Character expansion of matrix integrals

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Abstract

We consider expansions of certain multiple integrals and BKP tau functions in characters of orthogonal and symplectic groups. In particular we consider character expansions of integrals over orthogonal and over symplectic matrices.

Key words: matrix integrals, $\beta = 2$ ensembles, integrable systems, tau functions, Pfaff lattice, BKP, DKP, $\mathbb{O}(N)$ and $\mathbb{S}p(N)$ characters, free fermions

1   Introduction

The character expansion of matrix models used in physics was first presented in the works [11], [27] and was used in [30], [40], [20], [19], [42], [21] for various problems, in particular in the context of relationships between matrix models and integrable systems. In all the works mentioned, expansions in terms of Schur functions was used. The importance of such representation was shown in a set of papers, for instance, for the study of matrix models [27], [37], [56], for random processes and random partitions [3], [22], in communications [53], [7] for counting problems, see [39], [15], [16], [6], [17], [38], [41] for relations of quantum and classical models [4], [55], [5], [25], [9] and some others. Here we write down expansions in the characters of orthogonal and symplectic groups. We hope that this will also be useful.

Some notations   Let us recall that the characters of the unitary group $\mathbb{U}(n)$ are labeled by partitions and coincide with the so-called Schur functions [35]. A partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a set of nonnegative integers λ_i which are called parts of λ and which are ordered as $\lambda_i \geq \lambda_{i+1}$. The number of non-vanishing parts of λ is called the length of the partition λ, and will be denoted by $\ell(\lambda)$. The number $|\lambda| = \sum_i \lambda_i$ is called the weight of λ. The set of all partitions will be denoted by $\mathbb{P}$.

The Schur function corresponding to λ is defined as the following symmetric function in variables $x = (x_1, \ldots, x_n)$:

$$s_\lambda(x) = \frac{\det\left[x_j^{\lambda_i - i + n}\right]_{i,j}}{\det\left[x_j^{-i+n}\right]_{i,j}} \tag{1}$$

in case $\ell(\lambda) \leq n$ and vanishes otherwise. One can see that $s_\lambda(x)$ is a symmetric homogeneous polynomial of degree $|\lambda|$ in the variables $x_1, \ldots, x_n$.

Remark 1. In case the set x is the set of eigenvalues of a matrix X, we also write $s_\lambda(X)$ instead of $s_\lambda(x)$.

There is a different definition of the Schur function as quasi-homogeneous non-symmetric polynomial of degree $|\lambda|$ in other variables, $\mathbf{p} = (p_1, p_2, \ldots)$, where $\deg p_m = m$:

$$s_\lambda(\mathbf{p}) = \det\left[s_{(\lambda_i - i + j)}(\mathbf{p})\right]_{i,j} \tag{2}$$

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and the Schur functions $s_{(i)}$ are defined by $e^{\sum_{m>0} \frac{1}{m} p_m z^m} = \sum_{m \geq 0} s_{(i)}(\mathbf{p}) z^i$. The Schur functions defined by (1) and by (2) are equal, $s_\lambda(\mathbf{p}) = s_\lambda(x)$, provided the variables $\mathbf{p}$ and x are related by

$$p_m = \sum_i x_i^m \tag{3}$$

From now on, we will use in case the argument of s_λ is written as a fat letter the definition (2), and we imply the definition (1) otherwise.

Remark 2. For functions $f(\mathbf{p}) = f(\mathbf{p}(A))$, where $p_m(A) := \operatorname{Tr} A^m , m = 1, 2, \dots$ and A is a given matrix we may equally write either $f(\mathbf{p}(A))$ or $f(A)$ where the capital letter implies a matrix. In particular under this convention we may write $s_\lambda(A)$ and $\tau(A)$ instead of $s_\lambda(\mathbf{p}(A))$ and $\tau(\mathbf{p}(A))$.

Integrals over the unitary group. Consider the following integral over the unitary group which depends on two semi-infinite sets of variables $\mathbf{p} = (p_1, p_2, \dots)$ and $\mathbf{p}^* = (p_1^*, p_2^*, \dots)$, which are free parameters

$$I_{\mathbb{U}(n)}(\mathbf{p}, \mathbf{p}^*) := \int_{\mathbb{U}(n)} e^{\operatorname{tr} V(\mathbf{p}, U) + \operatorname{tr} V(\mathbf{p}^*, U^{-1})} d_* U = \tag{4}$$

$$\frac{1}{(2\pi)^n} \int_{0 \leq \theta_1 \leq \dots \leq \theta_n \leq 2\pi} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{-i\theta_k}|^2 \prod_{j=1}^n e^{\sum_{m>0} \frac{1}{m} \left(p_m e^{im\theta_j} + p_m^* e^{-im\theta_j} \right)} d\theta_j \tag{5}$$

$$V(\mathbf{p}, x) := \sum_{n>0} \frac{1}{n} p_n x^n \tag{6}$$

Here $d_* U$ is the Haar measure of the group $\mathbb{U}(n)$, see (184) in Appendix, and $e^{i\theta_1}, \dots, e^{i\theta_n}$ are the eigenvalues of $U \in \mathbb{U}(n)$. The exponential factors inside the integral may be treated as a perturbation of the Haar measure and parameters $\mathbf{p}$, $\mathbf{p}^*$ are called coupling constants.

Using the Cauchy-Littlewood identity

$$\tau(\mathbf{p} | \mathbf{p}^*) := e^{\sum_{m=1}^\infty \frac{1}{m} p_m^* p_m} = \sum_{\lambda \in \mathbb{P}} s_\lambda(\mathbf{p}^*) s_\lambda(\mathbf{p}) \tag{7}$$

and the orthogonality of the irreducible characters of the unitary group

$$\int s_\lambda(U) s_\mu(U^{-1}) d_* U = \delta_{\lambda, \mu} \tag{8}$$

we obtain that

$$I_{\mathbb{U}(n)}(\mathbf{p}, \mathbf{p}^*) = \sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \leq n}} s_\lambda(\mathbf{p}) s_\lambda(\mathbf{p}^*) \tag{9}$$

which express the integral over unitary matrices as the "perturbation series in coupling constants".

The formula (9) first appeared in [37] in the context of the study of Brezin-Gross-Witten model. It was shown there that the integral $I_{\mathbb{U}(n)}(\mathbf{p}, \mathbf{p}^*)$ may be related to the Toda lattice tau function of [24] and [54] under certain restriction. Then, the series in the Schur functions (9) may be related to the double Schur functions series found in [51] and [52].

In this paper we want to express integrals over the symplectic and over the orthogonal groups, $I_{\mathbb{S}p(N)}(\mathbf{p})$ and $I_{\mathbb{O}(N)}(\mathbf{p})$ respectively, as sums of product of characters of the orthogonal and of symplectic groups, i.e. to obtain the analogues of the relation (9) and relate these integrals and sums to integrable systems. On the one hand we shall relate $I_{\mathbb{S}p(N)}(\mathbf{p})$ and $I_{\mathbb{O}(2n)}(\mathbf{p})$ to the DKP[1], and we shall relate $I_{\mathbb{O}(2n+1)}(\mathbf{p})$ to BKP tau functions, introduced respectively in [24] and [26] and obtain Pfaffian representation for these integrals. On the other hand one can relate these integrals to the Toda lattice (TL) tau function [24], [54] which yields the determinant representation.

We show that the so-called $\beta = 1, 2, 4$ ensembles may be written as formal series in characters.

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[1] We need to note that the DKP hierarchy has other names. It was rediscovered in [2] using the approach different of [24] and called Pfaff lattice. It was also called coupled KP equation in [23]

2 Polynomials $o_\lambda(\mathbf{p})$ and $sp_\lambda(\mathbf{p})$ and TL tau functions $\tau_\pm(\mathbf{p}|\mathbf{p}^*)$

The orthogonal and symplectic characters are also labeled by partitions. They are given by the following expressions

$$o_\lambda = \frac{\det\left[x_j^{\lambda_i+n-i+1} - x_j^{-\lambda_i-n+i}\right]_{1\le i,j\le n}}{\det\left[x_j^{n-i+1} - x_j^{-n+i}\right]_{1\le i,j\le n}} \tag{10}$$

and

$$sp_\lambda = \frac{\det\left[x_j^{\lambda_i+n-i+1} - x_j^{-\lambda_i-n+i-1}\right]_{1\le i,j\le n}}{\det\left[x_j^{n-i+1} - x_j^{-n+i-1}\right]_{1\le i,j\le n}} \tag{11}$$

respectively. See [13] or Appendix A.5 for more information. Baker [8] realized that these characters can be obtained from the corresponding Schur functions s_λ by action of some operator. For this, it will be convenient to use the Schur functions in terms of the power sums p_m. As usual, write $\tilde\partial = (\partial_{p_1}, 2\partial_{p_2}, 3\partial_{p_3}, \ldots)$. Let

$$\Omega_\mp(\mathbf{p}) = \sum_{m>0}\left(-\frac{1}{2m}p_m^2 \mp \frac{1}{2m}p_{2m}\right), \quad \Omega_\mp = \Omega_\mp(\tilde\partial) := \sum_{m>0}\left(-\frac{m}{2}(\partial_m)^2 \mp \partial_{2m}\right) \tag{12}$$

then

$$o_\lambda(\mathbf{p}) = e^{\Omega_-(\tilde\partial)}\cdot s_\lambda(\mathbf{p}), \quad sp_\lambda(\mathbf{p}) = e^{\Omega_+(\tilde\partial)}\cdot s_\lambda(\mathbf{p}) \tag{13}$$

Hence, if we let the operator $\Omega_\mp(\tilde\partial^*)$ act on the Cauchy- Littelewood identity (7), we obtain

$$\tau_-(\mathbf{p}|\mathbf{p}^*) = \sum_\lambda o_\lambda(\mathbf{p}^*)s_\lambda(\mathbf{p}) \tag{14}$$

and

$$\tau_+(\mathbf{p}|\mathbf{p}^*) = \sum_\lambda sp_\lambda(\mathbf{p}^*)s_\lambda(\mathbf{p}) \tag{15}$$

where

$$\tau_\mp(\mathbf{p}|\mathbf{p}^*) = e^{-\frac{1}{2}\sum_{m=1}^\infty \frac{1}{m}p_m^2 \mp \sum_{m=1}^\infty \frac{1}{2m}p_{2m} + \sum_{m=1}^\infty \frac{1}{m}p_m p_m^*} \tag{16}$$

Remark 3. Note that

$$\tau_\mp(\mathbf{p}|\mathbf{p}^*) = e^{\Omega_\mp(\mathbf{p})}\tau_0(\mathbf{p}|\mathbf{p}^*) \tag{17}$$

where $\tau_0(\mathbf{p}|\mathbf{p}^*) = e^{\sum_{m>1}\frac{1}{m}p_m p_m^*}$ is known to be the simplest tau function of the TL hierarchy (this simplest tau function does not depend on the discrete TL time p_0).

It is well known that the function $\tau_0(\mathbf{p}|\mathbf{p}^*)$, for the variables $\mathbf{p} = (p_1, p_2, \ldots)$, is a solution of the Hirota bilinear equations for the KP hierarchy:

$$\oint \frac{dz}{2\pi i}e^{V(\mathbf{p}'-\mathbf{p},z)}\tau_0(\mathbf{p}'-[z^{-1}]|\mathbf{p}^*)\tau_0(\mathbf{p}+[z^{-1}]|\mathbf{p}^*) = 0 \tag{18}$$

Here V is given by (6) and the variables $\mathbf{p}^* = (p_1^*, p_2^*, \ldots)$ play the role of auxiliary parameters. Here and below the notation $[a]$ serves to denote the following set of power sums: $(a, a^2, a^3, \ldots)$. The action of $e^{\Omega_\mp(\tilde\partial^*)}e^{\Omega_\mp(\tilde\partial'^*)}$ on (18) gives

$$\oint \frac{dz}{2\pi i}e^{V(\mathbf{p}'-\mathbf{p},z)}\tau_\mp(\mathbf{p}'-[z^{-1}]|\mathbf{p}^*)\tau_\mp(\mathbf{p}+[z^{-1}]|\mathbf{p}^*) = 0 \tag{19}$$

hence $\tau_\mp(\mathbf{p}|\mathbf{p}^*)$ is also a tau function of the KP hierarchy. Then it follows from Remark 3 that both $\tau_\pm(\mathbf{p}|\mathbf{p}^*)$ are TL tau functions where $\mathbf{p}$ and $\mathbf{p}^*$ are two sets of the higher times. These tau functions do not depend on the discrete TL variable t_0 because $\tau_0(\mathbf{p}|\mathbf{p}^*)$ does not depend on it.

According to Sato [47] a KP tau function may be related to an element of an infinite dimensional Grassmannian as a series in the Schur functions

$$\text{tau}^{\text{KP}}(\mathbf{p}) = \sum_\lambda \pi_\lambda s_\lambda(\mathbf{p})$$

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where π_λ are the Plücker coordinates of the element. Hence according to (7), (14) and (15), the functions $s_\lambda(\mathbf{p}^*)$, $o_\lambda(\mathbf{p}^*)$ and $sp_\lambda(\mathbf{p}^*)$ are the Plücker coordinates of $\tau_0(\mathbf{p}|\mathbf{p}^*)$, $\tau_-(\mathbf{p}|\mathbf{p}^*)$ and $\tau(\mathbf{p}|\mathbf{p}^*)$, respectively. The related elements of the Grassmannian is written down in Appendix A.9.

The Plücker coordinates $o_\lambda(\mathbf{p}^*)$ and $sp_\lambda(\mathbf{p}^*)$ may be evaluated respectively as follows

$$o_\lambda(\mathbf{p}^*) = \left(s_\lambda(\tilde{\partial}) \cdot e^{-\frac{1}{2}\sum_{m=1}^\infty \frac{1}{m}p_m^2 - \sum_{m=1}^\infty \frac{1}{2m}p_{2m} + \sum_{m=1}^\infty \frac{1}{m}p_m p_m^*} \right) \Big|_{\mathbf{p}=0} \tag{20}$$

and

$$sp_\lambda(\mathbf{p}^*) = \left(s_\lambda(\tilde{\partial}) \cdot e^{-\frac{1}{2}\sum_{m=1}^\infty \frac{1}{m}p_m^2 + \sum_{m=1}^\infty \frac{1}{2m}p_{2m} + \sum_{m=1}^\infty \frac{1}{m}p_m p_m^*} \right) \Big|_{\mathbf{p}=0} \tag{21}$$

which may be compared with the identity for the Schur functions

$$s_\lambda(\mathbf{p}^*) = \left(s_\lambda(\tilde{\partial}) \cdot e^{\sum_{m=1}^\infty \frac{1}{m}p_m p_m^*} \right) \Big|_{\mathbf{p}=0} \tag{22}$$

As in the previous section, let us assign the weight k to p_k. We recall that the polynomials s_λ are quasi-homogeneous in the variables p_m of the weight $|\lambda|$. As we see from (20) and (21) polynomials o_λ and sp_λ are not quasi-homogeneous: they both may be presented as s_λ plus polynomials of minor weights.

For instance

$$o_{(1)}(\mathbf{p}) = sp_{(1)}(\mathbf{p}) = s_{(1)}(\mathbf{p}) = p_1$$

$$o_{(2)}(\mathbf{p}) = s_{(2)}(\mathbf{p}) - 1 = \frac{1}{2}p_2 + \frac{1}{2}p_1^2 - 1, \quad sp_{(2)}(\mathbf{p}) = s_{(2)}(\mathbf{p}) = \frac{1}{2}p_2 + \frac{1}{2}p_1^2$$

$$o_{1^2}(\mathbf{p}) = s_{1^2}(\mathbf{p}) = -\frac{1}{2}p_2 + \frac{1}{2}p_1^2, \quad sp_{1^2}(\mathbf{p}) = s_{1^2}(\mathbf{p}) - 1 = -\frac{1}{2}p_2 + \frac{1}{2}p_1^2 - 1$$

Next from $\tau_-(-\mathbf{p}|-\mathbf{p}^*) = \tau_+(\mathbf{p}|\mathbf{p}^*)$ and from $s_\lambda(\mathbf{p}) = (-)^{|\lambda|}s_{\lambda^{\mathrm{tr}}}(-\mathbf{p})$, we get

$$sp_\lambda(\mathbf{p}) = (-)^{|\lambda|}o_{\lambda^{\mathrm{tr}}}(-\mathbf{p}) \tag{23}$$

where $-\mathbf{p} = (-p_1, -p_2, -p_3, \dots)$.

From the following well-known formulas (see [35], pages 76 and 77 or [34] page 238)

$$e^{\frac{1}{2}\sum_{m>0}\frac{1}{m}p_m^2 + \sum_{m>0,\mathrm{odd}}\frac{1}{m}p_m} = \sum_{\mu \in \mathrm{P}} s_\mu(\mathbf{p}), \quad e^{-\Omega_+(\mathbf{p})} = \sum_{\mu \in \mathrm{P}} s_{\mu \cup \mu}(\mathbf{p}), \text{ and } e^{-\Omega_-(\mathbf{p})} = \sum_{\mu \in \mathrm{P}_{\mathrm{even}}} s_\mu(\mathbf{p})$$

where $\mathrm{P}_{\mathrm{even}}$ is the set of all partitions with even parts (including (0)), one deduces

Lemma 1.

$$\sum_{\mu \in \mathrm{P}} z^{|\mu|} s_\mu(\tilde{\partial}) \;=\; e^{\frac{1}{2}\sum_{m=1}^\infty mz^{2m}\partial_m^2 + \sum_{m>0,\mathrm{odd}} z^m \partial_m} \tag{24}$$

$$\sum_{\mu \in \mathrm{P}} z^{2|\mu|} s_{\mu \cup \mu}(\tilde{\partial}) \;=\; e^{\frac{1}{2}\sum_{m=1}^\infty mz^{2m}\partial_m^2 - \sum_{m>0,\,\mathrm{even}} z^m \partial_m} = e^{-\Omega_+(z^m\tilde{\partial}_m)} \tag{25}$$

$$\sum_{\mu \in \mathrm{P}_{\mathrm{even}}} z^{|\mu|} s_\mu(\tilde{\partial}) \;=\; e^{\frac{1}{2}\sum_{m=1}^\infty mz^{2m}\partial_m^2 + \sum_{m>0,\,\mathrm{even}} z^m \partial_m} = e^{-\Omega_-(z^m\tilde{\partial}_m)} \tag{26}$$

where $s_\lambda(\tilde{\partial})$ is defined as in (20)-(21).

(see also [43] where in (25) and (26) there is the opposite sign for linear term in exponents which is a misprint).

Remark 4. From Lemma 1 a number of relations may be obtained. We present two examples:

$$e^{-\sum_{m>0}\frac{1}{2m}p_m^2 - \sum_{m>0,\,\mathrm{odd}}\frac{1}{m}p_m} \sum_{\mu \in \mathrm{P}} s_{\mu/\lambda}(\mathbf{p}) = \sum_{\mu \in \mathrm{P}} s_{\lambda/\mu}(\mathbf{p}) = e^{\sum_{m=1}^\infty (m\partial_m^2 + \partial_m)} \cdot o_\lambda(\mathbf{p})$$

where the first equality is obtained from Ex 27(a) in I.5 of [35]. The second example follows from (13), (24) and from $s_\mu(\tilde{\partial}) \cdot s_\lambda(\mathbf{p}) = s_{\lambda/\mu}(\mathbf{p})$. The second example,

$$\sum_{\mu \in \mathrm{P}} s_{\lambda/\mu \cup \mu}(\mathbf{p}) = e^{\sum_{m>1} m\partial_m^2} \cdot o_\lambda(\mathbf{p}) \tag{27}$$

we obtain from (13) and (25). Note, that the constant term $\left[e^{\sum_{m>1} m\partial_m^2} \cdot o_\lambda(\mathbf{p}) \right]_{\mathbf{p}=0}$ of (27) is equal to 1 for any λ of form $\mu \cup \mu$, and vanishes otherwise.

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Relation to irreducible characters of the orthogonal and symplectic groups. We shall use notations explained in Remark 2 with $p_m(U) = \operatorname{Tr} U^m$.

In this notation we write

$$\tau_+(U|\mathbf{p}^*) = \prod_{i<j}(1 - x_i x_j) \prod_{k=1}^{n} e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m^* x_k^m} = \sum_\lambda sp_\lambda(\mathbf{p}^*) s_\lambda(U) \tag{28}$$

$$\tau_-(U|\mathbf{p}^*) = \prod_{i\leq j}(1 - x_i x_j) \prod_{k=1}^{n} e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m^* x_k^m} = \sum_\lambda o_\lambda(\mathbf{p}^*) s_\lambda(U) \tag{29}$$

where $x_1, \ldots, x_n$ are the eigenvalues of $U \in \mathbb{U}(n)$. Let us note that

$$\tau_-(U|\mathbf{p}^*) = \tau_+(U|\mathbf{p}^*)\det(1 - U^2) \tag{30}$$

Now take $Z \in \mathbb{S}p(2n)$ and let $z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}$ be the eigenvalues of Z. Then the formula (28) reads

$$\prod_{i<j\leq n}(1 - x_i x_j) \prod_{i,j=1}^{n}(1 - x_i z_j)^{-1}(1 - x_i z_j^{-1})^{-1} = \sum_{\lambda \in \mathbb{P}} sp_\lambda(Z) s_\lambda(U), \quad Z \in \mathbb{S}p(2n),\ U \in \mathbb{U}(n) \tag{31}$$

This relation is known as the Cauchy identity for the irreducible characters of $\mathbb{S}p(2n)$, see [34]. Thus, from the completeness of the Schur functions s_λ in the space of the symmetric functions in $x_1, \ldots, x_n$, it follows that the polynomials $sp_\lambda(\mathbf{p}(Z)) = sp_\lambda(Z)$ coincide with the characters of $\mathbb{S}p(2n)$.

Similarly, for $Z \in \mathbb{O}(2n)$ with the eigenvalues $z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}$, and for $U \in \mathbb{U}(n)$ with eigenvalues $x_1, \ldots, x_n$, we obtain

$$\prod_{i\leq j\leq n}(1 - x_i x_j) \prod_{i,j=1}^{n}(1 - x_i z_j)^{-1}(1 - x_i z_j^{-1})^{-1} = \sum_{\lambda \in \mathbb{P}} o_\lambda(Z) s_\lambda(U), \quad Z \in \mathbb{O}(2n),\ U \in \mathbb{U}(n) \tag{32}$$

While for $Z \in \mathbb{O}(2n+1)$ with the eigenvalues $z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}, 1$, we obtain

$$\prod_{i\leq j\leq n}(1-x_i x_j) \prod_{i,j=1}^{n}(1-x_i z_j)^{-1}(1-x_i z_j^{-1})^{-1} \prod_{i=1}^{n}(1-x_i)^{-1} = \sum_{\lambda \in \mathbb{P}} o_\lambda(Z) s_\lambda(U), \quad Z \in \mathbb{O}(2n+1),\ U \in \mathbb{U}(n) \tag{33}$$

Relations (32), (33) are known as the Cauchy identities for the orthogonal group, the polynomials $o_\lambda(Z)$ are irreducible characters of the orthogonal group.

We have

$$\tau_+(U|Z) = \sum_\lambda sp_\lambda(Z) s_\lambda(U), \quad Z \in \mathbb{S}p(2n), \quad U \in \mathbb{U}(n)$$

and

$$\tau_-(U|Z) = \sum_\lambda o_\lambda(Z) s_\lambda(U), \quad Z \in \mathbb{O}(N), \quad U \in \mathbb{U}(n),\ n = \left[\frac{N}{2}\right]$$

where sp_λ and o_λ are characters respectively of symplectic and orthogonal groups.

The content of this Section may be compared with [29], [8] where universal characters of classical groups were considered.

3   Characters and fermions

Since all Schur functions s_λ are in the GL_∞ group orbit, they are KP tau functions, i.e., they satisfy the bilinear identity:

$$Res_z \psi(z)\tau \otimes \psi^\dagger(z)\tau = 0 \tag{34}$$

where $\psi(z) = \sum_{i\in\mathbb{Z}} \psi_i z^i$ and $\psi^\dagger(z) = \sum_{i\in\mathbb{Z}} \psi_i^\dagger z^{-i-1}$, are free fermionic fields (see [24]), whose Fourier components anti-commute as follows $\psi_i\psi_j + \psi_j\psi_i = \psi_i^\dagger\psi_j^\dagger + \psi_j^\dagger\psi_i^\dagger = 0$ and $\psi_i\psi_j^\dagger + \psi_j^\dagger\psi_i = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker symbol. We put

$$\psi_i|0\rangle = \psi_{-1-i}^\dagger|0\rangle = \langle 0|\psi_{-1-i} = \langle 0|\psi_i^\dagger = 0, \quad \text{for } i < 0, \tag{35}$$

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where $\langle 0|$ and $|0\rangle$ are left and right vacuum vectors of the fermionic Fock space, $\langle 0| \cdot 1 \cdot |0\rangle = 1$. Let

$$\langle n| = \begin{cases} \langle 0|\psi_0^\dagger \cdots \psi_{n-1}^\dagger & \text{if} \quad n > 0 \\ \langle 0|\psi_{-1} \cdots \psi_{-n} & \text{if} \quad n < 0 \end{cases}, \quad |n\rangle = \begin{cases} \psi_{n-1} \cdots \psi_0 |0\rangle & \text{if} \quad n > 0 \\ \psi_{-n}^\dagger \cdots \psi_{-1}^\dagger |0\rangle & \text{if} \quad n < 0 \end{cases} \tag{36}$$

then $\langle n| \cdot 1 \cdot |m\rangle = \delta_{n,m}$. Note that $e^{\Omega_\mp}$ is an automorphism of the Fock space. It maps any charge sector into itself and maps Schur functions into orthogonal and symplectic characters, see (13). Thus o_λ (for e^{Ω_-}) and sp_λ (for e^{Ω_+}) satisfies

$$Res_z e^{\Omega_\mp(\tilde{\alpha})} \psi(z) e^{-\Omega_\mp(\tilde{\alpha})} \sigma \otimes e^{\Omega_\mp(\tilde{\alpha})} \psi^\dagger(z) e^{-\Omega_\mp(\tilde{\alpha})} \sigma = 0, \tag{37}$$

where $\sigma = e^{\Omega_\mp} \tau$. We now want to calculate $\Psi_\mp(z) = e^{\Omega_\mp(\tilde{\alpha})} \psi(z) e^{-\Omega_\mp(\tilde{\alpha})}$ and $\Psi_\mp^\dagger(z) = e^{\Omega_\mp(\tilde{\alpha})} \psi^\dagger(z) e^{-\Omega_\mp(\tilde{\alpha})}$. We use the vertex operator expression for $\psi(z)$ and $\psi^\dagger(z)$

$$\psi(z) = e^{\alpha_0} z^{\alpha_0} e^{-\sum_{i<0} \frac{\alpha_i}{z^i}} e^{-\sum_{i>0} \frac{\alpha_i}{z^i}} \tag{38}$$

$$\psi^\dagger(z) = e^{-\alpha_0} z^{-\alpha_0} e^{\sum_{i<0} \frac{\alpha_i}{z^i}} e^{\sum_{i>0} \frac{\alpha_i}{z^i}} \tag{39}$$

where

$$\alpha_m = \sum_{i \in \mathbb{Z}} : \psi_i \psi_{i+m}^\dagger : \tag{40}$$

for future use we also introduce

$$\Gamma(\mathbf{p}) := e^{\sum_{m=1}^\infty t_m \alpha_m}, \quad \Gamma^\dagger(\mathbf{p}) := e^{\sum_{m=1}^\infty t_m \alpha_{-m}} \tag{41}$$

Note that $[\alpha_i, \alpha_j] = i\delta_{i,-j}$, hence they form a Heisenberg algebra. Now use the standard realization of the Heisenberg algebra

$$\alpha_k = \partial_k, \quad \alpha_{-k} = kt_k, \quad \alpha_0 = q\partial_q, \quad e^{\alpha_0} = q$$

Then (38), respectively (39) turn into

$$\psi(z) = qz^{q\partial_q} e^{\sum_{i=1}^\infty t_i z^i} e^{-\sum_{i=1}^\infty \partial_i \frac{z^{-i}}{i}} \tag{42}$$

$$\psi^\dagger(z) = q^{-1} z^{-q\partial_q} e^{-\sum_{i=1}^\infty t_i z^i} e^{\sum_{i=1}^\infty \partial_i \frac{z^{-i}}{i}} \tag{43}$$

Using the following formulas which can easily be deduced from the Cambell-Baker-Hausdorff formula:

$$e^{a\partial_x} e^{bx} = e^{bx} e^{a(b+\partial_x)}, \qquad e^{a\partial_x^2} e^{bx} = e^{bx} e^{a(b+\partial_x)^2}$$

one thus obtains:

$$\Psi_\mp(z) = (1 - z^2)^{\frac{1}{2} \pm \frac{1}{2}} qz^{q\partial_q} e^{\sum_{i=1}^\infty t_i z^i} e^{-\sum_{i=1}^\infty \partial_i \frac{z^{-i} + z^i}{i}} \tag{44}$$

$$\Psi_\mp^\dagger(z) = (1 - z^2)^{\frac{1}{2} \mp \frac{1}{2}} q^{-1} z^{-q\partial_q} e^{-\sum_{i=1}^\infty t_i z^i} e^{\sum_{i=1}^\infty \partial_i \frac{z^{-i} + z^i}{i}} \tag{45}$$

Hence the orthogonal and symplectic characters satisfy the bilinear equation:

$$Res_z \Psi_\mp(z) \sigma_\mp \otimes \Psi_\mp^\dagger(z) \sigma_\mp = 0, \tag{46}$$

or equivalently

$$Res_z (1 - z^2) e^{\sum_{i=1}^\infty (t_i - s_i) z^i} \sigma_\mp(t - [z] - [z^{-1}]) \sigma_\mp(s + [z] + [z^{-1}]) = 0, \tag{47}$$

which is equation (5.4) of Baker [8].

Now note that if we write $\Psi_\mp(z) = \sum_{i \in \mathbb{Z}} \Psi_{\mp i} z^i$ and $\Psi_\mp^\dagger(z) = \sum_{i \in \mathbb{Z}} \Psi_{\mp i}^\dagger z^{-i-1}$, then the modes still satisfy the usual relations.

$$\Psi_{\mp i} \Psi_{\mp j} + \Psi_{\mp j} \Psi_{\mp i} = 0 = \Psi_{\mp i}^\dagger \Psi_{\mp j}^\dagger + \Psi_{\mp j}^\dagger \Psi_{\mp i}^\dagger \qquad \Psi_{\mp i} \Psi_{\mp j}^\dagger + \Psi_{\mp j}^\dagger \Psi_{\mp i} = \delta_{ij}$$

Using the above vertex operators on the vacuum $|0\rangle = q^0$ one still has

$$\Psi_{\mp i}|0\rangle = 0 = \Psi_{\mp(-i-1)}^\dagger|0\rangle \qquad i < 0$$

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Another approach is as follows, equation (46) still generates the GL_∞ goup orbit of the vacuum, however one has to take a different realization of gl_∞, viz. $\Psi_{\mp i}\Psi^\dagger_{\mp j}$ $(i, j \in \mathbb{Z})$ still forms a basis of gl_∞, it is the coefficients of $z^i y^{-j-1}$ in the expansion

$$X_{\mp}(y, z) = e^{\Omega_{\mp}(\tilde\alpha)}\psi(z)\psi^\dagger(y)e^{-\Omega_{\mp}(\tilde\alpha)} = \Psi_{\mp}(z)\Psi^\dagger_{\mp}(y)$$

Using the above vertex operators we find

$$X_{\mp}(y, z) = \frac{(1-z^2)^{\frac{1}{2}\pm\frac{1}{2}}(1-y^2)^{\frac{1}{2}\mp\frac{1}{2}}}{(z-y)(1-zy)}(z/y)^{q\partial_q}e^{\sum_{i=1}^\infty t_i(z^i-y^i)}e^{-\sum_{i=1}^\infty \partial_i \frac{z^{-i}+z^i-y^{-i}-y^i}{i}}$$

Clearly also the standard Heisenberg algebra changes.

Now define $\beta^\mp = e^{\Omega_\mp}\alpha_k e^{\Omega_\mp}$, then the β_k still have the standard commutation relations, $[\beta_i^\mp, \beta_j^\mp] = i\delta_{i,-j}$, however these elements arealized in a different way. Using

$$e^{a\partial_x}x = (x+a)e^{a\partial_x}, \qquad e^{a\partial_x^2}x = (x+2a\partial_x)e^{a\partial_x^2}$$

Hence,

$$\beta_k^\mp = \partial_k \quad \beta_{-k}^\mp = kt_k - \partial_k \mp \delta_{k,\text{even}}, \quad \beta_0^\mp = q\partial_q$$

And clearly

$$\Psi_{\mp}(z) = e^{\beta_0^\mp}z^{\beta_0^\mp}e^{-\sum_{i<0}\frac{\beta_i^\mp}{i}z^i}e^{-\sum_{i>0}\frac{\beta_i^\mp}{i}z^i}$$

$$\Psi^\dagger_{\mp}(z) = e^{-\beta_0^\mp}z^{-\beta_0^\mp}e^{\sum_{i<0}\frac{\beta_i^\mp}{i}z^i}e^{\sum_{i>0}\frac{\beta_i^\mp}{i}z^i}$$

or equivalently

$$\Psi_{\mp}(z) = qz^{q\partial_q}e^{\sum_{i=1}^\infty\left(t_i - \frac{\partial_i \pm \delta_{i,\text{even}}}{i}\right)z^i}e^{-\sum_{i=1}^\infty \partial_i \frac{z^{-i}}{i}} \tag{48}$$

$$\Psi^\dagger_{\mp}(z) = q^{-1}z^{-q\partial_q}e^{-\sum_{i=1}^\infty\left(t_i - \frac{\partial_i \pm \delta_{i,\text{even}}}{i}\right)z^i}e^{\sum_{i=1}^\infty \partial_i \frac{z^{-i}}{i}} \tag{49}$$

Note that one obtains (44), respectively (45) from (48), respectively (49), if one moves the differential operator part to the right. One can use

$$e^{a\partial_x + bx} = e^{\frac{ab}{2}}e^{bx}e^{a\partial_x}$$

Note that

$$X_{\mp}(y, z) = \frac{1}{(z-y)}(z/y)^{q\partial_q}e^{\sum_{i=1}^\infty\left(t_i - \frac{\partial_i \pm \delta_{i,\text{even}}}{i}\right)(z^i-y^i)}e^{-\sum_{i=1}^\infty \partial_i \frac{z^{-i}-y^{-i}}{i}}$$

The above suggest that we can take the normal Clifford algebra in ψ_i and $\psi^\dagger_j$, but choose another realization of the Heisenberg algebra, viz., the ones given by the $\beta_i^\mp$, such that the fields $\psi(z)$ and $\psi^\dagger(z)$ are given by (48), respectively (49). Then the tau function, which is in the KP hierarchy given by

$$\tau(\mathbf{p}) = \langle 0|e^{\sum_{i>1}t_i\alpha_i}g|0\rangle \qquad \text{for } g \in Gl_\infty$$

changes into $\sigma_\mp(\mathbf{p}) = e^{\Omega_\mp}\tau(\mathbf{p})$, which is equal to

$$\sigma_\mp(\mathbf{p}) = \langle 0|e^{\sum_{i>1}t_i\alpha_i - \frac{1}{2i}\alpha_i^2 \mp \frac{1}{2i}\alpha_{2i}}g|0\rangle \qquad \text{for } g \in Gl_\infty$$

which corresponds to the modified Hamiltonian of [8], Section 3, Approach I. Next calculate

$$\langle 0|e^{\sum_{i>1}t_i\alpha_i - \frac{1}{2i}\alpha_i^2 \mp \frac{1}{2i}\alpha_{2i}}e^{\sum_{i>1}t_i^*\alpha_{-i}}|0\rangle = \tau_\mp(\mathbf{p}^*|\mathbf{p})$$

Hence, it makes sense to look at

$$\langle 0|e^{\sum_{i>1}t_i\alpha_i - \frac{1}{2i}\alpha_i^2 \mp \frac{1}{2i}\alpha_{2i}}ge^{\sum_{i>1}t_i^*\alpha_{-i}}|0\rangle \qquad \text{for } g \in Gl_\infty$$

Remark 5. Actually we have

$$\tau(\mathbf{p}, \mathbf{p}^*) \to \tau^{\pm}(\mathbf{p}, \mathbf{p}^*) = e^{\Omega_{\pm}(\tilde{\delta})} \cdot \tau(\mathbf{p}, \mathbf{p}^*) \tag{50}$$

If

$$\tau(\mathbf{p}, \mathbf{p}^*) = \sum_{\lambda, \mu \in \mathrm{P}} s_{\lambda}(\mathbf{p}) \pi_{\lambda, \mu} s_{\mu}(\mathbf{p}^*)$$

where

$$\pi_{\lambda, \mu} = \langle 0 | \, s_{\lambda}(\tilde{\alpha}) \, g \, s_{\mu}(\tilde{\alpha}^*) \, | 0 \rangle$$

then

$$\tau^{\pm}(\mathbf{p}, \mathbf{p}^*) = \sum_{\lambda, \mu \in \mathrm{P}} s_{\lambda}(\mathbf{p}) \pi_{\lambda, \mu}^{\pm} s_{\mu}(\mathbf{p}^*),$$

where

$$\pi_{\lambda, \mu}^{+} = \langle 0 | \, sp_{\lambda}(\tilde{\alpha}) \, g \, s_{\mu}(\tilde{\alpha}^*) \, | 0 \rangle, \quad \pi_{\lambda, \mu}^{-} = \langle 0 | \, o_{\lambda}(\tilde{\alpha}) \, g \, s_{\mu}(\tilde{\alpha}^*) \, | 0 \rangle$$

Similarly, one can consider $\tau^{a,b}$ with $a, b = \pm$.

4 Integrals over symplectic group and over orthogonal groups

Haar measures and generating functions for characters. Lemma 6 in Appendix A.1 and formulae of the Appendix A.8 results in the following lemmas we shall need:

Lemma 2. *The Haar measures of the symplectic group $\mathbb{S}p(2n)$ and of the unitary group $\mathbb{U}(n)$ are related as follows*

$$
\begin{aligned}
e^{\operatorname{tr} V(S, \mathbf{p})} d_* S &= 2^{-n} \tau_-(U | \mathbf{p}) \tau_-(U^{-1} | \mathbf{p}) d_* U \tag{51} \\
&= 2^{-n} \tau_+(U | \mathbf{p}) \tau_+(U^{-1} | \mathbf{p}) \det(1 - U^2) \det(1 - U^{-2}) d_* U \tag{52} \\
&= 2^{-n} \tau_+(U | \mathbf{p}) \tau_-(U^{-1} | \mathbf{p}) \det(1 - U^2) d_* U \tag{53}
\end{aligned}
$$

where $e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_n}$ are eigenvalues of $S \in \mathbb{S}p(2n)$ while $e^{i\theta_1}, \ldots, e^{i\theta_n}$ are eigenvalues of $U \in \mathbb{U}(n)$.

Lemma 3. *The Haar measures of the orthogonal group $\mathbb{O}(2n)$ and of the unitary group $\mathbb{U}(n)$ are related as follows*

$$
\begin{aligned}
e^{\operatorname{tr} V(O, \mathbf{p})} d_* O &= 2^{-n} \tau_+(U | \mathbf{p}) \tau_+(U^{-1} | \mathbf{p}) d_* U \tag{54} \\
&= 2^{-n} \tau_-(U | \mathbf{p}) \tau_-(U^{-1} | \mathbf{p}) \det(1 - U^2)^{-1} \det(1 - U^{-2})^{-1} d_* U \tag{55} \\
&= 2^{-n} \tau_-(U | \mathbf{p}) \tau_+(U^{-1} | \mathbf{p}) \det(1 - U^2)^{-1} d_* U \tag{56}
\end{aligned}
$$

Lemma 4. *The Haar measures of the orthogonal group $\mathbb{O}(2n + 1)$ and of the unitary group $\mathbb{U}(n)$ are related as follows*

$$
\begin{aligned}
e^{\operatorname{tr} V(O, \mathbf{p})} d_* O &= 2^{-n} \tau_+(U | \mathbf{p}) \tau_+(U^{-1} | \mathbf{p}) \det(1 - U) \det(1 - U^{-1}) d_* U \tag{57} \\
&= 2^{-n} \tau_-(U | \mathbf{p}) \tau_-(U^{-1} | \mathbf{p}) \det(1 + U)^{-1} \det(1 + U^{-1})^{-1} d_* U \tag{58} \\
&= 2^{-n} \tau_-(U | \mathbf{p}) \tau_+(U^{-1} | \mathbf{p}) \det \frac{1 - U^{-1}}{1 + U} d_* U \tag{59}
\end{aligned}
$$

where $e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_n}, 1$ are eigenvalues of $O \in \mathbb{O}(2n + 1)$ while $e^{i\theta_1}, \ldots, e^{i\theta_n}$ are eigenvalues of $U \in \mathbb{U}(n)$.

4.1 Integrals over symplectic group.

Consider the following integral over the symplectic group

$$I_{\mathbb{S}p(2n)}(\mathbf{p}) = \int_{S \in \mathbb{S}p(2n)} e^{\sum_{m=1}^{\infty} t_m \operatorname{tr} S^m} d_* S \tag{60}$$

where $d_* S$ is the corresponding Haar measure. Explicitly

$$I_{\mathbb{S}p(2n)}(\mathbf{p}) = \frac{2^{n^2}}{\pi^n} \int_{0 \le \theta_1 \le \cdots \le \theta_n \le \pi} \prod_{i<j}^{n} (\cos \theta_i - \cos \theta_j)^2 \prod_{i=1}^{n} e^{2 \sum_{m=1}^{\infty} t_m \cos m\theta_i} \sin^2 \theta_i d\theta_i \tag{61}$$

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where $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_n}$ are the eigenvalues of S.

By analogy with matrix models studied in physics we call the parameters $\mathbf{p} = (p_1, p_2, \ldots)$ coupling constants, and (asymptotic) series in these parameters are called perturbation series.

Perturbation series for integrals over symplectic group as series in characters $o_\lambda(\mathbf{p})$, $sp_\lambda(\mathbf{p})$, $s_\lambda(\mathbf{p})$.

Proposition 1.

$$I_{\mathbb{S}p(2n)}(\mathbf{p}) = \sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \leq n}} s_{\lambda \cup \lambda}(\mathbf{p}) \tag{62}$$

$$= 2^{-n} \sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \leq n}} \left(o_\lambda(\mathbf{p}) \right)^2 = 2^{-n} \sum_{\substack{\lambda \in \mathbb{P} \\ \lambda_1 \leq n}} \left(sp_\lambda(-\mathbf{p}) \right)^2 \tag{63}$$

Formula (62) may be derived using of Cauchy-Littlewood formula

$$e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m \mathrm{tr} S^m} = \sum_\lambda s_\lambda(S) s_\lambda(\mathbf{p})$$

and the known relation (see for instance (6.13)-(6.15) in Sect 6, VII in [35])

$$\int_{S \in \mathbb{S}p(2n)} s_\lambda(S) d_* S = \begin{cases} 1 & \lambda^{tr} \text{ is even} \\ 0 & \text{otherwise} \end{cases} \tag{64}$$

where λ^{tr} is the partition conjugated to λ, see [35]. (This relation may be easily obtained by the evaluation of the Schur function $s_\lambda(z)$ where $z_i = x_i + x_i^{-1}$ inside the integral over symplectic group).

The left hand side of (63) is obtained from (51), (29) and (8), and the right hand side of (63) is the result of (23).

Integrals over symplectic group as DKP tau functions.   Here we need the fermionic language of Section 3, see also Appendix A.3.

Proposition 2.

$$I_{\mathbb{S}p(2n)}(\mathbf{p}) = \frac{1}{n!} \langle 2n | \Gamma(\mathbf{p}) e^{\frac{1}{4\pi i} \oint \psi(x^{-1})\psi(x)(x-x^{-1})\frac{dx}{x}} |0\rangle \tag{65}$$

$$= \frac{1}{n!} \langle 2n | \Gamma(\mathbf{p}) e^{\sum_{i \in \mathbb{Z}} \psi_i \psi_{i-1}} |0\rangle \tag{66}$$

Proof. The second equality follows from

$$\frac{1}{4\pi i} \oint \psi(x^{-1})\psi(x)(x-x^{-1})\frac{dx}{x} = \sum_{i \in \mathbb{Z}} \psi_i \psi_{i-1} \tag{67}$$

Let us consider the Taylor series of the exponential in the right hand side. The first equality follows from

$$\langle N | \psi(x_1) \cdots \psi(x_N) |0\rangle = \prod_{i<j} (x_i - x_j) =: \Delta_N(x) \tag{68}$$

and

$$\Delta_{2n}(e^{-i\theta_1}, e^{i\theta_1}, \ldots, e^{-i\theta_n}, e^{i\theta_n}) = (-i)^n 2^{n^2} \prod_{k<j} (\cos\theta_k - \cos\theta_j)^2 \prod_{k=1}^{n} \sin\theta_k \tag{69}$$

Let us mark that formula (62) follows from (67) and from results of [43] (see formulae for $S_4^{(1)}$ in [43]).

Pfaffian representation. Here we need Appendix A.6. Let us note that thank to the Wick's rule we directly obtain the Pfaffian representation of the integral (60) as follows

Proposition 3.

$$I_{\mathbb{S}p(2n)}(\mathbf{p}) = \mathrm{Pf}\,[M_{kj}(\mathbf{p})]_{k,j=1,\ldots,2n} \tag{70}$$

where M is the following Toeplitz matrix

$$M_{kj}(\mathbf{p}) = -M_{jk}(\mathbf{p}) = \frac{1}{4\pi i} \oint \left(x^{j-k} - x^{k-j}\right)\left(x - x^{-1}\right) e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m (x^m + x^{-m})} \frac{dx}{x} \tag{71}$$

Relation of the integral over $\mathbb{S}p(2n)$ to an integral over $\mathbb{U}(2n)$. We can either refer to results of [33] about the interplay between BKP and two-component KP tau functions, or do the following. Consider

$$I_{\mathbb{U}(2n)}(\mathbf{p}) := \int_{U \in \mathbb{U}(2n)} \det\left(U - U^\dagger\right) e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m \left(\mathrm{tr} U^m + \mathrm{tr} U^{-m}\right)} d_* U \tag{72}$$

Written as an integral over eigenvalues it is

$$I_{\mathbb{U}(2n)}(\mathbf{p}) = \frac{1}{(2n)!(2\pi)^{2n}} \oint \Delta_{2n}(x)\Delta_{2n}(x^{-1}) \prod_{i=1}^{2n} (x_i - x_i^{-1}) e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m \left(x_i^m + x_i^{-m}\right)} \frac{dx_i}{x_i} \tag{73}$$

We have

Lemma 5.

$$I_{\mathbb{U}(2n)}(\mathbf{p}) = \langle 2n, -2n | e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m \left(\alpha_m^{(1)} - \alpha_m^{(2)}\right)} e^{\frac{1}{2\pi i} \oint \psi^{(1)}(x^{-1}) \psi^{\dagger(2)}(x)(x - x^{-1}) \frac{dx}{x}} | 0, 0 \rangle \tag{74}$$

For the two-component fermions see Appendix, $\alpha_n^{(a)} := \sum_{i \in \mathbb{Z}} \psi_i^{(a)} \psi_{i+n}^{\dagger(a)}$, $n > 0$.

The Lemma is the direct result of [21] (see Proposition 4 there). We also have

Proposition 4.

$$\left(2^n \int_{S \in \mathbb{S}p(2n)} e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m \mathrm{tr} S^m} d_* S\right)^2 = \int_{U \in \mathbb{U}(2n)} \det\left(U - U^\dagger\right) e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m \left(\mathrm{tr} U^m + \mathrm{tr} U^{-m}\right)} d_* U \tag{75}$$

Proof. One way to prove it is to present $I_{\mathbb{U}(2n)}$ as determinant of M of (71). It is easy using the Wick's rule for the vacuum expectation value in Lemma 5. Via Wick's rule we directly obtain

$$I_{\mathbb{U}(2n)}(\mathbf{p}) = \det\,[2M_{ij}]_{i,j=1,\ldots,2n} \tag{76}$$

with the same matrix M as in Pfaffian representation (70). This proves (4).

The other way is to apply the results of [33] (see Proposition 4 there). Then we have

$$\left(2^n I_{\mathbb{S}p(2n)}(\mathbf{p})\right)^2 = \langle 2n, -2n | e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m \left(\alpha_m^{(1)} - \alpha_m^{(1)}\right)} e^{\frac{1}{4\pi i} \oint \psi^{(1)}(x^{-1}) \psi^{\dagger(2)}(x)(x - x^{-1}) \frac{dx}{x}} | 0, 0 \rangle \tag{77}$$

which coincides with the right hand side of (74). The end of proofs.

Different fermionic representation and Toda chain-AKNS tau function Denote $z = x + x^{-1}$. Introduce variables $\tilde{\mathbf{p}} = (\tilde{p}_1, \tilde{p}_2, \ldots)$ with the help

$$\sum_{n=1}^{\infty} \frac{1}{n} p_n (x^n + x^{-n}) = \sum_{n=1}^{\infty} \frac{1}{n} \tilde{p}_n z^n - c(\tilde{\mathbf{p}}), \quad c(\tilde{\mathbf{p}}) = \sum_n \frac{(2n)!}{(n!)^2} \frac{\tilde{p}_{2n}}{2n} \tag{78}$$

where the sets of $\{\tilde{p}_{2n},\, n > 0\}$ and of $\{\tilde{p}_{2n+1},\, n \geq 0\}$ may be expressed via the sets $\{p_{2n},\, n > 0\}$ and of $\{p_{2n+1},\, n \geq 0\}$ respectively by triangle transformations with binomial entries.

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Then thanks to (150), (154) in Appendix A.1 we have

$$I_{\mathbb{S}p(2n)}(\mathbf{p}(\tilde{\mathbf{p}})) = \int_{-2}^{2} \cdots \int_{-2}^{2} \prod_{i<j} (z_i - z_j)^2 \prod_{i=1}^{n} e^{\sum_{m=1}^{\infty} \frac{1}{m} \tilde{p}_m z_i^m} (4 - z_i^2) dz_i \tag{79}$$

The last integral is an example of well-studied $\beta = 2$ ensemble and may be presented in form of a determinant:

$$I_{\mathbb{S}p(2n)}(\mathbf{p}(\tilde{\mathbf{p}})) = \det[N_{ij}(\tilde{\mathbf{p}})]_{i,j=1,\ldots,n} \tag{80}$$

where N_{ij} are the so-called moments. In our case

$$N_{ij}(\tilde{\mathbf{p}}) = \int_{-2}^{2} z^{i+j} e^{\sum_{m=1}^{\infty} \frac{1}{m} \tilde{p}_m z^m} (4 - z^2) dz \tag{81}$$

The fermionic representation for $\beta = 2$ ensembles is known, see , in our case

$$=$$

4.2 Integrals over orthogonal group.

Consider the following integral over the orthogonal group $\mathbb{O}(2n)$

$$I_{\mathbb{O}(N)}(\mathbf{p}) = \int_{O \in \mathbb{O}(2n)} e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m \operatorname{tr} O^m} d_* O \tag{82}$$

where $d_* O$ is the corresponding Haar measure. Explicitly

$$I_{\mathbb{O}(2n)}(\mathbf{p}) = \frac{2^{(n-1)^2}}{\pi^n n!} \int_{0 \le \theta_1 \le \cdots \le \theta_n \le \pi} \prod_{i<j}^{n} (\cos \theta_i - \cos \theta_j)^2 \prod_{i=1}^{n} e^{2 \sum_{m=1}^{\infty} \frac{1}{m} p_m \cos m \theta_i} d\theta_i \tag{83}$$

where $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_n}$ are the eigenvalues of O. and

$$I_{\mathbb{O}(2n+1)}(\mathbf{p}) = \frac{2^{n^2}}{\pi^n n!} \int_{\theta_1 \le \cdots \le \theta_n \le \pi} \prod_{i<j}^{n} (\cos \theta_i - \cos \theta_j)^2 \prod_{i=1}^{n} \sin^2 \frac{\theta_i}{2} e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m (1 + 2 \cos m \theta_i)} d\theta_i \tag{84}$$

where $e^{\theta_1}, e^{-\theta_1}, \ldots, e^{\theta_n}, e^{-\theta_n}, 1$ are eigenvalues of $O(2n+1)$.

Remark 6. Then

$$I_{\mathbb{O}(2n)}(\mathbf{p}) = 2 I_{\mathbb{S}p(2n)}(\mathbf{p} + [1] + [-1]) \tag{85}$$

$$I_{\mathbb{O}(2n+1)}(\mathbf{p}) = \frac{1}{2} B^+(1) I_{\mathbb{O}(2n)}(\mathbf{p}) := \frac{1}{2} e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m} I_{\mathbb{O}(2n)}(\mathbf{p} - [1]) \tag{86}$$

According to (54) and according to (29), (28) we obtain

$$I_{\mathbb{O}(2n)}(\mathbf{p}) = 2^{1-n} \int_{\mathbb{U}(n)} \tau_+(U|\mathbf{p}) \tau_+(U^{-1}|\mathbf{p}) d_* U \tag{87}$$

$$I_{\mathbb{O}(2n)}(\mathbf{p}) = 2^{1-n} \int_{\mathbb{U}(n)} \tau_-(U|\mathbf{p}) \tau_-(U^{-1}|\mathbf{p}) \det(U - U^\dagger)^{-2} d_* U \tag{88}$$

Perturbation series for integrals over orthogonal group as series in characters $sp_\lambda(\mathbf{p})$, $o_\lambda(\mathbf{p})$, $s_\lambda(\mathbf{p})$.

Proposition 5.

$$\begin{aligned} I_{\mathbb{O}p(2n)}(\mathbf{p}) &= I_{\mathbb{O}(N)}(\mathbf{p}) = \sum_{\substack{\lambda \text{ even} \\ \ell(\mu) \le N}} s_\lambda(\mathbf{p}) \tag{89} \\ &= 2^{1-n} \sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \le n}} (sp_\lambda(\mathbf{p}))^2 = 2^{1-n} \sum_{\substack{\lambda \in \mathbb{P} \\ \lambda_1 \le n}} (o_\lambda(-\mathbf{p}))^2 \tag{90} \end{aligned}$$

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Equality (89) follows from the Cauchy-Littlewood formula

$$e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m \mathrm{tr} O^m} = \sum_{\lambda} s_\lambda(O) s_\lambda(\mathbf{p})$$

and (see for instance (3.19)-(3.21) in Sect 3, VII in [35])

$$\int_{O \in \mathbb{O}(N)} s_\lambda(O) d_* O = \begin{cases} 1 & \lambda \text{ is even} \\ 0 & \text{otherwise} \end{cases} \tag{91}$$

The left hand side of (90) follows from (54) , (28)) and (8), then the right hand side is the result of (23).

Integrals over orthogonal group as BKP tau functions.

Proposition 6.

$$n! \, I_{\mathbb{O}(2n)}(\mathbf{p}) \;=\; \langle 2n | \, \Gamma(\mathbf{p}) \, e^{\frac{1}{4\pi i} \oint \psi(x^{-1}) \psi(x) (x^{-1} - x)^{-1} \frac{dx}{x}} \, |0\rangle \tag{92}$$

$$\;=\; \langle 2n | \, \Gamma(\mathbf{p}) \, e^{\sum_{k \geq 0} \sum_{i \in \mathbb{Z}} \psi_i \psi_{i-1-2k}} |0\rangle \tag{93}$$

$$n! \, I_{\mathbb{O}(2n+1)}(\mathbf{p}) \;=\; e^{\sum_{m>0} \frac{1}{m} p_m} \langle 2n+1 | \, \Gamma(\mathbf{p}) \, e^{\frac{1}{4\pi i} \oint \psi(x^{-1}) \psi(x) \frac{x-1}{x+1} \frac{dx}{x}} \, |0\rangle \tag{94}$$

$$\;=\; e^{\sum_{m>0} \frac{1}{m} p_m} \langle 2n+1 | \, \Gamma(\mathbf{p}) \, e^{\sum_{i>j} (-)^{i+j} \psi_i \psi_j} \, |0\rangle \tag{95}$$

or, it can be written in an unified way as

$$n! \, I_{\mathbb{O}(N)}(\mathbf{p}) \;=\; \langle N | \, \Gamma(\mathbf{p}) \, e^{\frac{1}{4\pi i} \oint \psi(x^{-1}) \psi(x) (x^{-1} - x)^{-1} \frac{dx}{x} + \frac{1}{\sqrt{2}} \phi \psi(1)} \, |0\rangle \tag{96}$$

$$\;=\; \langle N | \, \Gamma(\mathbf{p}) \, e^{- \sum_{k \geq 0} \sum_{i \in \mathbb{Z}} \psi_i \psi_{i-1-2k} + \frac{1}{\sqrt{2}} \phi \sum_{i \in \mathbb{Z}} \psi_i} \, |0\rangle \tag{97}$$

where N may be even or odd.

The simplest way to prove this Proposition is just to apply Lemma 6 to Proposition 2. However let us prove it directly from the fermionic expressions like we did in the proof of Proposition 2.

It is convenient to re-write the fermionic exponent in (92):

$$\frac{1}{4\pi i} \oint \psi(x^{-1}) \psi(x) (x^{-1} - x) \frac{dx}{x} = \frac{i}{\pi} \int_0^\pi \frac{\psi(e^{-i\theta}) \psi(e^{i\theta})}{\sin \theta} d\theta \tag{98}$$

where we note that $\frac{\psi(e^{-i\theta}) \psi(e^{i\theta})}{\sin \theta} = i(\psi(1)\psi'(1) - \psi'(1)\psi(1))$ is not singular at $\theta = 0$. The proof basically repeats the the proof of Proposition 2. For the vacuum expectation value in the right hand side of (92) we obtain

$$\frac{1}{n!} \left(\frac{i^n}{\pi^n n!} \int_0^\pi \cdots \int_0^\pi \Delta_{2n}(e^{-i\theta_1}, e^{i\theta_1}, \ldots, e^{-i\theta_n}, e^{i\theta_n}) \prod_{i=1}^n e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m \cos m\theta_i} \frac{d\theta_i}{\sin \theta_i} \right) \tag{99}$$

Then formula (69) proves (92). The formula (93) then follows from

$$\frac{1}{4\pi i} \oint \psi(x^{-1}) \psi(x) (x^{-1} - x)^{-1} \frac{dx}{x} = \frac{1}{4\pi i} \oint \psi(x^{-1}) \psi(x) \frac{dx}{1 - x^2} = \frac{1}{2} \sum_{k \geq 0} \sum_{i \in \mathbb{Z}} \psi_i \psi_{i-1-2k}$$

Formula (94) and the unifying Formula (96) follows from (86) and the bosonization relations, see Appendix A.2. At last

$$\frac{1}{4\pi i} \oint \psi(x^{-1}) \psi(x) \frac{x-1}{x+1} \frac{dx}{x} = \frac{1}{2\pi i} \oint \psi(x^{-1}) \psi(x) (x - x^2 + x^3 - \cdots) \frac{dx}{x} = \sum_{i>j} (-)^{i+j} \psi_i \psi_j$$

Thus each relation of Proposition 6 is proven.

Pfaffian representation. Let us mark that thanks to the Wick's rule applied to calculate (92) we directly obtain the Pfaffian representation of the integral (83) as follows

Proposition 7.
$$I_{\mathbb{O}(2n)}(\mathbf{p}) = \mathrm{Pf}\,[M_{kj}(\mathbf{p})]_{k,j=1,\ldots,2n+2} \tag{100}$$

where in case N even

$$M_{kj}(\mathbf{p}) = \frac{1}{4\pi i}\oint \left(x^{j-k} - x^{k-j}\right)\left(x - x^{-1}\right)^{-1} e^{\sum_{m=1}^{\infty}\frac{1}{m}p_m(x^m + x^{-m})}\frac{dx}{x} \tag{101}$$

In case $N = 2n+1$ the entries M_{kj} $k,j \leq 2n+1$ as before and

$$M_{k,2n+2} = -M_{2n+2,k} = \frac{1}{2\pi i}\oint x^k e^{\sum_{m=1}^{\infty}\frac{1}{m}p_m(x^m + x^{-m})}\frac{dx}{x} \tag{102}$$

For the proof we notice that from Wick's rule we get

$$M_{kj}(\mathbf{p}) = \frac{1}{2\pi i}\langle 0|\psi_k^\dagger \psi_j^\dagger \oint \frac{\psi(x^{-1})\psi(x)}{x^{-1} - x}\frac{dx}{x}|0\rangle$$

Relation of the integral over $\mathbb{O}(N)$ to an integral over $\mathbb{U}(2n)$. Next turn to its two-component KP counterpart. According to Proposition 4 in [33] we have

$$\left(I_{\mathbb{O}(N)}(\mathbf{p})\right)^2 =$$

$$\langle N, -N|e^{\sum_{m=1}^{\infty}\frac{1}{m}p_m\left(\alpha_m^{(1)} - \alpha_m^{(2)}\right)}e^{\frac{1}{4\pi i}\oint \psi^{(1)}(x^{-1})\psi^{\dagger(2)}(x)(x-x^{-1})^{-1}\frac{dx}{x} + \frac{1}{\sqrt{2}}\left(\psi^{(1)}(1)\psi^\dagger - \psi\psi^{\dagger(2)}(1)\right)}|0,0\rangle \tag{103}$$

where we use the additional fermions $\psi, \psi^\dagger$ which anticommute with $\psi^{(i)}(x)$ and $\psi^{\dagger(i)}(x)$, also $\psi\psi^\dagger + \psi^\dagger\psi = 1$ and $\psi^\dagger|*,*\rangle = 0$, see Section 2 in [33]. Other the notations are the same as in Lemma 5.

As 1D TL-NLS tau function. Denote $z = x + x^{-1}$ and introduce variables $\tilde{\mathbf{p}} = (\tilde{p}_1, \tilde{p}_2, \ldots)$ with the help of (78). Then similar to (79) we get

$$I_{\mathbb{O}(2n)}(\mathbf{p}(\tilde{\mathbf{p}})) = \int_{-2}^{2}\cdots\int_{-2}^{2}\prod_{i<j}(z_i - z_j)^2 \prod_{i=1}^{n} e^{\sum_{m=1}^{\infty}\frac{1}{m}\tilde{p}_m z_i^m - c(\tilde{\mathbf{p}})}dz_i \tag{104}$$

$$I_{\mathbb{O}(2n+1)}(\mathbf{p}(\tilde{\mathbf{p}})) = \int_{-2}^{2}\cdots\int_{-2}^{2}\prod_{i<j}(z_i - z_j)^2 \prod_{i=1}^{n} e^{\sum_{m=1}^{\infty}\frac{1}{m}\tilde{p}_m z_i^m - c(\tilde{\mathbf{p}})}(2 - z_i)dz_i \tag{105}$$

4.3 On some integrals over unitary matrices.

(1) Here we consider a Cauchy-like integral:

$$(-1)^{\frac{1}{2}(N^2-N)}\int_{\mathbb{U}(N)} e^{\sum_{n>0}\frac{1}{n}p_n \mathrm{tr}U^n}\det(1 - U^\dagger)^{-1}\left(\det U\right)^{N-1} d_*U = (-1)^{\frac{1}{2}(N^2-N)}\exp\sum_{n>0}\frac{p_n}{n} \tag{106}$$

$$= \sum_{\substack{\lambda \\ \ell(\lambda)\leq N}} sp_\lambda(\mathbf{p}) = \sum_{\substack{\lambda \\ \lambda_1 \leq N}} (-1)^{|\lambda|}o_\lambda(-\mathbf{p}) \tag{107}$$

where we imply $\det(1 - U^\dagger)^{-1} = \sum_\lambda s_\lambda(U^\dagger)s_\lambda(\mathbf{p}_o)$, $\mathbf{p}_o = (1,1,\ldots)$ (Cauchy-Littlewood identity).

To prove (106) we need

$$(2\pi i)^{-N}\oint \det\left(z_i^{h_j}\right)_{i,j}\det\left(z_i^{-\tilde{h}_j}\right)_{i,j}\prod_i^N z_i^L \frac{dz_i}{z_i} = \prod_j \delta_{h_j,\tilde{h}_j-L} \rightarrow \int_{\mathbb{U}(N)} s_\lambda(U)s_{\tilde{\lambda}}(U^\dagger)\det U^L d_*U = \delta_{\lambda+L,\tilde{\lambda}} \tag{108}$$

which generalizes (8). Let us substitute Cauchy-Littlewood series for the exponential and for the determinant $\det(1-U^\dagger)^{-1}$ inside the integral. These seies are respectively $\sum_\lambda s_\lambda(\mathbf{p})s_\lambda(U)$ and $\sum_\lambda s_\lambda(U^\dagger)s_\lambda(\mathbf{p}_o)$.

Using (108) where $L = N-1$, up to the sign factor we obtain $\sum_\lambda s_\lambda(\mathbf{p}) s_{\tilde{\lambda}(\lambda)}(\mathbf{p}_o)$ where $\tilde{\lambda}_i(\lambda) = \lambda_i + N - 1$. Notice that each $s_\lambda(\mathbf{p}_o) = 1$ for $\lambda = (n)$, $n = 0, 1, \ldots$ and vanish otherwise. Then the first equality is true, because $\sum_{n \geq 0} s_{(n)}(\mathbf{p}) = \exp \sum_{n > 0} \frac{p_n}{n}$.

The second equality may be obtained with the help of the Schur-Littlewood relation (see Ch. I, Sect. 5, Ex. 4 in [35] page 76)

$$\sum_\lambda s_\lambda(U) = \prod_{i<j}(1 - z_i z_j)^{-1} \prod_k (1 - z_k)^{-1}$$

where z_i, $i = 1, \ldots, N$ are eigenvalues of U, written in form

$$\prod_k (1 - z_k^{-1}) \prod_{i<j}(1 - z_i^{-1} z_j^{-1}) \sum_\lambda s_\lambda(U^\dagger) = 1$$

and further re-written in form

$$(-1)^{\frac{1}{2}(N^2 - N)} (\det U)^{1-N} \det\left(1 - U^\dagger\right) \left(\prod_{i<j}(1 - z_i z_j) \sum_\lambda s_\lambda(U^\dagger)\right) = 1$$

where z_i, $i = 1, \ldots, N$ are eigenvalues of U. Then, inserting the left hand side inside the integral and using first (28) and then (8) we obtain (107). The third equality follows from (23). Relation (106) proves that sums of characters (107) are the elementary KP tau function with $\mathbf{p}$ being the KP higher times.

(2) Similarly for $N > 1$

$$0 = (-1)^{\frac{1}{2}(N^2 - N)} \int_{\mathbb{U}(N)} e^{\sum_{n>0} \frac{1}{m} p_n \text{tr} U^n} \det\left(1 - U^2\right) (\det U)^{N-1} d_* U = \tag{109}$$

$$\sum_{\substack{\lambda \\ \ell(\lambda \cup \lambda) \leq N}} o_{\lambda \cup \lambda}(\mathbf{p}) = \sum_{\substack{\lambda \text{ even} \\ \lambda_1 \leq N}} sp_\lambda(-\mathbf{p}) \tag{110}$$

Proof. To get the second equality we use

$$\sum_{\lambda^{\text{tr even}}} s_\lambda(U) = \prod_{i<j}(1 - z_i z_j)^{-1}$$

(see Ch I, Sect 5, Ex 5(b) in [35] page 77) witten in form

$$(-1)^{\frac{1}{2}(N^2 - N)} \det U^{1-N} \det(1 - U^2)^{-1} \left(\prod_{i \leq j}(1 - z_i z_j) \sum_\lambda s_{\lambda \cup \lambda}(U^\dagger)\right) = 1 \tag{111}$$

and relations (29) and (8). The third equality results from (23). The first equality in (109) is obtained using $\det(1 - U^2) = \sum_\lambda s_\lambda(U) s_\lambda(\mathbf{p}'_o)$, $\mathbf{p}'_o = (0, 1, 0, 1, \ldots)$ and (108) which gives 0 for each N.

5   The character expansion for $\beta = 2$ ensembles

The character expansions. Lemma 6 in Appendix A.1 and series (28), (29) induce the character expansion for a number of matrix integrals. Relations (147)-(149) may be used in the study of the so-called $\beta = 2$ ensembles. Then (154) yields a link of these ensembles with the BKP tau function [26].

Let

$$V(z, T) = \sum_{m>1} \frac{1}{m} z^m T_m \tag{112}$$

and variables $T = (T_1, T_2, \ldots)$ and $\mathbf{p} = (p_1, p_2, \ldots)$ are linearly dependent and related as follows

$$V(z, T) - c(T) = V(x, \mathbf{p}) + V(x^{-1}, \mathbf{p}) \qquad c(T) = \sum_{n=1}^\infty \frac{(2n)!}{n!n!} \frac{T_{2n}}{2n} \tag{113}$$

For instance, $T_1 = t_1 - \sum_{n=1}^\infty (2n+1) t_{2n+1}$.

Then we have

Proposition 8.

$$I_n(T) = \int \Delta(z_1, \ldots, z_n)^2 \prod_{i=1}^{n} z_i^N e^{V(z_i, T)} d\mu(z_i) = \tag{114}$$

$$= \sum_{h_1 > \cdots > h_n \geq 0} s_{\{h\}}(T)\, \pi_{\{h\}}(N) \tag{115}$$

$$= \sum_{\substack{h_1 > \cdots > h_n \geq 0 \\ \tilde{h}_1 > \cdots > \tilde{h}_n \geq 0}} s_{\{h\}}(\mathbf{p}^{(1)})\, \pi^{00}_{\{h,\tilde{h}\}}(N)\, s_{\{\tilde{h}\}}(\mathbf{p}^{(2)}) \tag{116}$$

$$= \sum_{\substack{h_1 > \cdots > h_n \geq 0 \\ \tilde{h}_1 > \cdots > \tilde{h}_n \geq 0}} e^{c(\mathbf{p})}\, o_{\{h\}}(\mathbf{p})\, \pi^{--}_{\{h,\tilde{h}\}}(N)\, o_{\{\tilde{h}\}}(\mathbf{p})\, e^{c(\mathbf{p})} \tag{117}$$

$$= \sum_{\substack{h_1 > \cdots > h_n \geq 0 \\ \tilde{h}_1 > \cdots > \tilde{h}_n \geq 0}} e^{c(\mathbf{p})}\, sp_{\{h\}}(\mathbf{p})\, \pi^{++}_{\{h,\tilde{h}\}}(N)\, sp_{\{\tilde{h}\}}(\mathbf{p})\, e^{c(\mathbf{p})} \tag{118}$$

$$= \sum_{\substack{h_1 > \cdots > h_n \geq 0 \\ \tilde{h}_1 > \cdots > \tilde{h}_n \geq 0}} e^{c(\mathbf{p})}\, o_{\{h\}}(\mathbf{p})\, \pi^{-+}_{\{h,\tilde{h}\}}(N)\, sp_{\{\tilde{h}\}}(\mathbf{p})\, e^{c(\mathbf{p})} \tag{119}$$

where $T = \mathbf{p}^{(1)} - \mathbf{p}^{(2)}$ (that is $T_m = p_m^{(1)} - p_m^{(2)}$), and

$$\pi_{\{h\}}(N) = n! \det \left[\pi^{00}_{n-i,h_j}(N) \right]_{i,j=1,\ldots,n} \tag{120}$$

$$\pi^{ab}_{\{h,\tilde{h}\}}(N) = n! \det \left[\pi^{ab}_{h_i,\tilde{h}_j} \right]_{i,j=1,\ldots,n}, \quad a,b = \pm, 0 \tag{121}$$

and the related moment matrices are

$$\pi^{00}_{i,j}(N) = \int z^{i+j+N} d\mu(z) \tag{122}$$

$$\pi^{++}_{i,j}(N) = \int x^{i-j}(x + x^{-1})^N d\mu(z(x)) \tag{123}$$

$$\pi^{--}_{i,j}(N) = \int x^{i-j}(x + x^{-1})^N \frac{d\mu(z(x))}{(1-x^2)^2} \tag{124}$$

$$\pi^{-+}_{i,j}(N) = \int x^{i-j}(x + x^{-1})^N \frac{d\mu(z(x))}{1-x^2} \tag{125}$$

Proof. We use Lemma 6 in the Appendix A.1 and relations (113), (28)-(29). The Schur functions which are involved in (28)-(29) we present as ratios of the determinants (1), at last we use the simple identity (sometimes called Andreif identity)

$$\int \det \left[\tau_i(x_j) \right]_{i,j=1,\ldots,n} \det \left[\tau_i(x_j) \right]_{i,j=1,\ldots,n} \prod_{i=1}^{n} d\mu(x_i) = n! \det \left[\int \tau_i(x)\tau_j(x) d\mu(x) \right]_{i,j=1,\ldots,n}$$

to get moment matrices (122)-(125).

Remark 7. Series (115) is a special case of (116) where $\mathbf{p}^{(2)} = 0$.

$\beta = 2$ ensembles as the BKP tau function.   Using (154) in Appendix one may verify that

$$I_n(T(\mathbf{p})) = n! e^{c(\mathbf{p})} \langle n| e^{\sum_{m>0} \frac{1}{m} p_m \alpha_m} e^{\int \frac{\psi(x^{-1})\psi(x)}{x^{-1}-x} d\mu(z(x)) + \sqrt{2} \int \psi(x) d\mu(x)\phi} |0\rangle \tag{126}$$

Then we can apple the Wick's rule to get the Pfaffian representation

$$I_n(T(\mathbf{p})) = n! \operatorname{Pf} \left[A_{ij}(\mathbf{p}) \right] \tag{127}$$

where, for n even

Remark 8. The integrals (114) may be related both to the BKP hierarchy where $\mathbf{p}$ are BKP higher times the one-dimensional Toda chain with higher times $\tilde{\mathbf{p}}$, see (78)

6 The character expansion of two-matrix models

Let

$$z = x + x^{-1}, \quad \tilde{z} = \tilde{x} + \tilde{x}^{-1} \tag{128}$$

In case we can express the variables $\mathbf{p}$ as a linear combination of the variables $\mathbf{p}^{(1)}$ and the variables $\tilde{\mathbf{p}}$ as a linear combination of the variables $\mathbf{p}^{(2)}$:

$$p_m = \sum_{n=1}^{\infty} D_{mn} p_n^{(1)}, \quad \tilde{p}_m = \sum_{n=1}^{\infty} \tilde{D}_{mn} p_n^{(2)}$$

in such a way that

$$V\left(\frac{az+b}{cz+d}, \mathbf{p}^{(1)}\right) = V(x, \mathbf{p}), \quad V\left(\frac{\tilde{a}\tilde{z}+\tilde{b}}{\tilde{c}\tilde{z}+\tilde{d}}, \mathbf{p}^{(1)}\right) = V(\tilde{x}, \tilde{\mathbf{p}}) \tag{129}$$

we can apply the same method based on (147)-(148) in Appendix A.1 to get character expansion of

$$I_n(\mathbf{p}^{(1)}, \mathbf{p}^{(2)}) = \int \Delta(z_1, \ldots, z_n) \Delta(\tilde{z}_1, \ldots, \tilde{z}_n) \prod_{i=1}^{n} z_i^N e^{V(z_i, \mathbf{p}^{(1)}) - V(\tilde{z}_i, \mathbf{p}^{(2)})} d\mu(z_i, \tilde{z}_i) \tag{130}$$

For the sake of simplicity let us take $d\mu(z, \tilde{z}) = 0$ if either $z \geq 1$ or $\tilde{z} \geq 1$, in (130) and in addition

$$V(z^{-1}, \mathbf{p}^{(1)}) = V(x, \mathbf{p}), \quad V(\bar{z}^{-1}, \mathbf{p}^{(1)}) = V(\tilde{x}, \tilde{\mathbf{p}})$$

such that $p_1 = p_1^{(1)}$, $p_2 = p_2^{(1)}$, $p_3 = p_3^{(1)} - t(1)_1$, $p_4 = p_4^{(1)} - 2t(1)_2$ and so on.

$$I_n(\mathbf{p}^{(1)}, \mathbf{p}^{(2)}) = \tag{131}$$

$$= \sum_{\substack{h_1 > \cdots > h_n \geq 0 \\ \tilde{h}_1 > \cdots > \tilde{h}_n \geq 0}} s_{\{h\}}(\mathbf{p}^{(1)}) \, G_{\{h,\tilde{h}\}}^{00} \, s_{\{\tilde{h}\}}(\mathbf{p}^{(2)}) \tag{132}$$

$$= \sum_{\substack{h_1 > \cdots > h_n \geq 0 \\ \tilde{h}_1 > \cdots > \tilde{h}_n \geq 0}} e^{c(\mathbf{p})} \, o_{\{h\}}(\mathbf{p}) \, G_{\{h,\tilde{h}\}}^{--}(\mathbf{p}, \tilde{\mathbf{p}}) \, o_{\{\tilde{h}\}}(\tilde{\mathbf{p}}) \, e^{c(\tilde{\mathbf{p}})} \tag{133}$$

$$= \sum_{\substack{h_1 > \cdots > h_n \geq 0 \\ \tilde{h}_1 > \cdots > \tilde{h}_n \geq 0}} e^{c(\mathbf{p})} \, sp_{\{h\}}(\mathbf{p}) \, G_{\{h,\tilde{h}\}}^{++}(\mathbf{p}, \tilde{\mathbf{p}}) \, sp_{\{\tilde{h}\}}(\tilde{\mathbf{p}}) \, e^{c(\tilde{\mathbf{p}})} \tag{134}$$

$$= \sum_{\substack{h_1 > \cdots > h_n \geq 0 \\ \tilde{h}_1 > \cdots > \tilde{h}_n \geq 0}} e^{c(\mathbf{p})} \, o_{\{h\}}(\mathbf{p}) \, G_{\{h,\tilde{h}\}}^{-+}(\mathbf{p}, \tilde{\mathbf{p}}) \, sp_{\{\tilde{h}\}}(\tilde{\mathbf{p}}) \, e^{c(\tilde{\mathbf{p}})} \tag{135}$$

$$= \sum_{\substack{h_1 > \cdots > h_n \geq 0 \\ \tilde{h}_1 > \cdots > \tilde{h}_n \geq 0}} s_{\{h\}}(\mathbf{p}^{(1)}) \, G_{\{h,\tilde{h}\}}^{0+}(\tilde{\mathbf{p}}) \, sp_{\{\tilde{h}\}}(\tilde{\mathbf{p}}) \, e^{c(\tilde{\mathbf{p}})} \tag{136}$$

$$= \sum_{\substack{h_1 > \cdots > h_n \geq 0 \\ \tilde{h}_1 > \cdots > \tilde{h}_n \geq 0}} s_{\{h\}}(\mathbf{p}^{(1)}) \, G_{\{h,\tilde{h}\}}^{0-}(\tilde{\mathbf{p}}) \, o_{\{\tilde{h}\}}(\tilde{\mathbf{p}}) \, e^{c(\tilde{\mathbf{p}})} \tag{137}$$

where

$$G_{\{h,\tilde{h}\}}^{ab} = \det\left[G_{h_i,\tilde{h}_j}^{ab}\right]_{i,j=1,\ldots,n}, \quad a,b = \pm, 0 \tag{138}$$

and the related moment matrices are

$$G_{i,j}^{00} = \int z^{i+N} \tilde{z}^{j+N} d\mu(z, \tilde{z}) \tag{139}$$

$$G_{i,j}^{++} = \int x^{i-\frac{n(n-1)}{2}} \tilde{x}^{j-\frac{n(n-1)}{2}} (x+x^{-1})^N (\tilde{x}+\tilde{x}^{-1})^N d\mu(z(x), \tilde{z}(\tilde{x})) \tag{140}$$

$$G_{i,j}^{--} = \int x^{i-\frac{n(n-1)}{2}} \tilde{x}^{j-\frac{n(n-1)}{2}} (x+x^{-1})^N (\tilde{x}+\tilde{x}^{-1})^N \frac{d\mu(z(x), \tilde{z}(\tilde{x}))}{(1-x^2)(1-\tilde{x}^2)} \tag{141}$$

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$$G_{i,j}^{-+} = \int x^{i-\frac{n(n-1)}{2}} \tilde{x}^{j-\frac{n(n-1)}{2}} (x+x^{-1})^N (\tilde{x}+\tilde{x}^{-1})^N \frac{d\mu(z(x),\tilde{z}(\tilde{x}))}{1-x^2} \tag{142}$$

$$G_{i,j}^{0+} = \int z^{i+N} \tilde{x}^{j-\frac{n(n-1)}{2}} (\tilde{x}+\tilde{x}^{-1})^N d\mu(z,\tilde{z}(\tilde{x})) \tag{143}$$

$$G_{i,j}^{0-} = \int z^{i+N} \tilde{x}^{j-\frac{n(n-1)}{2}} (\tilde{x}+\tilde{x}^{-1})^N \frac{d\mu(z,\tilde{z}(\tilde{x}))}{1-\tilde{x}^2} \tag{144}$$

Example. Take $\tilde{z} = \bar{z}$, $x = \bar{x}$, $t^{(1)} = \bar{t}^{(2)}$, $t_m = \bar{t}_m$, and $N = 0$. Take also $d\mu(z,\tilde{z}) = f(|z|)\frac{dz d\tilde{z}}{|1-x^2|^2}$, where z and x are related by (128). Then

$$I_n(\mathbf{p}^{(1)}, \mathbf{p}^{(2)}) = \sum_\lambda r_\lambda sp_\lambda(\mathbf{p}) sp(\bar{\mathbf{p}}) \tag{145}$$

Acknowledgements

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A   Appendices

A.1   Rewriting Vandermonde determinants

We have usefull elementary

Lemma 6. *Let*

$$z = x + x^{-1}, \quad x = \frac{z}{2} \pm \frac{1}{2}\sqrt{z^2 - 4} \tag{146}$$

(Joukowsky transform).
   Then

$$\prod_{1 \le k < j \le n} (z_k - z_j) = \prod_{k < j \le n} (1 - x_k x_j) \prod_{1 \le k < j \le n} (x_k^{-1} - x_j^{-1}) \tag{147}$$

$$= \tau_+(X|0) \prod_{1 \le k < j \le n} (x_k^{-1} - x_j^{-1}) \tag{148}$$

$$= \tau_-(X|0) \prod_{1 \le k < j \le n} (x_k^{-1} - x_j^{-1}) \prod_{j=1}^{n} \frac{1}{1 - x_j^2} \tag{149}$$

And as a result

$$\prod_{1 \le k < j \le n} (z_k - z_j)^2 = \prod_{1 \le k < j \le n} (1 - x_k x_j)(1 - x_k^{-1} x_j^{-1}) \prod_{1 \le k < j \le n} (x_k - x_j)(x_k^{-1} - x_j^{-1}) \tag{150}$$

$$= \tau_+(X|0)\tau_+(X^{-1}|0) \prod_{1 \le k < j \le n} (x_k - x_j)(x_k^{-1} - x_j^{-1}) \tag{151}$$

$$= \tau_-(X|0)\tau_-(X^{-1}|0) \prod_{1 \le k < j \le n} (x_k - x_j)(x_k^{-1} - x_j^{-1}) \prod_{j=1}^{n} \frac{1}{(1 - x_j^2)(1 - x_j^{-2})} \tag{152}$$

$$= \tau_-(X|0)\tau_+(X^{-1}|0) \prod_{1 \le k < j \le n} (x_k - x_j)(x_k^{-1} - x_j^{-1}) \prod_{j=1}^{n} \frac{1}{1 - x_j^2} \tag{153}$$

$$= \Delta_{2n}(x_1^{-1}, x_1, \ldots, x_n^{-1}, x_n) \prod_{j=1}^{n} \frac{1}{x_j^{-1} - x_j} \tag{154}$$

   Indeed, we have

$$(1 - x_j^{-1} x_k^{-1})(x_j - x_k) = (1 - x_j x_k)(x_j^{-1} - x_k^{-1}) = z_j - z_k$$

and obtain (150)-(151).

A.2   Vertex operators

Vertex operators we need are as follows

$$\hat{X}(L, \mathbf{p}, \lambda) := e^{\sum_{n=1}^{\infty} \lambda^n t_n} \lambda^L e^{-\sum_{n=1}^{\infty} \frac{1}{n\lambda^n} \frac{\partial}{\partial t_n}}, \quad \hat{X}^\dagger(L, \mathbf{p}, \lambda) := e^{-\sum_{n=1}^{\infty} \lambda^n t_n} \lambda^{-L} e^{\sum_{n=1}^{\infty} \frac{1}{n\lambda^n} \frac{\partial}{\partial t_n}} \tag{155}$$

$$\hat{Y}(L, \mathbf{s}, \lambda) := e^{-\sum_{n=1}^{\infty} \lambda^{-n} s_n} \lambda^L e^{\sum_{n=1}^{\infty} \frac{\lambda^n}{n} \frac{\partial}{\partial s_n}}, \quad \hat{Y}^\dagger(L, \mathbf{s}, \lambda) := e^{\sum_{n=1}^{\infty} \lambda^{-n} s_n} \lambda^{-L} e^{-\sum_{n=1}^{\infty} \frac{\lambda^n}{n} \frac{\partial}{\partial s_n}} \tag{156}$$

Interesting historical fact that the formula which relates fermions to bosons first was found in [46].

The following bosonization relation is useful

$$\langle L + N|\Gamma(\mathbf{p} + \sum_{i=1}^{N}[p_i]) = \frac{\langle L|\psi^\dagger(p_1^{-1})\cdots\psi^\dagger(p_N^{-1})\Gamma_+(\mathbf{p})}{\prod_{i=1}^{N} p_i^{(L+1)(N-1)} \prod_{i>j}(p_i - p_j)} \tag{157}$$

Introduce

$$\hat{\Omega}_n(L, \mathbf{p}) = \mathop{\mathrm{res}}_\lambda \left(\frac{\partial^n \hat{X}(L, \mathbf{p}, \lambda)}{\partial \lambda^n} \hat{X}^\dagger(L, \mathbf{p}, \lambda) \right), \quad \hat{\Omega}_n^*(L, \mathbf{s}) = \mathop{\mathrm{res}}_\lambda \left(\hat{Y}^\dagger(L, \mathbf{s}, \lambda) \frac{\partial^n \hat{Y}(L, \mathbf{s}, \lambda)}{\partial \lambda^n} \right) \tag{158}$$

A.3 Fermions

We shall remind some facts and notations of [24]. Introduce free fermionic fields $\psi(z) = \sum_{i\in\mathbb{Z}}\psi_i z^i$, $\psi^\dagger(z) = \sum_{i\in\mathbb{Z}}\psi_{-i-1}^\dagger z^i$ whose Fourier components anti-commute as follows $\psi_i\psi_j + \psi_j\psi_i = \psi_i^\dagger\psi_j^\dagger + \psi_j^\dagger\psi_i^\dagger = 0$ and $\psi_i\psi_j^\dagger + \psi_j^\dagger\psi_i = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker symbol. We put

$$\psi_i|0\rangle = \psi_{-i}^\dagger|0\rangle = \langle 0|\psi_{-i} = \langle 0|\psi_i^\dagger = 0 \tag{159}$$

where $\langle 0|$ and $|0\rangle$ are left and right vacuum vectors of the fermionic Fock space, $\langle 0| \cdot 1 \cdot |0\rangle = 1$. Also introduce

$$\langle n| = \begin{cases} \langle 0|\psi_0^\dagger\cdots\psi_{n-1}^\dagger & \text{if} \quad n > 0 \\ \langle 0|\psi_{-1}\cdots\psi_{-n} & \text{if} \quad n < 0 \end{cases}, \quad |n\rangle = \begin{cases} \psi_{n-1}\cdots\psi_0|0\rangle & \text{if} \quad n > 0 \\ \psi_{-n}^\dagger\cdots\psi_{-1}^\dagger|0\rangle & \text{if} \quad n < 0 \end{cases} \tag{160}$$

Then $\langle n| \cdot 1 \cdot |m\rangle = \delta_{n,m}$.

Following [26] we introduce an additional Fermi mode which we shall denote by ϕ with properties [2]

$$\phi\psi_i + \psi_i\phi = \phi\psi_i^\dagger + \psi_i^\dagger\phi = 0, \quad \phi^2 = \frac{1}{2} \tag{161}$$

$$\phi|0\rangle = |0\rangle\frac{1}{\sqrt{2}}, \qquad \langle 0|\phi = \frac{1}{\sqrt{2}}\langle 0| \tag{162}$$

such that $\langle L|\phi|L\rangle = \frac{(-)^L}{\sqrt{2}}$.

Now, two-component Fermi fields used in Section 6 are defined as

$$\psi^{(i)}(z) = \sum_{n\in\mathbb{Z}} z^n \psi_{2n+i}, \quad \psi^{(i)\dagger}(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}\psi_{2n+i}^\dagger \tag{163}$$

where $i = 1, 2, 3$. Other details about multi-component fermions may be found in [24], and in relation to matrix models in [21].

We have

$$\hat{X}(L, \mathbf{p}, \lambda)X^\dagger(L, \mu)\langle N + L|\Gamma(\mathbf{p})g\Gamma^\dagger(\mathbf{s})|L\rangle = \langle N + L|\Gamma(\mathbf{p})\psi(\lambda)\psi^\dagger(\mu)g\Gamma^\dagger(\mathbf{s})|L\rangle \tag{164}$$

$$\hat{Y}^\dagger(L, \mathbf{s}, \mu)Y(\lambda)\langle N + L|\Gamma(\mathbf{p})g\Gamma^\dagger(\mathbf{s})|L\rangle = \langle N + L|\Gamma(\mathbf{p})g\psi(\lambda)\psi^\dagger(\mu)\Gamma^\dagger(\mathbf{s})|L\rangle \tag{165}$$

Then it follows that

$$\hat{\Omega}_n(L, \mathbf{p})\langle N + L|\Gamma(\mathbf{p})g\Gamma^\dagger(\mathbf{s})|L\rangle = \langle N + L|\Gamma(\mathbf{p})\tilde{\Omega}_n g\Gamma^\dagger(\mathbf{s})|L\rangle \tag{166}$$

$$\hat{\Omega}_n(L, \mathbf{p})\langle N + L|\Gamma(\mathbf{p})g\Gamma^\dagger(\mathbf{s})|L\rangle = \langle N + L|\Gamma(\mathbf{p})g\tilde{\Omega}_n\Gamma^\dagger(\mathbf{s})|L\rangle \tag{167}$$

where

$$\tilde{\Omega}_n = \mathop{\mathrm{res}}_\lambda \left(\frac{\partial^n \psi(\lambda)}{\partial \lambda^n} \psi^\dagger(\lambda) \right) \tag{168}$$

Using the fermionic representation one may verify that tau functions related to the considered ensembles (dd-OE, dd-GinOE, dd-SE, dd-GinSE) obey the constraints

$$\left(\hat{\Omega}_n(L, \mathbf{p}) - \hat{\Omega}_n^*(L, \mathbf{s}) \right)\tau(L, \mathbf{p}', \mathbf{s}') = 0, \quad n \geq 1 \quad \text{odd} \tag{169}$$

where $t_k' = t_k - \frac{1}{2}\delta_{2,k}$ $s_k' = s_k - \frac{1}{2}\delta_{2,k}$ (this shift appears due to the Gauss measure in undeformed ensembles).

---

[2] In notations of [26] our ψ_n, $\psi_n^\dagger$ and ϕ read respectively as $\psi_{n+\frac{1}{2}}$, $\psi_{n+\frac{1}{2}}^\dagger$ and ψ_0

A.4 The Schur function

Consider polynomials $h_n(\mathbf{p})$ defined by $e^{\sum_{n=1}^{\infty} z^n t_n} = \sum_{n=0}^{\infty} z^n h_n(\mathbf{p})$. Then the Schur function labeled by a partition $\lambda = (\lambda_1, \ldots, \lambda_k > 0)$ may be defined as $s_\lambda(\mathbf{p}) = \det(h_{\lambda_i - i + j}(\mathbf{p}))_{i,j=1,\ldots,k}$. The notation $s_\lambda(X)$ denotes (A.4) where $\mathbf{p} = \sum_i [x_i]$ where x_i are eigenvalues of X.

A.5 Characters of classical Lie groups/algebras

We recall some information about characters of classical Lie algebras as it is presented in [13] Chapter 24. The character of a simple lie algebra is given by the Weyl character formula. For a dominant integral weight λ the character ch_λ is equal to

$$ch_\lambda = \frac{A_{\lambda+\rho}}{A_\rho}, \quad \text{where } A_\mu = \sum_{w \in W} \text{sgn}(w) e(w(\mu)). \tag{170}$$

Here W is the Weyl group of the simple Lie algebra. and ρ the sum of all the fundamental weights, or equivalently half the sum of all positive roots R^+. One has

$$A_\rho = \prod_{\alpha \in R^+} (e(\alpha/2) - e(-\alpha/2)) = e(\rho) \prod_{\alpha \in R^+} (1 - e(-\alpha)) = \prod_{\alpha \in R^+} e(-\rho)(e(\alpha) - 1)$$

First case: sl_n or rather gl_n. We identify the standard Cartan subalgebra, the diagonal matrices of gl_n, with its dual via the trace form $(a, b) = \text{trace}(ab)$. Let $\epsilon_i = E_{ii}$, and assume $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$. In this case R^+ consists of all elements $\epsilon_i - \epsilon_j$, with $i < j$ and

$$\rho = \frac{1}{2} \sum_{i<j} (\epsilon_i - \epsilon_j) = \sum_{i=1}^n \left(\frac{n+1}{2} - i\right)\epsilon_i$$

N.B. Fulton and Harris [13] have a different formula for ρ, they claim $\rho = \sum_i (n-i)\epsilon_i$ which is wrong. The Weyl group is the group S_n, the permutation group that permutes the elements ϵ_i. Denote by $x_i = e(\epsilon_i)$, then

$$A_{\lambda+\rho} = \sum_{w \in S_n} \text{sgn}(w) x_{w(1)}^{\lambda_1 + \frac{n+1}{2} - 1} x_{w(2)}^{\lambda_2 + \frac{n+1}{2} - 2} \cdots x_{w(n)}^{\lambda_n + \frac{n+1}{2} - n} = \det\left[x_j^{\lambda_i + \frac{n+1}{2} - i}\right]_{1 \le i,j \le n} \tag{171}$$

Taking $\lambda = 0$ in (171), we obtain $A_\rho = \det\left[x_j^{\frac{n+1}{2} - i}\right]$. Thus

$$ch_\lambda = s_\lambda = \frac{\det\left[x_j^{\lambda_i + \frac{n+1}{2} - i}\right]_{1 \le i,j \le n}}{\det\left[x_j^{\frac{n+1}{2} - i}\right]_{1 \le i,j \le n}} = \frac{\det\left[x_j^{\lambda_i + n - i}\right]_{1 \le i,j \le n}}{\det\left[x_j^{n-i}\right]_{1 \le i,j \le n}}$$

Another presentation of the character is the so called Giambelli or determinantal formula. One expresses the character in the elementary Schur functions functions $S_{(k)}(x)$ defined in (A.4):

$$s_\lambda(x) = \det\left[s_{(\lambda_i + j - i)}(x)\right]_{1 \le i,j \le n}$$

Second case: sp_{2n}. The positive roots aren now $\epsilon_i - \epsilon_j$, with $1 \le i < j \le n$ and $\epsilon_i + \epsilon_j$ with $i \le j$. The element $\rho = \sum_{i=1}^n (n + 1 - i)\epsilon_i$ and the Weyl group is the group that permutes all ϵi allowing also all possible sign changes, i.e., $\epsilon_i \mapsto \pm \epsilon_j$, hence it is the semi-direct product of S_n and $\mathbb{Z}_2^n$. Then

$$A_\mu = \sum_{w \in S_n} \sum_{\sigma \in \mathbb{Z}_2^n} \text{sgn}(w)\text{sgn}(\sigma) e\left(\sum_i (-)^{\sigma_i} \mu_i w(\epsilon_i)\right).$$

Here $\sigma = (\sigma_1, \ldots \sigma_n)$ and $\text{sgn}(\sigma) = (-)^{\sum_i \sigma_i}$. Using again $x_i = e(\epsilon_i)$, it is straightforward to check

$$A_\mu = \sum_{w \in S_n} \text{sgn}(w)(x_{\sigma(i)}^{\mu_i} - x_{\sigma(i)}^{-\mu_i}) \tag{172}$$

Hence

$$ch_\lambda = sp_\lambda = \frac{\det\left[x_j^{\lambda_i+n-i+1} - x_j^{-\lambda_i-n+i-1}\right]_{1\leq i,j\leq n}}{\det\left[x_j^{n-i+1} - x_j^{-n+i-1}\right]_{1\leq i,j\leq n}}$$

The determinantal formula in this case is due to Koike en Terada [31]

$$sp_\lambda(x) = \frac{1}{2}\det\left[s_{(\lambda_i-i+j)}(x) + s_{(\lambda_i-i-j+2)}(x)\right]_{i,j=1,\ldots,n}$$

where we substitute $x_{n+i} = x_i^{-1}$

Third case: so_{2n+1}**.** The positive roots aren now $\epsilon_i - \epsilon_j$, with $1 \leq i < j \leq n$ and $\epsilon_i + \epsilon_j$ with $i < j$ and all ϵ_i. The element $\rho = \sum_{i=1}^n (n + \frac{1}{2} - i)\epsilon_i$. The Weyl group is the same as in the sp_{2n} case. Hence, also A_μ is the same, viz. (172), as for sp_{2n}. Hence

$$ch_\lambda = o_\lambda = \frac{\det\left[x_j^{\lambda_i+n-i+\frac{1}{2}} - x_j^{-\lambda_i-n+i-\frac{1}{2}}\right]_{1\leq i,j\leq n}}{\det\left[x_j^{n-i+\frac{1}{2}} - x_j^{-n+i-\frac{1}{2}}\right]_{1\leq i,j\leq n}} = \frac{\det\left[x_j^{\lambda_i+n-i+1} - x_j^{-\lambda_i-n+i}\right]_{1\leq i,j\leq n}}{\det\left[x_j^{n-i+1} - x_j^{-n+i}\right]_{1\leq i,j\leq n}}$$

The determinantal formula is equal to

$$o_\lambda(x) = \det\left[s_{(\lambda_i-i+j)}(x) - s_{(\lambda_i-i-j)}(x)\right]_{i,j=1,\ldots,n} \tag{173}$$

here $x_{n+i} = x_i^{-1}$ and $x_{2n+1} = 1$. In term of irreducible charcters $s_{(k)}^\circ(x) = s_{(k)}(x) - s_{(k-2)}(x)$, one can rewrite (173) to

$$o_\lambda(x) = \frac{1}{2}\det\left[s_{(\lambda_i-i+j)}^\circ(x) + s_{(\lambda_i-i-j+2)}^\circ(x)\right]_{i,j=1,\ldots,n} \tag{174}$$

Fourth case: so_{2n}**.** The positive roots aren now $\epsilon_i - \epsilon_j$, with $1 \leq i < j \leq n$ and $\epsilon_i + \epsilon_j$ with $i < j$ and ρ is the same as for gl_n, viz. $\rho = \sum_{i=1}^n (n-i)\epsilon_i$. However the Weyl group is a subgroup of the Weyl group of so_{2n+1}, one only allows an even number of sign changes. This leads to

$$A_\mu = \frac{1}{2}\left(\det\left[x_j^{\mu_i} - x_j^{-\mu_i}\right]_{1\leq i,j\leq n} + \det\left[x_j^{\mu_i} + x_j^{-\mu_i}\right]_{1\leq i,j\leq n}\right)$$

and hence

$$ch_\lambda = \frac{\det\left[x_j^{\lambda_i+n-i} - x_j^{-\lambda_i-n+i}\right]_{1\leq i,j\leq n} + \det\left[x_j^{\lambda_i+n-i} + x_j^{-\lambda_i-n+i}\right]_{1\leq i,j\leq n}}{\det\left[x_j^{n-i} + x_j^{-n+i}\right]_{1\leq i,j\leq n}}$$

The determinantal formulas (173) and (174) also hold in this case with the restriction that in this case $x_{n+i} = x_i^{-1}$, the element x_{2n+1} does not exist.

A.6 Pfaffians

If A an anti-symmetric matrix of an odd order its determinant vanishes. For even order, say k, the following multilinear form in $A_{ij}, i < j \leq k$

$$\mathrm{Pf}[A] := \sum_\sigma \mathrm{sgn}(\sigma)\, A_{\sigma(1),\sigma(2)} A_{\sigma(3),\sigma(4)} \cdots A_{\sigma(k-1),\sigma(k)} \tag{175}$$

where sum runs over all permutation restricted by

$$\sigma : \sigma(2i-1) < \sigma(2i), \quad \sigma(1) < \sigma(3) < \cdots < \sigma(k-1), \tag{176}$$

coincides with the square root of $\det A$ and is called the *Pfaffian* of A, see, for instance [36]. As one can see the Pfaffian contains $1 \cdot 3 \cdot 5 \cdots (k-1) =: (k-1)!!$ terms.

Wick's relations. Let each of w_i be a linear combination of Fermi operators:

$$\hat{w}_i = \sum_{m \in \mathbb{Z}} v_{im} \psi_m + \sum_{m \in \mathbb{Z}} u_{im} \psi_m^\dagger, \quad i = 1, \ldots, n$$

Then the Wick's formula is

$$\langle l | \hat{w}_1 \cdots \hat{w}_n | l \rangle = \begin{cases} \mathrm{Pf}\,[A]_{i,j=1,\ldots,n} & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases} \tag{177}$$

where A is n by n antisymmetric matrix with entries $A_{ij} = \langle l | \hat{w}_i \hat{w}_j | l \rangle, \quad i < j$.

A.7 Hirota equations

KP Hirota equation:

$$\oint \frac{dz}{2\pi i} e^{V(\mathbf{p'}-\mathbf{p},z)} \tau_{\mp}(\mathbf{p'} - [z^{-1}]|\mathbf{p}^*) \tau_{\mp}(\mathbf{p} + [z^{-1}]|\mathbf{p}^*) = 0 \tag{178}$$

For $\tau_{\mp}$ defined by (16) we have for the left hand side

$$\oint \frac{dz}{2\pi i} \left(1 - z^{-2}\right) e^{\sum_{k=1}^{\infty} \frac{1}{k} p'_k \left(z^k + z^{-k}\right)} e^{-\sum_{k=1}^{\infty} \frac{1}{k} p_k \left(z^k + z^{-k}\right)}$$

which is equal to zero. Thus, $\tau_{\mp}(\mathbf{p}|\mathbf{p}^*)$ is the KP tau function with respect to the $\mathbf{p}$ variables.

Now in view of Giambelli relations for characters [8]

$$o_\lambda(\mathbf{p}) = \det\left(o_{(\alpha_i|\beta_j)}(\mathbf{p})\right), \qquad sp_\lambda(\mathbf{p}) = \det\left(sp_{(\alpha_i|\beta_j)}(\mathbf{p})\right), \qquad \lambda = (\alpha|\beta) \tag{179}$$

we can write

$$\tau_-(\mathbf{p}|\mathbf{p}^*) = \langle 0 | e^{\sum_{k>0} \frac{1}{k} J_k p_k} e^{\sum_{i,j \geq 0} (-)^j o_{(i|j)}(\mathbf{p}^*) \psi_i \psi_{-1-j}^\dagger} | 0 \rangle \tag{180}$$

$$\tau_+(\mathbf{p}|\mathbf{p}^*) = \langle 0 | e^{\sum_{k>0} \frac{1}{k} J_k p_k} e^{\sum_{i,j \geq 0} (-)^j sp_{(i|j)}(\mathbf{p}^*) \psi_i \psi_{-1-j}^\dagger} | 0 \rangle \tag{181}$$

Hirota equations for the large BKP hierarchy were written in [26]. For 2-BKP hierarchy Hirota equations are as follows [43]

$$\oint \frac{dz}{2\pi i} z^{N'+l'-N-l-2} e^{V(\mathbf{p'}-\mathbf{p},z)} \tau_{N'-1}(l', \mathbf{p'} - [z^{-1}], \mathbf{s'}) \tau_{N+1}(l, \mathbf{p} + [z^{-1}], \mathbf{s})$$

$$+ \oint \frac{dz}{2\pi i} z^{N+l-N'-l'-2} e^{V(\mathbf{p}-\mathbf{p'},z)} \tau_{N'+1}(l', \mathbf{p'} + [z^{-1}], \mathbf{s'}) \tau_{N-1}(l, \mathbf{p} - [z^{-1}], \mathbf{s})$$

$$= \oint \frac{dz}{2\pi i} z^{l'-l} e^{V(\mathbf{s'}-\mathbf{s},z^{-1})} \tau_{N'-1}(l'+1, \mathbf{p'}, \mathbf{s'} - [z]) \tau_{N+1}(l-1, \mathbf{p}, \mathbf{s} - [z])$$

$$+ \int \frac{dz}{2\pi i} z^{l-l'} e^{V(\mathbf{s'}-\mathbf{s},z^{-1})} \tau_{N'+1}(l'-1, \mathbf{p'}, \mathbf{s'} + [z]) \tau_{N-1}(l+1, \mathbf{p}, \mathbf{s} + [z])$$

$$+ \frac{(-1)^{l'+l}}{2}(1 - (-1)^{N'+N}) \tau_{N'}(l', \mathbf{p'}, \mathbf{s'}) \tau_N(l, \mathbf{p}, \mathbf{s}) \tag{182}$$

The difference Hirota equation may be obtained from the previous one [43]

$$-\frac{\beta}{\alpha-\beta} \tau_N(l, \mathbf{p} + [\beta^{-1}]) \tau_{N+1}(l, \mathbf{p} + [\alpha^{-1}]) - \frac{\alpha}{\beta-\alpha} \tau_N(l, \mathbf{p} + [\alpha^{-1}]) \tau_{N+1}(l, \mathbf{p} + [\beta^{-1}])$$

$$+ \frac{1}{\alpha\beta} \tau_{N+2}(l, \mathbf{p} + [\alpha^{-1}] + [\beta^{-1}]) \tau_{N-1}(l, \mathbf{p}) = \tau_{N+1}(l, \mathbf{p} + [\alpha^{-1}] + [\beta^{-1}]) \tau_N(l, \mathbf{p}). \tag{183}$$

A.8 Integrals over the unitary, orthogonal and symplectic groups

For $\mathbb{U}(n)$ the Haar measure is

$$d_*U = \frac{1}{(2\pi)^n} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{-i\theta_k}|^2 \prod_{j=1}^n d\theta_j, \quad -\pi \le \theta_1 < \ldots \theta_n \le \pi \tag{184}$$

where $e^{\theta_1}, \ldots, e^{\theta_n}$ are eigenvalues of $U \in \mathbb{U}(n)$.

For $\mathbb{O}(2n)$ the Haar measure is

$$d_*O = \frac{2^{(n-1)^2}}{\pi^n} \prod_{i<j}^n (\cos\theta_i - \cos\theta_j)^2 \prod_{i=1}^n d\theta_i, \quad 0 \le \theta_1 \le \cdots \le \theta_n \le \pi \tag{185}$$

where $e^{\theta_1}, e^{-\theta_1}, \ldots, e^{\theta_n}, e^{-\theta_n}$ are eigenvalues of $O \in \mathbb{O}(2n)$.

For $\mathbb{O}(2n+1)$ the Haar measure is

$$d_*O = \frac{2^{n^2}}{\pi^n} \prod_{i<j}^n (\cos\theta_i - \cos\theta_j)^2 \prod_{i=1}^n \sin^2\frac{\theta_i}{2} d\theta_i, \quad 0 \le \theta_1 \le \cdots \le \theta_n \le \pi \tag{186}$$

where $e^{\theta_1}, e^{-\theta_1}, \ldots, e^{\theta_n}, e^{-\theta_n}, 1$ are eigenvalues of $O \in \mathbb{U}(2n+1)$.

For $\mathbb{S}p(2n)$ the Haar measure is

$$d_*S = \frac{2^{n^2}}{\pi^n} \prod_{i<j}^n (\cos\theta_i - \cos\theta_j)^2 \prod_{i=1}^n \sin^2\theta_i d\theta_i, \quad 0 \le \theta_1 \le \cdots \le \theta_n \le \pi \tag{187}$$

where $e^{\theta_1}, e^{-\theta_1}, \ldots, e^{\theta_n}, e^{-\theta_n}$ are eigenvalues of $S \in \mathbb{S}p(2n)$.

A.9 Characters as vacuum expectation values

We can write the determinatal expression (2) for $s_\lambda(\mathbf{p})$ as a (vacuum) expectation value [24]

$$s_\lambda(\mathbf{p}) = \langle k|\Gamma(\mathbf{p})\psi_{k+\lambda_1-1}\psi_{k+\lambda_2-2}\cdots\psi_{k+\lambda_n-n}|k-n\rangle \tag{188}$$

Now using the commutation relations of $\Gamma(\mathbf{p})$ with the fermions ψ_j, we define

$$\psi_j(\mathbf{p}) := \Gamma(\mathbf{p})\psi_j\Gamma(\mathbf{p})^{-1} = \sum_{i=0}^\infty s_{(i)}(\mathbf{p})\psi_{j-i}, \qquad \psi_j^\dagger(\mathbf{p}) := \Gamma(\mathbf{p})\psi_j^\dagger\Gamma(\mathbf{p})^{-1} = \sum_{i=0}^\infty s_{(i)}(-\mathbf{p})\psi_{j+i}^\dagger \tag{189}$$

Hence,

$$s_\lambda(\mathbf{p}) = \langle k|\psi_{k+\lambda_1-1}(\mathbf{p})\psi_{k+\lambda_2-2}(\mathbf{p})\cdots\psi_{k+\lambda_n-n}(\mathbf{p})|k-n\rangle \tag{190}$$

which is exactly the determinant of (2). On the other hand, if we write λ in the Frobenius notation $\lambda = (a_1, a_2, \ldots, a_r|b_1, b_2, \ldots, b_r)$ where $a_i = \lambda_i - i \ge 0$ for $1 \le i \le r$ and $\lambda_{r+1} - r - 1 < 0$, and $b_i = \lambda_i' - i \ge 0$ for $1 \le i \le r$ and $\lambda_{r+1}' - r - 1 < 0$ for λ' the conjugate partition of λ. If we drop the condition that $a_i = \lambda_i - i \ge 0$ and $b_i = \lambda_i' - i \ge 0$, then we can see

$$\lambda = (a_1, a_2, \ldots | b_1, b_2, \ldots) \qquad \text{with } a_i = \lambda_i - i, \text{ and } b_i = \lambda_i' - i$$

as two infinite sequences. It is then well known (see e.g. [48]) that the sets

$$A = \{a_1, a_2, a_3, \ldots\} \quad \text{and } B = \{-b_1 - 1, -b_2 - 1, -b_3 - 1, \ldots\}$$

form a partition of $\mathbb{Z}$, thus

$$\psi_{k+\lambda_1-1}\psi_{k+\lambda_2-2}\cdots\psi_{k+\lambda_n-n}|k-n\rangle = (-)^{b_1+\cdots+b_r}\psi_{k+a_1}\psi_{k+a_2}\cdots\psi_{k+a_r}\psi_{k-b_r-1}^\dagger\psi_{k-b_{r-1}-1}^\dagger\cdots\psi_{k-b_1-1}^\dagger|k\rangle \tag{191}$$

and

$$s_\lambda(\mathbf{p}) = (-)^{b_1+\cdots+b_r+\frac{r(r-1)}{2}} \langle k|\psi_{k+a_1}(\mathbf{p})\psi_{k+a_2}(\mathbf{p})\cdots\psi_{k+a_r}(\mathbf{p})\psi^\dagger_{k-b_1-1}(\mathbf{p})\psi^\dagger_{k-b_2-1}(\mathbf{p})\cdots\psi^\dagger_{k-b_r-1}(\mathbf{p})|k\rangle \tag{192}$$

Now using Wick's relation, see Appendix, this is equal to the Pfaffian expression

$$(-)^{b_1+\cdots+b_r+\frac{r(r-1)}{2}} \operatorname{Pf}[A_\lambda]_{i,j=1,\ldots,2r}, \quad \text{where } A_\lambda = \begin{pmatrix} 0 & B \\ -B^{tr} & 0 \end{pmatrix}, \quad \text{for } B_{ij} = \langle k|\psi_{k+a_i}(\mathbf{p})\psi^\dagger_{k-b_j-1}(\mathbf{p})|k\rangle \tag{193}$$

Now

$$\operatorname{Pf}[A_\lambda]_{i,j=1,\ldots,2r} = (-)^{\frac{r(r-1)}{2}} \det[B]_{i,j=1,\ldots,r}$$

and

$$B_{ij} = \langle k|\Gamma(\mathbf{p})\psi_{k+a_i}\psi^\dagger_{k-b_j-1}|k\rangle = (-)^{b_j} s_{(a_i|b_j)}(\mathbf{p})$$

Thus

$$s_\lambda(\mathbf{p}) = \det\left[s_{(a_i|b_j)}(\mathbf{p})\right]_{i,j=1,\ldots,r} \tag{194}$$

which is a well known result of Littlewood [34] (see also [48]).

The orthogonal and symplectic characters can also be expressed as determinants (see e.g. [31], Theorem 1.3.2 and 1.3.3)

$$o_\lambda(\mathbf{p}) = \det\left[s_{(\lambda_i-i+j)}(\mathbf{p}) - s_{(\lambda_i-i-j)}(\mathbf{p})\right]_{i,j=1,\ldots,n}, \qquad sp_\lambda(\mathbf{p}) = \frac{1}{2}\det\left[s_{(\lambda_i-i+j)}(\mathbf{p}) + s_{(\lambda_i-i-j+2)}(\mathbf{p})\right]_{i,j=1,\ldots,n} \tag{195}$$

and similarly one has a formula ala (194), see [48]

$$o_\lambda(\mathbf{p}) = \det\left[o_{(a_i|b_j)}(\mathbf{p})\right]_{i,j=1,\ldots,r} \quad \text{and} \quad sp_\lambda(\mathbf{p}) = \det\left[sp_{(a_i|b_j)}(\mathbf{p})\right]_{i,j=1,\ldots,r} \tag{196}$$

Using (196), we calculate $o_{(a+1,1^b)}(\mathbf{p}) = o_{(a|b)}(\mathbf{p})$ explicitly

$$o_{(a|b)} = \det\begin{bmatrix} s_{(a+1)} - s_{(a-1)} & s_{(a+2)} - s_{(a-2)} & s_{(a+3)} - s_{(a-3)} & \cdots & s_{(a+b+1)} - s_{(a-b-1)} \\ s_{(0)} & s_{(1)} & s_{(2)} & \cdots & s_{(b)} \\ 0 & s_{(0)} & s_{(1)} & \cdots & s_{(b-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s_{(1)} \end{bmatrix} \tag{197}$$

$$= s_{(a|b)} - s_{(a-2|b)} + \det\begin{bmatrix} 0 & s_{(a)} - s_{(a-2)} & s_{(a+1)} - s_{(a-3)} & \cdots & s_{(a+b-1)} - s_{(a-b-1)} \\ s_{(0)} & s_{(1)} & s_{(2)} & \cdots & s_{(b)} \\ 0 & s_{(0)} & s_{(1)} & \cdots & s_{(b-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s_{(1)} \end{bmatrix}$$

$$= s_{(a|b)} - s_{(a-2|b)} - o_{(a-1|b-1)} = \sum_{j=0}^{b} (-)^j s_{(a-j|b-j)} - s_{(a-j-2|b-j)}$$

A similar calculation, using again (196), shows that $sp_{((a+1,1^b)}(\mathbf{p}) = sp_{(a|b)}(\mathbf{p})$ is equal to

$$sp_{(a|b)} = s_{(a|b)} - s_{(a-1|b-1)} - o_{(a|b-2)} = \sum_{j=0}^{b} (-)^j s_{(a-j|b-j)} - s_{(a-j|b-j-2)} \tag{198}$$

which is also equal to

$$sp_{(a|b)} = \sum_{kj0}^{a} (-)^j s_{(a-j|b-j)} - s_{(a-j|b-j-2)} \tag{199}$$

We can write both (197) and (199) as vacuum expectation values, viz.,

$$o_{(a|b)}(\mathbf{p}) = (-)^b \sum_{j=0}^{\infty} \langle k|\Gamma(\mathbf{p})(\psi_{k+a-j} - \psi_{k+a-j-2})\psi^\dagger_{k-b+j-1}|k\rangle \tag{200}$$

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$$= (-)^b \mathrm{Res}_{z=0} \mathrm{Res}_{w=0} \frac{(z^{-a-1} - z^{-a+1})w^{-b-1}}{1 - zw} \left(\frac{w}{z}\right)^k \langle k|\Gamma(\mathbf{p})\psi(z)\psi^\dagger(w)|k\rangle,$$

where

$$\psi(z) = \sum_{j\in\mathbb{Z}} \psi_j z^j, \qquad \psi^\dagger(z) = \sum_{j\in\mathbb{Z}} \psi_j^\dagger z^{-j-1} \tag{201}$$

And

$$sp_{(a|b)}(\mathbf{p}) = (-)^b \sum_{j=0}^{\infty} \langle k|\Gamma(\mathbf{p})\psi_{k+a-j}(\psi_{k-b+j-1}^\dagger - \psi_{k-b+j+1}^\dagger)|k\rangle \tag{202}$$

$$= (-)^b \mathrm{Res}_{z=0} \mathrm{Res}_{w=0} \frac{z^{-a-1}(w^{-b-1} - w^{-b+1})}{1 - zw} \left(\frac{w}{z}\right)^k \langle k|\Gamma(\mathbf{p})\psi(z)\psi^\dagger(w)|k\rangle$$

Now let $\lambda = (a_1, a_2, \cdots a_r | b_1, b_2, \cdots b_r)$, then

$$o_\lambda(\mathbf{p}) = (-)^{\sum b_j} \prod_{i=1}^{r} \mathrm{Res}_{z_i=0} \mathrm{Res}_{w_i=0}(z_i^{-a_i-1} - z_i^{-a_i+1})w_i^{-b_i-1} \det\left[\left(\frac{w_j}{z_i}\right)^k \langle k|\Gamma(\mathbf{p})\frac{\psi(z_i)\psi^\dagger(w_j)}{1 - z_i w_j}|k\rangle\right]_{i,j=1,\ldots,r}$$
$$\tag{203}$$

And similarly

$$sp_\lambda(\mathbf{p}) = (-)^{\sum b_j} \prod_{i=1}^{r} \mathrm{Res}_{z_i=0} \mathrm{Res}_{w_i=0} z_i^{-a_i-1}(w_i^{-b_i-1} - w_i^{-b_i+1}) \det\left[\left(\frac{w_j}{z_i}\right)^k \langle k|\Gamma(\mathbf{p})\frac{\psi(z_i)\psi^\dagger(w_j)}{1 - z_i w_j}|k\rangle\right]_{i,j=1,\ldots,r}$$
$$\tag{204}$$

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