ISSN 1064-2307, Journal of Computer and Systems Sciences International, 2015, Vol. 54, No. 1, pp. 27–38. © Pleiades Publishing, Ltd., 2015.
Original Russian Text © O.G. Andrianova, A.A. Belov, A.P. Kurdyukov, 2015, published in Izvestiya Akademii Nauk. Teoriya i Sistemy Upravleniya, 2015, No. 1, pp. 29–40.

Conditions of Anisotropic Norm Boundedness for Descriptor Systems[1]

O. G. Andrianova, A. A. Belov, and A. P. Kurdyukov

Moscow Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
e-mail: karsettt@yandex.ru

Received April 8, 2013; in final form May 19, 2014

Abstract—A class of systems, described by algebraic-difference equations, is under consideration. Such systems are called descriptor (singular). For these systems the conditions of anisotropic norm boundedness are obtained. Anisotropic norm describes the root-mean-square gain of the system with respect to random Gaussian stationary disturbances, which are characterized by mean anisotropy. The conditions are formulated in the form of the theorem, detailed proof is given. Numerical example, illustrating anisotropic norm computation method for descriptor systems based of the proven theorem, is considered.

DOI: 10.1134/S1064230714060021

INTRODUCTION

Physical control systems are affected by external random disturbances. The problem of disturbance attenuation is one of the fundamental control problems. The Linear Quadratic Gaussian (LQG) control theory deals with Gaussian white noise sequence as an input disturbance. However, noises, affecting the real control systems, are "colored", that's why LQG-controllers, designed on the assumption the input signal is the white noise sequence, can not reach desirable performance. On the other hand, H_∞-control theory (H_∞ means the norm of complex-valued matrix functions with index $p = \infty$), which gives the ability to solve disturbance attenuation problem, assumes that the input disturbance is a square integrable (summable) sequence. The disadvantage of H_∞-control theory is that the systems, closed by H_∞-controllers, are too conservative (they require a lot of control energy) if the input disturbance is uncorrelated or weakly correlated random signal.

The concept of mean anisotropy of random sequences [1, 2] made it possible to introduce a class of random "colored" noises, limited by some numerical parameter, called the mean anisotropy level. This allowed to develop methods for analysis and synthesis of stochastic control systems, that have robust performance with respect to the stochastic nature of input signals. In this case, the performance criterion is connected with anisotropic norm. The problems of optimal and suboptimal (minimizing anisotropic norm of the closed-loop system) control for ordinary linear discrete-time systems were solved in [3, 4]. Anisotropy-based suboptimal control is more flexible than optimal control as it is not necessary to design control law, which minimizes anisotropic norm of the closed-loop system. So, in case of optimal control problem the solution is the control law, that minimizes the performance criterion. In case of suboptimal control it is possible to choose the control law, which satisfies some designer's requirements. Evidently, there is an infinite set of such controllers. It allows to limit oneself to designer's requirements and to apply some additional criteria. But mathematical models of control systems cannot always be described only by ordinary differential or difference equations. Control systems, which mathematical models are described in physical state variables, may contain algebraic equations as constraints between state variables. Such systems are called descriptor systems. Descriptor systems found their application in modeling of aircraft dynamics [5], circuit engineering [6, 7], technical systems [8], economic systems [9], and electrical power engineering [10, 11]. That is why the descriptor systems have been widely studied in the last two decades. Because of the algebraic constraints between state variables, the system gets a specific behaviour, different from ordinary systems, that's why it is necessary to generalize mathematical methods, developed for ordinary systems. Evidently, the generalization of anisotropy-based analysis methods developed for ordinary systems [12] on descriptor systems will allow to consider a wider set of control objects, and it is an actual

[1] The article was translated by the authors.

problem. This paper is devoted to some conditions (in terms of generalized Riccati equations), under which anisotropic norm is bounded by a given scalar value.

1. BASICS OF DESCRIPTOR SYSTEMS THEORY AND ANISOTROPY-BASED ANALYSIS

In this section, some necessary definitions and important results on descriptor systems and anisotropy-based theory are represented.

1.1. Descriptor Systems

Before stating and solving the problem we consider how descriptor models of dynamical systems appear, and define basic concepts of discrete-time descriptor systems theory, such as regularity, causality, stability, and admissibility. More information about descriptor systems theory can be found in [13, 14].

A state-space representation of discrete-time descriptor systems is given by the following equations

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k) + Du(k), \end{aligned} \tag{1.1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control input, $y(k) \in \mathbb{R}^p$ is the output signal, $k = 0, 1, 2, \ldots$ is the discrete time. The matrix $E \in \mathbb{R}^{n\times n}$ may be singular, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$ are system's parameters.

We assume that for the system (1.1) the condition $\operatorname{rank} E = r \le n$ holds. Such systems are called singular or descriptor. Now we consider some basic definitions of discrete-time descriptor systems theory, that will be important for further explanation [14].

D e f i n i t i o n 1. The pair (E, A) is said to be regular if there exists a scalar λ such that $\det(\lambda E - A) \neq 0$.

Regularity of the pair (E, A) is a necessary and sufficient condition of existence and uniqueness of the system's (1.1) solution. The following lemma [14] provides necessary and sufficient conditions of regularity for the system (1.1).

L e m m a 1. The pair (E, A) is regular if and only if there exist invertible matrices $\tilde{Q}$ and $\tilde{R}$ such that

$$\tilde{Q}E\tilde{R} = \operatorname{diag}(I_r, N), \quad \tilde{Q}A\tilde{R} = \operatorname{diag}(A_1, I_{n-r}), \tag{1.2}$$

where $A_1 \in \mathbb{R}^{r\times r}$ is a known matrix, $I_r \in \mathbb{R}^{r\times r}, I_{n-r} \in \mathbb{R}^{(n-r)\times(n-r)}$ are identity matrices, and $N \in \mathbb{R}^{(n-r)\times(n-r)}$ is a nilpotent matrix.

According to Lemma 1, the system in a state-space representation (1.1) may be rewritten in the following form:

$$\begin{aligned} x_1(k+1) &= A_1x_1(k) + B_1u(k), \\ Nx_2(k+1) &= x_2(k) + B_2u(k), \end{aligned} \tag{1.3}$$

where $[B_1^{\mathrm{T}} \ \ B_2^{\mathrm{T}}]^{\mathrm{T}} = WB$.

D e f i n i t i o n 2. The matrix N is called a nilpotent matrix of index h if $N^h = 0$, and $N^{h-i} \neq 0$, $i = \overline{1, h}$.

D e f i n i t i o n 3. The index of the system (1.1) in the equivalent form (1.3) is called the index of nilpotency of the matrix N.

D e f i n i t i o n 4. The system (1.3) is called the Weierstrass canonical form of (1.1) [14].

Descriptor systems don't have solutions for any initial conditions.

D e f i n i t i o n 5. Initial conditions $x(0)$, for which a regular system has solutions, are called consistent. Consistent initial conditions satisfy the following equality:

$$(0 \ \ I)\,\tilde{R}^{-1}x(0) = \sum_{i=0}^{h-1} N^i B_2 u(i). \tag{1.4}$$

D e f i n i t i o n 6. The system (1.1) is called causal if its solution $x(k)$ depends only on $u(k), \ldots, u(0)$ and $x(k-1), \ldots, x(0)$ for any consistent initial conditions. It is true if the index of the nilpotency of the matrix N is equal to 1.

Now we introduce the concepts of stability and admissibility for the descriptor system [15].

D e f i n i t i o n 7. The generalized spectral radius for the system (1.1) or for the pair (E, A) is

$$\rho(E, A) = \max_{\lambda \in z|\det(zE - A) = 0} |\lambda|. \tag{1.5}$$

Besides, for simplicity we'll use the following notation $\rho(A) = \rho(I, A)$ which is used to denote the normal spectral radius of the matrix.

D e f i n i t i o n 8. The system (1.1) is called stable if $\rho(E, A) < 1$.

D e f i n i t i o n 9. The system (1.1) is called admissible if the pair (E, A) is regular, and the system is causal and stable.

D e f i n i t i o n 10. The transfer function of the system (1.1) is given by the complex-valued function, defined by the following expression:

$$G(z) = C(zE - A)^{-1}B + D, \quad z \in \mathbb{C}. \tag{1.6}$$

D e f i n i t i o n 11. Let $\mathbb{L}_2^{p\times m}(\Gamma)$ (where Γ is a unit circle on the complex plane) be the Hilbert space of matrix-valued functions $F : \Gamma \to \mathbb{C}^{p\times m}$ that have bounded $\mathbb{L}_2^{p\times m}(\Gamma)$-norm

$$\|F\|_{\mathbb{L}_2^{p\times m}(\Gamma)} = \left(\frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}(F^*(e^{i\omega})F(e^{i\omega}))d\omega\right)^{\frac{1}{2}}. \tag{1.7}$$

Here $F^*(e^i) = F^{\mathrm{T}}(e^{-i})$ is a conjugate system.

A subspace of $\mathbb{L}_2^{p\times m}(\Gamma)$ which consists of all rational transfer functions that have no poles in the exterior of the closed unit disk is denoted by $H_2^{p\times m}$.

D e f i n i t i o n 12. H_2-norm of the transfer function $G(z) \in H_2^{p\times m}$ is defined by the expression

$$\|G\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}(G^*(e^{i\omega})G(e^{i\omega}))d\omega\right)^{\frac{1}{2}}.$$

If $G(z)$ is strictly proper, and the system (1.1) is stable ($\rho(E, A) < 1$), then $G(z) = C(zE - A)^{-1}B \in H_2^{p\times m}$. On the other hand, if $G(z) \in H_2^{p\times m}$, then $G(z)$ is strictly proper, but not necessarily stable.

D e f i n i t i o n 13. Let $\mathbb{L}_\infty^{p\times m}(\Gamma)$ be a space of matrix-valued functions $F : \Gamma \to \mathbb{C}^{p\times m}$ that are (strongly) bounded on Γ. The subspace of $\mathbb{L}_\infty^{p\times m}(\Gamma)$ denoted by $H_\infty^{p\times m}$ consists of all rational transfer functions that are analytic in the exterior of the closed unit disk. Then the H_∞-norm of the transfer function $G(z) \in H_\infty^{p\times m}$ is defined by

$$\|G\|_\infty = \sup_{\omega \in [0, 2\pi]} \sigma_{\max}(G(e^{i\omega})) = \sup_{\omega \in [0, 2\pi]} \left\|G(e^{i\omega})\right\|_2,$$

where $\sigma_{\max}(G(e^{i\omega}))$ is a maximal singular value.

Evidently, H_∞-norm of the transfer function $G(z)$ is finite only if $G(z) \in \mathbb{L}_\infty^{p\times m}(\Gamma)$ is proper.

1.2. Mean Anisotropy of the Sequence and Anisotropic Norm of the System

In this section, basic concepts of anisotropy-based analysis are given. In [2, 16], mean anisotropy of a random sequence and anisotropic norm of a linear system are defined. Mean anisotropy and anisotropic norm are fully described in [2].

Let $W = \{W_k\}_{k\geq 0}$ be a stationary sequence of square integrable m-dimensional random vectors which is interpreted as a discrete-time random signal. Assembling the elements of W, associated with the interval $[0, N - 1]$, into a random vector

$$W_0^{N-1} = \left[W_0^{\mathrm{T}} \ \ldots \ W_{N-1}^{\mathrm{T}}\right]^{\mathrm{T}}, \tag{1.8}$$

We assume that W_0^{N-1} is absolutely continuously distributed for every $N > 0$.

D e f i n i t i o n 14. The anisotropy W_0^{N-1} is defined by:

$$\mathbf{A}(W_0^{N-1}) = \frac{m}{2}\ln\left(\frac{2\pi}{m}\mathbf{E}(|W_0^{N-1}|^2)\right) - h(W_0^{N-1}),$$

where $\mathbf{E}(\cdot)$ is mathematical expectation, $h(W_0^{N-1})$ is differential entropy

$$h(W_0^{N-1}) = \mathbf{E}_w(-\ln f(W_0^{N-1})) = -\int_{\mathbb{R}^{mN}} f(w)\ln f(w)dw,$$

where $f : \mathbb{R}^{mN} \to \mathbb{R}_+$ is the probability density function of the mN-valued vector W_0^{N-1}, and $w \in \mathbb{R}^{mN}$.

D e f i n i t i o n 15. Mean anisotropy of the sequence W is defined by the expression

$$\overline{\mathbf{A}}(W) = \lim_{N\to+\infty}\frac{\mathbf{A}(W_0^{N-1})}{N}. \tag{1.9}$$

It is shown in [2], that

$$\overline{\mathbf{A}}(W) = \mathbf{A}(W_0) + \overline{\mathbf{I}}(W_0;\{W_k\}_{k<0}), \tag{1.10}$$

where $\overline{\mathbf{I}}(W_0;\{W_k\}_{k<0}) = \lim_{s\to-\infty}\mathbf{I}(W_0;W_s^{-1})$ is the Shannon mutual information [17] between W_0 and the past history $\{W_k\}_{k<0}$ of the sequence W.

Now we assume that W is a discrete-time stationary Gaussian sequence, then

$$\overline{\mathbf{I}}(W_0;\{W_k\}_{k<0}) = \frac{1}{2}\ln\det(\mathbf{cov}(W_0)\mathbf{cov}^{-1}(\tilde{W}_0)), \tag{1.11}$$

where

$$\tilde{W}_0 = W_0 - \mathbf{E}(W_0|\{W_k\}_{k<0}) \tag{1.12}$$

is the error of mean-square optimal prediction of W_0 by the past history $\{W_k\}_{k<0}$, provided by the conditional expectation.

Let W be generated by the shaping filter with a transfer function $G(z)$ and the impulse response $g(k) \in \mathbb{R}^{m\times m}$ from a discrete-time Gaussian white noise sequence V with zero mean $\mathbf{E}V_k = 0$ and identity covariance matrix $\mathbf{E}(V_kV_k^{\mathrm{T}}) = I_m$, i.e.

$$W_j = \sum_{k=0}^{+\infty} g(k)V_{j-k}, \quad j \in \mathbb{Z}.$$

G is supposed to denote a linear operator of the shaping filter with the transfer function $G(z)$, and $\|G\|$ stands for the norm of this operator.

The transfer function of the filter

$$G(z) = \sum_{k=0}^{+\infty} g(k)z^{-k}$$

is supposed to belong to the Hardy space $H_2^{m\times m}$, analytic in the unit disc $|z| < 1$ on the complex plane, and has finite H_2-norm:

$$\|G\|_2 = \left(\sum_{k=0}^{+\infty}\mathrm{tr}(g^T(k)g(k))\right)^{1/2} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{tr}(\hat{G}^*(\omega)\hat{G}(\omega))d\omega\right)^{1/2} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{tr}\ S(\omega)d\omega\right)^{1/2}, \tag{1.13}$$

where $\hat{G}(\omega) = \lim_{l\to1} G(le^{i\omega})$ is the boundary value of the transfer function $G(z)$, $S(\omega) = \hat{G}^*(\omega)\hat{G}(\omega)$, $-\pi \le \omega \le \pi$ is the spectral density of W.

The covariance matrix of the prediction error (1.12) and the spectral density $S(\omega)$ are related by the Szegö-Kolmogorov formula:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\det S(\omega)d\omega = \ln\det \mathbf{cov}(\tilde{W}_0). \tag{1.14}$$

By using (1.10)–(1.12), the Szegö limit theorem [18], and (1.14), the mean anisotropy (1.9) of the stationary Gaussian random sequence $W = GV$ may be computed in terms of the spectral density $S(\omega)$ and H_2-norm of the shaping filter G as

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det\frac{mS(\omega)}{\|G\|_2^2}d\omega = -\frac{1}{4\pi}\ln\det\frac{m\mathbf{cov}(\tilde{W}_0)}{\|G\|_2^2}. \tag{1.15}$$

Mean anisotropy characterizes the spectral color of the signal, which is its divergence from the Gaussian white noise sequence. In case of W is the white noise sequence, mean anisotropy is equal to zero, if the signal is deterministic, its anisotropy equals to infinity. According to (1.15), $\overline{\mathbf{A}}(W)$ is completely determined by parameters of the shaping filter G, so we'll use $\overline{\mathbf{A}}(G)$ instead of $\overline{\mathbf{A}}(W)$ in what follows (see details in [1, 2]).

Let $Y = FW$ be an output of the linear system $F \in H_\infty^{p\times m}$, its transfer function $F(z)$ is analytic in the disc $|z| < 1$ and has a finite H_∞-norm.

D e f i n i t i o n 16. For a given $a \geq 0$ the a-anisotropic norm of the system F is defined by

$$|\!|\!|F|\!|\!|_a = \sup_{G\in\mathbf{G}_a}\frac{\|FG\|_2}{\|G\|_2}, \tag{1.16}$$

which is interpreted as the maximum gain (the ratio of power norms of the output Y and the input W) against the class of shaping filters

$$\mathbf{G}_a = \left\{G \in H_2^{m\times m} : \overline{\mathbf{A}}(G) \leq a\right\}.$$

The shaping filter $G \in \mathbf{G}_a$, for which the supremum from (1.16) is reached, is called the worst-case shaping filter, and the disturbance W, generated by this filter, is the worst-case input disturbance of the system F.

So, a-anisotropic norm $|\!|\!|F|\!|\!|_a$ characterizes the robustness of the system F against the random disturbance W, uncertainty of its statistical properties is described by the parameter a.

2. PROBLEM STATEMENT. ANISOTROPY-BASED BOUNDED REAL LEMMA

Proceed to anisotropy-based analysis problem statement for descriptor systems. The system is given by the following equations:

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bw(k),\\ y(k) &= Cx(k) + Dw(k), \end{aligned} \tag{2.1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^m$ is the input signal, $y(k) \in \mathbb{R}^p$ is the output signal. Hereafter, the values of random signals $w(k)$ and $v(k)$ have the same meaning as W_k and V_k in the Section 1.2.

We identify the system (2.1) with the system P, given by its transfer function

$$P = C(zE - A)^{-1}B + D. \tag{2.2}$$

We also use the following representation for the system (2.2) [13]:

$$P = \left[E, \begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

The system P is admissible. Suppose W is a stationary Gaussian random sequence, whose mean anisotropy does not exceed $a \geq 0$, i.e. W is generated from the m-dimensional Gaussian white noise V with zero mean and an identity covariance matrix by an unknown shaping filter G which belongs to the set $\mathbf{G}_a = \left\{G \in H_2^{m\times m} : \overline{\mathbf{A}}(G) \le a\right\}$.

The problem is to check the condition $|||P|||_a \le \gamma$ for the given system P, mean anisotropy level of the input disturbance $a \ge 0$ and the scalar value $\gamma > 0$.

Before formulating bounded real lemma for descriptor systems, we consider the following theorem.

Let a linear discrete-time stationary system $F \in H_\infty^{p\times m}$ with n-dimensional state vector x, m-dimensional input signal w and p-dimensional output y be given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k), \\ y(k) &= Cx(k) + Dw(k), \end{aligned} \tag{2.3}$$

where A, B, C, D are known real matrices of appropriate dimensions, the matrix A is stable ($\rho(A) < 1$).

T h e o r e m 1 [12]. Let $F \in H_\infty^{p\times m}$ be the system with a state-space representation (2.3) where $\rho(A) < 1$; a-anisotropic norm of the system F is strictly bounded by a given value $\gamma > 0$, i.e.

$$|||F|||_a < \gamma$$

if and only if there exists $q \in [0, \min(\gamma^{-2}, \|F\|_\infty^{-2}))$, such that the inequality

$$-\frac{1}{2}\ln\det((1 - q\gamma^2)\Sigma) \ge a$$

is true for the matrix Σ, which is defined by stabilizing solution ($\rho(A + BL) < 1$) $\hat{R} = \hat{R}^{\mathrm{T}} \ge 0$ of the following Riccati equation:

$$\begin{aligned} \hat{R} &= A^{\mathrm{T}}\hat{R}A + qC^{\mathrm{T}}C + L^{\mathrm{T}}\Sigma^{-1}L, \\ L &= \Sigma(B^{\mathrm{T}}\hat{R}A + qD^{\mathrm{T}}C), \\ \Sigma &= (I_m - B^{\mathrm{T}}\hat{R}B - qD^{\mathrm{T}}D)^{-1}. \end{aligned}$$

To proof the main result it's necessary to give the following theoretical results.

D e f i n i t i o n 17. A system with a transfer function $P(z)$ such that $P^*(z)P(z) = I_m$ is called inner or all-pass system.

The following lemma is true.

L e m m a 2 [13]. The system (2.1) is all-pass system if there exists $\hat{R} = \hat{R}^{\mathrm{T}}$ which satisfies the conditions $E^{\mathrm{T}}\hat{R}E \ge 0$ and

$$\begin{aligned} B^{\mathrm{T}}\hat{R}B + D^{\mathrm{T}}D &= I, \\ B^{\mathrm{T}}\hat{R}A + D^{\mathrm{T}}C &= 0, \\ A^{\mathrm{T}}\hat{R}A + C^{\mathrm{T}}C - E^{\mathrm{T}}\hat{R}E &= 0. \end{aligned}$$

P r o p o s i t i o n. Suppose, that one of the following conditions holds for the system (2.1)

(1) $\operatorname{rank}\begin{pmatrix} A & B \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$,

(2) $\operatorname{rank}\begin{pmatrix} E & B \end{pmatrix} = \operatorname{rank} E$.

T h e o r e m 2. Let $P \in H_\infty^{p\times m}$ be an admissible system with the state-space representation (2.1) where $\rho(E, A) < 1$. For the given scalar quantities $a \ge 0$ and $\gamma > 0$ the a-anisotropic norm is bounded by γ, that is

$$|||P|||_a \le \gamma,$$

if there exists the stabilizing solution $\hat{R} = \hat{R}^{\mathrm{T}}$ of the algebraic Riccati equation

$$E^{\mathrm{T}}\hat{R}E = A^{\mathrm{T}}\hat{R}A + qC^{\mathrm{T}}C + L^{\mathrm{T}}\Sigma^{-1}L, \tag{2.4}$$

$$L = \Sigma(B^{\mathrm{T}}\hat{R}A + qD^{\mathrm{T}}C), \tag{2.5}$$

$$\Sigma = (I_m - B^{\mathrm{T}}\hat{R}B - qD^{\mathrm{T}}D)^{-1}, \tag{2.6}$$

such that

$$E^{\mathrm{T}}\hat{R}E \geq 0,$$

where $q \in [0, \min(\gamma^{-2}, \|P\|_{\infty}^{-2}))$ satisfies the inequality

$$-\frac{1}{2}\ln\det((1 - q\gamma^2)\Sigma) \geq a. \tag{2.7}$$

P r o o f. The power norm ratio $\|PG\|_2 / \|G\|_2$ on the right-hand side of (1.16) and the mean anisotropy $\overline{\mathbf{A}}(G)$ in (1.15) are both invariant with respect to the shaping filter G. For the system P they are completely specified by the normalized spectral density [12]:

$$\Pi(\omega) = \frac{mS(\omega)}{\|G\|_2^2} = \frac{2\pi mS(\omega)}{\int_{-\pi}^{\pi}\mathrm{tr}S(\nu)d\nu}, \tag{2.8}$$

then

$$\overline{\mathbf{A}}(G) = \alpha(\Pi) = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det\Pi(\omega)d\omega, \tag{2.9}$$

$$\frac{\|PG\|_2}{\|G\|_2} = \nu(\Pi) = \left(\frac{1}{2\pi m}\int_{-\pi}^{\pi}\mathrm{tr}(\Lambda(\omega)\Pi(\omega))d\omega\right)^{1/2} \tag{2.10}$$

where the function $\Pi(\omega)$, defined by the Eq. (2.8) on the interval $[-\pi, \pi)$, takes values in the set of positive defined Hermitian matrices of the order m and satisfies the condition

$$\int_{-\pi}^{\pi}\mathrm{tr}\Pi(\omega)d\omega = 2\pi m,$$

and the function $\Lambda(\omega)$ is given by the expression

$$\Lambda(\omega) = \hat{P}^*(\omega)\hat{P}(\omega). \tag{2.11}$$

Note that the squared functional $\nu^2(\Pi)$ is linear on $\Pi(\omega)$. The strict convexity of $\alpha(\Pi)$ follows from the strict concavity of $\ln\det(\cdot)$ considered on a convex cone of positive defined matrices [19]. The strict convexity of $\alpha(\Pi)$ can also be obtained directly from the positive definiteness of its second variation

$$\begin{aligned}\delta^2\alpha(\Pi) &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\mathrm{tr}(\Pi^{-1}(\omega)\delta\Pi(\omega)\Pi^{-1}(\omega)\delta\Pi(\omega))d\omega \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\left\|\Pi^{-1/2}(\omega)\delta\Pi(\omega)\Pi^{-1/2}(\omega)\right\|^2 d\omega\end{aligned} \tag{2.12}$$

where $\delta\Pi(\omega)$ is the variation of $\Pi(\omega)$, and $\|M\| = \mathrm{tr}^{1/2}(M^*M)$ denotes the Frobenius norm of a matrix. In the Eq. (2.12) we use the formula for computation of the first variation of the inverse nonsingular matrix

$\delta(\Xi^{-1}) = -\Xi^{-1}(\delta\Xi)\Xi^{-1}$ and the property of matrix trace $\ln\det\Xi = \mathrm{tr}(\ln\Xi)$. The minimum value of the mean anisotropy of the disturbance W should achieve the given level $\gamma > 0$ for the power norm ratio

$$\min_{\nu(\Pi)\ge\gamma} \alpha(\Pi) = -\frac{1}{4\pi}\max_{\nu^2(\Pi)\ge\gamma^2}\int_{-\pi}^{\pi}\ln\det\Pi(\omega)d\omega. \tag{2.13}$$

By using the method of Lagrange multipliers, the first minimum in (2.13) is shown to be achieved at a spectral density which is proportional to

$$S_q(\omega) = (I_m - q\Lambda(\omega))^{-1}, \tag{2.14}$$

where q is a subsidiary variable satisfying the condition $0 \le q < \|P\|_\infty^{-2}$.

Accordingly, the functions

$$\mathcal{A}(q) = \alpha(\Pi_q), \quad \mathcal{N}(q) = \nu(\Pi_q) \tag{2.15}$$

are defined by evaluating the functionals $\alpha(\Pi)$ and $\nu(\Pi)$ from (2.9) and (2.10) at the normalized spectral density

$$\Pi_q(\omega) = \frac{2\pi m S_q(\omega)}{\int_{-\pi}^{\pi}\mathrm{tr}S_q(\nu)d\nu}, \tag{2.16}$$

associated with (2.14) and (2.8). Except the trivial case when the function Λ in (2.11) is a constant matrix, $\mathcal{A}(q)$ and $\mathcal{N}(q)$ are both strictly increasing in q (see [20, 21]). This allows the minimum required mean anisotropy in (2.13) to be computed as $\mathcal{A}(\mathcal{N}^{-1}(\gamma))$ where $\mathcal{N}^{-1}(\gamma)$ denotes the functional inverse of $\mathcal{N}(q)$. Therefore, the inequality $|||P|||_a \le \gamma$ is equivalent to $\mathcal{A}(\mathcal{N}^{-1}(\gamma)) \ge a$. Now, (2.14) implies that $\Lambda(\omega) = (I_m - S_q(\omega)^{-1})/q$ and

$$\frac{1}{2\pi m}\int_{-\pi}^{\pi}\mathrm{tr}\left(\Lambda(\omega)S_q(\omega)\right)d\omega = \frac{1}{q}\left(\frac{1}{2\pi m}\int_{-\pi}^{\pi}\mathrm{tr}S_q(\omega)d\omega - 1\right), \tag{2.17}$$

which, in combination with the definition of the function $\mathcal{N}(q)$ via (2.10), (2.15) and (2.16) gives

$$\frac{1}{2\pi m}\int_{-\pi}^{\pi}\mathrm{tr}S_q(\omega)d\omega = \frac{1}{1-q\mathcal{N}^2(q)}. \tag{2.18}$$

By substituting (2.9), (2.16) and (2.18) into the definition of $\mathcal{A}(q)$ from (2.15), it follows that the function can be represented as

$$\mathcal{A}(q) = \mathfrak{A}(q, \mathcal{N}(q)), \tag{2.19}$$

where

$$\mathfrak{A}(q,\gamma) = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det S_q(\omega)d\omega - \frac{m}{2}\ln(1-q\gamma^2). \tag{2.20}$$

Since $-\ln(1-q\gamma^2)$ is monotonically increasing in $\gamma \in [0;1/\sqrt{q})$, then so is $\mathfrak{A}(q,\gamma)$. A distinctive feature of the function $\mathfrak{A}(q,\gamma)$ is that it achieves its maximum with respect to q at the point $q = \mathcal{N}^{-1}(\gamma)$ where, in view of (2.19), it coincides with the function $\mathcal{A}(q)$:

$$\max_{0\le q<\|P\|_\infty^{-2}} \mathfrak{A}(q,\gamma) = \mathfrak{A}(\mathcal{N}^{-1}(\gamma),\gamma) = \mathcal{A}(\mathcal{N}^{-1}(\gamma)). \tag{2.21}$$

The significance of this property for establishing a criterion for the inequality $|||P|||_a \le \gamma$ is explained by that (2.21) implies the equivalence between $\mathcal{A}(\mathcal{N}^{-1}(\gamma)) \ge a$ and the existence of the parameter $q \in [0, \|P\|_\infty^{-2})$, satisfying $\mathfrak{A}(q,\gamma) \ge a$. Therefore, $|||P|||_a \le \gamma$ if $\mathfrak{A}(q,\gamma) \ge a$ for some q.

The property (2.21) is verified by differentiating the function $\mathfrak{A}(q,\gamma)$ from (2.20) with respect to its first argument:

$$\frac{\partial \mathfrak{A}(q,\gamma)}{\partial q} = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\partial \ln\det(I_m - q\Lambda(\omega))}{\partial q} d\omega + \frac{m\gamma^2}{2(1-q\gamma^2)}$$

$$- \frac{1}{4\pi}\int_{-\pi}^{\pi} \operatorname{tr}\left(\Lambda(\omega)S_q(\omega)\right) d\omega + \frac{m\gamma^2}{2(1-q\gamma^2)} - \frac{m\mathcal{N}^2(q)}{2(1-q\mathcal{N}^2(q))} + \frac{m\gamma^2}{2(1-q\gamma^2)} \quad (2.22)$$

$$= \frac{m(\gamma^2 - \mathcal{N}^2(q))}{2(1-q\gamma^2)(1-q\mathcal{N}^2(q))}.$$

The function $\mathcal{N}(q)$ is strictly monotonic, the representation (2.21) implies that $\partial\mathfrak{A}(q,\gamma)/\partial q$ is positive for $q < \mathcal{N}^{-1}(\gamma)$ and negative for $q > \mathcal{N}^{-1}(\gamma)$. It now remains to represent the inequality $\mathfrak{A}(q,\gamma) \ge a$ for the function (2.20) in the state space representation of the system P. We denote that (2.14) describes the parametric set of the worst-case spectral densities of the input disturbance W for the admissible values of q. Since the subsidiary variable q is fixed for the rest of the proof, we use the notation

$$S_*(\omega) = (I_m - q_*\Lambda(\omega))^{-1}, \quad (2.23)$$

where $q_* = \mathcal{A}^{-1}$.

We will obtain a state-space representation of the worst-case disturbance W_* with a spectral density S_*. In view of (2.11), the Eq. (2.23) is equivalent to

$$\hat{\Theta}^*(\omega)\hat{\Theta}(\omega) = I_m, \qquad -\pi \le \omega < \pi, \quad (2.24)$$

$$\hat{\Theta} = \begin{bmatrix} \sqrt{q_*}\hat{P}(\omega) \\ \hat{G}_*^{-1}(\omega) \end{bmatrix}. \quad (2.25)$$

Here $\hat{G}_*$ is the worst-case shaping filter. The Eq. (2.24) means, that the system $\hat{\Theta}$ is the all-pass system.

Let $\Sigma \in \mathbb{R}^{m\times m}$ be a positive defined symmetric matrix, and the matrix $L \in \mathbb{R}^{m\times n}$ is such that the pair $(E, A + BL)$ is admissible. We consider the input disturbance $W_* = G_*V$, which can be generated as

$$w_*(k) = Lx(k) + \Sigma^{1/2}\nu(k). \quad (2.26)$$

Find such matrices L and Σ, for which the input disturbance W_* is the worst-case. The state-space representation of the shaping filter G_* is

$$G_* = \left[E, \begin{array}{c|c} A+BL & B\Sigma^{1/2} \\ \hline L & \Sigma^{1/2} \end{array}\right]. \quad (2.27)$$

Since G_* is invertible, its inverse is described by

$$G_*^{-1} = \left[E, \begin{array}{c|c} A & B \\ \hline -\Sigma^{-1/2}L & \Sigma^{-1/2} \end{array}\right]. \quad (2.28)$$

The state-space realization of the closed-loop system Θ is

$$\Theta = \left[E,\ \begin{array}{c|c} A & B \\ \hline q^{1/2}C & q^{1/2}D \\ -\Sigma^{-1/2}L & \Sigma^{-1/2} \end{array} \right]. \tag{2.29}$$

According to Lemma 2, there exists a matrix $\hat{R} = \hat{R}^{\mathrm{T}}$, satisfying the condition $E^{\mathrm{T}}\hat{R}E \geq 0$ such that

$$B^{\mathrm{T}}\hat{R}B + \left[q^{1/2}D^{\mathrm{T}}\ \ (\Sigma^{-1/2})^{\mathrm{T}}\right]\begin{bmatrix} q^{1/2}D \\ \Sigma^{-1/2} \end{bmatrix} = I, \tag{2.30}$$

$$B^{\mathrm{T}}\hat{R}A + \left[q^{1/2}D^{\mathrm{T}}\ \ (\Sigma^{-1/2})^{\mathrm{T}}\right]\begin{bmatrix} q^{1/2}C \\ -\Sigma^{-1/2}L \end{bmatrix} = 0, \tag{2.31}$$

$$A^{\mathrm{T}}\hat{R}A + \left[q^{1/2}C^{\mathrm{T}}\ \ -L^{\mathrm{T}}(\Sigma^{-1/2})^{\mathrm{T}}\right]\begin{bmatrix} q^{1/2}C \\ -\Sigma^{-1/2}L \end{bmatrix} - E^{\mathrm{T}}\hat{R}E = 0. \tag{2.32}$$

As Σ is a positive defined symmetric matrix, from (2.30) and (2.31) we get

$$\Sigma = (I - B^{\mathrm{T}}\hat{R}B - qD^{\mathrm{T}}D)^{-1}, \tag{2.33}$$

$$L = \Sigma(B^{\mathrm{T}}\hat{R}A + qD^{\mathrm{T}}C). \tag{2.34}$$

These equations coincide with the Eqs. (2.5) and (2.6).

The expression (2.32) may be brought to (2.4) if it is rewritten in the form

$$E^{\mathrm{T}}\hat{R}E = A^{\mathrm{T}}\hat{R}A + qC^{\mathrm{T}}C + L^{\mathrm{T}}\Sigma^{-1}L. \tag{2.35}$$

Since the worst-case input disturbance is described by (2.26), where $\nu(k)$ is a white noise sequence with the identity covariance matrix and zero mean, the prediction error (1.12) takes the form $\tilde{w}(0) = \Sigma^{1/2}\nu(0)$ and, hence, $\mathbf{cov}(\tilde{w}(0)) = \Sigma$. Therefore, in combination with the Szegö-Kolmogorov formula (1.14), we find

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det S_*(\omega)d\omega = \ln\det\Sigma.$$

By substituting this equation in (2.20), we obtain

$$\mathfrak{A}(q,\gamma) = -\frac{1}{2}\ln\det\left((1-q\gamma^2)\Sigma\right).$$

Hence, the condition $\mathfrak{A}(q,\gamma) \geq a$ is equivalent to the inequality (2.7) for the matrix Σ, associated with generalized Riccati Eqs. (2.4)–(2.6). This completes the proof.

3. NUMERICAL EXAMPLE

Consider the system (2.1) with the following parameters

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0.3500 & 1.0000 & -0.7800 \\ 0.6300 & -0.1100 & 0.9700 \\ 0.6177 & 0.8038 & 0.7851 \end{pmatrix}, \quad B = \begin{pmatrix} -0.3 \\ 0.1 \\ 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0.70 & 2.00 & -1.56 \end{pmatrix}, \quad D = 0.6.$$

The results of conditions check for $a = 0.1$

γ	3.170	3.160	3.150
$[0, \min(\gamma^{-2}, \|P\|_\infty^{-2}))$	[0, 0.0214)		
q	0.0209	0.0213	$-0.0001i + 0.0214$
$E^{\mathrm{T}}\hat{R}E$	$\begin{pmatrix} 0.2508 & 0.4703 & 0 \\ 0.4703 & 3.3776 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0.2538 & 0.4818 & 0 \\ 0.4818 & 3.5087 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -0.0018i + 0.2577 & -0.0123i + 0.4993 & 0 \\ -0.0123i + 0.4993 & -0.1636i + 3.7194 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The system is admissible, $\rho(E, A) = 0.9799$ the second rank condition holds. H_∞-norm $\|P\|_\infty$ of the transfer function is equal to 6.8364.

To satisfy the conditions of Theorem 2 the value of parameter q should lie in interval $q \in [0, \min(\gamma^{-2}, \|P\|_\infty^{-2}))$ for the given mean anisotropy level a and parameter γ, where $\|P\|_\infty^{-2} = 0.0214$, and the inequality $E^{\mathrm{T}}\hat{R}E \geq 0$ should be true for the matrix $\hat{R}$.

Consider numerical experiments for different values of γ.

For $a = 0.1$ the following results are obtained (see table). The exact value of anisotropic norm is $|||P|||_a = 3.1537$.

As we can see, the conditions of theorem 2 are satisfied for $|||P|||_a < \gamma$. For $|||P|||_a > \gamma$ the conditions get broken not only on q, but also on $\hat{R}$.

Therefore, the conditions of the theorem 2 can be used for anisotropic norm computation with any set accuracy.

CONCLUSIONS

Anisotropy-based bounded real lemma for descriptor systems defines the conditions on anisotropic norm boundedness for admissible descriptor systems. These conditions consist of solvability of generalized Riccati equation under the inequality constraint. Note that obtained conditions are equivalent to the conditions from [12] for ordinary systems when $E = I$. The anisotropy-based bounded real lemma for descriptor systems can be useful for the anisotropic norm computation with given accuracy and for solving suboptimal control design problem.

REFERENCES

1. I. G. Vladimirov, A. P. Kurdyukov, and A. V. Semenov, "Anisotropy of signals and entropy of linear stationary systems," Dokl. Math. **51** (3), 388–390.
2. I. G. Vladimirov, P. Diamond, and P. Kloeden, "Anisotropy-based robust performance analysis of finite horizon linear discrete timevarying systems," Automat. Remote Control **67** (8), 1265–1282.
3. I. G. Vladimirov, A. P. Kurdjukov, and A. V. Semyonov, "State-space solution to anisotropy-based stochastic H_∞-optimization problem," in *Proc. 13th IFAC World Congress* (CA, San-Francisco, 1996), pp. 427–432.
4. M. M. Tchaikovsky, A. P. Kurdyukov, and V. N. Timin, "Synthesis of anisotropic suboptimal controllers by convex optimization," Submitted to European J. Contr. Preprint is available from http://arxiv.org/abs/1108.4982, 2011.
5. B. L. Stevens and F. L. Lewis, *Aircraft Modeling, Dynamics and Control* (Wiley, N.Y., 1991).
6. R. W. Newcomb, "The semistate description of nonlinear time-variable circuits," IEEE Trans. Circuits Syst. **28**, 62–71 (1981).
7. R. W. Newcomb and B. Dziurla, "Some circuits and systems applications of semistate theory," Circuit, Syst., Sig. Process. **8**, 235–260 (1989).
8. H. Hemami and B. F. Wyman, "Modeling and control of constrained dynamic systems with application to biped locomotion in the frontal plane," IEEE Trans. Automat. Control. **24**, 526–535 (1979).
9. A. Luenberger and D. G. Arbel, "Singular dynamic Leontief systems," Econometrica **45**, 991–995 (1977).

10. M. Sh. Misrikhanov and V. N. Ryabchenko, "A new criteria of static stability of electrical power system with algebraic-differential mathematical model," Vestnik Ivanovo State Power Univ., No. 2, 76–81 (2009).

11. M. Sh. Misrikhanov and V. N. Ryabchenko, "Iterative criteria of static stability of electrical power system in algebraic-differential form," Stochastic Optimization Informatics **5** (1-1), 209–225 (2009).

12. A. P. Kurdyukov, E. A. Maximov, and M. M. Tchaikovsky, "Anisotropy-based bounded real lemma," in *Proc. 19th Intern. Sympos. of Mathematical Theory of Networks and Systems* (Budapest, Hungary, 2010), pp. 2391–2397.

13. A. A. Belov, "Anisotropy-based control synthesis for descriptor systems," *Ph. D. Thesis* (M., 2011).

14. L. Dai, *Singular Control Systems, Lecture Notes in Control and Information Sciences* (Springer-Verlag, N.Y., 1989).

15. S. Xu and J. Lam, *Robust Control and Filtering of Singular Systems. Lecture Notes in Control and Information Sciences* (Springer-Verlag, Berlin, 2006).

16. A. V. Semyonov, I. G. Vladimirov, and A. P. Kurdjukov, "Stochastic approach to H_∞-optimization," in *Proc. 33rd IEEE Conf. Decision and Control* (Florida, 1994), pp. 2249–2250.

17. R. M. Gray, *Entropy and Information Theory* (Springer-Verlag, N.Y., 1991).

18. U. Grenander and G. Szegö, *Toeplitz Forms and Their Applications* (University of California Press, Berkeley, 1958).

19. R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambrige University Press, N.Y., 2007).

20. P. Diamond, I. G. Vladimirov, A. P. Kurdjukov, and A. V. Semyonov, "Anisopropy-based performance analysis of linear discrete-time-invariant control systems," Int. J. Control. **74**, 28–42 (2001).

21. I. G. Vladimirov, A. P. Kurdjukov, and A. V. Semyonov, "On computing the anisotropic norm of linear discrete-time-invariant systems," in *Proc. 13th IFAC World Congr.* (CA, San-Francisco, 1996), pp. 179–184.