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Anisotropy-Based Suboptimal State-Feedback Control Design Using Linear Matrix Inequalities

A. A. Belov[*,**,a] **and O. G. Andrianova**[**,b]

[*]*ITMO University, St. Petersburg, Russia*
[**]*Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia*
e-mail: [a]*a.a.belov@inbox.ru,* [b]*andrianovaog@gmail.com*

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Abstract—A computationally efficient method for the design of a suboptimal anisotropic controller for discrete descriptor systems based on convex optimization methods is proposed. Numerical examples are given.

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1. INTRODUCTION

While performing the mathematical modeling of control systems in terms of physical variables, one may obtain models combining differential and algebraic equations. Such systems, known as descriptor systems, found its applications in different fields of science and technology [1–3].

Despite the obvious ease of mathematical modeling, the analysis and synthesis methods for such systems substantially differ from the classical counterparts, being rather complex for implementation. The methods developed for the ordinary systems are difficult to generalize to this class of systems due to the existing algebraic constraint equations that assign fundamentally new properties to the descriptor systems. Among such properties, we mention the system's unsolvability with respect to the derivative, the need to use sufficiently smooth signals as the system inputs, as well as noncausal (in the discrete case) or impulsive (in the continuous case) behavior.

Some problems solved for the ordinary systems are still urgent for the descriptor systems. A problem of this group is solved in the current paper, namely, the design of a suboptimal anisotropy-based controller for a descriptor system based on linear matrix inequalities (LMIs).

Anisotropy-based theory dates back to the pioneering papers [4, 5]. This approach proceeds from the information-theoretical representation of random signals. Anisotropy-based theory studies the response of a system of the signal with respect to "colored noises" affecting it. Here "color" is considered as the Kullback–Leibler divergence from the Gaussian white noise [4]. In this case, the performance criterion is the anisotropic norm of the system. The anisotropy-based analysis problem for the ordinary systems was solved in [6] using linear matrix inequalities. This result was generalized to the descriptor systems in [7]. However, the conditions derived in [7] are nonstrict and, moreover, nonconvex due to the existing singular matrices in the constraints. In addition, the solution of the control design problem using the conditions from [7] yields nonlinear matrix inequalities, which makes the obtaining of numerical results more complicated. The present paper formulates a new bounded real lemma in terms of linear matrix inequalities, leading to a numerically effective algorithm for the analysis of the descriptor systems. The analysis algorithm is then applied to design a suboptimal anisotropy-based controller.

In the sequel, we adopt the following notation: I_r as the identity matrix of dimensions $(r \times r)$, $\|P\|$ as the norm of a transfer function $P(z)$, $\sigma_{\max}(\cdot)$ as the maximal singular value of a transfer matrix, and $\rho(E, A) = \max|\lambda|_{\lambda \in \{z | \det(zE-A)=0\}}$ as the generalized spectral radius of the matrix pair (E, A).

This work has the following structure. Section 2 gives some preliminary background on anisotropy-based analysis and basics of discrete descriptor systems theory. Next, Section 3 establishes the new anisotropy-based bounded real lemma in terms of linear matrix inequalities for the ordinary and descriptor systems. In Section 4 the suboptimal anisotropy-based control design problem based on the proved bounded real lemma is stated and solved. And finally, numerical examples are given.

2. THEORETIC BACKGROUND

2.1. Descriptor Systems

The state-space dynamics of a discrete descriptor system has the form

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bf(k), \\ y(k) &= Cx(k) + Df(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ denotes the state vector, $f(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ are the input and output vectors, respectively. In addition, E, A, B, C, and D specify constant matrices of appropriate dimensions, $E \in \mathbb{R}^{n \times n}$ being singular ($\operatorname{rank} E = r < n$).

In theory of descriptor systems, a key notion is the regularity of system (1), which coincides with the regularity of the matrix pencil $(\lambda E - A)$ where λ indicates an arbitrary scalar. The regularity of the system stands for the existence of a unique solution under fixed initial conditions [3].

Definition 1. System (1) is called regular if $\exists \lambda \neq 0 : \det(\lambda E - A) \neq 0$.

Throughout the paper, it is assumed that the systems under consideration are regular. Here are some definitions required for further exposition.

Definition 2. The transfer function of system (1) is described by

$$P(z) = C(zE - A)^{-1}B + D, \quad z \in \mathbb{C}. \tag{2}$$

With this definition, the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of the descriptor system have the form

$$\|P\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} \operatorname{Trace}\left(P^*(\mathrm{e}^{i\omega}) P(\mathrm{e}^{i\omega}) \right) d\omega \right)^{\frac{1}{2}} = \left(\frac{1}{2\pi} \int_0^{2\pi} \|P(\mathrm{e}^{i\omega})\| d\omega \right)^{\frac{1}{2}},$$

$$\|P\|_\infty = \sup_{\omega \in [0, 2\pi]} \sigma_{\max}\left(P(\mathrm{e}^{i\omega}) \right).$$

Definition 3. The descriptor system (1) is called admissible if it is regular, causal ($\deg \det(zE - A) = \operatorname{rank} E$) and stable ($\rho(E, A) < 1$). For details, see [1, 8].

For the regular system (1), there exist two nonsingular matrices $\overline{W}$ and $\overline{V}$ such that $\overline{W} E \overline{V} = \operatorname{diag}(I_r, 0)$, see [1].

Consider the coordinate transformation

$$\overline{V}^{-1} x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \tag{3}$$

where $x_1(k) \in \mathbb{R}^r$ and $x_2(k) \in \mathbb{R}^{n-r}$.

Then, premultiplying the state equation of system (1) by the matrix $\overline{W}$ and using transformation (3), we rewrite system (1) in the form

$$\begin{aligned} x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) + B_1 f(k), \\ 0 &= A_{21}x_1(k) + A_{22}x_2(k) + B_2 f(k), \\ y(k) &= C_1 x_1(k) + C_2 x_2(k) + Df(k), \end{aligned} \tag{4}$$

where

$$\overline{W}A\overline{V} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad \overline{W}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad C\overline{V} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \tag{5}$$

Again, the details can be found in [1].

The matrices $\overline{W}$ and $\overline{V}$ are calculated from the singular value decomposition

$$E = U\mathrm{diag}(S, 0)Y^\top,$$

where U and Y are real orthogonal matrices, and S is diagonal matrix of order r induced by the nonzero singular values of the matrix E, i.e.,

$$\overline{W} = \mathrm{diag}(S^{-1}, I_{n-r})U^\top, \qquad \overline{V} = Y.$$

Form (4) is known as the SVD canonical form [1]. In addition, it should be noted that the system is causal if $\det(A_{22}) \neq 0$, and stable if $\rho(A_{11} - A_{12}A_{22}^{-1}A_{21}) < 1$ [8].

While solving control problems for the descriptor systems, one has to guarantee the stability of a dynamic subsystem, moreover, eliminating the undesired noncausal behavior. Therefore, for the descriptor systems, causal controllability and stabilizability. Consider them in detail are differentiated.

Choose the state-feedback control law

$$f(k) = F_c x(k) + h(k), \tag{6}$$

where $F_c \in \mathbb{R}^{m\times n}$ is a constant matrix and $h(k)$ acts as a new input. Then the closed-loop system takes the form

$$Ex(k+1) = (A + BF_c)x(k) + Bh(k). \tag{7}$$

Definition 4. System (1) is called causally controllable if there exist the state-feedback control (6) making the closed-loop system (7) causal.

The causal controllability can be verified using the following rank criterion [1].

Theorem 1. *System* (1) *is causally controllable if*

$$\mathrm{rank}\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = \mathrm{rank}\,(E) + n.$$

Stabilizabitily of descriptor system is a possibility to control unstable modes in dynamical subsystem.

Definition 5. System (1) is called stabilizable if there exists a state-feedback control law of the form $f(k) = F_{st}x(k)$ such that $\rho(E, A + BF_{st}) < 1$.

2.2. Mean Anisotropy of the Sequence and Anisotropic Norm of the System

Let $W = \{w(k)\}_{k\in\mathbb{Z}}$ represent a stationary sequence of square summable random m-dimensional vectors. The sequence W can be generated from the Gaussian white noise (further denoted by $V = \{v(k)\}_{k\in\mathbb{Z}}$) with the zero mean and the identity covariance matrix using an admissible shaping filter with the transfer function $G(z) = C_G(zE_G - A_G)^{-1}B_G + D_G$. The mean anisotropy of the sequence is the Kullback–Leibler divergence of the signal's probability density function from the probability density function of the Gaussian white noise.

The mean anisotropy of the sequence can be defined via the shaping filter parameters, i.e.,

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det\frac{mS(\omega)}{\|G\|_2^2}d\omega,$$

where

$$S(\omega) = \widehat{G}(\omega)\widehat{G}^*(\omega) \quad (-\pi \leqslant \omega \leqslant \pi), \quad \widehat{G}(\omega) = \lim_{l\to 1} G(le^{i\omega})$$

gives the boundary value of the function $G(z)$.

Remark 1. The random sequence W is completely described by the shaping filter G, which makes the notations $\overline{\mathbf{A}}(G)$ and $\overline{\mathbf{A}}(W)$ equivalent.

The mean anisotropy characterizes the "color" of the random signal, i.e., the difference between given the signal and the Gaussian white noise. If the mean anisotropy of the signal is $\overline{\mathbf{A}}(W) = 0$, then it represents the Gaussian white noise; in the case of $\overline{\mathbf{A}}(W) \to \infty$, the sequence defines a deterministic signal. For details, see [4, 9].

Denote by $Y = PW$ the output of a linear discrete-time system $P \in \mathcal{H}_\infty^{p\times m}$ whose transfer function $P(z)$ is analytical in the open unit circle $|z| < 1$, having a finite $\mathcal{H}_\infty$-norm.

Definition 6. Given $a \geqslant 0$, the a-anisotropic norm of the system P is defined by

$$|\!|\!| P |\!|\!|_a = \sup\left\{\|PG\|_2/\|G\|_2 : \ G \in \mathbf{G}_a\right\}, \tag{8}$$

i.e., as the maximum gain of the system with respect to the class of shaping filters

$$\mathbf{G}_a = \left\{G \in \mathcal{H}_2^{m\times m} : \ \overline{\mathbf{A}}(G) \leqslant a\right\}.$$

Thus, the a-anisotropic norm $|\!|\!| P |\!|\!|_a$ describes the stochastic gain of the system P with respect to the input sequence W.

Remark 2. If the mean anisotropy of the signal is $\overline{\mathbf{A}}(W) = 0$, then $|\!|\!| P |\!|\!|_a = \frac{\|P\|_2}{\sqrt{m}}$; in the case of $\overline{\mathbf{A}}(W) \to \infty$, we have $|\!|\!| P |\!|\!|_a = \|P\|_\infty$ [4, 9].

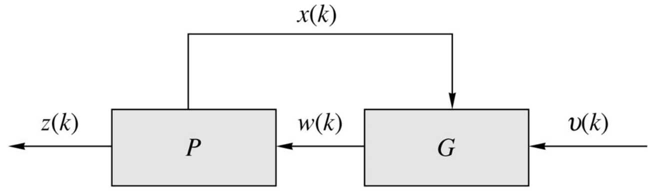

Fig. 1. Computation of a-anisotropic norm.

3. NEW ANISOTROPY-BASED BOUNDED REAL LEMMA AND COMPUTATION OF A-ANISOTROPIC NORM OF THE SYSTEM

3.1. New Anisotropy-Based Bounded Real Lemma for Ordinary Systems

Problem statement

Consider an ordinary discrete-time system written in the form

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k), \\ y(k) &= Cx(k) + Dw(k), \end{aligned} \tag{9}$$

where $x(k) \in \mathbb{R}^n$ denotes the state vector, $w(k) \in \mathbb{R}^m$ is a random stationary sequence with a given mean anisotropy $\overline{\mathbf{A}}(W) = a$, and $y(k) \in \mathbb{R}^p$ is output. Here A, B, C, and D are constant matrices of appropriate dimensions. The transfer function of the system (9) is defined as

$$T(z) = C(zI - A)^{-1}B + D.$$

Assume that the system (9) is stable and the scalars $a \geqslant 0$ and $\gamma > 0$ are given. It is necessary to to check if the following inequality is satisfied

$$|\!|\!| T |\!|\!|_a < \gamma.$$

The following lemma provides the answer, see [6].

Lemma 1. *Let system* (9) *with the transfer function* $T(z) \in \mathcal{H}_\infty^{p\times m}$ *be stable. For given scalars* $a \geqslant 0$ *and* $\gamma > 0$*, the a-anisotropic norm of the system is bounded above by* γ*, i.e.,*

$$|\!|\!| T |\!|\!|_a < \gamma,$$

if there exist a scalar $\eta > \gamma^2$ *and a* $n \times n$ *matrix* $\Phi = \Phi^\top > 0$ *satisfying the inequalities*

$$\eta - \left(\mathrm{e}^{-2a} \det(\eta I_m - B^\top \Phi B - D^\top D)\right)^{1/m} < \gamma^2, \tag{10}$$

$$\begin{bmatrix} A^\top \Phi A - \Phi + C^\top C & A^\top \Phi B + C^\top D \\ B^\top \Phi A + D^\top C & B^\top \Phi B + D^\top D - \eta I_m \end{bmatrix} < 0. \tag{11}$$

Now, we formulate a theorem determining new boundedness conditions for the a-anisotropic norm of the ordinary system.

Theorem 2. *Let system* (9) *with the transfer function* $T(z) \in \mathcal{H}_\infty^{p\times m}$ *be stable. If for given scalars* $a \geqslant 0$ *and* $\gamma > 0$ *there exist a scalar* $\eta > \gamma^2$*, a matrix* $\Phi = \Phi^\top > 0$ *of dimensions* $n \times n$ *and an arbitrary matrix* Y *of dimensions* $n \times n$ *satisfying the inequalities*

$$\eta - \left(\mathrm{e}^{-2a} \det(\eta I_m - B^\top \Phi B - D^\top D)\right)^{1/m} < \gamma^2, \tag{12}$$

$$\begin{bmatrix} -\frac{1}{2}Y - \frac{1}{2}Y^\top & YA & YB & \Phi^\top - Y^\top - \frac{1}{2}Y & 0 \\ A^\top Y^\top & -\Phi & 0 & A^\top Y^\top & C^\top \\ B^\top Y^\top & 0 & -\eta I_m & B^\top Y^\top & D^\top \\ \Phi - Y - \frac{1}{2}Y^\top & YA & YB & -Y - Y^\top & 0 \\ 0 & C & D & 0 & -I_p \end{bmatrix} < 0, \tag{13}$$

then the a-anisotropic norm of the system is bounded above by γ, i.e.,

$$|||T|||_a < \gamma.$$

The proof of this result is given in to the Appendix.

Remark 3. To eliminate the product $D^\top D$ from inequality (12), introduce a new variable Ψ such that

$$\Psi < \eta I_m - B^\top \Phi B - D^\top D. \tag{14}$$

Using Schur's complement lemma, transform (14) in the following way:

$$\Psi - \eta I_m + B^\top \Phi B - D^\top (-I_p) D < 0,$$

$$\begin{bmatrix} \Psi - \eta I_m + B^\top \Phi B & D^\top \\ D & -I_p \end{bmatrix} < 0.$$

Therefore, inequality (12) can be rewritten as the system of two inequalities

$$\eta - \left(\mathrm{e}^{-2a} \det(\Psi)\right)^{1/m} < \gamma^2$$

and

$$\begin{bmatrix} \Psi - \eta I_m + B^\top \Phi B & D^\top \\ D & -I_p \end{bmatrix} < 0.$$

3.2. New Anisotropy-Based Bounded Real Lemma for Descriptor Systems

Problem statement

Problem 1. Consider a descriptor system in the state-space representation

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bw(k), \\ y(k) &= Cx(k) + Dw(k), \end{aligned} \tag{15}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^m$ forms a random stationary sequence with a known mean anisotropy $\overline{\mathbf{A}}(W) = a$, and $y(k) \in \mathbb{R}^p$ denotes the observable output. Here E, A, B, C, and D are known constant matrices of appropriate dimensions, $\operatorname{rank} E = r < n$. The transfer function $P(z)$ of the system is defined by

$$P(z) = C(zE - A)^{-1}B + D.$$

The system (15) is assumed to be and the scalars $a \geqslant 0$ and $\gamma > 0$ are specified. It is necessary to to check if the following inequality is satisfied

$$|||P|||_a < \gamma.$$

The system is regular, and hence there exist two matrices $\overline{W}$ and $\overline{V}$ transforming system (15) to the equivalent form (4). Introduce the following notation:

$$E_d = \overline{W} E \overline{V}, \quad A_d = \overline{W} A \overline{V}, \quad B_d = \overline{W} B, \quad C_d = C\overline{V}, \quad D_d = D.$$

To proceed, we state the anisotropy-based bounded real lemma for the descriptor system (15).

Theorem 3. *Let system* (15) *with the transfer function* $P(z) \in \mathcal{H}_\infty^{p\times m}$ *be admissible. In addition, assume that* $\operatorname{rank} E = \operatorname{rank}[E \ \ B]$. *For given scalars* $a \geqslant 0$ *and* $\gamma > 0$, *the a-anisotropic norm of the system is bounded above by* γ, *i.e.,*

$$\|P\|_a < \gamma,$$

if there exist matrices $L \in \mathbb{R}^{r\times r}$, $L > 0$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, *and* $\Psi \in \mathbb{R}^{m\times m}$, *as well as scalars* $\eta > \gamma^2$ *and* $\alpha > 0$ *satisfying the inequalities*

$$\eta - \left(\mathrm{e}^{-2a}\det(\Psi)\right)^{1/m} < \gamma^2, \tag{16}$$

$$\begin{bmatrix} \Psi - \eta I_m + B_d^\top \Theta B_d & D_d^\top \\ D_d & -I_p \end{bmatrix} < 0, \tag{17}$$

and

$$\begin{bmatrix} -\frac{1}{2}Q - \frac{1}{2}Q^\top & \Gamma A_d & \Gamma B_d & L^\top - Q^\top - \frac{1}{2}Q & 0 \\ A_d^\top \Gamma^\top & \Pi A_d + A_d^\top \Pi^\top - \Theta & \Pi B_d & A_d^\top \Gamma^\top & C_d^\top + \alpha A_d^\top \Pi^\top C_d^\top \\ B_d^\top \Gamma^\top & B_d^\top \Pi^\top & -\eta I_m & B_d^\top \Gamma^\top & D_d^\top + \alpha B_d^\top \Pi^\top C_d^\top \\ L - Q - \frac{1}{2}Q^\top & \Gamma A_d & \Gamma B_d & -Q - Q^\top & 0 \\ 0 & C_d + \alpha C_d \Pi A_d & D_d + \alpha C_d \Pi B_d & 0 & -I_p \end{bmatrix} < 0, \tag{18}$$

where

$$\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}, \quad \Gamma = \begin{bmatrix} Q & R \end{bmatrix}.$$

The proof of this theorem is given in the Appendix.

Set $\xi = \gamma^2$. To calculate the a-anisotropic norm of the descriptor system, it is necessary to solve the following optimization problem: find $\xi_* = \inf \xi$ on the collection $\{L, Q, R, S, \Psi, \eta, \xi\}$ that satisfies inequalities (16), (17), and (18). If the minimum value ξ_* is obtained, then the a-anisotropic norm of the system P can be approximately calculated by the formula

$$\|P\|_a \approx \sqrt{\xi_*}.$$

Here α is a predetermined quantity.

Example 1. Choose the system matrices

$$E = \begin{bmatrix} 3 & 0 & 2 & -5 \\ 0 & 3 & -2 & 2 \\ 2 & 2 & 0 & -2 \\ 2 & -4 & 4 & -6 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.7 & -3.25 & -0.7 & 0 \\ 1.8 & 0.4 & -6.4 & 2.6 \\ 1.0 & -1.9 & -5.4 & 2.4 \\ -0.6 & -2.7 & 5.4 & -2.8 \end{bmatrix},$$

$$B = \begin{bmatrix} 3.2 & -3.5 \\ 2.5 & -7.9 \\ 3.8 & -7.6 \\ -1.2 & 8.2 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.2 & 0.4 & 0.45 & 0.6 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.2 & 1.0 \end{bmatrix}, \quad \operatorname{rank}(E) = 2.$$

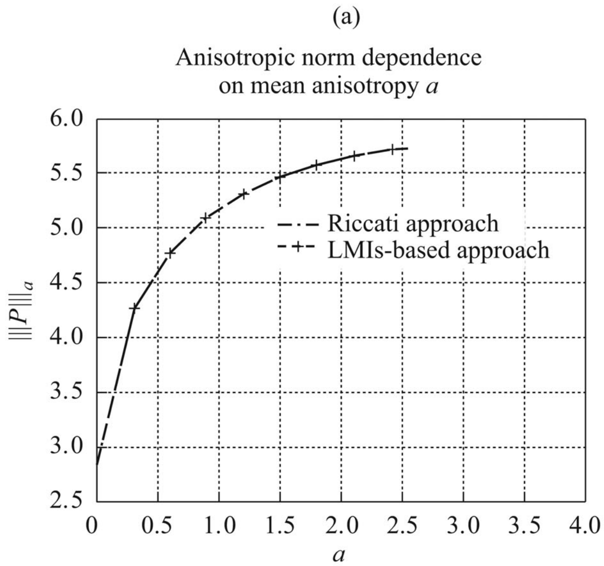

Fig. 2. The a-anisotropic norm of the descriptor system.

As easily verified, the system is causal and stable. Figure 2a illustrates the a-anisotropic norm calculated using the bounded real lemma. And Fig. 2b shows the absolute error of the calculated anisotropic norm with respect to its counterpart yielded by the Riccati approach [11].

4. SUBOPTIMAL ANISOTROPY-BASED CONTROL DESIGN FOR DESCRIPTOR SYSTEMS

Problem statement

Problem 2. Consider a discrete-time descriptor system of the form

$$\begin{aligned} Ex(k+1) &= Ax(k) + B_1 w(k) + B_2 u(k), \\ z(k) &= Cx(k) + D_1 w(k), \end{aligned} \tag{19}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^{m_1}$ represents a random stationary sequence with a given mean anisotropy $\overline{\mathbf{A}}(W) = a \geqslant 0$, $z(k) \in \mathbb{R}^q$ denotes the controllable output, and $u(k) \in \mathbb{R}^{m_2}$ indicates the control vector. In addition, E, A, B_1, B_2, C, and D_1 are known constant matrices of appropriate dimensions.

Assume that

1) the state vector is completely observable;

2) system (19) is causally controllable;

3) system (19) is stabilizable;

4) the scalar $\gamma > 0$ has a given value.

The problem is to find a state-feedback control law $u(k) = Fx(k)$ making the closed-loop system P_{cl} (19)

1) causal;

2) stable;

3) satisfying the inequality $|||P_{cl}|||_a < \gamma$.

The system is regular, and hence there exist two matrices $\overline{W}$ and $\overline{V}$ transforming system (19) to the equivalent form (4). Introduce the following notation: $E_d = \overline{W}E\overline{V}$, $A_d = \overline{W}A\overline{V}$, $B_{1d} = \overline{W}B_1$, $B_{2d} = \overline{W}B_2$, $C_d = C\overline{V}$, $D_{1d} = D_1$.

Theorem 4. *Let the rank conditions* $\operatorname{rank} E = \operatorname{rank}[E \;\; B_1]$ *and* $\operatorname{rank} E^\top = \operatorname{rank}\begin{bmatrix}E^\top & C^\top\end{bmatrix}$ *hold. For the given scalar $\gamma > 0$ and mean anisotropy $a \geqslant 0$, Problem 2 is solvable if there exist matrices $L \in \mathbb{R}^{r\times r}$, $L > 0$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $Z \in \mathbb{R}^{n\times m_1}$, and $\Psi \in \mathbb{R}^{m_1\times m_1}$, as well as a scalar $\eta > \gamma^2$ and a sufficiently large scalar $\alpha > 0$ satisfying the inequalities*

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{21}^\top & \Lambda_{31}^\top & \Lambda_{41}^\top & 0 \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{32}^\top & \Lambda_{21} & \Lambda_{52}^\top \\ \Lambda_{31} & \Lambda_{32} & -\eta I_q & \Lambda_{31} & \Lambda_{53}^\top \\ \Lambda_{41} & \Lambda_{21}^\top & \Lambda_{31}^\top & -(Q+Q^\top) & 0 \\ 0 & \Lambda_{52} & \Lambda_{53} & 0 & -I_{m_1} \end{bmatrix} < 0, \tag{20}$$

$$\eta - \left(\mathrm{e}^{-2a}\det(\Psi)\right)^{1/m_1} < \gamma^2, \tag{21}$$

$$\begin{bmatrix} \Psi - \eta I_{m_1} + B_{1d}^\top \Theta B_{1d} & D_{1d}^\top \\ D_{1d} & -I_q \end{bmatrix} < 0, \tag{22}$$

where

$$\Lambda_{11} = -\frac{1}{2}Q - \frac{1}{2}Q^\top, \quad \Lambda_{21} = A_d\Gamma^\top + B_{1d}Z^\top\Omega^\top, \quad \Lambda_{31} = C_d\Gamma^\top, \; \Lambda_{41} = L - Q - \frac{1}{2}Q^\top,$$

$$\Lambda_{22} = LA_d^\top + A_dL^\top + \Phi Z B_{1d} + B_{1d}Z^\top\Phi^\top - \Theta, \quad \Lambda_{32} = C_d\Pi^\top,$$

$$\Lambda_{52} = B_{1d}^\top + \alpha B_{1d}^\top \Pi A_d^\top + \alpha B_{1d}^\top \Phi^\top Z B_{2d}^\top, \quad \Lambda_{53} = D_{1d} + \alpha B_{1d}^\top \Pi C_d^\top.$$

In addition,

$$\Theta = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix},$$

$$\Omega = \begin{bmatrix} I_r & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} Q & R \end{bmatrix}.$$

The feedback gain is calculated from the expression

$$F_2 = Z^\top \begin{bmatrix} Q^{-\top} & 0 \\ -S^{-\top}R^\top Q^{-\top} & S^{-\top} \end{bmatrix} \overline{V}^{-1}. \tag{23}$$

The proof of this theorem can be found in the Appendix.

Remark 4. To construct the γ-optimal control, it is required to solve the following problem: find $\xi_* = \inf \xi$ on the set $\{L, Q, R, S, Z, \Psi, \eta, \xi\}$ satisfying inequalities (20)–(22). Here $\xi = \gamma^2$, and $\alpha > 0$ is a given scalar.

Consider some examples illustrating the performance of the algorithm.

Example 2. The system has the following parameters:

$$E = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 4.5 & 0.1 \\ 1.7 & 0.8 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad D_1 = 0.05.$$

Table 1. Controller design for the system from Example 2

$\overline{\mathbf{A}}(W)$	0	0.05	5
$\lVert P_{cl} \rVert_a$	0.3048	0.3075	0.3081
$\rho(E,A)$	0.0141	0.100	0.1623
F	$\begin{bmatrix} -2.257 & -0.064 \end{bmatrix}$	$\begin{bmatrix} -2.300 & -0.150 \end{bmatrix}$	$\begin{bmatrix} -2.331 & -0.212 \end{bmatrix}$

Table 2. Controller design for the system from Example 3

$\overline{\mathbf{A}}(W)$	0	0.1	0.2	0.5	1	2	4.5
$\lVert P_{cl} \rVert_a$	0.2739	0.3266	0.3430	0.3662	0.3796	0.3855	0.3866
$\rho(E,A)$	0.9999	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998

This system is causal yet instable ($\rho(E, A) = 1.3192$). Table 1 combines the results yielded by the algorithm in the case $\alpha = 100$.

Also the proposed algorithm is numerically effective for a large-scale systems.

Example 3. The system parameters are

$$E = \begin{bmatrix} 0.3 & 0.5 & 0.1 & 0 & 0.5 \\ 0.7 & 0.8 & 3.3 & 0 & 0.6 \\ 0.6 & 0.8 & 0.3 & 0 & 0.8 \\ 0.7 & 0.5 & 0.9 & 0 & 1 \\ 0.6 & 0.7 & 0.3 & 0 & 0.4 \end{bmatrix}, \quad A = \begin{bmatrix} 0.3 & 0.5001 & 0.1002 & 0.0005 & 0.5006 \\ 0.7 & 0.7941 & 3.2909 & 0.0006 & 0.6002 \\ 0.6 & 0.8 & 0.2999 & 0.0008 & 0.8004 \\ 0.7 & 0.4989 & 0.8978 & 0.001 & 1.0003 \\ 0.6 & 0.7 & 0.2998 & 0.0004 & 0.4013 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.0003 & -0.0002 \\ -0.0058 & 0.0019 \\ 0.0002 & -0.0013 \\ -0.0013 & -0.0015 \\ 0.0001 & 0.0017 \end{bmatrix}, \quad B_2 = 10^{-3} \begin{bmatrix} 0.1 & -0.125 \\ 0.2333 & 0.2 \\ 0.2 & 0.2 \\ 0.2333 & 0.125 \\ 0.2 & 0.175 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.3 \end{bmatrix}.$$

This system is noncausal ($\deg\det(zE - A) = 3$, $\operatorname{rank} E = 4$) lying on the stability boundary ($\rho(E, A) = 1.000$).

The results are presented by Table 2.

5. CONCLUSION

The paper has established the boundedness conditions for the a-anisotropic norm of a descriptor system in terms of linear matrix inequalities. These conditions are strict, and the system analysis method based on them is numerically effective. Relied on these conditions, anisotropy-based suboptimal state-space control problem for descriptor systems is solved. The performance of the proposed algorithm has been illustrated by two numerical examples. Also note that the developed method generalizes the suboptimal control design suggested in [12] for $a \to \infty$ and $D \neq 0$.

APPENDIX

Proof of Theorem 2. Assume that inequalities (12) and (13) hold. Rewrite (13) as

$$\Xi + \Upsilon^\top Y^\top \Delta + \Delta^\top Y \Upsilon < 0, \tag{A.1}$$

where $\Delta = \begin{bmatrix} I_n & 0 & 0 & I_n \end{bmatrix}$, $\Upsilon = \begin{bmatrix} -\frac{1}{2}I_n & A & B & -I_n \end{bmatrix}$, while a symmetrical matrix Ξ has the form

$$\Xi = \begin{bmatrix} 0 & 0 & 0 & \Phi \\ 0 & C^\top C - \Phi & C^\top D & 0 \\ 0 & D^\top C & D^\top D - \eta I_m & 0 \\ \Phi & 0 & 0 & 0 \end{bmatrix}.$$

By the projection lemma [10], the matrix inequality (A.1) is solvable for the matrix Y of dimensions $n \times n$ if and only if

$$M^\top \Xi M < 0 \quad \text{and} \quad N^\top \Xi N < 0.$$

Here

$$M^\top = \begin{bmatrix} 0 & I_n & 0 & 0 \\ 0 & 0 & I_m & 0 \\ -I_n & 0 & 0 & I_n \end{bmatrix} \quad \text{and} \quad N^\top = \begin{bmatrix} I_n & 0 & 0 & -\frac{1}{2}I_n \\ 0 & I_n & 0 & A^\top \\ 0 & 0 & I_m & B^\top \end{bmatrix}.$$

Moreover, the columns of the matrix N form the basis of the null space of the matrix Υ, while the columns of the matrix M the basis of the null space of the matrix Δ. Note that

$$M^\top \Xi M = \begin{bmatrix} C^\top C - \Phi & C^\top D & 0 \\ D^\top C & D^\top D - \eta I_m & 0 \\ 0 & 0 & -2\Phi \end{bmatrix} < 0, \tag{A.2}$$

$$N^\top \Xi N = \begin{bmatrix} -\Phi & \Phi A & \Phi B \\ A^\top \Phi & C^\top C & C^\top D \\ B^\top \Phi & D^\top C & D^\top D \end{bmatrix} < 0. \tag{A.3}$$

Since $\Phi = \Phi^\top > 0$, using Schur's complement lemma, inequality (A.3) can be transformed in the following way:

$$\begin{bmatrix} C^\top C & C^\top D \\ D^\top C & D^\top D \end{bmatrix} - \begin{bmatrix} A^\top \\ B^\top \end{bmatrix} \Phi(-\Phi)^{-1}\Phi \begin{bmatrix} A & B \end{bmatrix} < 0,$$

which directly gives

$$\begin{bmatrix} A^\top \Phi A - \Phi + C^\top C & A^\top \Phi B + C^\top D \\ B^\top \Phi A + D^\top C & B^\top \Phi B + D^\top D - \eta I_m \end{bmatrix} < 0.$$

Hence, the conditions of the theorem are reduced to those of Lemma 1 stated and proved in [6].

The proof of Theorem 2 is finished.

Proof of Theorem 3. Assume that inequalities (16)–(18) hold. Using the equivalent form (4) of system (15), rewrite inequality (18) as

$$Z = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & 0 \\ z_{12}^\top & -L & z_{23} & 0 & z_{25} & z_{26} \\ z_{13}^\top & z_{23}^\top & z_{33} & z_{34} & z_{35} & z_{36} \\ z_{14}^\top & 0 & z_{34}^\top & -\eta I_m & z_{45} & z_{46} \\ z_{15}^\top & z_{25}^\top & z_{35}^\top & z_{45}^\top & z_{55} & 0 \\ 0 & z_{26}^\top & z_{36}^\top & z_{46}^\top & 0 & -I_p \end{bmatrix} < 0,$$

where

$$z_{11} = -\frac{1}{2}Q - \frac{1}{2}Q^\top, \quad z_{12} = QA_{11} + RA_{21},$$

$$z_{13} = QA_{12} + RA_{22}, \quad z_{14} = QB_1 + RB_2,$$

$$z_{15} = L^\top - Q^\top - \frac{1}{2}Q, \quad z_{23} = A_{21}^\top S^\top,$$

$$z_{25} = A_{11}^\top Q^\top + A_{21}^\top R^\top, \quad z_{26} = C_1^\top + \alpha A_{21}^\top S^\top C_2^\top,$$

$$z_{33} = SA_{22} + A_{22}^\top S^\top, \quad z_{34} = SB_2,$$

$$z_{35} = A_{12}^\top Q^\top + A_{22}^\top R^\top, \quad z_{36} = C_2^\top + \alpha A_{22}^\top S^\top C_2^\top,$$

$$z_{45} = B_1^\top Q^\top + B_2^\top R^\top, \quad z_{46} = D^\top + \alpha B_2^\top S^\top C_2^\top,$$

$$z_{55} = -Q - Q^\top.$$

As $Z < 0$, it appears that $KZK^\top < 0$ for a nonsingular matrix K. Choose

$$K = \begin{bmatrix} I_r & 0 & 0 & 0 & 0 & 0 \\ 0 & I_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & 0 & 0 & I_p \\ 0 & 0 & I_{n-r} & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$KZK^\top = \begin{bmatrix} z_{11} & z_{12} & z_{14} & z_{15} & 0 & z_{13} \\ z_{12}^\top & -L & 0 & z_{25} & z_{26} & z_{23} \\ z_{14}^\top & 0 & -\eta I_m & z_{45} & z_{46} & z_{34}^\top \\ z_{15}^\top & z_{25}^\top & z_{45}^\top & z_{55} & 0 & z_{35}^\top \\ 0 & z_{26}^\top & z_{46}^\top & 0 & -I_p & z_{36}^\top \\ z_{13}^\top & z_{23}^\top & z_{34} & z_{35} & z_{36} & z_{33} \end{bmatrix} < 0.$$

Consider the equality $KZK^\top = W + W^\top$, which implies that

$$W = \begin{bmatrix} w_{11} & 0 & 0 & 0 & 0 & 0 \\ w_{21} & w_{22} & 0 & w_{24} & w_{25} & w_{26} \\ w_{31} & 0 & w_{33} & w_{34} & w_{35} & w_{36} \\ w_{41} & 0 & 0 & w_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & w_{55} & 0 \\ w_{61} & 0 & 0 & w_{64} & w_{65} & w_{66} \end{bmatrix},$$

$$w_{11} = -\frac{1}{2}Q, \quad w_{21} = w_{24} = A_{11}^\top Q^\top + A_{21}^\top R^\top,$$

$$w_{22} = -\frac{1}{2}L, \quad w_{25} = C_1^\top + \alpha A_{21}^\top S^\top C_2^\top,$$

$$w_{26} = A_{21}^\top S^\top, \quad w_{31} = w_{34} = B_1^\top Q^\top + B_2^\top R^\top,$$

$$w_{33} = -\frac{\eta}{2}I_m, \quad w_{35} = D^\top + \alpha B_2^\top S^\top C_2^\top,$$

$$w_{36} = B_2^\top S^\top, \quad w_{41} = L - Q - \frac{1}{2}Q^\top,$$

$$w_{44} = -Q, \quad w_{55} = -\frac{1}{2}I_p, \quad w_{65} = C_2^\top + \alpha A_{22}^\top S^\top C_2^\top,$$

$$w_{61} = w_{64} = A_{12}^\top Q^\top + A_{22}^\top R^\top, \quad w_{66} = A_{22}^\top S^\top.$$

Due to the accepted notation, we have the inequality

$$W + W^\top < 0. \tag{A.4}$$

Since $z_{33} = A_{22}^\top S^\top + SA_{22} < 0$ in the matrix $Z < 0$, both matrices A_{22} and S are nonsingular. System (15) is causal, and hence its input-output operator can be reduced to the equivalent input-output operator in the form of an ordinary system $\widehat{T}$ with the representation

$$\widehat{x}(k+1) = \widehat{A}\widehat{x}(k) + \widehat{B}w(k),$$
$$\widehat{y}(k) = \widehat{C}\widehat{x}(k) + \widehat{D}w(k),$$

where $x(k) \in \mathbb{R}^r$,

$$\widehat{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad \widehat{B} = B_1 - A_{12}A_{22}^{-1}B_2,$$
$$\widehat{C} = C_1 - C_2A_{22}^{-1}A_{21}, \quad \widehat{D} = D - C_2A_{22}^{-1}B_2.$$

Now, we demonstrate that $\widehat{A}$ is a Schur matrix and $|\!|\!|\widehat{T}|\!|\!|_a < \gamma$. The blocks SA_{22} and $A_{22}^\top S^\top$ represent invertible matrices with the properties $A_{22}^\top S^\top < 0$ and $SA_{22} < 0$. Application of Schur's complement lemma to (A.4) yields

$$\begin{bmatrix} -\frac{1}{2}Q - \frac{1}{2}Q^\top & Q\widehat{A} & Q\widehat{B} & L^\top - Q^\top - \frac{1}{2}Q & 0 \\ \widehat{A}^\top Q^\top & -L & 0 & \widehat{A}^\top Q^\top & \widehat{C}^\top \\ \widehat{B}^\top Q^\top & 0 & -\eta I_m & \widehat{B}^\top Q^\top & \widehat{D}^\top \\ L - Q - \frac{1}{2}Q^\top & Q\widehat{A} & Q\widehat{B} & -Q - Q^\top & 0 \\ 0 & \widehat{C} & \widehat{D} & 0 & -I_p \end{bmatrix} < 0.$$

By Theorem 2 for the ordinary systems, we have $\rho(\widehat{A}) < 1$ and $\|\widehat{T}\|_a < \gamma$.

The proof of Theorem 3 is finished.

Proof of Theorem 4. Show that the controller resolving the design problem for the transformed system does so for the original one. Really, the transfer function of the closed-loop system can be rewritten in the form

$$\begin{aligned} P_{cl}(z) &= C\overline{V}\,\overline{V}^{-1}(zE - A - B_2F_2)^{-1}\overline{W}^{-1}\overline{W}B_1 + D_1 \\ &= C\overline{V}(z\overline{W}E\overline{V} - \overline{W}A\overline{V} - \overline{W}B_2F_2\overline{V})^{-1}\overline{W}B_1 + D_1 \\ &= C_d(zE_d - A_d - B_{2d}F_d)^{-1}B_{1d} + D_{1d}, \end{aligned}$$

where $F_d = F_2\overline{V}$.

Suppose that inequalities (20)–(22) hold. Then it follows from block (1,1) that the matrix Q is invertible. In addition, let the matrix S be invertible. (Otherwise, one can choose $\epsilon \in (0, 1)$ so that inequality (20) remains in force for the variable $\bar{S} = S + \epsilon I_{n-r}$. In this case, S is obviously substituted by $\bar{S}$.)

Replacing the variable Z with $\begin{bmatrix} Q & R \\ 0 & S \end{bmatrix} F_d^\top$ in (20), we obtain the conditions of Theorem 3 for the system dual to (19). Therefore, the closed-loop system (19) appears admissible, and the a-anisotropic norm of the transfer function is bounded by γ.

If the control design problem is solvable, then the closed-loop system (19) satisfies the conditions of Theorem 3. This is also applied to the dual system. Using the linear change of variables $\begin{bmatrix} Q & R \\ 0 & S \end{bmatrix} F_d^\top = Z$, which yields $\begin{bmatrix} Q & R \end{bmatrix} F_d^\top = \begin{bmatrix} I_r & 0 \end{bmatrix} Z$ and $\begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} F_d^\top = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Z$, we obtain inequality (20).

It was mentioned earlier, that the variables Q and S are invertible. Then the feedback gain for the closed-loop system (19) takes the form

$$F_d = Z^\top \begin{bmatrix} Q^{-\top} & 0 \\ -S^{-\top}R^\top Q^{-\top} & S^{-\top} \end{bmatrix}.$$

Performing the inverse transformation of coordinates, we get the gain F_2 from (23).

The proof of Theorem 4 is finished.

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