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CONTROL IN STOCHASTIC SYSTEMS AND UNDER UNCERTAINTY CONDITIONS

Anisotropic Norm Computation for Descriptor Systems with Nonzero-Mean Input Signals[1]

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Abstract—Linear stationary discrete-time descriptor systems with input sequences of random Gaussian nonzero-mean vectors with bounded mean anisotropy are under consideration. Conditions of anisotropic norm boundedness for such systems are given in terms of generalized discrete-time algebraic Riccati equations (GDARE) and linear matrix inequalities (LMI). On basis of these results, the algorithm of anisotropic norm computation using convex optimization techniques is developed. Numerical examples illustrate methods of anisotropic norm computation.

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INTRODUCTION

Stochastic anisotropy-based robust control theory was established in Russia in 1994 [1–3]. This theory, lying in some sense between the classical H_2- and H_∞-control theories, allows to design control laws, which minimize anisotropic norm of the closed-loop system. The main concepts of anisotropy-based control theory are anisotropy of the random vector, mean anisotropy of the input sequence and anisotropic norm of the system. Anisotropy of the random vector (or its spectral color) is the measure of divergence between the probability density function (pdf) of the vector and the pdf of Gaussian random vector with zero mean and scalar covariance matrix. Mean anisotropy of the sequence is time-averaged anisotropy of the extended vector, which consists of the sequence's elements. Anisotropic norm of the system is the maximal value of the root mean-square (RMS) gain of the system in respect of all possible input signals with mean anisotropy levels less than a given scalar value.

In classical anisotropy-based analysis and design problem statements, stationary ergodic sequences of Gaussian random zero-mean vectors are considered as input disturbances [4, 5]. However, in real problems, at different faults in equipment or in presence of nontrivial input signals, mean values of disturbance vectors are different from zero. Such disturbances are called nonzero-mean. Therefore, in the framework of anisotropy-based theory it makes sense to consider the input signal, containing a nonzero deterministic component.

Mathematical models of contol systems cannot be always described only difference or differential equations. The systems, mathematical models of which are given in physical state variables, may contain algebraic equations of connection between state variables. Such systems are called descriptor (differential-algebraic, difference-algebraic, singular and so on). Because of the algebraic equations, descriptor systems can acquire properties that are not usual for normal systems, this fact entangles the use of classical mathematical methods. Methods of analysis and design, developed for descriptor systems, can be successfully applied to normal systems, which are a special case descriptor if the latter have no connection equations.

Solving the design problem for such systems is based on checking boundedness of the quality function by a given positive scalar value, it is directly connected with computation of anisotropic norm of the closed-loop system. In this paper, the problem of computing anisotropic norm for descriptor systems with nonzero-mean input signals is solved using convex optimization techniques. This research is founded on the following results for zero-mean input signals: conditions of anisotropic norm boundedness for descriptor systems in terms of GDARE (anisotropy-based bounded real lemma) [6], similar results in LMI-representation [7].

This paper consists of the following sections. In Section 1, main concepts of descriptor systems theory and anisotropy-based control theory are given. Conditions of anisotropic norm boundedness in terms of

[1] The article was translated by the authors.

GDARE ans LMI are formulated in Section 2. Using these results, the method of anisotropic norm computation for normal and descriptor systems with nonzero-mean input signals is worked out. Numerical examples are given in Section 3.

1. BASICS OF DESCRIPTOR SYSTEMS THEORY AND ANISOTROPY-BASED CONTROL THEORY

This section provides us with basic information on descriptor systems and main definitions of anisotropy-based theory, necessary for further discussion.

1.1. Descriptor Systems

A stationary discrete-time descriptor system P is given in the form

$$P \sim \begin{cases} Ex(k+1) = Ax(k) + Bw(k), \\ z(k) = Cx(k) + Dw(k), \end{cases} \tag{1.1}$$
$$x(0) = x_0,$$

where $x(k) \in \mathbb{R}^n$ is the object's state, $w(k) \in \mathbb{R}^m$ is the input signal, $z(k) \in \mathbb{R}^p$ is the controllable output, A, B, C, D, E are known matrices of appropriate dimensions and $\operatorname{rank} E \le n$. If $\operatorname{rank} E = n$, the system (1.1) can be reduced to the normal system of the same dimension. In this paper, the same symbol P is used to denote both normal and descriptor systems. The system (1.1) is equal to the transfer function

$$P \sim P(z) = D + C(zE - A)^{-1}B, \quad z \in \mathbb{C}.$$

Definition 1. Let $\mathbb{L}_2^{p\times m}(\Gamma)$, $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ be a space of matrix-valued functions $P : \Gamma \to \mathbb{C}^{p\times m}$, which have finite $\mathbb{L}_2^{p\times m}(\Gamma)$-norms:

$$\|P\|_{\mathbb{L}_2^{p\times m}(\Gamma)} = \left(\frac{1}{2\pi} \int_0^{2\pi} \operatorname{tr}(P^*(e^{i\omega})P(e^{i\omega}))d\omega \right)^{\frac{1}{2}} < \infty,$$

where $P^*(e^{i\omega}) = P^{\mathrm{T}}(e^{-i\omega})$ is a conjugate function with respect to $P(e^{i\omega})$. A subspace of $\mathbb{L}_2^{p\times m}(\Gamma)$ which consists of all rational transfer functions with no poles in the exterior of the unit disk on the complex plane is denoted by $H_2^{p\times m}$. So, the $\mathbb{L}_2^{p\times m}(\Gamma)$-norm of the transfer function $P(z)$ from the subspace $H_2^{p\times m}$ is called the H_2-norm and is designated by $\|P\|_2$.

Definition 2. Let $\mathbb{L}_\infty^{p\times m}(\Gamma)$ be a space of matrix-valued functions $P : \Gamma \to \mathbb{C}^{p\times m}$ that are essentially bounded on $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$, i.e. the space of functions that are not bounded on the set of zero measure. The subspace of $H_\infty^{p\times m}$ denoted by $\mathbb{L}_\infty^{p\times m}(\Gamma)$ consists of all the rational transfer functions that are analytic in the exterior of the closed unit disk. The H_∞-norm of the transfer function $P \in H_\infty^{p\times m}$ is given by

$$\|P\|_\infty = \sup_{\omega\in[0,2\pi]} \sigma_{\max}(P(e^{i\omega})) = \sup_{\omega\in[0,2\pi]} \sqrt{\lambda_{\max}(P^*(e^{i\omega})(P(e^{i\omega}))},$$

where $\lambda_{\max}(M)$ is the maximal eigenvalue of the square matrix M.

Definition 3. A descriptor system P is called admissible if it is regular (i.e. $\exists\lambda \ne 0$: $\det(\lambda E - A) \ne 0$), causal (i.e. $\deg\det(zE - A) = \operatorname{rank} E$) and stable (i.e. $\rho(E, A) = \max_{z\in\mathbb{C}}\{|z| : \det(zE - A) = 0\} < 1$). For more information about descriptor systems theory see [8, 9].

If the descriptor system (1.1) is admissible, and $\operatorname{rank}(E) = r < n$, then there exist two nonsingular matrices of change of variables, which transform the system to the form

$$P_{\mathrm{SVD}} \sim \begin{cases} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(k+1) \\ \tilde{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} w(k), \\ z(k) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix} + Dw(k), \end{cases} \tag{1.2}$$

called the SVD canonical form [8, 10] of the system (1.1). Here $\tilde{x}_1(k) \in \mathbb{R}^r$, $\tilde{x}_2(k) \in \mathbb{R}^{n-r}$, I_r is used to denote $r \times r$-dimensional identity matrix.

1.2. Anisotropy-Based Control Theory

In this section, we formulate the main concepts of anisotropy-based control theory. For more information, see [1, 2, 11, 12].

Let $\{w(k)\}_{k \ge 0}$ be a nonzero-mean stationary ergodic sequence of Gaussian m-dimensional random vectors, generated by a shaping filter in the following descriptor form:

$$G \sim \begin{cases} E_g x_g(k+1) = A_g x_g(k) + B_g(\mathbf{v}(k) + \mu), \\ w(k) = C_g x_g(k) + D_g(\mathbf{v}(k) + \mu), \end{cases} \tag{1.3}$$

where the sequence $\{\mathbf{v}(k)\}_{k \ge 0}$ is a Gaussian m-dimensional white noise, $\mu \in \mathbb{R}^m$ is a constant vector, A_g, B_g, C_g, D_g, E_g are known matrices. $W_{0:N-1} = (w^{\mathrm{T}}(0) \ldots w^{\mathrm{T}}(N-1))^{\mathrm{T}} \in \mathbb{R}^{mN}$ is an extended random vector, which consists of the first N elements of the sequence $\{w(k)\}_{k \ge 0}$.

D e f i n i t i o n 4. Anisotropy of the random vector $W_{0:N-1}$ is defined by

$$\mathrm{A}(W_{0:N-1}) = \frac{m}{2} \ln\left(\frac{2\pi e}{m} \mathrm{E}[|W_{0:N-1}|^2]\right) + \int_{\mathbb{R}^{mN}} f(x) \ln f(x) dx,$$

where $f(x)$ is a probability density function of the vector $W_{0:N-1}$, $\mathrm{E}[\cdot]$ is a mean operator.

D e f i n i t i o n 5. Mean anisotropy of the sequence $\{w(k)\}_{k \ge 0}$ is

$$\bar{\mathrm{A}}(W) = \lim_{N \to \infty} \frac{\mathrm{A}(W_{0:N-1})}{N}.$$

T h e o r e m 1 [10]. Mean anisotropy of the input disturbance $\{w(k)\}_{k \ge 0}$, generated by the shaping filter in the form (1.3), is given by

$$\bar{\mathrm{A}}(W) = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \det\left(\frac{mS(\omega)}{\|G\|_2^2 + |\mathcal{M}|^2}\right) d\omega,$$

where $\mathcal{M}$ is a mean value of the vector $w(k)$ for $k \to \infty$, $S(\omega) = G^*(e^{i\omega})G(e^{i\omega})$ is a spectral density of the shaping filter.

R e m a r k 1. The vector of mean value $\mathcal{M}$ for the steady-state condition is given by the expression $\mathcal{M} = (D_g + C_g(E_g - A_g)^{-1} B_g)\mu$ and is connected with matrices of the SVD canonical form (1.2) for the shaping filter (1.3) by the formula $\mathcal{M} = (\hat{D} + \hat{C}(I_n - \hat{A})^{-1} \hat{B})\mu$ where $\hat{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $\hat{B} = B_1 - A_{12}A_{22}^{-1}B_2$, $\hat{C} = C_1 - C_2 A_{22}^{-1} A_{21}$, $\hat{D} = D_g - C_2 A_{22}^{-1} B_2$.

Let the input of the system (1.1) be a sequence $\{w(k)\}_{k \ge 0}$ of the form (1.3) with a bounded mean anisotropy $\bar{\mathrm{A}}(W) \le a$.

D e f i n i t i o n 6. Anisotropic norm of the system P is given by

$$|||P|||_a = \sup_{W : \bar{\mathrm{A}}(W) \le a} Q(P, W) = \sup_{W : \bar{\mathrm{A}}(W) \le a} \sqrt{\frac{\|PG\|_2^2 + |\mathcal{P}\mathcal{M}|^2}{\|G\|_2^2 + |\mathcal{M}|^2}},$$

where $\mathcal{P}\mathcal{M} = \left(D + C(E - A)^{-1}B\right)\mathcal{M}$ is a mean value of the output $z(k)$ for $k \to \infty$.

Theorem 2 [10]. Anisotropic norm of the descriptor system (1.1) with nonzero-mean input signal may be computed as

$$|||P|||_a = \sup_{q \in [0, \|P\|_\infty^{-2})} \{\mathcal{N}(q) \mid \mathcal{A}(q) \le a\},$$

where

$$\overline{\mathrm{A}}(W) = \mathcal{A}(q) = \frac{m}{2}\left(\ln\left(\Phi(q) + \frac{1}{m}|\mathcal{M}|^2\right) - \Psi(q)\right),$$

$$Q(P, W) = \mathcal{N}(q) = \sqrt{\frac{\Phi(q) - 1 + \frac{q}{m}|\mathcal{P}\mathcal{M}|^2}{q\Phi(q) + \frac{q}{m}|\mathcal{M}|^2}},$$

the functions $\Phi(q)$ and $\Psi(q)$ are defined by expressions

$$\Phi(q) = \frac{1}{2\pi m}\int_{-\pi}^{\pi} \mathrm{tr}(I_m - q\Lambda(\omega))^{-1} d\omega,$$

$$\Psi(q) = \frac{1}{2\pi m}\int_{-\pi}^{\pi} \ln\det(I_m - q\Lambda(\omega))^{-1} d\omega,$$

where $\Lambda(\omega) = \hat{P}^*(\omega)\hat{P}(\omega)$, $\hat{P}(\omega) = \lim_{l \to 1-0} P(le^{i\omega})$ is the value of the transfer function $P(z)$ when $|z|$ goes to the boundary of the unit circle.

This theorem gives us an opportunity to find the supremum of the scalar function $\mathcal{N}(q)$ for $\mathcal{A}(q) \le a$ in order to compute anisotropic norm of the system.

Remark 2. In general case, the functions $\mathcal{A}(q)$ and $\mathcal{N}(q)$ are not monotonic [13], but for the condition $\|G\|_2^2 + |\mathcal{M}|^2 = 1$ they get monotony and become

$$\overline{\mathbf{A}}(W) = \mathcal{A}(q) = \frac{m}{2}\left(\ln\left(\frac{\Phi(q)}{1 - |\mathcal{M}|^2}\right) - \Psi(q)\right), \tag{1.4}$$

$$Q(P, W) = \mathcal{N}(q) = \sqrt{\frac{\Phi(q) - 1}{q\Phi(q)}(1 - |\mathcal{M}|^2) + |\mathcal{P}\mathcal{M}|^2}. \tag{1.5}$$

Moreover, the constraint $\mathcal{A}(q) \le a$ gives a convex set.

1.3. Anisotropy-Based Bounded Real Lemma for Descriptor Systems

Now we formulate the conditions of anisotropic norm boundedness for descriptor systems with zero-mean input signal. These conditions are obtained in [6].

Let the system P be admissible, the following rank condition is true for the system P [7]:

$$\mathrm{rank}([E \;\; B]) = \mathrm{rank}(E).$$

The input disturbance $\{w(k)\}_{k \ge 0}$ is a stationary ergodic zero-mean ($\mu = 0$) sequence of Gaussian random vectors with bounded mean anisotropy $\overline{\mathbf{A}}(W) \le a$. It is generated by (1.3). The following theorem gives us sufficient conditions of anisotropic norm boundedness by a given scalar value $\gamma > 0$ for descriptor systems (1.1).

Theorem 3 [6]. Let $P \in H_\infty^{p\times m}$ be an admissible descriptor system with a state-space representation (1.1), let the input signal be a nonzero-mean sequence (1.3) where $\mu = 0$. For the given scalar values $a \ge 0$ and $\gamma > 0$ anisotropic norm of the system is bounded by the value γ, i.e.

$$|||P|||_a \le \gamma,$$

if there exists a pair $(q, \hat{R})$, which consists of the stabilizing solution $\hat{R} = \hat{R}^{\mathrm{T}}$ of the GDARE

$$E^{\mathrm{T}}\hat{R}E = A^{\mathrm{T}}\hat{R}A + qC^{\mathrm{T}}C + L^{\mathrm{T}}\Sigma^{-1}L,$$

$$L = \Sigma(B^{\mathrm{T}}\hat{R}A + qD^{\mathrm{T}}C),$$

$$\Sigma = (I_m - B^{\mathrm{T}}\hat{R}B - qD^{\mathrm{T}}D)^{-1},$$

where $E^{\mathrm{T}}\hat{R}E \ge 0$, and the value $q \in [0, \min(\gamma^{-2}, \| P \|_\infty^{-2}))$, which satisfies the inequality

$$-\frac{1}{2}\ln\det((1 - q\gamma^2)\Sigma) \ge a.$$

For normal systems there exists a similar theorem ($E = I_n$) [14].

2. PROBLEM STATEMENT AND MAIN RESULTS

In this section, two main problems are stated and solved: to find the conditions of anisotropic norm boundedness in the form $|||P|||_a \le \gamma$ for the desciptor system (1.1) with nonzero-mean input signal, generated by the shaping filter (1.3); using the obtained results to develop a method of anisotropic norm computation.

In order to get the conditions of anisotropic norm boundeness for nonzero-mean input signals we give some preliminary results.

Definition 7. The system with a transfer function $P(z)$, which satisfies the condition $\hat{P}^*(\omega)\hat{P}(\omega) = I_m$, is called the all-pass system (the inner system or the inner).

Lemma 1 [12]. The system (1.1) is the all-pass system if there exists a matrix $\hat{R} = \hat{R}^{\mathrm{T}}$, which satisfies the condition $E^{\mathrm{T}}\hat{R}E \ge 0$ and the equations

$$B^{\mathrm{T}}\hat{R}B + D^{\mathrm{T}}D = I,$$
$$B^{\mathrm{T}}\hat{R}A + D^{\mathrm{T}}C = 0,$$
$$A^{\mathrm{T}}\hat{R}A + C^{\mathrm{T}}C - E^{\mathrm{T}}\hat{R}E = 0.$$

Now we give conditions of anisotropic norm boundedness for descriptor systems with nonzero-mean input signals.

Theorem 4. Let $P \in H_\infty^{p\times m}$ be an admissible descriptor system (1.1) with an input signal $\{w(k)\}_{k\ge 0}$ (1.3), that has bounded mean anisotropy level $\overline{\mathbf{A}}(W) \le a$. The values $|\mathcal{M}|$ and H_2 are known, i.e. the condition $\|G\|_2^2 + |\mathcal{M}|^2 = 1$ is supposed to be satisfied. Anisotropic norm of the system is bounded by the value $\gamma > 0$, i.e. $|||P|||_a \le \gamma$, if and only if there exists such value $q \in [\max\{0, q_1\}, \min\{q_2, \|P\|_\infty^{-2}\})$

$$q_1 = \frac{1 - |\mathcal{M}|^2 - e^{-\frac{2a}{m}}}{\gamma^2 - |\mathcal{P}\mathcal{M}|^2}, \quad q_2 = \frac{1 - |\mathcal{M}|^2}{\gamma^2 - |\mathcal{P}\mathcal{M}|^2},$$

that the inequality

$$\det(\mathbb{S})^{\frac{1}{m}} \ge (1 - |\mathcal{M}|^2 + q|\mathcal{P}\mathcal{M}|^2 - q\gamma^2)^{-1}e^{-\frac{2a}{m}} \tag{2.1}$$

is satisfied for the matrix $\mathbb{S} = (I_m - B^{\mathrm{T}}RB - qD^{\mathrm{T}}D)^{-1}$, which coincides with the stabilizing solution $R = R^{\mathrm{T}}$ of the GDARE

$$E^{\mathrm{T}}RE = A^{\mathrm{T}}RA + qC^{\mathrm{T}}C + L^{\mathrm{T}}\mathbb{S}^{-1}L,$$
$$L = \mathbb{S}(B^{\mathrm{T}}RA + qD^{\mathrm{T}}C).$$

Moreover, $E^{\mathrm{T}}RE \geq 0$.

P r o o f. If the values $|\mathcal{M}|$ and $\|G\|_2$ are known, the expressions for mean anisotropy and root mean-square gain may be rewritten in the forms (1.4) and (1.5) where q is a scalar parameter, which defines the spectral density of the shaping filter

$$S_q(\omega) = \sigma(I_m - q\Lambda(\omega))^{-1}, \tag{2.2}$$

for $\sigma = \dfrac{1-|\mathcal{M}|^2}{m\Phi(q)}$ [15].

As the functions $\mathcal{N}(q)$ are $\mathcal{A}(q)$ monotone increasing, the inequality $|\!|\!|P|\!|\!|_a \leq \gamma$ is equivalent to the condition $\mathcal{A}(\mathcal{N}^{-1}(\gamma)) \geq a$. $\Phi(q)$ can be found from the formula (1.5) in the following way:

$$\Phi(q) = \frac{1-|\mathcal{M}|^2}{1-|\mathcal{M}|^2 + q|\mathcal{P}\mathcal{M}|^2 - q\mathcal{N}^2(q)}.$$

Taking in account the expression (2.2) for the spectral density $S_q(\omega)$, put $\Phi(q)$ into (1.4):

$$\mathcal{A}(q) = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det\left(I_m - q\Lambda(\omega)\right)^{-1}d\omega - \frac{m}{2}\ln(1-|\mathcal{M}|^2 + q|\mathcal{P}\mathcal{M}|^2 - q\mathcal{N}^2(q)) = \mathfrak{A}(q,\mathcal{N}(q)).$$

So, the inequality $\mathcal{A}(\mathcal{N}^{-1}(\gamma)) \geq a$ for some value of $\mathfrak{A}(q,\gamma) \geq a$ is equivalent to $q \in [0,\|P\|_\infty^{-2})$. More precisely [6],

$$\mathcal{A}(\mathcal{N}^{-1}(\gamma)) \geq a \Leftrightarrow \mathfrak{A}(\mathcal{N}^{-1}(\gamma),\gamma) \geq a \Leftrightarrow \sup_{q\in[0,\|P\|_\infty^{-2})}\mathfrak{A}(q,\gamma) \geq a$$
$$\Leftrightarrow \quad \exists\, q \in [0,\|P\|_\infty^{-2}): \quad \mathfrak{A}(q,\gamma) \geq a.$$

In other words, the existence of q, for which the inequality $\mathfrak{A}(q,\gamma) \geq a$ is satisfied, guarantees $|\!|\!|P|\!|\!|_a \leq \gamma$.

We formulate this result in terms of Riccati equations. The spectral density of the shaping filter G may be written in the form (2.2) if and only if the system

$$\Theta \sim \begin{Bmatrix} \sqrt{q}P \\ \sqrt{\sigma}G^{-1} \end{Bmatrix}$$

is the all-pass system, i.e. $\hat{\Theta}^*(\omega)\hat{\Theta}(\omega) = I$, $-\pi \leq \omega < \pi$. Let G be represented as

$$G \sim \begin{cases} Ex_g(k+1) = (A+BL)x_g(k) + B\Sigma^{1/2}(\boldsymbol{v}_k + \mu), \\ w(k) = Lx_g(k) + \Sigma^{1/2}(\boldsymbol{v}_k + \mu) \end{cases}$$

or

$$G \sim \left[E, \begin{array}{c|c} A+BL & B\Sigma^{1/2} \\ \hline L & \Sigma^{1/2} \end{array}\right].$$

The inverse shaping filter G^{-1} is given by

$$G^{-1} \sim \left[E, \begin{array}{c|c} A & B \\ \hline -\Sigma^{-1/2}L & \Sigma^{-1/2} \end{array}\right].$$

The representation of the closed-loop system Θ in the state space is

$$\Theta \sim \left[E, \begin{array}{c|c} A & B \\ \hline \sqrt{q}C & \sqrt{q}D \\ -\sqrt{\sigma}\Sigma^{-1/2}L & \sqrt{\sigma}\Sigma^{-1/2} \end{array} \right].$$

We introduce a new matrix-valued variable $\mathbb{S} = \sigma^{-1}\Sigma$ and represent Θ as

$$\Theta \sim \left[E, \begin{array}{c|c} A & B \\ \hline \sqrt{q}C & \sqrt{q}D \\ -\mathbb{S}^{-1/2}L & \mathbb{S}^{-1/2} \end{array} \right].$$

According to the formula of Kolmogorov-Szegö type [15], the following formula takes place

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det S_q(\omega) d\omega = \ln\det\Sigma,$$

so, the expression

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det\left(I_m - q\Lambda(\omega)\right)^{-1} d\omega = \ln\det\mathbb{S}$$

is true.

Applying the conditions of lemma 1 to the all-pass system Θ, we get the expressions which coincide with the equations from the theorem. The condition $\mathfrak{A}(q,\gamma) \geq a$ becomes

$$-\frac{1}{2}\ln\det((1 - |\mathcal{M}|^2 + q|\mathcal{P}\mathcal{M}|^2 - q\gamma^2)\mathbb{S}) \geq a. \tag{2.3}$$

So, $\det(\mathbb{S})^{\frac{1}{m}} \geq (1 - |\mathcal{M}|^2 + q|\mathcal{P}\mathcal{M}|^2 - q\gamma^2)^{-1} e^{-\frac{2a}{m}}$. The theorem is proved.

Remark 3. Now we explain why the variable q belongs to the interval $[\max\{0, q_1\}, \min\{q_2, \|P\|_\infty^{-2}\})$ where

$$q_1 = \frac{1 - |\mathcal{M}|^2 - e^{-\frac{2a}{m}}}{\gamma^2 - |\mathcal{P}\mathcal{M}|^2}, \quad q_2 = \frac{1 - |\mathcal{M}|^2}{\gamma^2 - |\mathcal{P}\mathcal{M}|^2}.$$

Firstly, $q \in [0, \|P\|_\infty^{-2})$, we have to explain the origin of the points q_1 and q_2. As

$$-\frac{1}{2}\ln\det((1 - |\mathcal{M}|^2 + q|\mathcal{P}\mathcal{M}|^2 - q\gamma^2)\mathbb{S}) \geq a \geq 0$$

and $\mathbb{S} \geq 0$, we get

$$1 - |\mathcal{M}|^2 + q|\mathcal{P}\mathcal{M}|^2 - q\gamma^2 \geq 0,$$

it follows that $q \leq q_2 = (1 - |\mathcal{M}|^2)(\gamma^2 - |\mathcal{P}\mathcal{M}|^2)^{-1}$. Secondly, as $0 \leq B^{\mathrm{T}}RB + qD^{\mathrm{T}}D \leq I_m$, i.e. $\ln\det\mathbb{S} \geq 0$, so the inequality

$$-\frac{1}{2}\ln\det((1 - |\mathcal{M}|^2 + q|\mathcal{P}\mathcal{M}|^2 - q\gamma^2)\mathbb{S}) \geq a$$

leads to

$$m\ln(1 - |\mathcal{M}|^2 + q|\mathcal{P}\mathcal{M}|^2 - q\gamma^2) \leq -2a.$$

Consequently, the parameter q also satisfies the relation

$$q \geq q_1 = \frac{1 - |\mathcal{M}|^2 - e^{-\frac{2a}{m}}}{\gamma^2 - |\mathcal{P}\mathcal{M}|^2}.$$

In order to get anisotropic norm computation formula for nonzero-mean input signals using convex optimization techniques, we have to provide some subsidiary statements.

Lemma 2 [14]. Let P a normal system, given by (1.1) where $E = I_n$, let the values of $\gamma > 0$ and $a \geq 0$ be fixed. Suppose that there exists a pair (q, R), which consists of the matrix $R = R^{\mathrm{T}} > 0$ and the scalar value $q \in [0, \|P\|_\infty^{-2})$, for which the following expressions hold true

$$R > A^{\mathrm{T}}RA + qC^{\mathrm{T}}C + (A^{\mathrm{T}}RB + qC^{\mathrm{T}}D)(I_m - B^{\mathrm{T}}RB - qD^{\mathrm{T}}D)^{-1}(A^{\mathrm{T}}RB + qC^{\mathrm{T}}D)^{\mathrm{T}},$$

$$I_m - B^{\mathrm{T}}RB - qD^{\mathrm{T}}D > 0,$$

$$\ln\det(I_m - B^{\mathrm{T}}RB - qD^{\mathrm{T}}D) \geq m\ln(1 - q\gamma^2) + 2a. \tag{2.4}$$

Then there exists a solution $\hat{R} = \hat{R}^T \geq 0$ of the algebraic Riccati equation

$$\hat{R} = A^{\mathrm{T}}\hat{R}A + qC^{\mathrm{T}}C + (A^{\mathrm{T}}\hat{R}B + qC^{\mathrm{T}}D)(I_m - B^{\mathrm{T}}\hat{R}B - qD^{\mathrm{T}}D)^{-1}(A^{\mathrm{T}}\hat{R}B + qC^{\mathrm{T}}D)^{\mathrm{T}},$$

besides

$$I_m - B^{\mathrm{T}}\hat{R}B - qD^{\mathrm{T}}D > 0,$$

$$\ln\det(I_m - B^{\mathrm{T}}\hat{R}B - qD^{\mathrm{T}}D) \geq m\ln(1 - q\gamma^2) + 2a,$$

and $R > \hat{R}$.

The existence of the solution R of inequalities from the lemma guarantees the existence of the solution $\hat{R}$ of Riccati equations. The conditions of anisotropic norm boundedness for discrete-time descriptor systems with zero-mean input signals are formulated in terms of GDARE in [6] and in terms of LMI in [7].

Lemma 3 [16]. Let $X = X^{\mathrm{T}} \geq 0$ be a real $m \times m$-matrix, p be a scalar which belongs to $0 \leq p \leq 1/m$. So, the function $-(\det X)^p$ is convex, and the set $\{(X, t) | X = X^{\mathrm{T}} \geq 0, t \leq (\det X)^p\}$ may be represented as

$$\left\{(X, t) \mid X = X^{\mathrm{T}} \geq 0, \begin{bmatrix} X & \Delta \\ \Delta^{\mathrm{T}} & \operatorname{diag}\Delta \end{bmatrix} \geq 0, t \leq (\det \operatorname{diag}\Delta)^p\right\},$$

where Δ is a subsidiary lower triangular matrix, $\operatorname{diag}\Delta$ is a diagonal matrix, which consists of diagonal elements of the matrix Δ.

This lemma allows to transform the expression $t \leq (\det X)^p$ to the expressions

$$\begin{bmatrix} X & \Delta \\ \Delta^{\mathrm{T}} & \operatorname{diag}\Delta \end{bmatrix} \geq 0 \quad \text{and} \quad t \leq (\det \operatorname{diag}\Delta)^p,$$

which gives a convex set of constraints.

Using the given results to the problem of checking anisotropic norm boundedness for nonzero-mean input signals, we get the following theorem.

Theorem 5. Let $P \in H_\infty^{p\times m}$ be an admissible descriptor or a stable normal system ($E = I_n$) with a state-space representation (1.1). The input signal $\{w(k)\}_{k\geq 0}$ (1.3) is a nonzero-mean sequence with a bounded mean anisotropy level $\overline{\mathbf{A}}(W) \leq a$, moreover, the condition $\|G\|_2^2 + |\mathcal{M}|^2 = 1$ is saisfied. Then anisotropic norm of the system is bounded by $\gamma > 0$, i.e. $|||P|||_a < \gamma$, if there exists a pair (η, Φ), which consists of the matrix $\Phi = \Phi^{\mathrm{T}}$, that satisfies the following LMI:

$$\begin{bmatrix} A^{\mathrm{T}}\Phi A - E^{\mathrm{T}}\Phi E + C^{\mathrm{T}}C & A^{\mathrm{T}}\Phi B + C^{\mathrm{T}}D \\ B^{\mathrm{T}}\Phi A + D^{\mathrm{T}}C & B^{\mathrm{T}}\Phi B + D^{\mathrm{T}}D - \eta I_m \end{bmatrix} < 0$$

and the scalar parameter

$$\eta \in \left(\max\left\{ \frac{\gamma^2 - |\mathcal{P}\mathcal{M}|^2}{1 - |\mathcal{M}|^2}, \|P\|_\infty^2 \right\}, \frac{\gamma^2 - |\mathcal{P}\mathcal{M}|^2}{1 - |\mathcal{M}|^2 - e^{-\frac{2a}{m}}} \right), \tag{2.5}$$

for which the inequality

$$\eta(1 - |\mathcal{M}|^2) + |\mathcal{P}\mathcal{M}|^2 - e^{-\frac{2a}{m}} \det(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}} D)^{\frac{1}{m}} < \gamma^2 \tag{2.6}$$

is true.

If the system P is normal ($E = I_n$), the matrix Φ should be positive defined, if it is descriptor, the inequality $E^{\mathrm{T}}\Phi E \geq 0$ should be satisfied.

P r o o f. The conditions of lemma 2 lead to the following statement: the inequality $|||P|||_a < \gamma$ holds true for the system in descriptor form if there exists a pair (q, R), which satisfies the system of matrix inequalities

$$E^{\mathrm{T}} RE > A^{\mathrm{T}} RA + qC^{\mathrm{T}} C + (A^{\mathrm{T}} RB + qC^{\mathrm{T}} D)(I_m - B^{\mathrm{T}} RB - qD^{\mathrm{T}} D)^{-1} (A^{\mathrm{T}} RB + qC^{\mathrm{T}} D)^{\mathrm{T}}, \tag{2.7}$$

$$I_m - B^{\mathrm{T}} RB - qD^{\mathrm{T}} D > 0, \tag{2.8}$$

$$\ln\det(I_m - B^{\mathrm{T}} RB - qD^{\mathrm{T}} D) > m\ln(1 - |\mathcal{M}|^2 + q|\mathcal{P}\mathcal{M}|^2 - q\gamma^2) + 2a. \tag{2.9}$$

The difference between the right-hand side of (2.9) and (2.4) is the presence of the expression $q|\mathcal{P}\mathcal{M}|^2 - |\mathcal{M}|^2$. It is connected with existence of nonzero deterministic constituent in the input disturbance $\{w(k)\}_{k \geq 0}$. Taking into account the formula (2.1) from theorem 4 makes it not difficult to get the appearance of this expression in the inequality (2.4).

As the matrix $\mathbb{S} = (I_m - B^{\mathrm{T}} RB - qD^{\mathrm{T}} D)^{-1}$ is positive defined, the following representation of the inequality (2.7) is true:

$$\begin{bmatrix} A^{\mathrm{T}} RA - E^{\mathrm{T}} RE + qC^{\mathrm{T}} C & A^{\mathrm{T}} RB + qC^{\mathrm{T}} D \\ B^{\mathrm{T}} RA + qD^{\mathrm{T}} C & B^{\mathrm{T}} RB + qD^{\mathrm{T}} D - I_m \end{bmatrix} < 0.$$

Multiplying this matrix inequality on $\eta = q^{-1}$ and denoting $\Phi = \eta R$, we get the following LMI:

$$\begin{bmatrix} A^{\mathrm{T}}\Phi A - E^{\mathrm{T}}\Phi E + C^{\mathrm{T}} C & A^{\mathrm{T}}\Phi B + C^{\mathrm{T}} D \\ B^{\mathrm{T}}\Phi A + D^{\mathrm{T}} C & B^{\mathrm{T}}\Phi B + D^{\mathrm{T}} D - \eta I_m \end{bmatrix} < 0.$$

It is equal to the LMI from the theorem.

Transform the inequality (2.3):

$$-\frac{1}{2}\ln\det((1 - |\mathcal{M}|^2 + q|\mathcal{P}\mathcal{M}|^2 - q\gamma^2)\mathbb{S}) > a \Leftrightarrow \ln\det(\mathbb{S}) < m\ln\left((1 - |\mathcal{M}|^2 + q|\mathcal{P}\mathcal{M}|^2 - q\gamma^2)e^{-\frac{2a}{m}} \right)$$

$$\Leftrightarrow \ln\det(I_m - B^{\mathrm{T}} RB - qD^{\mathrm{T}} D) > m\ln\left((1 - |\mathcal{M}|^2 + q|\mathcal{P}\mathcal{M}|^2 - q\gamma^2)e^{-\frac{2a}{m}} \right)$$

$$\Leftrightarrow \det(I_m - B^{\mathrm{T}} RB - qD^{\mathrm{T}} D) > m(1 - |\mathcal{M}|^2 + q|\mathcal{P}\mathcal{M}|^2 - q\gamma^2)e^{-\frac{2a}{m}}.$$

Multiplying the obtained inequality on $\eta = q^{-1}$ and taking into account the denotation $\Phi = \eta R$, we get

$$\eta(1 - |\mathcal{M}|^2) + |\mathcal{P}\mathcal{M}|^2 - e^{-\frac{2a}{m}} \det(\eta I_m - B^{\mathrm{T}}\Phi B - D^{\mathrm{T}} D)^{\frac{1}{m}} < \gamma^2.$$

The theorem is proved.

R e m a r k 4. In order to get the interval, which η belongs to, it is sufficient to take into account that $\eta = q^{-1}$ and to transform the interval from remark 3.

The constraints from theorem 5, which the pair (η, Φ) should satisfy, are represented as LMI-conditions. So, we can compute anisotropic norm of the linear system with nonsero-mean input signal, using convex optimization techniques. Anisotropic norm of the system P can be approximately computed as $|||P|||_a \approx \sqrt{\gamma_*}$ where γ_* is a solution of the following convex optimization problem:

$$\gamma_* = \underset{(\eta,\Phi,\hat{\gamma}):\,(2.5)-(2.9)}{\arg\inf \hat{\gamma}}.$$

Here $\hat{\gamma} = \gamma^2$. The obtained method of anisotropic norm computation is much simplier in comparison with homotopy method [14].

3. NUMERICAL EXAMPLE

Consider a normal system $P\,(E = I_n)$ with a state-space representation (1.1), given by matrices

$$A = \begin{bmatrix} 0.23596 & -0.85560 & -0.68160 \\ -0.77840 & 0.00756 & -0.26010 \\ 1.09960 & -0.93760 & -0.22880 \end{bmatrix}, \quad B = \begin{bmatrix} -0.52480 & 1.85510 \\ 1.12830 & -0.27730 \\ 0.55014 & 1.06660 \end{bmatrix},$$

$$C = \begin{bmatrix} -2.09920 & 0.37147 & 0.69535 \\ 0.63848 & -0.37140 & 0.87763 \end{bmatrix}, \quad D = \begin{bmatrix} 1.03360 & 0.60107 \\ 0.41979 & -0.67400 \end{bmatrix}.$$

We suppose that the input signal is represented by the sequence of random Gaussian vectors (1.3) with a known mean value $\mathcal{M}$, and H_2-norm of the shaping filter is $\|G\|_2 = \sqrt{0.75}$. For $a = 0.375$ we find anisotropic norm of the system $|||P|||_a$ in the following cases: only $|\mathcal{M}| = 0.5$ is known; both $|\mathcal{M}| = 0.5$ and $\|G\|_2 = \sqrt{0.75}$ are known.

In the first case, the functions $A(q)$ and $N(q)$ are not monotonic [13], so we cannot compute anisotropic norm of the system, using convex optimization techniques. We can only approximately compute the values of $A(q)$ and $N(q)$ for the points from the interval $q \in [0, \|P\|_\infty^{-2})$ with a fixed step d. Then we choose such pairs $(A(q), N(q))$, that the function $A(q)$ satisfies the condition $|A(q) - a| \le d/2$, and find the maximal value of $N(q_*)$, $q_* = \arg\sup\{\mathcal{N}(q) : |A(q) - a| \le d/2\}$. The accuracy of anisotropic norm computation is

$$\varepsilon \sim \left|\frac{N(q_* + d) - N(q_* - d)}{2}\right| \sim \left|\frac{\partial N(q)}{\partial A(q)}\right|_{q_*} c_d, \quad \text{where} \quad c_d \sim \left|\frac{\partial A(q)}{\partial q}\right|_{q_*} d.$$

The relation between $N(q)$ and $A(q)$ is represented on the Fig. 1a. As we can see, to find anisotropic norm for $\overline{\mathbf{A}}(W) \le 0.375$ we have to choose two pairs (0.375, 5.789) and (0.375, 6.194) from the set $(A(q), N(q))$, that are too close to the inflection point where $\left|\frac{\partial N(q)}{\partial A(q)}\right| \gg 1$. If the step d is too large, this condition may lead to incorrect results.

In the second case, when both $|\mathcal{M}| = 0.5$ and $\|G\|_2 = \sqrt{0.75}$ are known, the functions $A(q)$ and $N(q)$ are monotonic (see Fig. 1b). So, we can use convex optimization techniques to compute anisotropic norm of the system. The advantages of this method are fast work and high accuracy. The methods are compared in Table 1.

The strings for the case A represent the results of computation with the same work time for both algorithms (M1 stands for "step" computation, M2 is convex optimization method). The case B is given for the same accuracy of algorithms (10^{-3}). We can see, that the method M1 needs more time for computation than M2, or gives bad accuracy for the same work time. For additional comparison of algorithms the results of anisotropic norm computation for $\overline{\mathbf{A}}(W) \le 0.5$ (in this case, the supremum $N(q)$ is far from the inflection point) are given. No principle difference is found.

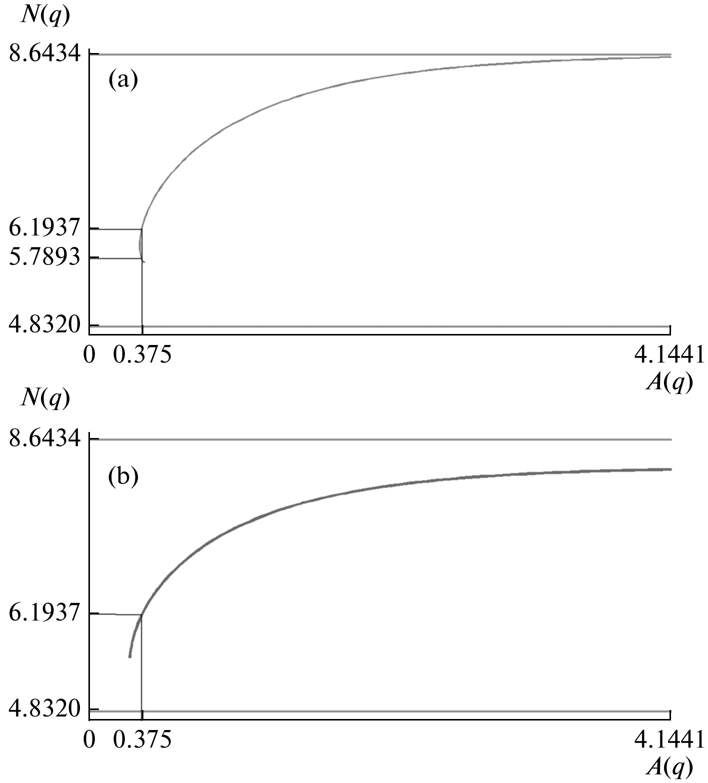

Fig. 1. The set of pairs $(A(q), N(q))$ for the normal system in cases: (a) the value $\|G\|_2$ is not known, (b) $\|\|G\|\|_2$ is known.

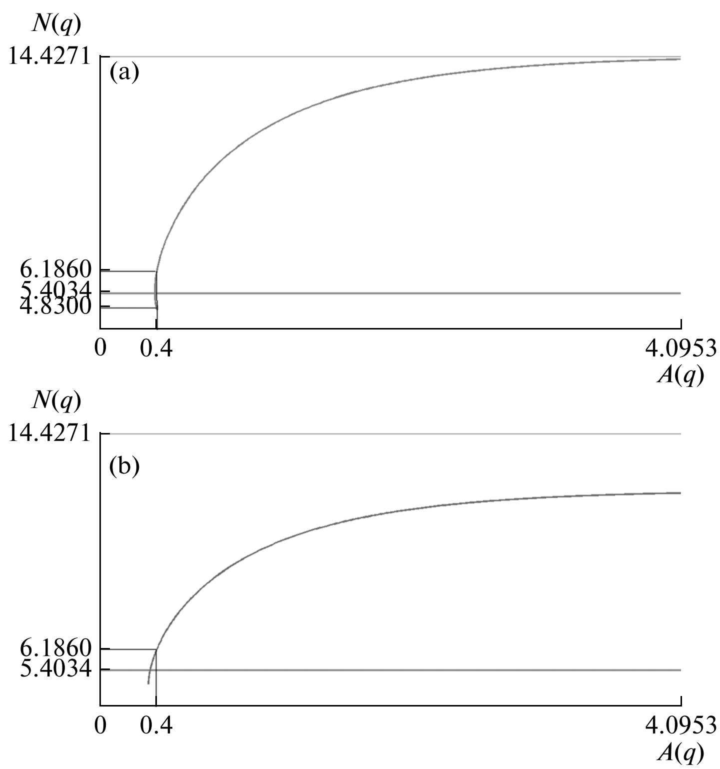

Fig. 2. The set of pairs $(A(q), N(q))$ for the descriptor system in cases: (a) the value is not known, (b) $\|\|G\|\|_2$ is known.

Table 1. Comparison of algorithms for the normal system

Case	Method	Working time, seconds	a	$\mathcal{A}(q_*)$	$\mathcal{N}(q_*)$	Accuracy ε
A	M1	4.1	0.375	0.3735	6.1807	0.013
			0.5	0.5171	6.7763	0.029
	M2	3.9	0.375	0.375	6.1858	0.007
			0.5	0.5	6.7591	0.012
Б	M1	330	0.375	0.3747	6.1913	10^{-3}
			0.5	0.4989	6.7482	10^{-3}
	M2	4.5	0.375	0.375	6.1929	10^{-4}
			0.5	0.5	6.7473	10^{-4}

Table 2. Comparison of algorithms for the descriptor system

Case	Method	Working time, seconds	a	$\mathcal{A}(q_*)$	$\mathcal{N}(q_*)$	Accuracy ε
A	M1	3.9	0.4	0.3969	6.0927	0.095
			0.7	0.6837	9.4377	0.202
	M2	3.8	0.4	0.4	6.1276	0.062
			0.7	0.7	9.5083	0.127
Б	M1	410	0.4	0.3997	6.1827	10^{-3}
			0.7	0.7017	9.6277	10^{-3}
	M2	4.2	0.4	0.4	6.1895	10^{-4}
			0.7	0.7	9.6350	10^{-4}

Now we check how the obtained algorithms work for a descriptor system (1.1) with the following matrices:

$$E = \begin{bmatrix} 3 & 0 & 2 & -5 \\ 0 & 3 & -2 & 2 \\ 2 & 2 & 0 & -2 \\ 2 & -4 & 4 & -6 \end{bmatrix}, \quad A = \begin{bmatrix} 0.70 & -3.25 & -0.70 & 0.00 \\ 1.80 & 0.40 & -6.40 & 2.60 \\ 1.00 & -1.90 & -5.40 & 2.40 \\ -0.60 & -2.70 & 5.40 & -2.80 \end{bmatrix}, \quad B = \begin{bmatrix} 3.2 & -3.5 \\ 2.5 & -7.9 \\ 3.8 & -7.6 \\ -1.2 & 8.2 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.20 & 0.40 & 0.45 & 0.60 \\ -0.20 & 0.10 & -0.20 & -1.00 \end{bmatrix}, \quad D = \begin{bmatrix} 0.2 & 1.0 \\ -1.0 & 0.3 \end{bmatrix}.$$

Suppose, that $|\mathcal{M}| = 0.304$ and $\|G\|_2 = \sqrt{0.907}$.

In the first case, the functions $A(q)$ and $N(q)$ are not monotonic, to compute anisotropic norm of the system, we have to find the values of these functions in the points, chosen with a set step. The relationship between $N(q)$ and $A(q)$ for the descriptor system is given on Fig. 2a. For the constraint $\overline{\mathbf{A}}(W) \le 0.4$ to find the supremum of the function $N(q)$ we choose two points (0.400, 4.830) and (0.400, 6.186), that are close to the inflection point, this fact decreases accuracy.

In the second case, when the values $|\mathcal{M}| = 0.304$ and $\|G\|_2 = \sqrt{0.907}$ are known, the functions $A(q)$ and $N(q)$ are monotonic, there are no difficulties in applying convex optimization method to compute anisotropic norm. Comparison of methods of anisotropic norm computation for the given descriptor system is represented in Table 2. As in the case of the normal system, the first method needs more time for computation and gives bad accuracy. In addition, you can see the results for $\overline{\mathbf{A}}(W) \le 0.7$. Table 2 gives us all the results of computation.

CONCLUSIONS

Two types of conditions of anisotropic norm boundedness for descriptor systems with nonzero-mean input signals are obtained. In the first case, in order to check the constraint it is necessary to solve the

GDARE and the special type inequality. In the second case, sufficient conditions of anisotropic norm boundedness are given in terms of LMI and special type inequality. The results, obtained for descriptor systems, can be used for normal systems when $E = I_n$. On basis of these results, the algorithm of anisotropic norm computation is developed for known values of $|\mathcal{M}|$ and $\|G\|_2$. This algorithm is founded on the solution of convex optimization problem in terms of LMI, which gives a convex set of constraints. The numerical example shows that the algorithm has a set of advantages against the algorithm of "step" computation.

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