us recall that a *fundamental domain of the action* g *on* X is a closed set $D_g \subset X$ such that there is a set $\tilde{D}_g$ with the following properties:

(1) $\mathrm{cl}(\tilde{D}_g) = D_g$;

2) $g^k(\tilde{D}_g) \cap \tilde{D}_g = \emptyset$ for all $k \in (\mathbb{Z} \setminus \{0\})$;

3) $\bigcup_{k\in\mathbb{Z}} g^k(\tilde{D}_g) = X$.

We say that g acts *discontinuously* on X if for every compact set $K \subset X$ the set of elements $k \in \mathbb{Z}$ such that $g^k(K) \cap K \neq \emptyset$ is finite. In a case of such action, the projection $p_{X/g}$ is a cover (see Statement 2.8 below) and then we can make the following construction. Suppose that the space X/g is connected, and denote by n_X the number of connected components of X and by $p^{-1}_{X/g}(\hat{x})$ the preimage of a point $\hat{x} \in X/g$ with respect to the cover $p_{X/g} : X \to X/g$ (it is an orbit of some point $x \in p^{-1}_{X/g}(\hat{x})$). Let $\hat{c}$ be a loop in X/g such that $\hat{c}(0) = \hat{c}(1) = \hat{x}$. Due to the monodromy theorem (see, e.g., [23, Corollary 16.6]), there is a unique path c in X with the beginning at the point x ($c(0) = x$), which is the lift of $\hat{c}$. Therefore, there is an element[5] $k \in n_X\mathbb{Z}$ independent on $\hat{c}$ such that $c(1) = g^k(x)$. Let $\eta_{X/g} : \pi_1(X/g) \to n_X\mathbb{Z}$ be a map sending $[\hat{c}]$ to k.

Statement 2.8

Let a diffeomorphism g acts discontinuously on n-manifold X. Then we have the following:

(1) The natural projection $p_{X/g} : X \to X/g$ is a cover.

(2) The quotient X/g is an n-manifold.

(3) For a fundamental domain D_g of action g on X, the orbit spaces D_g/g and X/g are homeomorphic.

(4) The map $\eta_{X/g} : \pi_1(X/g) \to n_X\mathbb{Z}$ is an epimorphism.

[5]Here $n_X\mathbb{Z}$ denotes the set of integers multiples by n_X.

Statement 2.9
Let diffeomorphisms g, g' act discontinuously on manifolds X, X', accordingly, and the manifolds X/g, X'/g' are connected. Then we have the following:

(1) If $h : X \to X'$ is a homeomorphism such that $hg = g'h$, then the map $\hat{h} : X/g \to X'/g'$ given by the formula $\hat{h} = p_{X'/g'} h p^{-1}_{X/g}$ is a homeomorphism and $\eta_{X/g} = \eta_{X'/g'} \hat{h}_*$.

(2) If $\hat{h} : X/g \to X'/g'$ is a homeomorphism such that $\eta_{X/g} = \eta_{X'/g'} \hat{h}_*$, then for some point $x \in X$ and $x' \in p^{-1}_{X'/g'}(\hat{h}(p_{X/g}(x)))$ there is a unique homeomorphism $h : X \to X'$ being a lift of $\hat{h}$ such that $hg = g'h$, $h(x) = x'$.

Let us illustrate the facts above on the orbit space $\hat{\mathcal{W}}^u_{q,\nu} = (\mathbb{R}^q \setminus O)/a^u_{q,\nu}$ of the action of the canonical expansion $a^u_{q,\nu}$ on $\mathbb{R}^q \setminus O$ for $q \in \{1, \dots, n\}$, $\nu \in \{+1, -1\}$. It is obvious that this action is discontinuous and its fundamental domain is the annulus $\{(x_1, \dots, x_q) \in \mathbb{R}^q : 1 \le x_1^2 + \cdots + x_q^2 \le 4\}$ (see Figure 9), which implies the following list of possibilities.

Statement 2.10

(1) The space $\hat{\mathcal{W}}^u_{1,-1}$ is homeomorphic to the circle.
(2) The space $\hat{\mathcal{W}}^u_{1,+1}$ is homeomorphic to the pair of circles.
(3) The space $\hat{\mathcal{W}}^u_{2,-1}$ is homeomorphic to the Klein bottle.
(4) The space $\hat{\mathcal{W}}^u_{2,+1}$ is homeomorphic to the torus $\mathbb{T}^2$.
(5) The space $\hat{\mathcal{W}}^u_{q,-1}$, $q \ge 3$ is homeomorphic to a generalized Klein bottle.[6]
(6) The space $\hat{\mathcal{W}}^u_{q,+1}$, $q \ge 3$ is homeomorphic to $\mathbb{S}^{q-1} \times \mathbb{S}^1$.

Now let r be a periodic point of $f \in MS(M^n)$ with the Morse index $q_r \ge 1$. Consider the orbit space $\hat{W}^u_{\mathcal{O}_r} = (W^u_{\mathcal{O}_r} \setminus \mathcal{O}_r)/f$. The next statement illustrates an interrelation between $\hat{W}^u_{\mathcal{O}_r}$ and the linear model.

Statement 2.11
Let r be a periodic point of a diffeomorphism $f \in MS(M^n)$ with the period m_r, the orientation type ν_r, and the Morse index $q_r \ge 1$. Then the natural projection $p_{\hat{W}^u_{\mathcal{O}_r}}$ is a cover which induces a structure of a smooth orientable q_r-manifold on the space $\hat{W}^u_{\mathcal{O}_r}$, and there is a homeomorphism $\hat{h}^u_{\mathcal{O}_r} : \hat{W}^u_{\mathcal{O}_r} \to \hat{\mathcal{W}}^u_{q_r,\nu_r}$ such that $\eta_{\hat{W}^u_{\mathcal{O}_r}}([\hat{c}]) = m_r \eta_{\hat{\mathcal{W}}^u_{q_r,\nu_r}}([\hat{h}^u_{\mathcal{O}_r}(\hat{c})])$ for every closed curve $\hat{c} \subset \hat{W}^u_{\mathcal{O}_r}$.

[6]A *generalized Klein bottle* is a topological space which is obtained from $\mathbb{S}^{q-1} \times [0, 1]$ by identification of its boundary with respect to the map $g : \mathbb{S}^{q-1} \times \{0\} \to \mathbb{S}^{q-1} \times \{1\}$ given by the formula $g(x_1, x_2, \dots, x_q, 0) = (-x_1, x_2, \dots, x_q, 1)$.

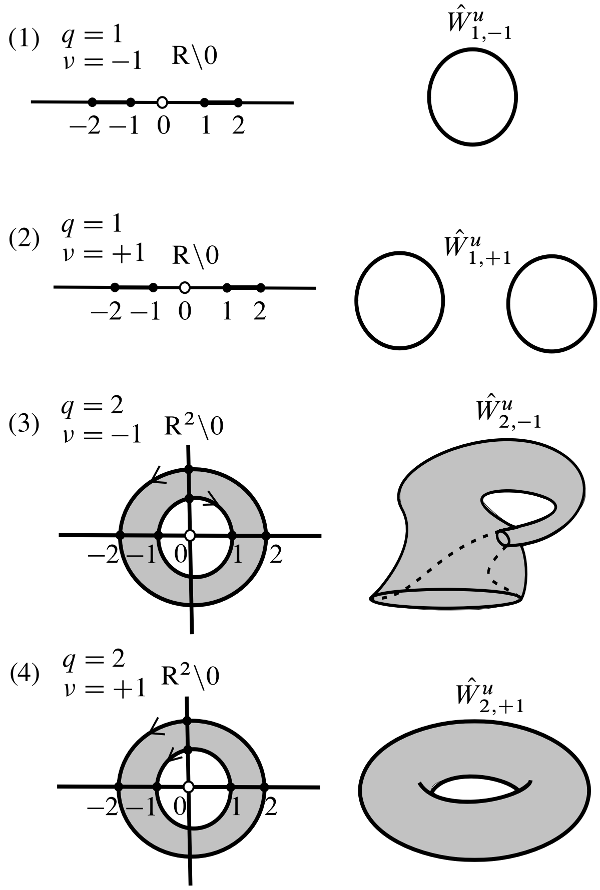

Figure 9. Orbit spaces of the canonical expansion.

The space orbit $\hat{\mathcal{W}}^s_{q,\nu} = (\mathbb{R}^{n-q} \setminus O)/a^s_{q,\nu}$ of the *canonical contraction* for $q \in \{0, \dots, n-1\}$, $\nu \in \{+1, -1\}$ and the space $\hat{W}^s_{\mathcal{O}_r} = (W^s_{\mathcal{O}_r} \setminus \mathcal{O}_r)/f$ for a periodic point r with the Morse index $q_r \leq (n-1)$ are defined similarly.

Figure 10(a) shows a Morse–Smale diffeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$, the nonwandering set of which consists of eight periodic points with the following periodic data: $\mathcal{O}_{\omega_1}(1,0,+1)$, $\mathcal{O}_{\omega_2}(1,0,+1)$, $\mathcal{O}_{\omega_3}(3,0,+1)$, $\mathcal{O}_{\sigma_1}(1,2,+1)$, $\mathcal{O}_{\sigma_2}(1,1,+1)$, $\mathcal{O}_{\sigma_3}(3,1,+1)$, $\mathcal{O}_{\alpha_1}(1,3,+1)$, and $\mathcal{O}_{\alpha_2}(1,3,+1)$. Figure 10(b) shows the fundamental domains of the action of the diffeomorphism f on $W^s_{\mathcal{O}_{\omega_i}} \setminus \mathcal{O}_{\omega_i}$, $i = 1,2,3$, $W^u_{\mathcal{O}_{\alpha_i}} \setminus \mathcal{O}_{\alpha_i}$, $i = 1,2$. Each fundamental domain is the 3-annulus from which the orbits spaces $\hat{W}^s_{\mathcal{O}_{\omega_i}}$, $i = 1,2,3$, $\hat{W}^u_{\mathcal{O}_{\alpha_i}}$, $i = 1,2$, are obtained by gluing the boundary spheres by the diffeomorphisms $f^{m_{\omega_i}}$, $i = 1,2,3$, $f^{m_{\alpha_i}}$, $i = 1,2$, respectively. The

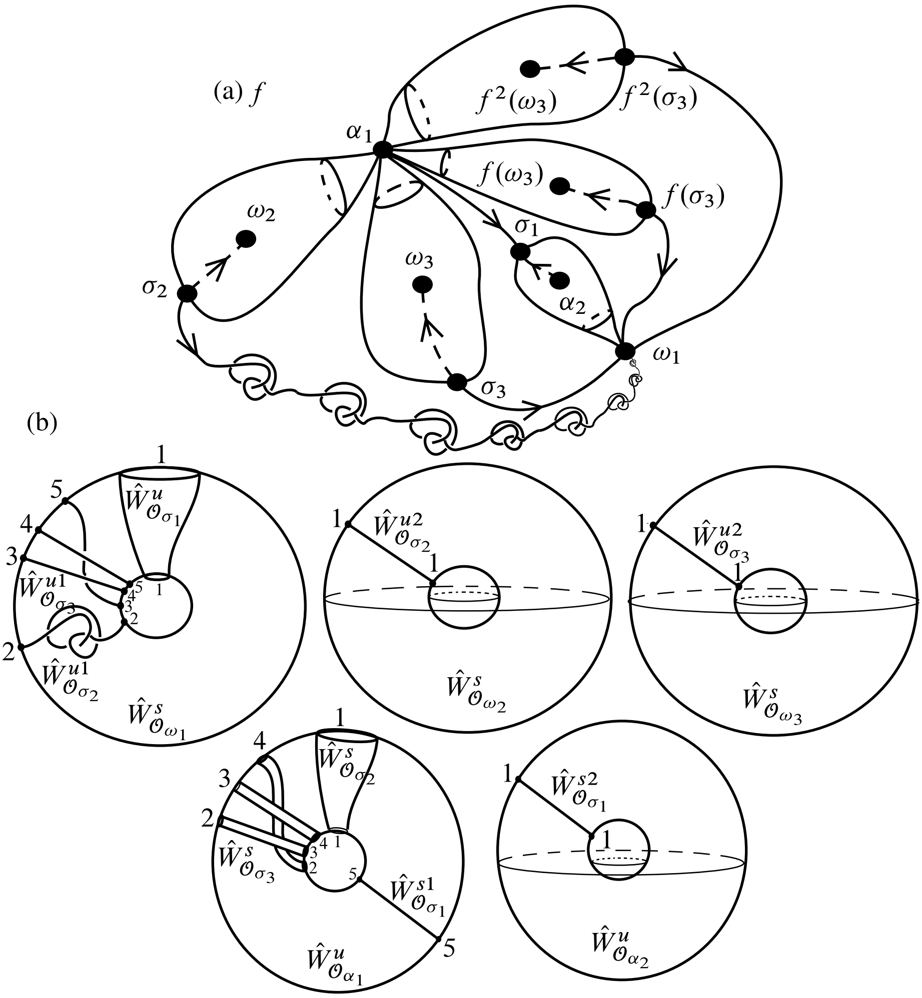

Figure 10. Orbit spaces of the separatrices of the periodic points.

orbits spaces $\hat{W}^s_{\mathcal{O}_{\sigma_i}}$, $\hat{W}^u_{\mathcal{O}_{\sigma_i}}$, $i = 1, 2, 3$ are obtained from the arcs and the cylinders by gluing the points with the same numbers and the circles with the same numbers.

Figure 11(a) shows a Morse–Smale diffeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$, the nonwandering set of which consists of five periodic points with the following periodic data: $\mathcal{O}_\omega(1, 0, +1)$, $\mathcal{O}_{\sigma_1}(1, 2, -1)$, $\mathcal{O}_{\sigma_2}(2, 2, +1)$, $\mathcal{O}_{\alpha_1}(2, 3, +1)$, and $\mathcal{O}_{\alpha_2}(2, 3, +1)$. Figure 11(b) shows the fundamental domains of the action of the diffeomorphism f on $W^s_{\mathcal{O}_\omega} \setminus \mathcal{O}_\omega$ and $W^u_{\mathcal{O}_{\alpha_i}} \setminus \mathcal{O}_{\alpha_i}$, $i = 1, 2$. Each fundamental domain is the 3-annulus from which the orbits spaces $\hat{W}^s_{\mathcal{O}_\omega}$, $\hat{W}^u_{\mathcal{O}_{\alpha_i}}$, $i = 1, 2$, are obtained by gluing the boundary

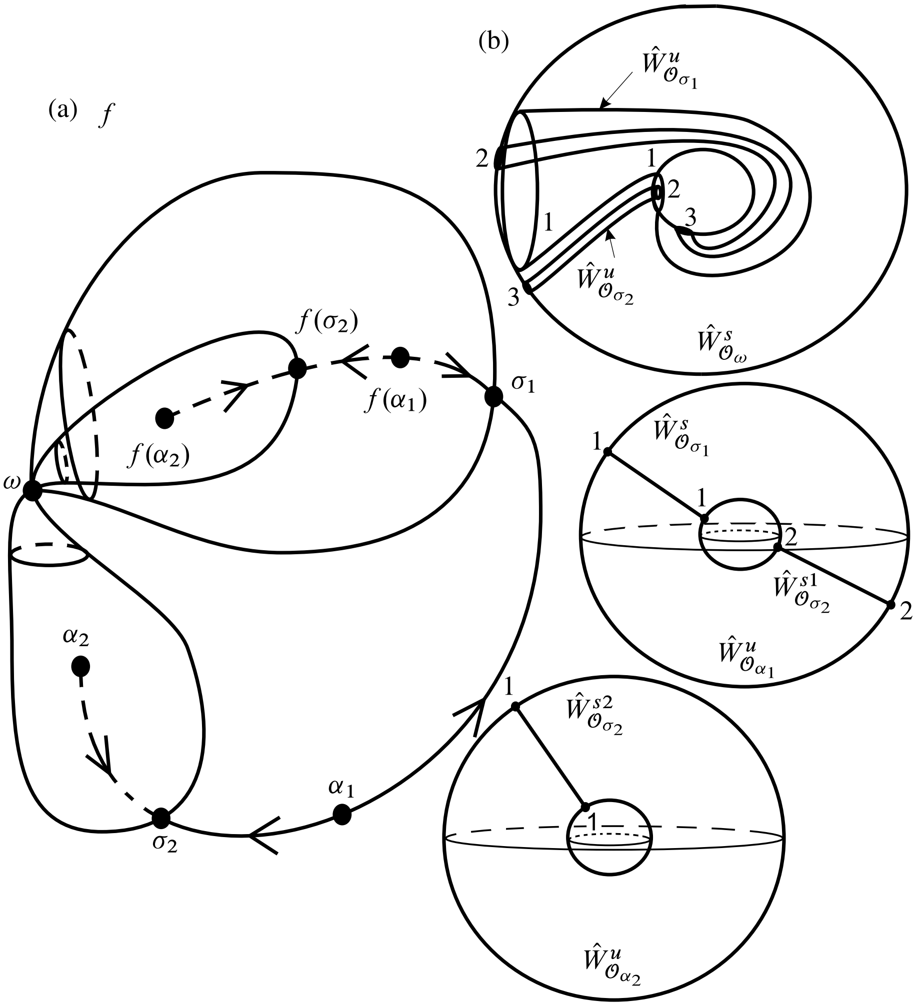

Figure 11. Orbit spaces of the separatrices of the periodic points.

spheres of the annulus by the the diffeomorphisms f^{m_ω}, $f^{m_{\alpha_i}}$, $i = 1, 2$, respectively. The orbits spaces $\hat{W}^s_{\mathcal{O}_{\sigma_i}}$, $\hat{W}^u_{\mathcal{O}_{\sigma_i}}$, $i = 1, 2$, are obtained from the arcs and the cylinders by gluing the points with the same numbers and the circles with the same numbers.

On the set $\mathcal{N}^u_q = \mathcal{N}_q \setminus W^s_O$, the action of the group $A_{q,\nu} = \{a^k_{q,\nu}, k \in \mathbb{Z}\}$ is discontinuous again. Due to Statement 2.8, the space orbit $\hat{\mathcal{N}}^u_{q,\nu} = (\mathcal{N}^u_q)/a_{q,\nu}$ is a smooth n-manifold. As $a_{q,\nu}|_{W^u_O \setminus O} = a^u_{q,\nu}|_{W^u_O \setminus O}$, we have that $\hat{\mathcal{N}}^u_{q,\nu}$ is a tubular neighborhood of the space $\hat{\mathcal{W}}^u_{q,\nu}$. Furthermore, $\hat{\mathcal{W}}^u_{q,+1}$ is homeomorphic to $\mathbb{S}^{q-1} \times \mathbb{S}^1 \times \{0\}$

and its tubular neighborhood $\hat{\mathcal{N}}^u_{q,+1}$ is homeomorphic to $\mathbb{S}^{q-1} \times \mathbb{S}^1 \times \mathbb{D}^{n-q}$. As $a^2_{q,-1} = a^2_{q,+1}$ and the diffeomorphisms $a^2_{q,+1}$ and $a_{q,+1}$ are topologically conjugated due to Statement 2.2, we have that the manifold $\hat{\mathcal{W}}^u_{q,+1}$ is the 2-fold cover of the manifold $\hat{\mathcal{W}}^u_{q,-1}$ and the manifold $\hat{\mathcal{N}}^u_{q,+1}$ is the 2-fold cover of the neighborhood $\hat{\mathcal{N}}^u_{q,-1}$.

Similarly, one defines the orbit space $\hat{\mathcal{N}}^s_{q,\nu} = \mathcal{N}^s_q / a^s_{q,\nu}$ (where $\mathcal{N}^s_q = \mathcal{N}_q \setminus W^u_O$), the covering map $p_{\hat{\mathcal{N}}^s_{q,\nu}} : \mathcal{N}^s_q \to \hat{\mathcal{N}}^s_{q,\nu}$, and the map $\eta_{\hat{\mathcal{N}}^s_{q,\nu}}$ from the union of the fundamental groups of the connected components of the manifold $\hat{\mathcal{N}}^s_{q,\nu}$ into the group $\mathbb{Z}$.

Figure 12 shows these objects for $n = 3$, $q = 1$, and $\nu = +1$. To make the structure of the orbit space $\hat{\mathcal{N}}^s_{q,\nu}$, $\hat{\mathcal{N}}^u_{q,\nu}$ more clear, we mark out the fundamental domain of the action of the canonical diffeomorphism $a_{q,\nu}$ on the sets $\mathcal{N}^s_q$, $\mathcal{N}^u_q$.

Now let σ be a saddle periodic point with Morse index q_σ of a diffeomorphism $f \in MS(M^n)$, and let $N_{\mathcal{O}_\sigma}$ be a linearizing neighborhood of the orbit $\mathcal{O}_\sigma$. Let

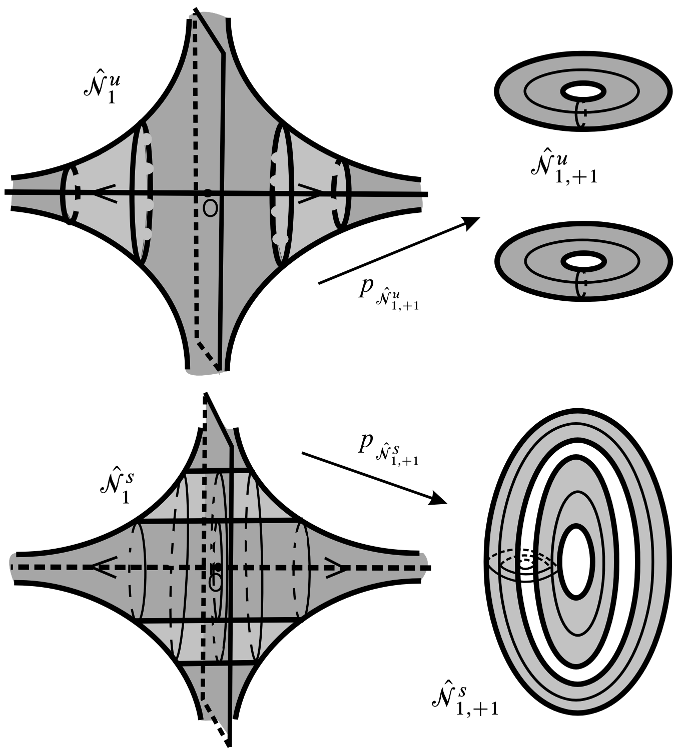

Figure 12. Neighborhoods of the orbit spaces of the canonical contraction and the expansion for $n = 3$.

$N^u_{\mathcal{O}_\sigma} = N_{\mathcal{O}_\sigma} \setminus W^s_{\mathcal{O}_\sigma}$. Consider the orbit space $\hat{N}^u_{\mathcal{O}_\sigma} = N^u_{\mathcal{O}_\sigma}/f$ of the action of the diffeomorphism f on $N^u_{\mathcal{O}_\sigma}$. Denote by $p_{\hat{N}^u_{\mathcal{O}_\sigma}} : N^u_{\mathcal{O}_\sigma} \to \hat{N}^u_{\mathcal{O}_\sigma}$ the natural projection. The following statement shows the connection between the orbit space $\hat{N}^u_{\mathcal{O}_\sigma}$ and the linear model.

Statement 2.12

Let σ be a saddle periodic point of period m_σ with orientation type ν_σ and Morse index q_σ for a diffeomorphism $f \in MS(M^n)$. Then the projection $p_{\hat{N}^u_{\mathcal{O}_\sigma}}$ is the covering map; it induces a structure of a smooth n-manifold on the orbit space $\hat{N}^u_{\mathcal{O}_\sigma}$ and it induces a map $\eta_{\hat{N}^u_{\mathcal{O}_\sigma}}$ from the union of the fundamental groups of the connected components of the manifold $\hat{N}^u_{\mathcal{O}_\sigma}$ into the group $\mathbb{Z}$ such that there is a homeomorphism $\hat{\mu}^u_{\mathcal{O}_\sigma} : \hat{N}^u_{\mathcal{O}_\sigma} \to \hat{\mathcal{N}}^u_{q_\sigma,\nu_\sigma}$ which satisfies $\eta_{\hat{N}^u_{\mathcal{O}_\sigma}}([\hat{c}]) = m_\sigma \eta_{\hat{\mathcal{N}}^u_{q_\sigma,\nu_\sigma}}([\hat{\mu}^u_{\mathcal{O}_\sigma}(\hat{c})])$ for any closed curve $\hat{c} \subset \hat{N}^u_{\mathcal{O}_\sigma}$.

Similarly, one defines the orbit space $\hat{N}^s_{\mathcal{O}_\sigma} = N^s_{\mathcal{O}_\sigma}/f$ of the action of f on $N^s_{\mathcal{O}_\sigma} = N_{\mathcal{O}_\sigma} \setminus W^u_{\mathcal{O}_\sigma}$, the covering map $p_{\hat{N}^s_{\mathcal{O}_\sigma}} : N^s_{\mathcal{O}_\sigma} \to \hat{N}^s_{\mathcal{O}_\sigma}$, and the map $\eta_{\hat{N}^s_{\mathcal{O}_\sigma}}$ consisting of nontrivial homomorphisms into the group $\mathbb{Z}$ of the fundamental group of each connected component of the manifold $\hat{N}^s_{\mathcal{O}_\sigma}$.

Below, for any $t \in (0,1]$, we denote $N^t_\sigma = (\mu_\sigma)^{-1}(\mathcal{N}^t_{q_\sigma})$, $N^t_{\mathcal{O}_\sigma} = \bigcup_{k=0}^{m_\sigma-1} f^k(N^t_\sigma)$, $\mathcal{N}^{ut}_{q_\sigma} = \mathcal{N}^t_{q_\sigma,\nu_\sigma} \setminus W^s_O$, $N^{ut}_\sigma = (\mu_\sigma)^{-1}(\mathcal{N}^{ut}_{q_\sigma})$, $N^{ut}_{\mathcal{O}_\sigma} = \bigcup_{k=0}^{m_\sigma-1} f^k(N^{ut}_\sigma)$, $\mathcal{N}^{st}_{q_\sigma} = \mathcal{N}^t_{q_\sigma} \setminus W^u_O$, $N^{st}_\sigma = (\mu_\sigma)^{-1}(\mathcal{N}^{st}_{q_\sigma})$, $N^{st}_{\mathcal{O}_\sigma} = \bigcup_{k=0}^{m_\sigma-1} f^k(N^{st}_\sigma)$, $\hat{\mathcal{N}}^{ut}_{q_\sigma,\nu_\sigma} = p_{\hat{\mathcal{N}}^{ut}_{q_\sigma,\nu_\sigma}}(\mathcal{N}^{ut}_{q_\sigma})$, and $\hat{\mathcal{N}}^{st}_{q_\sigma,\nu_\sigma} = p_{\hat{\mathcal{N}}^{st}_{q_\sigma,\nu_\sigma}}(\mathcal{N}^{st}_{q_\sigma})$.

Statement 2.13

For every $t \in (0,1)$, the neighborhood N^t_σ is linearizing. Generically among the t's, the boundary of N^t_σ does not contain any heteroclinic point.

Let us recall that we divided the set of the saddle orbits of f into two parts Σ_A, Σ_R such that $W^s_{\Sigma_R} \cap W^u_{\Sigma_A} = \emptyset$ and we set $A = W^u_{\Sigma_A} \cup \Omega_0$, $R = W^s_{\Sigma_R} \cup \Omega_3$, $V = M^n \setminus (A \cup R)$.

Statement 2.14

The orbit space $\hat{V} = V/f$ is a closed orientable n-manifold.

3. Compatible foliations

In this section, for any diffeomorphism $f \in MS(M^3)$ the existence of a compatible system of neighborhoods is proved. This system is a key point for the construction of a conjugating homeomorphism. Before the definition, let us introduce some notation.

Let $f \in MS(M^3)$. It follows from Statement 2.7 that if the set Ω_2 is empty, then R_f consists of a unique source. If $\Omega_2 \neq \emptyset$, then denote n the length of the longest chain connecting some p, q from Ω_2. Divide the set Ω_2 into f-invariant disjoint subsets $\Sigma_0, \Sigma_1, \dots, \Sigma_n$ inductively as follows: let Σ_0 consist of those orbits $\mathcal{O} \in \Omega_2$ such that $W^u_{\mathcal{O}} \cap W^s_{\mathcal{O}'} = \emptyset$ for any orbit $\mathcal{O}' \subset (\Omega_2 \setminus \mathcal{O})$, and, for $i \geq 0$, let Σ_{i+1} consist of those orbits $\mathcal{O} \subset (\Omega_2 \setminus (\Sigma_0 \cup \dots \cup \Sigma_i))$ such that $W^u_{\mathcal{O}} \cap W^s_{\mathcal{O}'} = \emptyset$ for any orbit $\mathcal{O}' \subset (\Omega_2 \setminus (\Sigma_0 \cup \dots \cup \Sigma_i \cup \mathcal{O}))$. A similar representation of the set Ω_1 can be obtained by going to the inverse map f^{-1}. We denote by $\Sigma^-_0, \Sigma^-_1, \dots, \Sigma^-_{n^-}$ the components of this splitting. Without loss of generality suppose that $n \geq n^-$.

Notation 3.1

- $W^u_i := W^u_{\Sigma_i}$, $W^s_i := W^s_{\Sigma_i}$.
- $N^t_i := \bigcup_{\mathcal{O} \in \Sigma_i} N^t_{\mathcal{O}}$ and μ_i composed by $\mu_{\mathcal{O}}$, $\mathcal{O} \in \Sigma_i$.
- For every point $x \in N_i$, denote by $F^u_{i,x}$ (resp., $F^s_{i,x}$) the leaf of the foliation F^u_i (resp., F^s_i) passing through x.
- For every point $x \in N_i$, set $x^u_i = W^u_i \cap F^s_{i,x}$ and $x^s_i = W^s_i \cap F^u_{i,x}$. Thus, we have $x = (x^u_i, x^s_i)$ in the coordinates defined by μ_i.
- For $i \in \{0, \dots, n\}$, set $A_i := A_f \cup \bigcup_{j=0}^{i} W^u_j$, $R_i := R_f \setminus \bigcup_{j=0}^{i} W^s_j$, $V_i := M \setminus (A_i \cup R_i)$, $\hat{V}_i := V_i/f$. Observe that f acts freely on V_i, and denote by $p_i : V_i \to \hat{V}_i$ the natural projection and by η_i the corresponding epimorphism.
- For completeness, we put $A_{-1} = A_f$, $R_{-1} = R_f$, $V_{-1} = V_f$, $\hat{V}_{-1} = \hat{V}_f$, $p_{-1} = p_f$, and $\eta_{-1} = \eta_f$.
- For $j, k \in \{0, \dots, n\}$ and $t \in (0, 1)$, set $\hat{W}^s_{j,k} = p_k(W^s_j \cap V_k)$, $\hat{W}^u_{j,k} = p_k(W^u_j \cap V_k)$, $\hat{N}^t_{j,k} = p_k(N^t_j \cap V_k)$.
- $L^u := \bigcup_{i=0}^{n} W^u_i$, $L^s := \bigcup_{i=0}^{n} W^s_i$, $L^u_i := L^u \cap V_i$, $L^s_i := L^s \cap V_i$, $\hat{L}^u_i := p_i(L^u_i)$, and $\hat{L}^s_i := p_i(L^s_i)$.

Definition 3.1
Let $f \in MS(M^3)$. A collection N_f of linearizing neighborhoods $N_{\mathcal{O}_1}, \dots, N_{\mathcal{O}_{k_f}}$ of all saddle orbits $\mathcal{O}_1, \dots, \mathcal{O}_{k_f}$ of f is called *compatible* and the corresponding foliations are called *compatible* if, for every saddle orbits $\mathcal{O}_i, \mathcal{O}_j$, the following properties hold:

(1) If $W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j} = \emptyset$ and $W^u_{\mathcal{O}_i} \cap W^s_{\mathcal{O}_j} = \emptyset$, then $N_{\mathcal{O}_i} \cap N_{\mathcal{O}_j} = \emptyset$.

(2) If $W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j} \neq \emptyset$, then $(F^s_{\mathcal{O}_j,x} \cap N_{\mathcal{O}_i}) \subset F^s_{\mathcal{O}_i,x}$ and $(F^u_{\mathcal{O}_i,x} \cap N_{\mathcal{O}_j}) \subset F^u_{\mathcal{O}_j,x}$ for $x \in (N_{\mathcal{O}_i} \cap N_{\mathcal{O}_j})$.

Remark 3.1
The compatible system of neighborhoods is a modification of the admissible system of tubular families introduced by J. Palis and S. Smale in [26] and [28]. But here we construct a compatible system of neighborhoods for an arbitrary diffeomorphism $f \in MS(M^3)$ independently (see Theorem 2).

Figure 13 shows a foliated neighborhood of a point A belonging to a heteroclinic curve. Some Morse–Smale diffeomorphisms with heteroclinic curves on $\mathbb{S}^3$ are represented below.

THEOREM 2
For each diffeomorphism $f \in MS(M^3)$ there is a compatible system of neighborhoods.

Proof
The proof consists of four steps.

Step 1. Due to [6, Lemma 3.5] there exist f-invariant neighborhoods $U^s_0, \dots, U^s_n$ of the sets $\Sigma_0, \dots, \Sigma_n$, respectively, equipped with 2-dimensional f-invariant foliations $F^u_0, \dots, F^u_n$ whose leaves are smooth such that the following properties hold for each $i \in \{0, \dots, n\}$:

(i) The unstable manifolds W^u_i are leaves of the foliation F^u_i, and each leaf of the foliation F^u_i is transverse to L^s_i.

(ii) For any $0 \leq i < k \leq n$ and $x \in U^s_i \cap U^s_k$, we have the inclusion $F^u_{k,x} \cap U^s_i \subset F^u_{i,x}$.

Denote by $F^u_{\Omega_2}$ the 2-dimensional foliation constructed on the f-invariant neighborhood $\bigcup_{i=0}^n U^s_i$. We have similarly the 2-dimensional foliation $F^s_{\Omega_1}$. Set

$$\hat{F}^u_{\Omega_2} = p_f(F^u_{\Omega_2}), \qquad \hat{F}^s_{\Omega_1} = p_f(F^s_{\Omega_1}).$$

Step 2. Let us construct an f-invariant neighborhood U_H of the union of the heteroclinic curves $H = W^s_{\Omega_1} \cap W^u_{\Omega_2}$ equipped by an f-invariant foliation G_H consisting of 2-dimensional disks which are transversal to H and to both 2-dimensional foliations $F^u_{\Omega_2}$ and $F^s_{\Omega_1}$.

Let $\hat{H} = \hat{\Gamma}^s_f \cap \hat{\Gamma}^u_f$. Let us divide the set H into the parts $H_0, \dots, H_n$ in the following way: $H_0 = W^s_{\Sigma^-_0} \cap W^u_{\Sigma_0}$, $H_1 = (W^s_{\Sigma^-_0} \cup W^s_{\Sigma^-_1}) \cap (W^u_{\Sigma_0} \cup W^u_{\Sigma_1}) \setminus H_0, \dots, H_{n^-} = (\bigcup_{j=0}^{n^-} W^s_{\Sigma^-_i} \cap \bigcup_{j=0}^{n^-} W^u_{\Sigma_i}) \setminus (\bigcup_{j=0}^{n^- -1} H_j), \dots, H_n = (\bigcup_{j=0}^{n^-} W^s_{\Sigma^-_i} \cap \bigcup_{j=0}^{n} W^u_{\Sigma_i}) \setminus (\bigcup_{j=0}^{n-1} H_j)$.

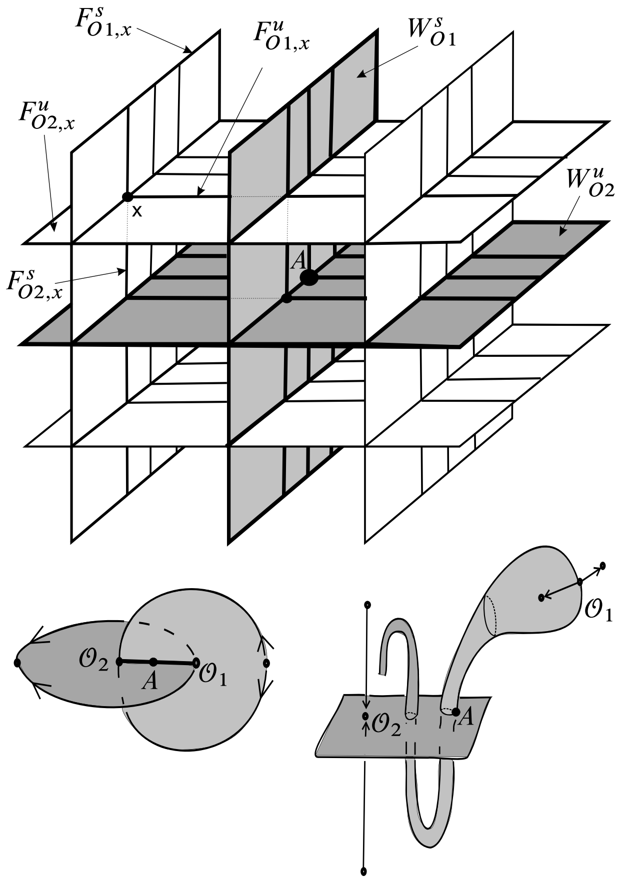

Figure 13. Foliated neighborhood of a point A on a heteroclinic curve and examples of heteroclinic curves.

Let $\hat{H} = p_f(H)$ and $\hat{H}_i = p_f(H_i)$. By construction, the set $\hat{H}$ is compact and consists of an at most countable set of curves being the projection of all heteroclinic curves of f. Moreover, the set $\hat{H}_0$ consists of a finite number of circles, as it is a transversal intersection of compact surfaces $p_f(W^s_{\Sigma_0^-})$ and $p_f(W^u_{\Sigma_0})$. Any curve $\hat{\gamma}_0 \subset \hat{H}_0$ belongs to the intersection $p_f(W^s_{\sigma_0^-}) \cap p_f(W^u_{\sigma_0})$ for some $\sigma_0^- \in \Sigma_0^-$, $\sigma_0 \in \Sigma_0$ (depending on $\hat{\gamma}_0$). Then there is a tubular neighborhood $U_{\hat{\gamma}_0}$ of the curve $\hat{\gamma}_0$ foliated by 2-dimensional disks $\hat{G}_{\hat{\gamma}_0} = \{\hat{d}_x,\, x \in \hat{\gamma}_0\}$ which are transversal to the leaves of the foliations $\hat{F}^s_{\sigma_0^-}$, $\hat{F}^u_{\sigma_0}$. Denote by $\hat{U}_0$ the union of $U_{\hat{\gamma}_0}$, $\hat{\gamma}_0 \subset \hat{H}_0$ and by $\hat{G}_0$ a foliation on $\hat{U}_0$ composed by $\hat{G}_{\hat{\gamma}_0}$, $\hat{\gamma}_0 \subset \hat{H}_0$. Denote by $\hat{G}_{0,\hat{x}}$ a leaf of the foliation $\hat{G}_0$ passing through a point $\hat{x} \in \hat{U}_0$.

By construction, the set $\hat{H}_1 \setminus \operatorname{int} \hat{U}_0$ consists of a finite number of compact curves. Any curve $\hat{\gamma}_1 \subset (\hat{H}_1 \setminus \operatorname{int} \hat{U}_0)$ belongs to the intersection $p_f(W^s_{\sigma_1^-}) \cap p_f(W^u_{\sigma_1})$ for some $\sigma_1^- \in (\Sigma_0^- \cup \Sigma_1^-)$, $\sigma_1 \in (\Sigma_0 \cup \Sigma_1)$ (depending on $\hat{\gamma}_1$). Then there is a tubular neighborhood $U_{\hat{\gamma}_1}$ of the curve $\hat{\gamma}_1$ foliated by 2-dimensional disks $\hat{G}_{\hat{\gamma}_1} = \{\hat{d}_x, x \in \hat{\gamma}_1\}$ which are transversal to the leaves of the foliations $\hat{F}^s_{\sigma_1^-}$, $\hat{F}^u_{\sigma_1}$. Denote by $\hat{U}_1$ the union of $U_{\hat{\gamma}_1}$, $\hat{\gamma}_1 \subset (\hat{H}_1 \setminus \operatorname{int} \hat{U}_0)$ and by $\hat{G}_1$ a foliation on $\hat{U}_1$ composed by $\hat{G}_{\hat{\gamma}_1}$, $\hat{\gamma}_1 \subset (\hat{H}_1 \setminus \operatorname{int} \hat{U}_0)$. Due to the compatibility of the 2-dimensional foliations $F^s_{\Omega_1}$, $F^u_{\Omega_2}$, we can construct these tubular neighborhoods with disk foliations satisfying the properties

$$(\hat{G}_{1,\hat{x}} \cap \hat{U}_0) \subset \hat{G}_{0,\hat{x}} \quad \text{for } \hat{x} \in (\hat{U}_0 \cap \hat{U}_1).$$

Continuing this process, we get a 2-dimensional foliation $\hat{G}$ which is formed by the 2-dimensional disks on a neighborhood $\hat{U}$ of the set $\hat{H}$. Then $G_H = p_f^{-1}(\hat{G})$ is a required foliation on $U_H = p_f^{-1}(\hat{U})$.

Denote by

$$\hat{F}^s_{\hat{H}} \quad (\text{resp., } \hat{F}^u_{\hat{H}})$$

a 1-dimensional foliation which is formed by the intersection of the leaves of the foliation $\hat{G}$ with the leaves of the foliation $\hat{F}^s_{\Omega_1}$ (resp., $\hat{F}^u_{\Omega_2}$), and denote by F^s_H (resp., F^u_H) a foliation on U_H consisting of the preimages with respect to the projection p_f of leaves of the foliation $\hat{F}^s_{\hat{H}}$ (resp., $\hat{F}^u_{\hat{H}}$). Without loss of generality we can suppose that the following map $\hat{\pi}^s_{\hat{H}} : U(\hat{H}) \to \hat{\Gamma}^u_f$ (resp., $\hat{\pi}^u_{\hat{H}} : U(\hat{H}) \to \hat{\Gamma}^s_f$) along the leaves $\hat{F}^s_{\hat{H}}$ (resp., $\hat{F}^u_{\hat{H}}$) is well defined.

We also have the following statement.

LEMMA 3.1

There exist f-invariant neighborhoods $U^u_0, \dots, U^u_n$ of the sets $\Sigma_0, \dots, \Sigma_n$, respectively, equipped with 1-dimensional f-invariant foliations $F^s_0, \dots, F^s_n$ with smooth leaves such that the following properties hold for each $i \in \{0, \dots, n\}$:

(iii) *The stable manifold W^s_i is a leaf of the foliation F^s_i, and each leaf of the foliation F^s_i is transverse to L^u_i.*

(iv) *For any $0 \le j < i$ and $x \in U^u_i \cap U^u_j$, we have the inclusion $(F^s_{j,x} \cap U^u_i) \subset F^s_{i,x}$.*

(v) *The intersection of a leaf of the foliation F^s_i with the set U_H is a leaf of the foliation F^s_H.*

Proof

The proof is done by an increasing induction from $i = 0$; it is skipped due to its similarity to Step 1. □

Step 3. We prove the following statement for each $i = 0, \dots, n$.

LEMMA 3.2

(vi) *There exists an f-invariant neighborhood $\tilde{N}_i$ of the set Σ_i contained in $U_i^s \cap U_i^u$ and such that the restrictions of the foliations F_i^u and F_i^s to $\tilde{N}_i$ are transverse.*

Proof
Let us choose a fundamental domain K_i^s of the restriction of f to $W_i^s \setminus \Sigma_i$ and take a tubular neighborhood $N(K_i^s)$ of K_i^s whose disk fibers are contained in leaves of F_i^u. By property (i), each leaf of the foliation F_i^u is transverse to L_i^s; hence F_i^u is transverse to W_i^s. Since every leaf of the foliation F_i^s is smooth and transversal to W_i^u then, due to the λ-lemma (see, e.g., [27, p. 85, Remarks]), F_i^s is C^1-closed to W_i^s. Thus, if $N(K_i^s)$ is small enough, the foliations F_i^s and F_i^u have transverse intersection in $N(K_i^s)$. Set

$$\tilde{N}_i := W_i^u \cup \bigcup_{k\in\mathbb{Z}} f^k\big(N(K_i^s)\big).$$

This is a neighborhood of σ_i; it satisfies condition (vi), and the previous properties (i)–(v) still hold. Moreover, we can choose $N(K_i^s)$ such that $\partial\tilde{N}_i$ is smooth. □

Step 4. To prove the theorem it remains to show the existence of linearizing neighborhoods $N_i \subset \tilde{N}_i$, $i = 0, \dots, n$, for which the required foliations are the restriction to N_i of the foliations F_i^u, F_i^s.

For each orbit of f in Σ_i, choose a unique point p. Let $\tilde{N}_p$ be a connected component of $\tilde{N}_i$ containing p. There is a homeomorphism $\varphi_p^u : W_p^u \to W_O^u$ (resp., $\varphi_p^s : W_p^s \to W_O^s$) conjugating the diffeomorphisms $f^{m_p}|_{W_p^u}$ and $a_{q_p,\nu_p}|_{W_O^u}$ (resp., $f^{m_p}|_{W_p^s}$ and $a_{q_p,\nu_p}|_{W_O^s}$). In addition, for any point $z \in \tilde{N}_p$ there is a unique pair of points $z_s \in W_p^s$, $z_u \in W_p^u$ such that $z = F_{i,z_u}^s \cap F_{i,z_s}^u$. We define a topological embedding $\tilde{\mu}_p : \tilde{N}_p \to \mathbb{R}^3$ by the formula $\tilde{\mu}_p(z) = (x_1, x_2, x_3)$, where $(x_1, x_2) = \varphi_p^u(z_u)$ and $x_3 = \varphi_p^s(z_s)$. Choose $t_0 \in (0, 1]$ such that $\mathcal{N}_{q_p}^{t_0} \subset \tilde{\mu}_p(\tilde{N}_p)$. Observe that $a_{q_p,\nu_p}|_{\mathcal{N}_{q_p}^{t_0}}$ is conjugate to $a_{q_p,\nu_p}|_{\mathcal{N}_{q_p}}$ by

$$h(x_1, x_2, x_3) = \Big(\frac{x_1}{\sqrt{t_0}}, \frac{x_2}{\sqrt{t_0}}, \frac{x_3}{\sqrt{t_0}}\Big).$$

Set $N_p = \tilde{\mu}_p^{-1}(\mathcal{N}_{q_p}^{t_0})$ and $\mu_p = h\tilde{\mu}_p : N_p \to \mathcal{N}_{q_p}$. Then, N_p is the wanted neighborhood with its linearizing homeomorphism μ_p. Set $N_{f^k(p)} = f^k(N_p)$, and denote by $\tilde{\mu}_i$ a map composed from $\tilde{\mu}_p$ for all $p \in \Sigma_i$ such that $\tilde{\mu}_{f^k(p)}(x) = \tilde{\mu}_p(f^{-k}(x))$ for $x \in N_{f^k(p)}$ and $k \in \{1, \dots, m_p\}$.

For saddle points with Morse index 1, it is possible to prove lemmas similar to 3.1 and 3.2 and, hence, construct compatible neighborhoods. □

4. The proof of the classification Theorem 1

Let us prove that the diffeomorphisms $f, f' \in MS(M^3)$ are topologically conjugated if and only if their schemes are equivalent.

Necessity. Let the diffeomorphisms $f, f' \in MS(M^3)$ be topologically conjugated by a homeomorphism $h : M^3 \to M^3$. Set $\varphi = h|_{V_f}$. Then the homeomorphism $\varphi : V_f \to V_{f'}$ conjugates the diffeomorphisms $f|_{V_f}$ and $f'|_{V_{f'}}$. As the natural projects $p_f, p_{f'}$ are covers and φ sends the invariant manifolds of the periodic points of f to the invariant manifolds of the periodic points of f' preserving dimension and stability, we have that, due to Statement 2.9, a map $\hat{\varphi} = p_{f'}\varphi p_f^{-1} : \hat{V}_f \to \hat{V}_{f'}$ is the required homeomorphism making the schemes $S_f, S_{f'}$ equivalent.

Sufficiency. For proving the sufficiency of the conditions in Theorem 1, let us consider a homeomorphism $\hat{\varphi} : \hat{V}_f \to \hat{V}_{f'}$ such that

(1) $\eta_f = \eta_{f'}\hat{\varphi}_*$;

(2) $\hat{\varphi}(\hat{\Gamma}^s_f) = \hat{\Gamma}^s_{f'}$ and $\hat{\varphi}(\hat{\Gamma}^u_f) = \hat{\Gamma}^u_{f'}$.

From now on, the dynamical objects attached to f' will be denoted by $L'^u, L'^s, \Sigma'_i, \dots$ with the same meaning that $L^u, L^s, \Sigma_i, \dots$ have with respect to f. Due to property (1), $\hat{\varphi}$ lifts to an *equivariant*[7] homeomorphism $\varphi : V_f \to V_{f'}$; that is, $f'|_{V_{f'}} = \varphi f \varphi^{-1}|_{V_{f'}}$.

If the homeomorphism φ admits a continuous extension to the 1-dimensional stable and unstable manifold of the saddle points, then it induces a conjugacy between f and f', ending the proof. However, in general, φ does not admit such a continuous extension on the 1-dimensional invariant manifolds: we will need to modify φ in order to get a new homeomorphism, extending to the 1-dimensional invariant manifolds. Before the detailed construction we give a sketch of such modification.

4.1. *Intuitive idea for the proof*

It is not hard to check that φ can be extended on the periodic saddles (see Lemma 4.1 below). Due to property (2), φ can be extended so that it maps Γ^u_f to $\Gamma^u_{f'}$ and Γ^s_f to $\Gamma^s_{f'}$ (see Lemma 4.2 below).

Each saddle point σ is equipped with an invariant linearizing neighborhood N_σ with invariant stable F^s_σ and unstable F^u_σ foliations. Furthermore, this system of local foliations is *compatible*: these invariant neighborhoods may intersect, and, on the

[7]For brevity, *equivariance* stands for (f, f')-equivariance.

intersection, the stable leaves of dimension 1 are contained in the 2-dimensional stable leaves and the stable foliations of the same dimensions coincide; the same holds similarly for the unstable foliations (see Theorem 2 above).

These locally invariant compatible foliations induce on the quotient $\hat{V}_f$ and $\hat{V}_{f'}$ compatible foliations defined in the neighborhood of the stable and unstable laminations $\hat{\Gamma}^s_f \cup \hat{\Gamma}^u_f$ and $\hat{\Gamma}^s_{f'} \cup \hat{\Gamma}^u_{f'}$, respectively. If one can modify $\hat{\varphi}\colon \hat{V}_f \to \hat{V}_{f'}$ in order that it preserves these foliations, we are done: each point in a 1-dimensional stable or unstable manifold is the intersection point of this manifold with a 2-dimensional leaf.

This approach works well when the diffeomorphisms f and f' have no heteroclinic intersections. In that case, the laminations $\hat{\Gamma}^s_f$ and $\hat{\Gamma}^u_f$ are disjoint and consist of finitely many tori or Klein bottles. Then we can modify $\hat{\varphi}$ in a neighborhood of the laminations $\hat{\Gamma}^s_f$ and $\hat{\Gamma}^u_f$ so that it preserves the system of compatible foliations, and we can glue this local modification with the old homeomorphism far from the laminations. This comes from the following general fact.

PROPOSITION 4.1
For a torus Γ embedded into a 3-manifold M^3 and a given orientation-preserving homeomorphism ψ defined in a neighborhood $N(\Gamma)$ of Γ and inducing the identity by restriction on Γ, there is a homeomorphism Ψ which is the identity map in a small neighborhood of Γ and which coincides with ψ in $\partial N(\Gamma)$.

We have not been able to get such a statement for the general and transversely intersecting laminations $\hat{\Gamma}^s_f$ and $\hat{\Gamma}^u_f$. For this reason, we give up this global conceptual approach and go back to a progressive, step-by-step approach. What we do is consider the saddle points one by one. First, we consider a saddle σ_0 whose 2-dimensional unstable manifold is not accumulated by others. In other words, we consider a saddle whose 2-dimensional unstable manifold does not intersect the 1-dimensional stable manifolds of different saddles. For every such saddle, we modify φ in a neighborhood of its unstable manifolds. Then we consider saddles σ_1 whose 2-dimensional unstable manifolds are only accumulated by the ones on which we have already done the modification. Then we perform a modification of φ preserving the modifications which have been already done, and we prepare the next modifications. In this way we consider one by one the saddles $\sigma_0, \dots, \sigma_n$ with 2-dimensional unstable manifolds. After we perform all the modifications for getting an extension along every 2-dimensional unstable manifold, we consider the saddle points with the other index, namely, the 2-dimensional stable manifold. We will check that we can perform the same kind of modifications without breaking the extensions which have been already done.

We explain, very roughly, how we modify the conjugacy homeomorphism φ in the neighborhood of a periodic orbit. First notice that the homeomorphism φ, in restriction to a given 2-dimensional unstable manifold, extends by continuity in a unique way on the heteroclinic points (which are at most countably many). Therefore φ can be considered as defined on the whole 2-dimensional unstable (or stable) manifold. We also define a conjugacy homeomorphism ψ^s on the union of the 1-dimensional stable manifold, so that ψ^s preserves the holonomies of the local 2-dimensional (compatible) unstable foliations. In any small linearizing neighborhood of an index-2 saddle point σ, one gets a local conjugacy homeomorphism ϕ_σ whose expression is the product of the restriction of φ to W^u_σ by the restriction ψ^s to W^s_σ. Thus $\xi_\sigma = \phi_\sigma^{-1}\varphi$ is a homeomorphism defined in a neighborhood of σ and commuting with the diffeomorphism f and inducing the identity map on W^u_σ.

The goal now is to build a homeomorphism commuting with f, coinciding with the identity map in a neighborhood of σ and coinciding with ξ_σ outside of a slightly larger neighborhood. The difficulty is that this homeomorphism needs to preserve the intersection of this neighborhood with the invariant manifold crossing the neighborhood. We want also to preserve the extension we already have done in a neighborhood of the other saddles. These neighborhoods are crossing the neighborhood of σ as vertical tubes B (see Figure 14), which we want to preserve. As W^u_σ locally disconnects the manifold, we work only on one side of W^u_σ. One can choose a fundamental domain of f on this side which is the product of an annulus (fundamental domain in W^u_σ) by a vertical segment.

The main topological difficulty we need to solve is to extend a homeomorphism defined in a neighborhood of this fundamental domain while preserving at the same time the 2-dimensional horizontal unstable leaves, some 2-dimensional stable leaves T (see Figure 14) also crossing the fundamental domain, and the homeomorphism in some crossing tube. This will be done by using a variation of Proposition 4.1. To solve this puzzle, we first try to simplify as much as possible the geometry of these intersections. Indeed, we reduce all constructions to $\mathbb{R}^3$ using linearizing maps. So, in the subsection below we prove the crucial Proposition 4.2 for the linear model, applying corollaries from classical topological results in Section 5. In Section 4.3, in a sequence of lemmas, we explain how to reduce a nonlinear situation to the linear one.

4.2. *Linear model*

Let us recall that we denoted by $a : \mathbb{R}^3 \to \mathbb{R}^3$ the canonical linear diffeomorphism with the unique fixed point $O = (0,0,0)$, which is a saddle point with the plane Ox_1x_2 as the unstable manifold and with the axis Ox_3 as the stable manifold; for simplicity, we assume that a has a sign $\nu = +1$. That is, $a(x_1,x_2,x_3) = (2x_1, 2x_2, x_3/2)$. Let

$$N = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : (x_1^2 + x_2^2)x_3^2 \le 1, x_3 \ge 0\}.$$

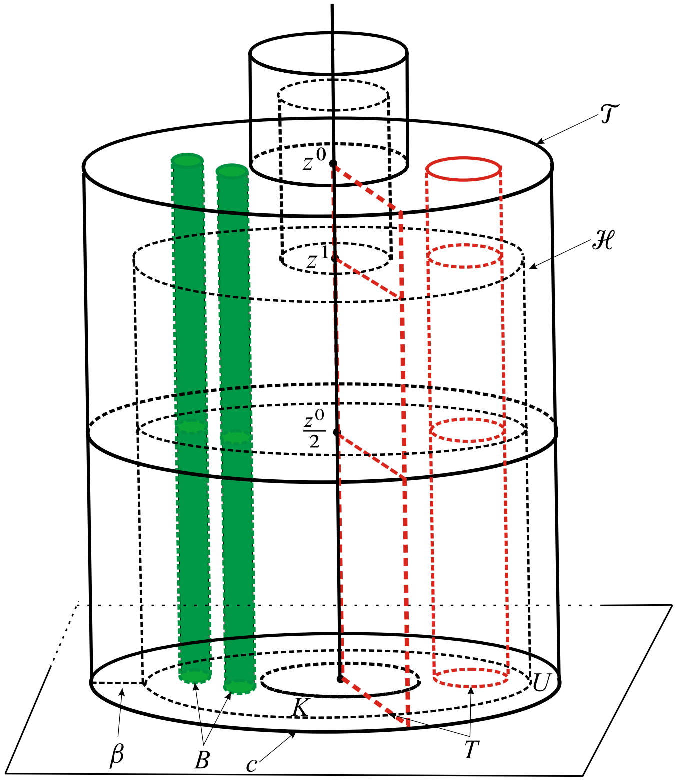

Figure 14. A linear model.

Let $\rho > 0$, $\delta \in (0, \frac{\rho}{4})$ and

$$d = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \le \rho^2, x_3 = 0\},$$
$$U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (\rho - \delta)^2 \le x_1^2 + x_2^2 \le \rho^2, x_3 = 0\},$$
$$c = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = \rho^2, x_3 = 0\},$$
$$c^0 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = \left(\rho - \frac{\delta}{2}\right)^2, x_3 = 0\right\},$$
$$c^1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = (\rho - \delta)^2, x_3 = 0\}.$$

Let $K = d \setminus \operatorname{int} a^{-1}(d)$, $V = (K \cup a(K)) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \ge 0, x_3 = 0\}$ and $\beta = U \cap Ox_1^+$, where $Ox_1^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3^2 + x_2^2 = 0, x_1 > 0\}$. Let $T \subset Ox_1x_2$ be an a-invariant closed 1-dimensional lamination such that either $c = U \cap T$ or T (maybe empty) is transversal to ∂U and every connected component of $T \cap U$

is a segment which has a unique intersection point with each connected component of ∂U.

Choose a point $Z^0 = (0,0,z^0) \in Ox_3^+$ such that $\rho^2 \cdot (z^0)^2 < \frac{1}{4}$. Then, choose a point $Z^1 = (0,0,z^1)$ in Ox_3^+ so that $z^0 > z^1 > \frac{z^0}{2}$. Let $\Pi(z) = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 = z\}$. For every set $A \subset Ox_1x_2$, let

$$\tilde{A} = A \times [0, z^0].$$

Denote by $\mathcal{W}$ a 3-ball bounded by the surface $\tilde{c}$ and the two planes $\Pi(z^0)$ and $\Pi(\frac{z^0}{2})$. Let Δ be a closed 3-ball bounded by the surface $\tilde{c}^1$ and the two planes Ox_1x_2 and $\Pi(z^1)$. Let

$$\mathcal{T} = \bigcup_{k\in\mathbb{Z}} a^k(\tilde{d}) \qquad \text{and} \qquad \mathcal{H} = \bigcup_{k\in\mathbb{Z}} a^k(\Delta).$$

Notice that the construction yields $\mathcal{H} \subset \mathcal{T}$ and makes $\mathcal{W}$ a fundamental domain for the action of a on $\mathcal{T} \setminus Ox_1x_2$ (see Figure 14).

PROPOSITION 4.2

Let $z^0 > z^1 > \cdots > z^m > \frac{z^0}{2} > 0$, and let $\xi : \mathcal{T} \setminus Ox_3 \to N$ be a topological embedding with the following properties:

(i) $\xi a = a\xi$;

(ii) *ξ is the identity on Ox_1x_2;*

(iii) *$\xi(\Pi(z^0) \cap \mathcal{T}) \subset \Pi(z^0)$ and $\xi(\Pi(z^i) \cap \partial\mathcal{T}) \subset \Pi(z^i)$, $i \in \{2,\dots,m\}$;*

(iv) *$\xi(\tilde{c}) \cap \tilde{c}^0 = \emptyset$, $\xi(\tilde{c}^1) \cap \tilde{c}^0 = \emptyset$ and $\xi(\tilde{\beta}) \subset \tilde{V}$;*

(v) *$\xi(\tilde{T} \cap \mathcal{T}) \subset \tilde{T}$ and $\xi^{-1}(\tilde{T}) \subset \tilde{T}$.*

Then there is a topological embedding $\zeta : \mathcal{T} \to N$ such that

(1) $\zeta a = a\zeta$;

(2) *ζ is the identity on $\mathcal{H}$;*

(3) *$\zeta(\Pi(z^i) \cap \mathcal{T}) \subset \Pi(z^i)$, $i \in \{0,2,\dots,m\}$*

(4) *ζ is ξ on $\partial\mathcal{T}$;*

(5) *$\zeta(\tilde{T} \cap \mathcal{T}) \subset \tilde{T}$ and $\zeta^{-1}(\tilde{T}) \subset \tilde{T}$.*

Moreover, if ξ is the identity on $\tilde{B}$ for a set $B \subset (K \setminus U)$, then ζ is also the identity on $\tilde{B}$.

Proof

By construction, the 3-ball $\mathcal{W}$ is a fundamental domain of the diffeomorphism a restricted to $\mathcal{T} \setminus Ox_1x_2$. Therefore, to prove the proposition, it is enough to construct the topological embedding $\zeta_{\mathcal{W}} = \zeta|_{\mathcal{W}}$ with the properties (1)–(5) on the set $\mathcal{W}$. We can extend ζ_W to $\mathcal{T}$ by the formula $\zeta(x) = a^{-k}(\zeta_W(a^k(x)))$, where $a^k(x) \in \mathcal{W}$ on $\mathcal{T} \setminus Ox_1x_2$, and we define ζ to be the identity on Ox_1x_2.

First, we define $\zeta_{\mathcal{W}}$ to be the identity on $\mathcal{W} \cap \mathcal{H}$. Divide the remaining solid torus $\mathcal{Q} = \mathcal{W} \setminus \mathcal{H}$ into the following pieces: a solid torus $Q_1 = \tilde{U} \cap \mathcal{Q}$, a solid torus $Q_2 = a^{-1}(\tilde{U}) \cap \mathcal{Q}$, and a solid torus $Q_3 = \mathcal{Q} \setminus (Q_1 \cup Q_2)$ (see Figure 14). Define a topological embedding $h_{Q_3} : Q_3 \to \mathcal{Q}$ in the following way. Let $\kappa : [z^1, z^0] \to [0, z^0]$ be a homeomorphism given by the formula $\kappa(z) = \frac{z^0(z-z^1)}{z^0-z^1}$ and $h_{Q_3} = (\mathrm{id} \times \kappa)^{-1}\xi(\mathrm{id} \times \kappa)$ on $Q_3 = (K \setminus \operatorname{int} U) \times [z^1, z^0]$. Thus we get the desired embedding $\zeta_{\mathcal{W}}$ on $(\mathcal{W} \cap \mathcal{H}) \cup Q_3$.

Let $z^{m+1} = z^0/2$. By property (iv) of the map ξ we have $\xi(\tilde{c}^1) \cap \tilde{c} = \emptyset$; then the surfaces $\tilde{c}$, $\zeta_{\mathcal{W}}(\tilde{c}^1)$, $\Pi(z^0)$, and $\Pi(z^{m+1})$ bound a closed 3-dimensional set, which we denote by $\check{Q}_1$. By property (iv) of the map ξ we have $\xi(\tilde{c}) \cap \tilde{c}^1 = \emptyset$; then the surfaces $\zeta_{\mathcal{W}}(a^{-1}(\tilde{c}) \cap \mathcal{W})$, $a^{-1}(\tilde{c}^1) \cap \mathcal{W}$, $\Pi(z^0)$, and $\Pi(z^1)$ bound a closed 3-dimensional set, which we denote by $\check{Q}_2$. Due to Proposition 5.1 below the sets $\check{Q}_1$, $\check{Q}_2$ are solid tori. Let

$$A^i = Q_1 \cap \Pi(z^i), \qquad \check{A}^i = \check{Q}_1 \cap \Pi(z^i), \quad i \in \{0, 2, \dots, m+1\}.$$

Let $S^i(\check{S}^i)$, $i \in \{2, \dots, m+1\}$ be the closure of a connected component of the set $Q_1 \setminus \bigcup_{j=2}^{m} \Pi(z^i)$ $(\check{Q}_1 \setminus \bigcup_{j=2}^{m} \Pi(z^i))$ bounded by $\Pi(z^i)$ below. By Proposition 5.1 below, $S^i(\check{S}^i)$ is a solid torus. Further, let us consider two possibilities distinctly. Case (1): $c = U \cap T$; Case (2): T is either empty or consists of curves joining two boundary components of U.

Case 1. Let us define a topological embedding $h_{A^i} : A^i \to \check{A}^i$ such that

- h_{A^i} is $h_{\mathcal{W}}$ on $\tilde{c}^1 \cap A^i$ and is ξ on $\tilde{c} \cap A^i$, $h_{A^i}(\tilde{\beta} \cap A^i) \subset \tilde{V}$, and $h_{A^i}(\tilde{T} \cap A^i) = \tilde{T} \cap \check{A}^i$ for $i \in \{2, \dots, m+1\}$;
- $h_{A^0} = \xi|_{A^0}$.

Moreover, we have a homeomorphism $h_{\partial S^i} : \partial S^i \to \partial \check{S}^i$ which coincides with h_{A^j} on $S^i \cap A^j$, with ξ on $S^i \cap \tilde{c}$, and with $\zeta_{\mathcal{W}}$ on $S^i \cap \tilde{c}^1$. By construction, a curve $\mu^i = \partial(S^i \cap \tilde{\beta})$ is a meridian of S^i and $\check{\mu}^i = h_{\partial S^i}(\mu^i)$ is a meridian of $\check{S}^i$. Then (see, e.g., [33]) there is a homeomorphism $h_{S^i} : S^i \to \check{S}^i$ such that $h_{S^i}|_{\partial S^i} = h_{\partial S^i}$.

Similarly, we have a homeomorphism $h_{\partial Q_2} : \partial Q_2 \to \partial \check{Q}_2$ such that $h_{\partial Q_2}|_{Q_2 \cap \Pi(z^1)} = \mathrm{id}|_{Q_2 \cap \Pi(z^1)}$, $h_{Q_2 \cap \Pi(z^0)} = a^{-1} h_{A_1^{m+1}} a|_{Q_2 \cap \Pi(z^0)}$, $h_{\partial Q_2}|_{Q_2 \cap a^{-1}(\tilde{c})} = \zeta_{\mathcal{W}}|_{Q_2 \cap a^{-1}(\tilde{c})}$, and $h_{\partial Q_2}|_{Q_2 \cap a^{-1}(\tilde{c}^1)} = \mathrm{id}|_{Q_2 \cap a^{-1}(\tilde{c}^1)}$. Hence there is a homeomorphism $h_{Q_2} : Q_2 \to \check{Q}_2$ such that $h_{Q_2}|_{\partial Q_2} = h_{\partial Q_2}$.

Thus the required homeomorphism is defined by the formula

$$\zeta|_W(x) = \begin{cases} x, & x \in (\mathcal{H} \cap \mathcal{W}); \\ h_{Q_3}(x), & x \in Q_3; \\ h_{S^i}(x), & x \in S^i, i \in \{2, \dots, m+1\}; \\ h_{Q_2}(x), & x \in Q_2. \end{cases}$$

Case 2. According to Proposition 5.2 and Remarks 5.1 there is a topological embedding $h_{A^i} : A^i \to \check{A}^i$ such that

- h_{A^i} is $h_{\mathcal{W}}$ on $\tilde{c}^1 \cap A^i$, is ξ on $\tilde{c} \cap A^i$ and $h_{A^i}(\tilde{T} \cap A^i) = \tilde{T} \cap \check{A}^i$ for $i \in \{2, \dots, m+1\}$.

Let $h_{A^0} = \xi|_{A^0}$. Thus we have a homeomorphism $h_{\partial S^i} : \partial S^i \to \partial \check{S}^i$ which coincides with h_{A^j} on $S^i \cap A^j$, with ξ on $S^i \cap \tilde{c}$, and with $\zeta_{\mathcal{W}}$ on $S^i \cap \tilde{c}^1$; also, $h_{\partial S^i}(\partial S^i \cap \tilde{T}) = \partial \check{S}^i \cap \tilde{T}$. According to Proposition 5.2 there is a homeomorphism $h_{S^i} : S^i \to S^i$ such that h_{S^i} is $h_{\partial S^i}$ on ∂S^i and $h_{S^i}(\tilde{T} \cap S^i) = \tilde{T} \cap \check{S}^i$.

Similarly, we can construct a homeomorphism $h_{\partial Q_2} : \partial Q_2 \to \partial \check{Q}_2$ such that $h_{\partial Q_2}|_{Q_2 \cap \Pi(z^1)} = \mathrm{id}|_{Q_2 \cap \Pi(z^1)}$, $h_{Q_2 \cap \Pi(z^0)} = a^{-1} h_{A_1^{m+1}} a|_{Q_2 \cap \Pi(z^0)}$, $h_{\partial Q_2}|_{Q_2 \cap a^{-1}(\tilde{c})} = \zeta_{\mathcal{W}}|_{Q_2 \cap a^{-1}(\tilde{c})}$, and $h_{\partial Q_2}|_{Q_2 \cap a^{-1}(\tilde{c}^1)} = \mathrm{id}|_{Q_2 \cap a^{-1}(\tilde{c}^1)}$. Hence there is a homeomorphism $h_{Q_2} : Q_2 \to \check{Q}_2$ such that $h_{Q_2}|_{\partial Q_2} = h_{\partial Q_2}$ and $h_{Q_2}(Q_2 \cap \tilde{T}) = \check{Q}_2 \cap \tilde{T}$.

Thus the required homeomorphism is defined by the formula

$$\zeta|_W(x) = \begin{cases} x, & x \in \mathcal{T}; \\ h_{Q_3}(x), & x \in Q_3; \\ h_{S^i}(x), & x \in S^i, i \in \{2, \dots, m+1\}; \\ h_{Q_2}(x), & x \in Q_2. \end{cases}$$ □

4.3. *Reduction to the linear model*

Recall the partition $\Sigma_0 \sqcup \cdots \sqcup \Sigma_n$ associated with the Smale order on the periodic points of index 2. Let $\hat{\varphi} : \hat{V}_f \to \hat{V}_{f'}$ be a homeomorphism which gives the equivalence of the schemes, and let $\varphi : V_f \to V_{f'}$ be a lift of $\hat{\varphi}$. First, we state needed results from [6] and then use Notation 3.1.

LEMMA 4.1 ([6, Lemma 4.1])
There is a unique continuous extension of φ to Ω_2 such that $\varphi(\Sigma_i) = \Sigma'_i$ for every $i = 0, \dots, n$. This extension is equivariant and bijective from Ω_2 to Ω'_2, which preserves the type of the orientation and the period of points.

Let us introduce the *radial functions* $r_i^u, r_i^s : N_i \to [0, +\infty)$ defined by

$$r_i^u(x) = \left\| \mu_i(x_i^u) \right\|^2 \qquad \text{and} \qquad r_i^s(x) = \left(\mu_i(x_i^s) \right)^2.$$

With this definition at hand, the neighborhood N_i^t of Σ_i is defined by the inequality

$$r_i^u(x).r_i^s(x) < t.$$

Observe that the radial function r_i^s endows each stable separatrix γ_p of $p \in \Sigma_p$ with a natural order (and similarly with $'$).

LEMMA 4.2 ([6, Lemma 4.2])
There is a unique continuous extension of $\varphi|_{\Gamma_f^u}$,

$$\varphi^{us} : L^u \to L'^u,$$

such that the following hold:
(1) *If* $x \in W_j^u \cap W_i^s$, $j > i$, *then* $\varphi^{us}(x) \in W_j'^u \cap W_i'^s$.
(2) *If* x *and* y *lie in* $\gamma_p \cap L^u$ *with* $r_p^s(x) < r_p^s(y)$, *then* $\varphi^{us}(x)$ *and* $\varphi^{us}(y)$ *lie in* $\gamma'_{\varphi(p)} \cap L'^u$ *with* $r'_{\varphi(p)}(\varphi^{us}(x)) < r'_{\varphi(p)}(\varphi^{us}(y))$.

Remark 4.1
Due to Lemma 2.13, we may assume that in all the lemmas below we choose values $t = \beta_i, a_i, \ldots$ such that the boundary of the linearizing neighborhood N_i^t does not contain any heteroclinic point.

LEMMA 4.3 ([6, Lemma 4.4])
There are numbers $\beta_0, \ldots, \beta_n \in (0, 1]$ *such that, for every* $i \in \{0, \ldots, n\}$, *for every point* $p \in \Sigma_i$ *and* $x \in N_p^{\beta_i} \cap L^u$, *the next inequality holds:*

$$r_i'^u\big(\varphi^{us}(x_i^u)\big).r_i'^s\big(\varphi^{us}(x_i^s)\big) < \frac{1}{2}.$$

Let us set $a_0 = \beta_0$ and $a_1 = \beta_1$.

LEMMA 4.4
If $n \geq 2$, *then there are numbers* $a_2 \in (0, \beta_2], \ldots, a_n \in (0, \beta_n]$ *with the following property for each* $i \in \{2, \ldots, n\}$: *for* $0 \leq k \leq i - 2$ *the intersection*

$$W_k^s \cap \Big(N_i^{a_i} \setminus \Big(\bigcup_{\mu=k+1}^{i-1} N_\mu^{a_\mu}\Big)\Big)$$

is either empty or consists of open arcs, each of which is a leaf of the foliation $F_i^s \cap N_i^{a_i}$.

Proof
We will construct the sequence by induction on $i = 2, \ldots, n$.

For $i = 2$, the unstable manifolds of points from Σ_2 have only heteroclinic intersections with the stable manifolds of saddles from Σ_k with $k < 2$. Thus, the projection $\hat{W}_{2,1}^u$ is a union of a finite number of pairwise disjoint smooth tori and Klein bottles in the manifold $\hat{V}_1$. The set $\hat{L}_1^s$ is a compact 1-dimensional lamination, and the intersection $\hat{W}_{2,1}^u \cap \hat{L}_1^s$ is transversal and consists of an at most countable set of points

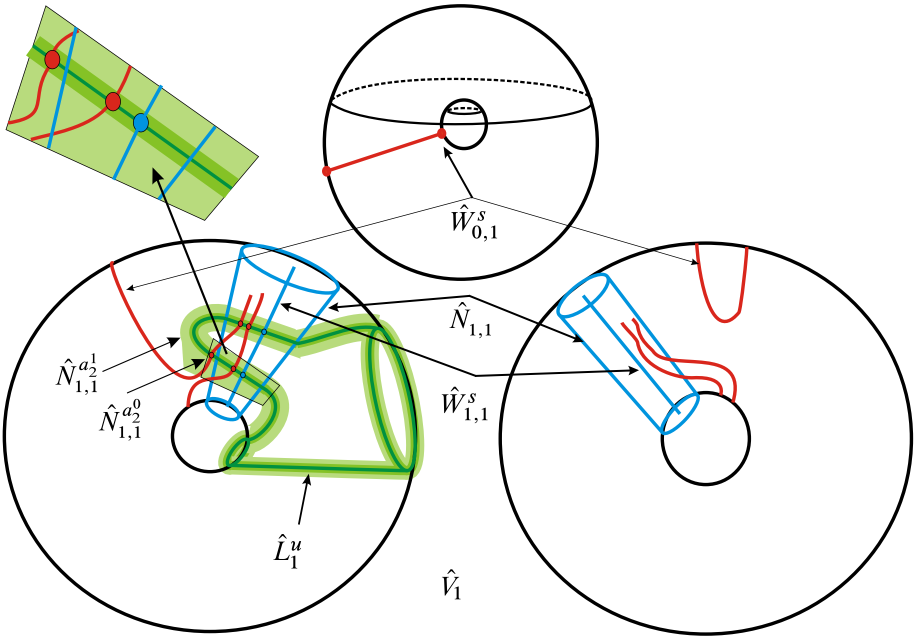

Figure 15. Illustration of the proof of Lemma 4.4 for $i = 2$.

which are the projection with respect to p_1 of the heteroclinic orbits from the unstable manifolds W^u_2. The set $\hat{W}^u_{2,1} \cap (\hat{W}^s_{0,1} \setminus \hat{N}^{a_1}_{1,1})$ is either empty or consists of a finite number of $q_0 \in \mathbb{N}$ points. Due to Remark 4.1, there is a number $a_2 \in (0, \beta_2]$ such that the intersection $\hat{N}^{a_2}_{2,1} \cap (\hat{W}^s_{0,1} \setminus \hat{N}^{a_1}_{1,1})$ is either empty or consists of exactly q_0 intervals, each of which is a leaf of the foliation $\hat{F}^s_2 \cap \hat{N}^{a_2}_2$ (see Figure 15, where $q_0 = 1$). Hence, the intersection $W^s_0 \cap (N^{a_2}_2 \setminus N^{a_1}_1))$ is either empty or consists of open arcs, each of which is a leaf of the foliation $F^s_2 \cap N^{a_2}_2$.

Let us describe how to find the number a_i, $i > 0$, supposing that the numbers a_j with the desired properties are already constructed for all $j = 0, \dots, i-1$.

The unstable manifold of points from Σ_i have only heteroclinic intersections with the stable manifolds of saddles from Σ_k with $k < i$. Thus, the projection $\hat{W}^u_{i,i-1}$ is a smooth torus in the manifold $\hat{V}_{i-1}$. The set $\hat{L}^s_{i-1}$ is a compact 1-dimensional lamination, the intersection $\hat{W}^u_{i,i-1} \cap \hat{L}^s_{i-1}$ is transversal and consists of an at most countable set of points which are the projection with respect to p_{i-1} of the heteroclinic orbits from the unstable manifolds W^u_i. For each $k = 0, \dots, i-2$, the intersection $\hat{W}^u_{i,i-1} \cap (\hat{W}^s_{k,i-1} \setminus \bigcup_{\mu=k+1}^{i-1} \hat{N}^{a_\mu}_{\mu,i-1})$ is either empty or consists of $q_k \geq 0$ points. Due to Remark 4.1 there is a number $a^k_i \in (0, \beta_i]$ such that the intersection

$\hat{N}^{a_i^k}_{i,i-1} \cap (\hat{W}^s_{k,i-1} \setminus \bigcup_{\mu=0}^{k-1} \hat{N}^{a_\mu}_{\mu,i-1})$ is either empty or consists of exactly q_k intervals, each of which is a leaf of the foliation $\hat{F}^s_i \cap \hat{N}^{a_i^k}_i$.

Thus $a_i = \min\{a_i^0, \dots, a_i^{i-2}\}$ as required. □

The corollary below immediately follows from Lemma 4.4.

COROLLARY 4.1

For each $k \in \{0, \dots, n-1\}$, the intersection $\hat{W}^s_{k,k} \cap (\bigcup_{i=k+1}^{n} \hat{N}^{a_i}_{i,k})$ consists of a finite number of open arcs $\hat{I}^k_1, \dots, \hat{I}^k_{r_k}$ such that $\hat{I}^k_l$ for each $l = 1, \dots, r_k$ is a connected component of the intersection $\hat{W}^s_{k,k} \cap \hat{N}^{a_i}_{i,k}$ for some $i > k$.

For brevity, for $i = 0, \dots, n$, we denote by φ^u_i the restriction $\varphi^{us}|_{W^u_i}$ in the rest of the proof of Theorem 1. Let $\psi^s_i : W^s_i \to W'^s_i$ be any equivariant homeomorphism which extends $\varphi^{us}|_{W^s_i \cap L^u}$, and let $t_i \in (0, 1)$ be a small enough number so that, for every $x \in N^{t_i}_i$, the next inequality holds:

$$(*)_i \quad r'^s\big(\varphi^u_i(x^u_i)\big).r'^u\big(\psi^s_i(x^s_i)\big) < 1.$$

In this setting, one derives an equivariant embedding, $\phi_{\varphi^u_i, \psi^s_i} : N^{t_i}_i \to N'_i$, which is defined by sending $x \in N^{t_i}_i$ to $(\varphi^u_i(x^u_i), \psi^s_i(x^s_i))$.

LEMMA 4.5

There is a homeomorphism $\psi^s : L^s \to L'^s$ consisting of conjugating homeomorphisms $\psi^s_0 : W^s_0 \to W'^s_0, \dots, \psi^s_n : W^s_n \to W'^s_n$ such that, for each $i \in \{0, \dots, n\}$, we have the following:

(1) $\psi^s_i|_{W^s_i \cap L^u} = \varphi^{us}|_{W^s_i \cap L^u}$.

(2) *The topological embedding $\phi_{\varphi^u_i, \psi^s_i}$ is well defined on $N^{a_i}_i$.*

(3) *If $x \in (W^s_i \cap N^{a_j}_j)$, $j > i$, then $\psi^s_i(x) = \phi_{\varphi^u_j, \psi^s_j}(x)$.*

Proof

We are going to construct ψ^s_i by a decreasing induction on i from $i = n$ up to $i = 0$.

The stable manifolds of the saddles from Σ_n do not have any heteroclinic intersection. The projection $\hat{W}^s_{n,n}$ is a smooth submanifold of the manifold $\hat{V}_n$ consisting of a finite number of connected components homeomorphic to circle. The same holds for $\hat{W}'^s_{n,n}$. Let us define a homeomorphism $\hat{\psi}^s_n : \hat{W}^s_{n,n} \to \hat{W}'^s_{n,n}$ in the following way. First notice that the extension of φ to Ω_2 obtained by Lemma 4.1 satisfies $\varphi(\Sigma_i) = \Sigma'_i$ by virtue of Lemma 4.2. Since it is equivariant, φ maps an f-orbit $\mathcal{O}$ in Σ_n to an f'-orbit $\mathcal{O}'$ in Σ'_n. If the orientation type of $\mathcal{O}$ is -1, then so is $\mathcal{O}'$, as can be seen by the nonorientability of the corresponding leaves of Γ^u and Γ'^u. In this case, the part of

$W^s_{n,n}$ corresponding to $\mathcal{O}$ is a circle $\hat{\gamma}$, as well as the part $\hat{\gamma}'$ of $W^s_{n,n}$ corresponding to $\mathcal{O}'$. Moreover, both circles are oriented by the dynamics. Define $\hat{\psi}^s_n$ on $\hat{\gamma}$ to be an orientation-preserving homeomorphism to $\hat{\gamma}'$. If the orientation type of $\mathcal{O}$ is $+1$, then the part of $W^s_{n,n}$ corresponding to $\mathcal{O}$ consists of two circles, and these circles are determined by the transverse orientation of the leaf of $\hat{\Gamma}^u$ corresponding to $\mathcal{O}$. The same is true for $\mathcal{O}'$. Considering how $\hat{\varphi}$ maps the transverse orientation, one can assign circles to circles. Again define $\hat{\psi}^s_n$ on these circles as homeomorphisms preserving the dynamically determined orientation.

Let $\tilde{\psi}^s_n : W^s_n \setminus \Sigma_n \to W'^s_n \setminus \Sigma'_n$ be a lift with respect to p_n of $\hat{\psi}^s_n$ on $W^s_n \setminus \Sigma_n$. For a circle $\hat{\gamma}$ in $W^s_{n,n}$ corresponding to an orbit $\mathcal{O}$, we have that $p_n^{-1}(\hat{\gamma})$ consists of several curves in $W^s_n \setminus \Sigma_n$. First consider the case where the orientation type of $\mathcal{O}$ is $+1$. In this case, these curves correspond bijectively to points in $\mathcal{O}$. The same is true for $\hat{\gamma}' = \hat{\psi}^s_n(\hat{\gamma})$. Therefore, the extended φ yields a correspondence between the curves. Define a lift ψ^s_n so as to map a curve to the curve corresponding to it. If the orientation type of $\mathcal{O}$ is -1, then the curves in $p_n^{-1}(\hat{\gamma})$ are determined by the points p in $\mathcal{O}$ plus the local transverse orientation around p of the corresponding leaves in $\hat{\Gamma}^u$. The same is true for $\hat{\gamma}' = \hat{\psi}^s_n(\hat{\gamma})$. But, also in this case, the map φ gives us the correspondence between the curves, and we can define ψ^s_n just as before. Notice that there is still ambiguity for the lift, which we shall make use of in the next step.

Let $p \in \Sigma_n$, and let γ_p be a connected component of $W^s_p \setminus p$. Let us choose a fundamental domain I_{γ_p} of $f^{m_{\gamma_p}}|_{\gamma_p}$, where m_{γ_p} is the period of the separatrix γ_p. Set $N_{I_{\gamma_p}} = \{x \in N^{a_n}_p \mid x^s_n \in I_{\gamma_p}\}$ and $\lambda'^u_{\gamma_p} = \sup\{r'^u_p(\varphi^{us}(x^u_n)) \mid x \in N_{I_{\gamma_p}}\}$. For $k \in \mathbb{Z}$, we set $\lambda'^s_{\gamma_p}(k) = \frac{1}{2^k} \cdot \sup\{r'^s_{p'}(\tilde{\psi}^s_n(x)) \mid x \in I_{\gamma_p}\}$. As $\lambda'^s_{\gamma_p}(k)$ tends to 0 as k tends to ∞, there is $k_* \in \mathbb{N}$ such that $\lambda'^s_{\gamma_p}(k_*) \cdot \lambda'^u_{\gamma_p} < 1$. Set $\psi^s_n|_{\gamma_p} = f'^{k_*}\tilde{\psi}^s_n|_{\gamma_p}$ for such k_* which is a multiple of $\operatorname{per}(\gamma_p)$. We define similarly ψ^s_n on other connected components of $W^s_p \setminus p$ (different from γ_p) and set $\psi^s_n(p) = p'$. Thus $r'^u_{p'}(\varphi^{us}(x^u_n)) \cdot r'^s_{p'}(\psi^s_n(x^s_n)) \le \lambda'^s_{\gamma_p}(k_*) \cdot \lambda'^u_{\gamma_p} < 1$ and, hence, the topological embedding $\phi_{\varphi^u_n, \psi^s_n}$ is well defined on $N^{a_n}_p$. Then we do the same for each point $p \in \Sigma_i$.

Let us describe a construction of the homeomorphism ψ^s_i, $i < n$, supposing that the homeomorphisms $\psi^s_n, \dots, \psi^s_{i+1}$ are already constructed.

The stable manifolds of points from Σ_i have heteroclinic intersections only with unstable manifolds of the saddles Σ_j with $j > i$. Thus, the projection $\hat{W}^s_{i,i}$ is a smooth submanifold of the manifold $\hat{V}_i$ consisting of a finite number of connected components homeomorphic to the circle. Due to Corollary 4.1, the intersection $\hat{W}^s_{i,i} \cap (\bigcup_{j=i+1}^n \hat{N}^{a_j}_j)$ consists of a finite number of open arcs $\hat{I}^i_1, \dots, \hat{I}^i_{r_i}$ such that $\hat{I}^i_l$ for each $l = 1, \dots, r_i$ is a connected component of the intersection $\hat{W}^s_{i,i} \cap \hat{N}^{a_j}_{j,i}$ for some $j > i$ (see Lemma 4.5).

Denote by I^i_l a connected component of the set $p_i^{-1}(\hat{I}^i_l)$. The arc I^i_l is an arc in $N^{a_j}_j$ intersecting W^u_j at a unique point x^i_l. Set $x'^i_l = \varphi^{us}(x^i_l)$, and denote by I'^i_l the

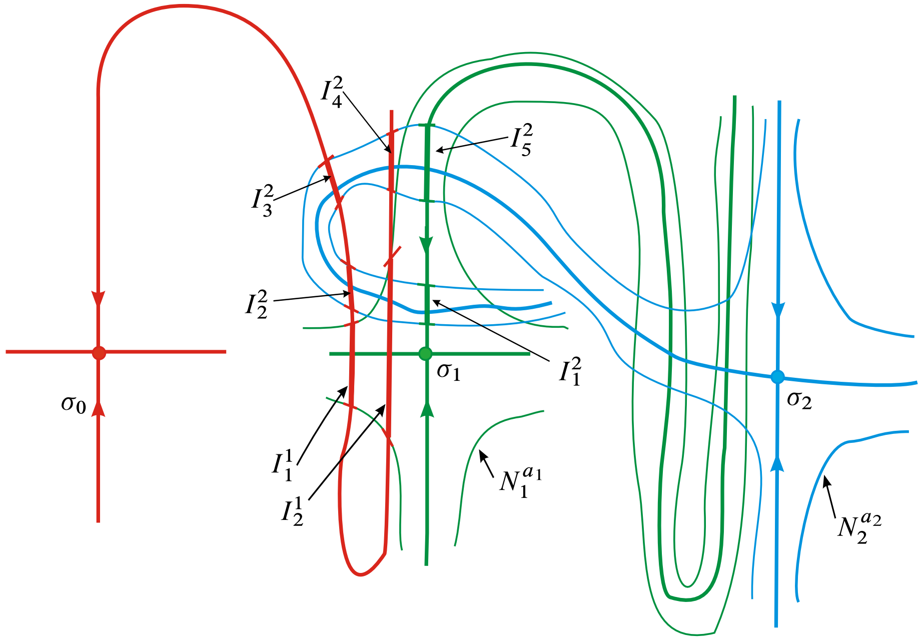

Figure 16. Illustration of the proof of Lemma 4.5.

connected component of $W_i'^s \cap N_j'$ passing through the point $x_l'^i$. By assumption of the induction on the set I_l^i, the homeomorphism $\phi_{\varphi_j^u,\psi_j^s}$ is well defined. Let us set $\psi_{I_l^i} = \phi_{\varphi_j^{us},\psi_j^s}|_{I_l^i} : I_l^i \to I_l'^i$.

Set

$$\hat{\psi}^s_{i,l} = p_i'\psi_{I_l^i}(p_i|_{I_l^i})^{-1} : \hat{I}_l^i \to \hat{W}'^s_{i,i}.$$

Notice that $\hat{\psi}^s_{i,l}$ does not depend on the choice of the lift I_l^i of $\hat{I}_l^i$. Indeed, if $\tilde{I}_l^i$ be a connected component of the set $p_i^{-1}(\hat{I}_l^i)$ different from I_l^i, then there is a unique $z \in \mathbb{Z}$ such that $\tilde{I}_l^i = f^z(I_l^i)$. As ψ_j^s conjugates $f|_{W_j^s}$ and $f'|_{W_j'^s}$, we have that $\psi_{\tilde{I}_l^i}(f^z(x)) = f'^z(\psi_{I_l^i}(x))$ for any $x \in I_l^i$. It means that $p_i'\psi_{I_l^i}(p_i|_{I_l^i})^{-1} = p_i'\psi_{\tilde{I}_l^i}(p_i|_{\tilde{I}_l^i})^{-1}$.

By construction, the map $\hat{\psi}^s_{i,l}$, $l = 1, \dots, r_i$ coincides on $\hat{I}_l^i \cap \hat{L}_i^u$ with $p_i'\varphi^{us}(p_i|_{\hat{I}_l^i \cap \hat{L}_i^u})^{-1}$. Due to Lemma 4.2, the map φ^{us} sends $W_i^s \cap L^u$ to $W_i'^s \cap L'^u$ while preserving the order on each connected component $W_i^s \setminus \Sigma_i$ and $W_i'^s \setminus \Sigma_i'$. Then there is a homeomorphism $\hat{\psi}_i^s : \hat{W}^s_{i,i} \to \hat{W}'^s_{i,i}$ coinciding with $\hat{\psi}^s_{i,l}$ on $\hat{I}_l^i$, $l = 1, \dots, r_i$. Moreover, $\hat{\psi}_i^s$ possesses the following property: Let $\hat{\gamma}$ be a connected component of $\hat{W}^s_{i,i}$, and let $U(\hat{\gamma})$ be its neighborhood; let $\hat{\gamma}' = \hat{\psi}_i^s(\hat{\gamma})$, and let $U(\hat{\gamma}')$ be a neighborhood of $\hat{\gamma}'$. Then $\hat{\varphi}_i(U(\hat{\gamma})) \cap U(\hat{\gamma}') \neq \emptyset$. Notice that $\hat{\varphi}_i$ can be defined on an open dense subset of $\hat{V}_i$. Denote by $\tilde{\psi}_i^s : W_i^s \to W_i'^s$ a homeomorphism which is a lift with respect to p_i of $\hat{\psi}_i^s$ on $W_i^s \setminus \Sigma_i$ such that it coincides with φ^{us} on $W_i^s \cap L^u$ and $\psi_i^s|_{\Sigma_i} = \varphi|_{\Sigma_i}$.

Let $p \in \Sigma_i$, and let γ_p be a connected component of $W_p^s \setminus p$ containing heteroclinic points. Let $N_{\gamma_p}^{a_i}$ be a connected component of $N_p^{a_i} \setminus W_p^u$ containing γ_p. Set $\gamma_{p'} = \tilde{\psi}_i^s(\gamma_p)$ and $\psi_i^s|_{\gamma_p} = \tilde{\psi}^s|_{\gamma_p}$. Let us show that $r_{p'}'^s(\varphi_i^u(x_i^u)) \cdot r_{p'}'^u(\psi_i^s(x_i^s)) < 1$ for each $x \in N_{\gamma_p}^{a_i}$. Indeed, let us choose a heteroclinic point $y \in \gamma_p$. Set $\lambda_{\gamma_p}'^u = \sup\{r_p'^u(\varphi^{us}(x_i^u)) \mid x \in (N_{\gamma_p}^{a_i} \cap F_{i,y}^u)\}$ and $\lambda_{\gamma_p}'^s = r_{p'}'^s(\varphi^{us}(y))$. By Lemma 4.3 $\lambda_{\gamma_p}'^u \cdot \lambda_{\gamma_p}'^s < \frac{1}{2}$. Denote by Q_{γ_p} a subset of $N_{\gamma_p}^{a_i}$ which is bounded by $\partial N_p^{a_i}$, $F_{i,y}^u$, and $f^{m_{\gamma_p}}(F_{i,y}^u)$. Notice that Q_{γ_p} is a fundamental domain of $f^{m_{\gamma_p}}|_{N_{\gamma_p}^{a_i}}$. By construction, $r_{p'}'^u(\varphi_i^u(x_i^u)) \leq 2\lambda_{\gamma_p}'^u$ and $r_{p'}'^s(\psi_i^s(x_i^s)) \leq \lambda_{\gamma_p}'^s$ for any $x \in Q_{\gamma_p}$ and, hence, $r_{p'}'^s(\varphi_i^u(x_i^u)) \cdot r_{p'}'^u(\psi_i^s(x_i^s)) \leq 2\lambda_{\gamma_p}'^u \cdot \lambda_{\gamma_p}'^s < 1$ for each $x \in Q_{\gamma_p}$. Thus $\phi_{\varphi_i^u,\psi_i^s}(N_p^{a_i}) \subset N_{p'}'$.

Let $p \in \Sigma_i$, and let ℓ_p be a connected component of $W_p^s \setminus p$ which does not contain heteroclinic points. Let $N_{\ell_p}^{a_i}$ be a connected component of $N_p^{a_i} \setminus W_p^u$ containing ℓ_p. Set $\ell_p' = \tilde{\psi}_i^s(\ell_p)$. Let us choose a fundamental domain I_{ℓ_p} of $f^{m_{\ell_p}}|_{\ell_p}$, where m_{ℓ_p} is the period of the separatrix ℓ_p. Set $N_{I_{\ell_p}} = \{x \in N_p^{a_i} \mid x^s \in I_{\ell_p}\}$ and $\lambda_{\ell_p}'^u = \sup\{r_{p'\_}'^u(\varphi_i^u(x_i^u)) \mid x \in N(I_{\ell_p})\}$. For $k \in \mathbb{Z}$, we set $\lambda_{\ell_p}'^s(k) = \frac{1}{2^k} \cdot \sup\{r_{p'}'^s(\tilde{\psi}_i^s(x)) \mid x \in I_{\ell_p}\}$. As $\lambda_{\ell_p}'^s(k)$ tends to 0 as k tends to ∞, there is $k_* \in \mathbb{N}$ such that $\lambda_{\ell_p}'^s(k_*) \cdot \lambda_{\ell_p}'^u < 1$. Set $\psi_i^s|_{\ell_p} = f'^{k_*}\tilde{\psi}_i^s|_{\ell_p}$. Thus $\phi_{\varphi_i^u,\psi_i^s}(N_{\ell_p}^{a_i}) \subset N_p'$.

Finally, a map $\psi^s : L^s \to L'^s$ consisting of the homeomorphisms $\psi_0^s : W_0^s \to W_0'^s, \ldots, \psi_n^s : W_n^s \to W_n'^s$ is a homeomorphism due to following property: if $x \in (W_i^s \cap N_j^{a_j})$, $j > i$, then $\psi_i^s(x) = \phi_{\varphi_j^u,\psi_j^s}(x)$. ☐

Let $n \geq 1$, $i \in \{0, \ldots, n-1\}$, and let G_i be the union of all stable 1-dimensional separatrices of saddle points in Σ_i which contain heteroclinic points. Let $\check{G}_i \subset G_i$ be the union of separatrices in G_i such that $G_i = \bigcup_{\gamma \in \check{G}_i} \text{orb}(\gamma)$ and, for every pair (γ_1, γ_2) of distinct separatrices in $\check{G}_i$ and every $k \in \mathbb{Z}$, we have $\gamma_2 \neq f^k(\gamma_1)$. For $\gamma \in G_i$ with the endpoint $p \in \Sigma_i$ and a point $q \in \Sigma_j$, $j > i$, let us consider a sequence of different periodic orbits $p = p_0 \prec p_1 \prec \cdots \prec p_k = q$ such that $\gamma \cap W_{p_1}^u \neq \emptyset$; the length of the longest such chain is denoted $\text{beh}(q \mid \gamma)$.

Let $\gamma \in \check{G}_i$ be a separatrix of $p \in \Sigma_i$, and let N_γ^t be the connected component of $N_p^t \setminus W_p^u$ which contains γ. We endow with the index γ (resp., p) the preimages in M (through the linearizing map μ_p) of all objects from the linear model $\mathcal{N}$ associated with the separatrix γ (resp., p); for precision, we decide that $\mu_p(\gamma) = Ox_3^+$. For a separatrix γ in $\check{G}_i$, let us fix a saddle point q_γ such that $\text{beh}(q_\gamma \mid \gamma) = 1$. Notice that the intersection $\gamma \cap W_{q_\gamma}^u$ consists of a finite number of heteroclinic orbits. Let $T_p = W_p^u \cap W_{\Omega_1}^s$.

LEMMA 4.6

Let $n \geq 1$, $i \in \{0, \dots, n-1\}$. *For every* $\gamma \in \check{G}_i$ *with* $\gamma \subset W^s_p$ *(in Section 4.2), there are positive numbers* ρ, δ *such that the following properties hold:*

(1) U_p *avoids all heteroclinic points.*

(2) *Either* $c_p = U_p \cap T_p$ *or the sets* ∂U_p *and* T_p *intersect transversally and each path-connected component of the intersection* $U_p \cap T_p$ *is a segment which intersects each from both connected components of* ∂U_p *at a unique point.*

For a chosen c_p, *there is a positive number* ε *such that for every heteroclinic point* $Z^0_\gamma \in (\gamma \cap W^u_{q_\gamma})$ *with* $z^0 < \varepsilon$ *the following properties hold:*

(3) $\varphi(\tilde{d}_p) \subset \phi_{\varphi^u_i, \psi^s_i}(N^{a_i}_i)$.

(4) $\varphi(\tilde{c}_p) \cap \phi_{\varphi^u_i, \psi^s_i}(\tilde{c}^0_p) = \emptyset$, $\varphi(\tilde{c}^1_p) \cap \phi_{\varphi^u_i, \psi^s_i}(\tilde{c}^0_p) = \emptyset$ *and* $\varphi(\tilde{\beta}_\gamma) \subset \phi_{\varphi^u_i, \psi^s_i}(\tilde{V}_\gamma)$.

Proof

Let $\gamma \in \check{G}_i$, $i \in \{0, \dots, n-1\}$. If W^u_p contains a compact heteroclinic curve which is noncontractible in $W^u_p \setminus p$, then we assign c_p to be this heteroclinic curve. In the opposite case, due to Lemma 2.13, there is a generic $\rho > 0$ such that the curve c_p avoids all heteroclinic points. Since W^s_l accumulates on W^s_k for every $l < k$, we have that $K_p \cap W^s_{i-1}$ is made of a finite number of heteroclinic points $y_1, \dots, y_r$ which we can cover by closed 2-disks $b_1, \dots, b_r \subset \operatorname{int} K_p$. In $K_p \setminus \operatorname{int}(b_1 \cup \dots \cup b_r)$ there is a finite number of heteroclinic points from W^s_{i-2} which we cover by the union of a finite number of closed 2-disks, and so on. Thus we get that all heteroclinic points in K_p belong to the union of finitely many closed 2-disks avoiding ∂K_p. Therefore, there is $\delta \in (0, \frac{\rho}{4})$ such that U_p avoids heteroclinic points. This proves item (1). As the set T_p is a closed 1-dimensional $C^{1,1}$-lamination, due to the theory of the general position there is a generic U_p with the property (2).

By assumption of Theorem 1, φ is defined on the complement of the stable manifolds and, by definition, $\phi_{\varphi^u_i, \psi^s_i}$ coincides with φ on $W^u_i \setminus L^s$, and hence on U_p. As φ and $\phi_{\varphi^u_i, \psi^s_i}$ are continuous, we can choose $\varepsilon > 0$ sufficiently small so that, if Z^0_γ is any heteroclinic point in the intersection $\gamma \cap W^u_{q_\gamma}$ with $z^0 < \varepsilon$, the requirements of (3) and (4) are fulfilled. $\square$

Let us fix U_p satisfying items (1)–(2) of Lemma 4.6, and let us define

$$U_i = \bigcup_{p \in \check{\Sigma}_i} \Big(\bigcup_{k=0}^{\operatorname{per}(p)-1} f^k(U_p) \Big), \qquad K_i = \bigcup_{p \in \check{\Sigma}_i} \Big(\bigcup_{k=0}^{\operatorname{per}(p)-1} f^k(K_p) \Big),$$

where $\check{\Sigma}_i$ is a subset of Σ_i obtained by the choice of one point from each orbit.

LEMMA 4.7
Let $n \geq 2$. For every $i \in \{0, \dots, n-2\}$ and $\gamma \in \check{G}_i$, there is a heteroclinic point $Z^0_\gamma \in \gamma$ satisfying the conditions of Lemma 4.6. In addition,

$$\mathcal{T}_\gamma \cap \tilde{U}_j = \emptyset \quad \textit{for } j \in \{i+1, \dots, n-1\}.$$

Proof
In this statement, it is meant that $\tilde{U}_{n-1}$ is associated with the points Z^0_γ, $\gamma \in \check{G}_{n-1}$ given by Lemma 4.6 and $\tilde{U}_j$ is associated with the points Z^0_γ, $\gamma \in \check{G}_j$ given by Lemma 4.7 for every $j > i$. Therefore, it makes sense to prove this lemma by decreasing induction on i from $i = n-2$ to 0. That is what is done below. It is also worth noticing that nothing is required with respect to Σ_n because the stable separatrices of Σ_n have no heteroclinic points.

Let us first prove the lemma for $i = n-2$. Let $\gamma \in \check{G}_{n-2}$, and let p be the saddle endpoint of γ. Notice that the intersection $\gamma \cap K_{n-1}$ consists of a finite number of points $a_1, \dots, a_l$ avoiding U_{n-1}. Let $d_1, \dots, d_l \subset K_{n-1}$ be compact disks with centers $a_1, \dots, a_l$ and radius r_* (in linear coordinates of N_p) avoiding U_{n-1}. Let us choose a number $n^* \in \mathbb{N}$ such that $\frac{\rho}{2^{n^*}} < r^*$. Let $Z^*_\gamma \subset \gamma$ be a point such that the segment $[p, Z^*_\gamma]$ of γ avoids $\tilde{K}_{n-1}$ and $\mu_p(Z^*_\gamma) = Z^* = (0, 0, z^*)$ where $z^* < \varepsilon$. Then every heteroclinic point z^0_γ, such that $z^0 < \frac{z^*}{2^{n^*}}$ possesses the property $\mathcal{T}_\gamma \cap \tilde{K}_{n-1}$, avoids $\tilde{U}_{n-1}$.

For the induction, let us assume now that the construction of the desired heteroclinic points is done for $i+1, i+2, \dots, n-2$. Let us do it for i. Let $\gamma \in \check{G}_i$. By assumption of the induction, $(\bigcup_{k=i+1}^{j-1} \mathcal{T}_k) \cap \tilde{U}_j = \emptyset$ for $j \in \{i+2, \dots, n-1\}$. Since W^s_{k-1} accumulates on W^s_k for every $k \in \{0, \dots, n\}$, we have that $(\bigcup_{k=i+1}^{j-1} \mathcal{T}_k) \cap K_j$ is a compact subset of K_j, and the intersection $(\gamma \setminus (\bigcup_{k=i+1}^{j-1} \mathcal{T}_k)) \cap K_j$ consists of a finite number of points $a_1, \dots, a_l$ avoiding U_j. Let $d_1, \dots, d_l \subset K_j$ be compact disks with centers $a_1, \dots, a_l$ and radius r_* (in linear coordinates of N_p) avoiding U_j and such that r_* is less than the distance between $\partial(K_j \setminus U_j)$ and $(\bigcup_{k=i+1}^{j-1} \mathcal{T}_k) \cap K_j$. Similar to the case $i = n-2$ it is possible to choose a heteroclinic point Z^0_γ sufficiently close to the saddle p where γ ends such that the set $(\mathcal{T}_\gamma \setminus (\bigcup_{k=i+1}^{j-1} \mathcal{T}_k)) \cap \tilde{K}_j$ avoids $\tilde{U}_j$. □

Everywhere below, we assume that for every $\gamma \subset \check{G}_i$ the neighborhood $\mathcal{T}_\gamma$ satisfies Lemmas 4.6 and 4.7. Let

$$\mathcal{T}_i = \bigcup_{\gamma \subset \check{G}_i} \Big(\bigcup_{k=0}^{\mathrm{per}(\gamma)-1} f^k(\mathcal{T}_\gamma) \Big).$$

For $\gamma \subset \check{G}_i$, $j > i$, let us denote by $\mathcal{J}_{\gamma,j}$ the union of all connected components of $W^u_j \cap \mathcal{T}_\gamma$ which do not lie in $\operatorname{int} \mathcal{T}_k$ with $i < k < j$. Let $\mathcal{J}_\gamma = \bigcup_{j=i+1}^n \mathcal{J}_{\gamma,j}$ and

$$\mathcal{J}_i = \bigcup_{\gamma \subset \check{G}_i} \Big(\bigcup_{k=0}^{\operatorname{per}(\gamma)-1} f^k(\mathcal{J}_\gamma) \Big).$$

Let $\mathcal{W}_\gamma$ be the fundamental domain of $f^{\operatorname{per}(\gamma)}|_{\mathcal{T}_\gamma \setminus W^u_p}$ limited by the plaques of the two heteroclinic points Z^0_γ and $f^{\operatorname{per}(\gamma)} Z^0_\gamma$. Notice that $\gamma \cap \mathcal{W}_\gamma$ is a fundamental domain of $f^{\operatorname{per}(\gamma)}|_\gamma$.

LEMMA 4.8
The set $\mathcal{J}_{\gamma,j} \cap \mathcal{W}_\gamma$ consists of a finite number of closed 2-disks.

Proof
Let $\hat{\gamma} = p_i(\gamma)$, $\hat{\mathcal{T}}_\gamma = p_i(\mathcal{T}_\gamma)$, and $\hat{\mathcal{T}}_{i,j} = p_i(\mathcal{T}_j)$. Since W^u_j accumulates on W^u_l only when $l < j$, we set $\hat{W}^u_j \setminus \bigcup_{l=i+1}^{j-1} \hat{\mathcal{T}}_{l,i}$ as a compact set. Due to Lemma 4.7, the intersection of $\hat{\mathcal{T}}_\gamma \cap \partial \hat{\mathcal{T}}_{l,i}$ consists of 2-disks which are projections with respect to p_i of the leaves of the foliation F^u_i. Thus the intersection $\hat{\gamma} \cap (\hat{W}^u_j \setminus \bigcup_{l=i+1}^{j-1} \hat{\mathcal{T}}_{l,i})$ consists of a finite number of closed 2-disks. □

Due to Lemma 4.8, the set $\mathcal{J}_\gamma \cap \gamma \cap \mathcal{W}_\gamma$ consists of a finite number of heteroclinic points, which are denoted by $Z^2_\gamma, \dots, Z^m_\gamma$ (m depends on γ). Finally, choose an arbitrary point $Z^1_\gamma \in \gamma$ so that the arc $(z^0_\gamma, z^1_\gamma) \subset \gamma$ does not contain heteroclinic points from $\mathcal{J}_\gamma$. Let us construct $\mathcal{H}_\gamma$ using the point $Z^1 = \mu_p(Z^1_\gamma)$. Without loss of generality we will assume that $\mu_p(Z^i_\gamma) = Z^i = (0,0,z^i)$ for $z^0 > z^1 > \cdots > z^m > \frac{z^0}{2}$ and $\mu_p(T_p) = T$. For $i = 0, \dots, n-1$, let

$$\mathcal{H}_i = \bigcup_{\gamma \subset \check{G}_i} \Big(\bigcup_{k=0}^{\operatorname{per}(\gamma)-1} f^k(\mathcal{H}_\gamma) \Big), \qquad \mathcal{M}_i = V_f \cup \bigcup_{k=0}^{i} (G_k \cup \Sigma_k), \qquad \text{and}$$

$$\mathcal{M}'_i = V_{f'} \cup \bigcup_{k=0}^{i} (G'_k \cup \Sigma'_k).$$

LEMMA 4.9
There is an equivariant homeomorphism $\varphi_0 : \mathcal{M}_0 \to \mathcal{M}'_0$ with the following properties:

(1) *φ_0 coincides with φ out of $\mathcal{T}_0$.*
(2) *$\varphi_0|_{\mathcal{H}_0} = \phi_{\psi^u_0,\psi^s_0}|_{\mathcal{H}_0}$, where $\psi^u_0 = \varphi|_{W^u_0}$.*

(3) $\varphi_0(W_1^u)=W_1'^u$ *and* $\varphi_0(W_k^u\setminus\bigcup_{j=1}^{k-1}\operatorname{int}\mathcal{T}_j)\subset W_k'^u$ *for every* $k\in\{2,\dots,n\}$.
4) $\varphi_0(W^s_{\Omega_1}\cap\mathcal{M}_0)=W^s_{\Omega_1'}\cap\mathcal{M}_0'$.

Proof
The desired φ_0 should be an interpolation between $\varphi:V_f\setminus\mathcal{T}_0\to M'$ and $\phi_{\varphi_0^u,\psi_0^s}|_{\mathcal{H}_0}$. Due to Lemma 4.6 (3) and the equivariance of the considered maps, the embedding

$$\xi_0=\phi^{-1}_{\psi_0^u,\psi_0^s}\varphi:\mathcal{T}_0\setminus W_0^s\to M$$

is well defined. Let $\gamma\subset\check{G}_0$ be a separatrix ending at $p\in\Sigma_0$, and let $\xi_\gamma=\xi_0|_{\mathcal{T}_\gamma}$. By construction, the topological embedding $\xi=\mu_p\xi_\gamma\mu_p^{-1}:\mathcal{T}\to N$ satisfies all conditions of Proposition 4.2. Let ζ be the embedding from the inclusion of that lemma, and let $\zeta_\gamma=\mu_p^{-1}\zeta\mu_p$. Independently, one does the same for every separatrix $\gamma\subset\check{G}_0$ and then extends it to all separatrices in G_0 by equivariance. As a result, we get a homeomorphism ζ_0 of $\mathcal{T}_0$ onto $\xi_0(\mathcal{T}_0)$ which coincides with ξ_0 on $\partial\mathcal{T}_0$. Now, define the homeomorphism $\varphi_0:\mathcal{M}_0\to\mathcal{M}_0'$ to be equal to $\phi_{\psi_0^u,\psi_0^s}\zeta_0$ on $\mathcal{T}_0$ and to φ on $\mathcal{M}_0\setminus\mathcal{T}_0$. One checks the following properties:
(1) φ_0 coincides with φ out of $\mathcal{T}_0$.
(2) $\varphi_0|_{\mathcal{H}_0}=\phi_{\psi_0^u,\psi_0^s}|_{\mathcal{H}_0}$.
(3) $\varphi_0(\mathcal{J}_0)\subset L^u$.
(4) $\varphi_0(W^s_{\Omega_1}\cap\mathcal{M}_0)=W^s_{\Omega_1'}\cap\mathcal{M}_0'$.

Property (3) and the definition of the set $\mathcal{J}_\gamma$ imply that $\varphi_0(W_1^u)=W_1'^u$ and $\varphi_0(W_k^u\setminus\bigcup_{j=1}^{k-1}\operatorname{int}\mathcal{T}_j)\subset W_k'^u$ for every $k\in\{2,\dots,n\}$. Thus φ_0 satisfies all required conditions of the lemma. ☐

LEMMA 4.10
Assume that $n\geq 2$, $i\in\{0,\dots,n-2\}$, *and assume that there is an equivariant topological embedding* $\varphi_i:\mathcal{M}_i\to M'$ *with the following properties:*
(1) φ_i *coincides with* φ_{i-1} *out of* $\mathcal{T}_i$.
(2) $\varphi_i|_{\mathcal{H}_i}=\phi_{\psi_i^u,\psi_i^s}$, *where* $\psi_i^u=\varphi_{i-1}|_{W_i^u}$ *and* $\varphi_{-1}=\varphi$.
(3) *There is an* f*-invariant union of tubes* $\mathcal{B}_i\subset(\mathcal{T}_i\cap\bigcup_{j=0}^{i-1}\mathcal{H}_j)$ *containing* $(\mathcal{T}_i\cap(\bigcup_{j=0}^{i-1}W_j^s))$, *where* φ_i *coincides with* φ_{i-1} *(we assume that* $\mathcal{B}_0=\emptyset$*).*
(4) $\varphi_i(W^u_{i+1})=W'^u_{i+1}$ *and* $\varphi_i(W_k^u\setminus\bigcup_{j=i+1}^{k-1}\operatorname{int}\mathcal{T}_j)\subset W_k'^u$ *for every* $k\in\{i+2,\dots,n\}$.
5) $\varphi_i(W^s_{\Omega_1}\cap\mathcal{M}_i)=W^s_{\Omega_1'}\cap\mathcal{M}_i'$.

Then there is a homeomorphism φ_{i+1} *with the same properties* (1)–(5).

Proof
The desired φ_{i+1} should be an interpolation between $\varphi_i:\mathcal{M}_{i+1}\setminus\mathcal{T}_{i+1}\to M'$ and

$\phi_{\psi^u_{i+1},\psi^s_{i+1}}|_{\mathcal{H}_{i+1}}$, where $\psi^u_{i+1} = \varphi_i|_{W^u_{i+1}}$. Let $\gamma \subset \check{G}_{i+1}$ be a separatrix ending at $p \in \Sigma_{i+1}$. It follows from the definition of the set $\mathcal{J}_i$ and the choice of the point q_γ that $(W^u_{q_\gamma} \cap \mathcal{T}_i) \subset \mathcal{J}_i$. Then, due to condition (4) for φ_i, we have $\varphi_i(W^u_{q_\gamma} \cap \mathcal{T}_i) \subset W^u_{q'}$. By property (1) of the homeomorphism φ_i and the properties of $\mathcal{T}_{i+1}$ from Lemmas 4.6(1) and 4.7, we get that $\varphi_i|_{\tilde{U}_p} = \varphi|_{\tilde{U}_p}$. Then $\phi_{\varphi^u_{i+1},\psi^s_{i+1}}|_{\tilde{U}_p} = \phi_{\psi^u_{i+1},\psi^s_{i+1}}|_{\tilde{U}_p}$. Thus it follows from property (3) in Lemma 4.6 that the following embedding is well defined: $\xi_\gamma = \phi^{-1}_{\psi^u_{i+1},\psi^s_{i+1}}\varphi_i : \mathcal{T}_\gamma \setminus (\gamma \cup p) \to M'$.

By construction, the topological embedding $\xi = \mu_p \xi_\gamma \mu_p^{-1}$ satisfies all conditions of Proposition 4.2. Let ζ be the embedding which is yielded by that proposition. Define $\zeta_\gamma = \mu_p^{-1}\zeta\mu_p$. Notice that, by property (3) of the homeomorphism ψ^s in Lemma 4.5 and by the properties $\psi^u_{i+1} = \varphi_i|_{W^u_i}$, we have that ζ_γ is the identity on a neighborhood $\tilde{B}_\gamma \subset (\mathcal{T}_\gamma \cap \bigcup_{j=0}^{i} \mathcal{H}_j)$ of $\mathcal{T}_\gamma \cap (\bigcup_{j=0}^{i} W^s_j)$. Independently, one does the same for every separatrix $\gamma \subset \check{G}_{i+1}$. Assuming that $\zeta_{f(\gamma)} = f'\zeta_\gamma f^{-1}$ and $\tilde{B}_{i+1} = \bigcup_{\gamma \subset \check{G}_{i+1}} (\bigcup_{k=0}^{\mathrm{per}(\gamma)-1} f^k(\tilde{B}_\gamma))$, we get a homeomorphism ζ_{i+1} on $\mathcal{T}_{i+1}$. Thus the required homeomorphism coincides with $\phi_{\psi^u_{i+1},\psi^s_{i+1}}$ on $\mathcal{H}_{i+1}$ and with φ_i out of $\mathcal{T}_{i+1}$. $\square$

Let G be the union of all stable 1-dimensional separatrices which do not contain heteroclinic points. We have $N^t_G = \bigcup_{\gamma \subset G} N^t_\gamma$ and

$$\mathcal{M} = \mathcal{M}_{n-1} \cup G.$$

We also have similar objects with prime for f'.

LEMMA 4.11
There are numbers $0 < \tau_1 < \tau_2 < 1$ and an equivariant embedding $h : \mathcal{M} \to M'$ with the following properties:

(1) *h coincides with φ_{n-1} out of $N^{\tau_2}_G$.*
(2) *h coincides with $\phi_{\varphi_{n-1},\psi^s}$ on $|_{\mathcal{N}^{\tau_1}_G}$, where $\psi^s : L^s \to L'^s$ is yielded by Lemma 4.5.*
(3) *There is an f-invariant neighborhood of the set $N_G \cap (G_0 \cup \cdots \cup G_{n-1})$ where h coincides with φ_{n-1}.*
(4) *$h(W^s_{\Omega_1} \cap \mathcal{M}) = W^s_{\Omega'_1} \cap \mathcal{M}'$.*

Proof
Let $\check{G} \subset G$ be a union of separatrices from G such that $\gamma_2 \neq f^k(\gamma_1)$ for every $\gamma_1, \gamma_2 \subset \check{G}$, $k \in \mathbb{Z} \setminus \{0\}$ and $G = \bigcup_{\gamma \in \check{G}} \mathrm{orb}(\gamma)$. Let $i \in \{0, \ldots, n\}$, $p \in \Sigma_i$, and $\gamma \subset G$.

Notice that $(N_\gamma \setminus (\gamma \cup p))/f^{\mathrm{per}(\gamma)}$ is homeomorphic to $\hat{X} \times [0,1]$, where $\hat{X}$ is a 2-torus and the natural projection $\pi_\gamma : N_\gamma \setminus (\gamma \cup p) \to \hat{X} \times [0,1]$ sends ∂N^t_γ to $\hat{X} \times \{t\}$

for each $t \in (0,1)$ and sends $W^u_p \setminus p$ to $\hat{X} \times \{0\}$. Let $\xi_\gamma = \phi^{-1}_{\varphi_{n-1}|_{W^u_i}, \psi^s_i} \varphi_{n-1}|_{N^{a_i}_\gamma \setminus (\gamma \cup p)}$ and $\hat{\xi}_\gamma = \pi_\gamma \xi_\gamma \pi^{-1}_\gamma|_{\hat{X} \times [0,a_i]}$. Due to item (3) of Lemma 4.10, the homeomorphism $\hat{\xi}_\gamma$ coincides with the identity in some neighborhood of $\pi_\gamma(N^{a_i}_\gamma \cap (G_0 \cup \cdots \cup G_{n-1}))$. Let us choose this neighborhood of the form $B_\gamma \times [0,a_i]$. Let $\hat{T}_\gamma = \pi_\gamma(T_p)$. Let us choose numbers $0 < \tau_{1,\gamma} < \tau_{2,\gamma} < a_i$ such that $\hat{\xi}_\gamma(\hat{X} \times [0,\tau_{2,\gamma}]) \subset \hat{X} \times [0,\tau_{1,\gamma})$. By construction, $\hat{\xi}_\gamma : \hat{X} \times [0,\tau_{2,\gamma}] \to X \times [0,1]$ is a topological embedding which is the identity on $\hat{X} \times \{0\}$, $\hat{\xi}_\gamma|_{B_\gamma \times [0,\tau_{2,\gamma}]} = \mathrm{id}|_{B_\gamma \times [0,\tau_{2,\gamma}]}$ and, due to item (4) of Lemma 4.10, $\hat{\xi}_\gamma(\hat{T}_\gamma \times [0,\tau_{2,\gamma}]) \subset \hat{T}_\gamma \times [0,1]$. Then, due to Proposition 5.3,

(1) There is a homeomorphism $\hat{\zeta}_\gamma : X \times [0,\tau_{2,\gamma}] \to \hat{\xi}(X \times [0,\tau_{2,\gamma}])$ such that $\hat{\zeta}_\gamma$ is the identity on $X \times [0,\tau_{1,\gamma}]$ and is $\hat{\xi}_\gamma$ on $X \times \{\tau_{2,\gamma}\}$.

(2) $\hat{\zeta}_\gamma|_{B_\gamma \times [0,\tau_{2,\gamma}]} = \mathrm{id}|_{B_\gamma \times [0,\tau_{2,\gamma}]}$.

(3) $\hat{\zeta}(\hat{T}_\gamma \times [0,\tau_{2,\gamma}]) \subset \hat{T}_\gamma \times [0,1]$.

Let ζ_γ be a lift of $\hat{\zeta}_\gamma$ on $N^{\tau_{2,\gamma}}_\gamma$ which is ξ_γ on $\partial N^{\tau_{2,\gamma}}_\gamma$. Thus $\varphi_\gamma = \phi_{\varphi_{n-1}|_{W^u_i}, \psi^s_i} \zeta_\gamma$ is the desired extension of φ_{n-1} to N_γ. Doing the same for every separatrix $\gamma \subset \check{G}$ and extending it to the other separatrices from G by equivariance, we get the required homeomorphism h for $\tau_1 = \min_{\gamma \subset \check{G}}\{\tau_{1,\gamma}\}$ and $\tau_2 = \min_{\gamma \subset \check{G}}\{\tau_{2,\gamma}\}$. □

So we get a homeomorphism $h : M \setminus (\Omega_0 \cup \Omega_3) \to M \setminus (\Omega'_0 \cup \Omega'_3)$ conjugating $f|_{M \setminus (\Omega_0 \cup \Omega_3)}$ with $f'|_{M \setminus (\Omega'_0 \cup \Omega'_3)}$. Notice that $M \setminus (W^s_{\Omega_1} \cup W^s_{\Omega_2} \cup \Omega_3) = W^s_{\Omega_0}$ and $M \setminus (W^s_{\Omega'_1} \cup W^s_{\Omega'_2} \cup \Omega'_3) = W^s_{\Omega'_0}$. Since $h(W^s_{\Omega_1}) = W^s_{\Omega'_1}$ and $h(W^s_{\Omega_2}) = W^s_{\Omega'_2}$, we have that $h(W^s_{\Omega_0} \setminus \Omega_0) = W^s_{\Omega'_0} \setminus \Omega'_0$. Thus for each connected component Y of $W^s_{\Omega_0} \setminus \Omega_0$ there is a sink $\omega \in \Omega_0$ such that $Y = W^s_\omega \setminus \omega$. Similarly, $h(Y)$ is a connected component of $W^s_{\Omega'_0} \setminus \Omega'_0$ such that $h(Y) = W^s_{\omega'} \setminus \omega'$ for a sink $\omega' \in \Omega'_0$. Then we can continuously extend h to Ω_0, assuming that $h(\omega) = \omega'$ for every $\omega \in \Omega_0$. A similar extension of h to Ω_3 finishes the proof.

5. Topological background

The following proposition is a corollary of Theorem 3.1 from [19]. In fact, in [19] the objects are required to be smooth, but actually the results are true for tame objects also.

PROPOSITION 5.1

Let P be homeomorphic to $K \times [0,1]$, where $K = \mathbb{S}^1 \times [0,1]$ and $Q \subset P$ is a tame embedded annulus such that $P \setminus Q$ is not connected and the annuli $K \times \{0\}$, $K \times \{1\}$ belong to the different connected components of $P \setminus Q$. Then the set $P \setminus Q$ consists of two connected components, the closure of each of which is homeomorphic to P (see Figure 17).

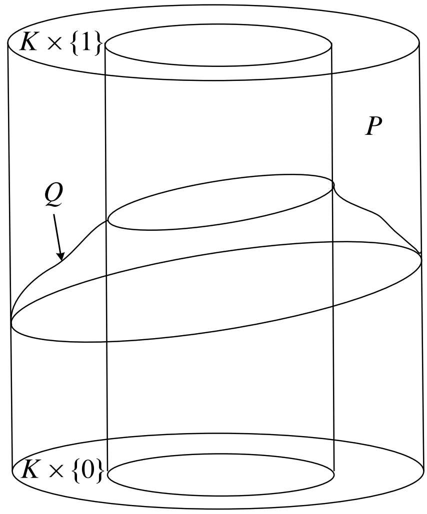

Figure 17. An illustration for Proposition 5.1.

PROPOSITION 5.2

We have the following:

- *Let* C *be a compact subset of* $[0, 1]$ *including* 0 *and* 1.
- *Let* $\mathcal{L}$ *be a lamination* $\{L_t = \mathbb{R}^2 \times \{t\}\}_{t\in C}$.
- *There is a tame topological embedding* $g : \partial(\mathbb{D}^2 \times [0, 1]) \to \mathbb{R}^2 \times [0, 1]$ *such that* $g(\partial(\mathbb{D}^2 \times \{t\})) \subset L_t$ *and* $g^{-1}(L_t) \subset L_t$ *for any* $t \in C$.

Then there is a homeomorphism $h : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2 \times [0, 1]$ *such that* $h(L_t) = L_t$ *for any* $t \in C$ *and* $h = g$ *on* $\partial(\mathbb{D}^2 \times [0, 1])$.

Proof

Let us introduce the canonical projection $p : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2$, where $p(r, t) = r$. Let us consider a homotopy $g_t : \partial\mathbb{D}^2 \to \mathbb{R}^2$, $t \in [0, 1]$ given by the formula $g_t(x) = p(g(x, t))$. By [14], there is an isotopy $\bar{g}_t : \partial\mathbb{D}^2 \to \mathbb{R}^2$, $t \in [0, 1]$ such that $\bar{g}_t = g_t$ for $t \in C$ and $\lim_{t\to C} \|g_t(x) - \bar{g}_t(x)\| = 0$. Let us extend this isotopy up to an isotopy $\bar{G}_t : \mathbb{R}^2 \to \mathbb{R}^2$. Let $\bar{G} : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2 \times [0, 1]$ be a homeomorphism given by the formula $\bar{G}(r, t) = (\bar{G}_t(r), t)$.

Let $Q = \partial(\mathbb{D}^2 \times [0, 1])$ and $\bar{Q} = \bar{G}^{-1}(g(\partial(\mathbb{D}^2 \times [0, 1])))$. Let us define $\psi : \partial Q \to \partial\bar{Q}$ by the formula $\psi = \bar{G}^{-1}g$. By construction, ψ is the identity on $\mathcal{L}$ and $\lim_{t\to C} \|\psi(x, t) - (x, t)\| = 0$. Let us show that there is a map $\Psi : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2 \times [0, 1]$ which coincides with ψ on ∂Q and is the identity on $\mathcal{L}$ such that $\lim_{t\to C} \|\Psi(x, t) - (x, t)\| = 0$; thus $h = \bar{G}\Psi$ will be the required homeomorphism.

For this aim, let us denote by $A_{a,b}$ a connected component of $Q \setminus \mathcal{L}$ bounded by the leaves L_a, L_b, $a, b \in C$. If the set C is finite, then, by the Alexander trick, there is a homeomorphism $\Psi_{a,b} : \mathbb{R}^2 \times [a,b] \to \mathbb{R}^2 \times [a,b]$ which is ψ on $A_{a,b}$ and is the identity on $L_a \cup L_b$. Then Ψ is composed by $\Psi_{a,b}$. If C is an infinite set, then, for a sequence of annuli A_{a_n,b_n} such that $\lim_{n\to\infty}(b_n - a_n) = 0$, we construct a homeomorphism $\Psi_n : \mathbb{R}^2 \times [a_n,b_n] \to \mathbb{R}^2 \times [a_n,b_n]$ which is ψ on A_{a_n,b_n} and is the identity on $L_{a_n} \cup L_{b_n}$ such that $\lim_{n\to\infty} \|\Psi_n(x,t) - (x,t)\| = 0$. This finishes the proof.

Let $\psi_n = \psi|_{A_{a_n,b_n}}$. As $\lim_{t\to C} \|\psi(x,t) - (x,t)\| = 0$, there is a sequence δ_n which tends to 0 as $n \to \infty$ and such that $b_n - a_n < \delta_n$, and ψ_n moves no point more than δ_n. Let U_n be a solid torus which is the one-sided δ_n-neighborhood of A_{a_n,b_n} and $\Sigma_n = \partial U_n$. Let $\phi_n : \Sigma_n \to \mathbb{R}^2 \times [a_n,b_n]$ be a topological embedding which is ψ_n on A_{a_n,b_n} and the identity on the other part of Σ_n.

Let $\Sigma'_n = \phi_n(\Sigma_n)$, and let U'_n be a solid torus bounded by Σ'_n (see Proposition 5.1). Let us choose in U_n an even number of vertical meridian disks $D_n^1, \dots, D_n^{2k_n}$ with distance between them less than $3\delta_n$ and such that $\phi_n(\partial D_n^{2i-1})$ avoids $\bigcup_{i=1}^{k_n} D_n^{2i}$. The closed curve $\phi_n(\partial D_n^{2i-1})$ is a meridian in the torus Σ'_n; hence, it is the boundary of a disk $D_n'^{2i-1}$ in U'_n whose interior avoids Σ'_n. By the standard pushing procedure, we can get that $D_n'^{2i-1}$ avoids $\bigcup_{i=1}^{k_n} D^{2i}$. Since every connected component of the sets $U_n \setminus \bigcup_{i=1}^{k_n} D_n^{2i-1}$ and $U'_n \setminus \bigcup_{i=1}^{k_n} D_n'^{2i-1}$ is a 3-ball, there is a homeomorphism $\Phi_n : U_n \to U'_n$ which sends D_n^{2i-1} to $D_n'^{2i-1}$. By construction, it does not move any point more than $4\delta_n$.

Doing the same for the other one-sided δ_n-neighborhood of A_{a_n,b_n} and extending by identity out of the δ_n-neighborhood of A_{a_n,b_n}, we get Ψ_n. ☐

Remark 5.1
A similar proposition is obviously true for a similar 1-dimensional lamination in $\mathbb{R}^1 \times [0,1]$.

PROPOSITION 5.3
Let $\hat{X}$ be a compact topological space, $0 < \tau_1 < \tau_2 < 1$, and let $\hat{\xi} : \hat{X} \times [0,\tau_2] \to \hat{X} \times [0,1]$ be a topological embedding which is the identity on $\hat{X} \times \{0\}$ and $\hat{X} \times [0,\tau_1] \subset \hat{\xi}(\hat{X} \times [0,\tau_2])$. Then we have the following:

(1) *There is a homeomorphism $\hat{\zeta} : \hat{X} \times [0,\tau_2] \to \xi(\hat{X} \times [0,\tau_2])$ such that $\hat{\zeta}$ is the identity on $\hat{X} \times [0,\tau_1]$ and is $\hat{\xi}$ on $\hat{X} \times \{\tau_2\}$.*

(2) *If for a set $\hat{B} \subset \hat{X}$ the equality $\hat{\xi}|_{\hat{B}\times[0,\tau_2]} = \mathrm{id}|_{\hat{B}\times[0,\tau_2]}$ is true, then $\hat{\zeta}|_{\hat{B}\times[0,\tau_2]} = \mathrm{id}|_{\hat{B}\times[0,\tau_2]}$.*

(3) *If for a set $\hat{T} \subset \hat{X}$ the inclusion $\hat{\xi}(\hat{T} \times [0,\tau_2]) \subset \hat{T} \times [0,1]$ is true, then $\hat{\zeta}(\hat{T} \times [0,\tau_2]) \subset \hat{T} \times [0,1]$.*

Proof
Let us choose $l \in (\tau_1, \tau_2)$ such that $\hat{X} \times [0, l] \subset \hat{\xi}(\hat{X} \times [0, \tau_2])$. Define a homeomorphism $\kappa : [\tau_1, 1] \to [0, 1]$ by the formula

$$\kappa(t) = \begin{cases} (x, \frac{l(t-\tau_1)}{l-\tau_1}), & t \in [\tau_1, l]; \\ (x, t), & t \in [l, 1]. \end{cases}$$

Let $\mathcal{K}(x, t) = (x, \kappa(t))$ on $\hat{X} \times [\tau_1, 1]$. Then the required homeomorphism can be defined by the formula

$$\hat{\xi}(x, t) = \begin{cases} (x, t), & t \in [0, \tau_1]; \\ \mathcal{K}^{-1}\xi(\mathcal{K}((x, s)))), & s \in [\tau_1, \tau_2]. \end{cases}$$

Properties (2) and (3) automatically follow from this formula. □

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References

[1] R. H. FOX and E. ARTIN, *Some wild cells and spheres in three-dimensional space* Ann. of Math. (2) **49** (1948), 979–990. MR 0027512. DOI 10.2307/1969408. *(2519)*

[2] A. N. BEZDENEZHYKH and V. Z. GRINES, *Dynamical properties and topological classification of gradient-like diffeomorphisms on two-dimensional manifolds, I*, Selecta Math. Soviet. **11** (1992), 1–11. MR 1155894. *(2508)*

[3] ———, *Dynamical properties and topological classification of gradient-like diffeomorphisms on two-dimensional manifolds, II*, Selecta Math. Soviet. **11** (1992), 13–17. MR 1155895. *(2508)*

[4] C. BONATTI, S. CROVISIER, G. VAGO, and A. WILKINSON, *Local density of diffeomorphisms with large centralizers*, Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), no. 6, 925–954. MR 2504109. DOI 10.24033/asens.2085. *(2509)*

[5] C. BONATTI and V. GRINES, *Knots as topological invariants for gradient-like diffeomorphisms of the sphere S^3*, J. Dynam. Control Systems **6** (2000), no. 4, 579–602. MR 1778215. DOI 10.1023/A:1009508728879. *(2510)*

[6] C. BONATTI, V. GRINES, F. LAUDENBACH, and O. POCHINKA, *Topological classification of Morse–Smale diffeomorphisms without heteroclinic curves on 3-manifolds*, Ergodic Theory Dynam. Systems **39** (2019), no. 9, 2403–2432. DOI 10.1017/etds.2017.129. *(2530, 2540, 2541)*

[7] C. BONATTI, V. GRINES, V. MEDVEDEV, and E. PECOU, *Three-manifolds admitting Morse–Smale diffeomorphisms without heteroclinic curves*, Topology Appl. **117**

(2002), no. 3, 335–344. MR 1874094. DOI 10.1016/S0166-8641(01)00028-1. *(2515)*

[8] ———, *Topological classification of gradient-like diffeomorphisms on* 3*-manifolds*, Topology **43** (2004), no. 2, 369–391. MR 2052968. DOI 10.1016/S0040-9383(03)00053-3. *(2514)*

[9] C. BONATTI, V. GRINES, and O. POCHINKA, *Classification of the Morse–Smale diffeomorphisms with a finite set of heteroclinic orbits on* 3*-manifolds* (in Russian), Tr. Mat. Inst. Steklova **250** (2005), 5–53; English translation in Proc. Steklov Inst. Math. **250** (2005), no. 3, 1–46. MR 2200906. *(2515)*

[10] ———, *Realization of Morse–Smale diffeomorphisms on* 3*-manifolds* (in Russian), Tr. Mat. Inst. Steklova **297** (2017), 46–61; English translation in Proc. Steklov Inst. Math. **297** (2017), no. 1, 35–49. MR 3695406. DOI 10.1134/S0371968517020030. *(2514)*

[11] C. BONATTI and R. LANGEVIN, *Difféomorphismes de Smale des surfaces*, Astérisque **250**, Soc. Math. France, Paris, 1998. MR 1650926. *(2508)*

[12] C. BONATTI and L. PAOLUZZI, 3*-manifolds which are orbit spaces of diffeomorphisms*, Topology **47** (2008), no. 2, 71–100. MR 2415770. DOI 10.1016/j.top.2007.06.003. *(2511, 2515)*

[13] C. E. BURGESS, *Embeddings of surfaces in euclidean three-space*, Bull. Amer. Math. Soc. **81** (1975), no. 5, 795–818. MR 0375319. DOI 10.1090/S0002-9904-1975-13834-1. *(2519)*

[14] D. B. A. EPSTEIN, *Curves on* 2*-manifolds and isotopies*, Acta Math. **115** (1966), no. 1, 83–107. MR 0214087. DOI 10.1007/BF02392203. *(2553)*

[15] V. GRINES, *Topological classification of Morse–Smale diffeomorphisms with a finite set of heteroclinic trajectories on surfaces* (in Russian), Mat. Zametki **54** (1993), no. 3, 3–17; English translation in Math. Notes **54** (1993), no. 3–4, 881–889. MR 1248280. DOI 10.1007/BF01209552. *(2508)*

[16] V. Z. GRINES, *On the topological classification of structurally stable diffeomorphisms of surfaces with one-dimensional attractors and repellers*, Mat. Sb. **188** (1997), no. 4, 537–569; English translation in Sb. Math. **188** (1997), no. 4–4, 537–569. MR 3060070. DOI 10.4213/SM2012v203n12ABEH004286. *(2508)*

[17] V. Z. GRINES, E. Y. GUREVICH, V. S. MEDVEDEV, and O. V. POCHINKA, *On the embedding of Morse–Smale diffeomorphisms on a* 3*-manifold in a topological flow* (in Russian), Mat. Sb. **203** (2012), no. 12, 81–104; English translation in Sb. Math. **203** (2012), no. 11–12, 1761–1784. MR 3060070. DOI 10.1070/SM2012v203n12ABEH004286. *(2509)*

[18] V. Z. GRINES, T. V. MEDVEDEV, and O. V. POCHINKA, *Dynamical Systems on* 2*- and* 3*-Manifolds*, Dev. Math. **46**, Springer, Cham, 2016. MR 3585813. DOI 10.1007/978-3-319-44847-3. *(2508, 2515)*

[19] V. Z. GRINES, V. S. MEDVEDEV, and E. V. ZHUZHOMA, *On surface attractors and repellers on* 3*-manifolds* (in Russian), Mat. Zametki **78** (2005), no. 6, 813–826; English translation in Math. Notes **78** (2005), no. 5–6, 757–767. MR 2249032. DOI 10.1007/s11006-005-0181-1. *(2552)*

[20] V. Z. GRINES and O. POCHINKA, *Morse–Smale cascades on* 3*-manifolds* (in Russian), Uspekhi Mat. Nauk **68** (2013), no. 1, 129–188; English translation in Russian Math. Surveys **68** (2013), 117–173. MR 3088080. DOI 10.1070/rm2013v068n01abeh004823. *(2508, 2515)*

[21] V. GRINES, E. ZHUZHOMA, V. MEDVEDEV, and O. POCHINKA, *Global attractor and repeller of Morse–Smale diffeomorphisms* (in Russian), Tr. Mat. Inst. Steklova **271** (2010), 111–133; English translation in Proc. Steklov Inst. Math. **271** (2010), no. 1, 103–124. MR 2841715. DOI 10.1134/S0081543810040097. *(2515)*

[22] O. G. HARROLD, JR., H. C. GRIFFITH, and E. E. POSEY, *A characterization of tame curves in three-space*, Trans. Amer. Math. Soc. **79** (1955), 12–34. MR 0091457. DOI 10.2307/1992834. *(2519)*

[23] C. KOSNIOWSKY, *A First Course in Algebraic Topology*, Cambridge Univ. Press, Cambridge, 1980. MR 0586943. *(2515, 2522)*

[24] E. LEONTOVICH and A. MAIER, *On a scheme determining the topological structure of the separation of trajectories*, Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 557–560. MR 0072305. *(2508)*

[25] E. E. MOISE, *Geometric Topology in Dimensions* 2 *and* 3, Grad. Texts in Math. **47**, Springer, New York, 1977. MR 0488059. *(2519)*

[26] J. PALIS, *On Morse-Smale dynamical systems*, Topology **8** (1969), 385–404. MR 0246316. DOI 10.1016/0040-9383(69)90024-X. *(2508, 2515, 2520, 2530)*

[27] J. PALIS and W. DE MELO, *Geometrical Theory of Dynamical Systems*, Springer, New York, 1982. MR 0669541. *(2515, 2516, 2533)*

[28] J. PALIS and S. SMALE, "Structural stability theorems" in *Global Analysis (Berkeley, CA, 1968)*, Proc. Sympos. Pure Math. **XIV**, Amer. Math. Soc., Providence, 1970, 223–231. MR 0267603. *(2508, 2530)*

[29] M. PEIXOTO, "On the classification of flows on 2-manifolds" in *Dynamical Systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, Academic Press, New York, 1973. 389–419. MR 0334289. *(2508)*

[30] D. PIXTON, *Wild unstable manifolds*, Topology **16** (1977), no. 2, 167–172. MR 0445552. DOI 10.1016/0040-9383(77)90014-3. *(2519)*

[31] A. O. PRISHLYAK, *Morse–Smale vector fields without closed trajectories on* 3*-manifolds* (in Russian), Mat. Zametki **71** (2002), no. 2, 254–260; English translation in Math. Notes **71** (2002), no. 1–2, 230–235. MR 1900798. DOI 10.1023/A:1013963315626. *(2510)*

[32] C. ROBINSON, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, Stud. Adv. Math., 2nd ed., CRC Press, Boca Raton, FL, 1999. MR 1792240. *(2515)*

[33] D. ROLFSEN, *Knots and Links*, Math. Lecture Ser. **7**, Publish or Perish, Houston, 1990. MR 1277811. *(2539)*

[34] S. SMALE, *On gradient dynamical systems*, Ann. of Math. (2) **74** (1961), no. 1, 199-206. MR 0133139. DOI 10.2307/1970311. *(2510)*

[35] ———, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817. MR 0228014. DOI 10.1090/S0002-9904-1967-11798-1. *(2515, 2520)*

[36] W. P. THURSTON, *Three-Dimensional Geometry and Topology, Vol. 1*, Princeton Math. Ser. **35**, Princeton Univ. Press, Princeton, 1997. MR 1435975. *(2515)*

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