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Метою видання журналу «Праці міжнародного геометричного центру» є консолідація та підтримка досліджень з геометрії, топології та динамічних систем. Тематична спрямованість журналу пов'язана з висвітленням найбільш важливих та актуальних проблем у галузі математики та її застосувань, зокрема у диференціальній геометрії, топології, математичній фізиці та динамічних системах.

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A calculation of periodic data of surface diffeomorphisms with one saddle orbit

Elena Nozdrinova, Olga Pochinka

Abstract. We prove that every orientable surface admits an orientation-preserving diffeomorphism with one saddle orbit. It distinguishes in principle the considered class of systems from source-sink diffeomorphisms existing only on the sphere. It is shown that diffeomorphisms with one saddle orbit of a positive type on any surface have exactly three node orbits. We also describe all possible types of periodic data for such diffeomorphisms, and found formulas expressing periods of sources via periods of the sink and the saddle.

Анотація. При вивченні дискретних динамічних систем важливу роль відіграють періодичні орбіти. Класичним прикладом є теорема Шарковського про відображення відрізку в себе, яка стверджує, що з існування орбіт періоду три "породжує хаос". В останні 40 років з'явилось багато робіт присвячених вивченню періодичних даних відображень поверхонь. Найбільш корисними інструментами для доведення існування нерухомих точок та, в більш загальному випадку, періодичних точок неперервного відображення компактного многовиду, є теорема Лефшеця про нерухому точку та її узагальнення. Дзета-функція Лефшеця спрощує вивчення періодичних точок дифеоморфізмів поверхонь з регулярною динамікою. Результати досліджень в даному напрямку можна знайти в роботах таких авторів як: П. Бланшар, С. Баттерсон, У. Жако, Дж. Френкс, С. Нарасімхан і ін. Опис періодичних даних градієнтно-подібних дифеоморфізмів поверхонь був отриманий А. Безденежних та В. Грінесом і спирався на класифікацію гомеоморфізмів поверхонь, отриману Дж. Нільсеном.

У роботі "A complete topological classification of Morse-Smale diffeomorphisms on surfaces: a kind of kneading theory in dimension two" В. Грінес, О. Починка, С. Ван Стрієн показали, що вивчення періодичних даних довільних дифеоморфізмів Морса-Смейла на поверхнях зводиться шляхом фільтрації до задачі обчислення періодичних даних дифеоморфізмів з єдиною седловою періодичною орбітою. Представлена робота присвячена вирішенню останньої задачі у випадку, коли орбіта сідлової точки

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має додатний тип орієнтації. В статті доведено, що на кожній орієнтовній поверхні існує дифеоморфізм, який зберігає орієнтацію і має єдину сідлову орбіту. Це принципово відрізняє розглянутий клас систем від дифеоморфізмів "джерело-стік", які існують лише на сфері. Показано, що дифеоморфізми з однією сідловою орбітою на будь-якій поверхні мають рівно три вузлових орбіти. Крім того, встановлено всі можливі типи періодичних даних для таких дифеоморфізмів, а саме: знайдено формули, що явно виражають періоди джерел через періоди стоку та сідла.

1. Introduction and a formulation of results

In the study of discrete dynamical systems, i.e. study of orbits of self-maps f defined on a given compact manifold, the periodic behavior plays an important role. During the last forty years there were a growing number of results showing that certain simple assumptions on f force qualitative and quantitative properties (like the set of periods) of a system. One of the best known result in this direction is the paper entitled "Period three implies chaos for the interval continuous self-maps" (see, for example, [8]). The effect described in [8] was discovered by A. Sharkovsky in [12]. The most useful tools for proving existence of fixed points, or more generally of periodic points for a continuous self-map f of a compact manifold, is the Lefschetz Fixed Point Theorem and its generalizations (see, for instance [11], [3]). The Lefschetz zeta function simplifies the study of periodic points of f. This is a generating function for all the Lefschetz numbers of all iterates of f.

Periodic data of diffeomorphisms with regular dynamics on surfaces is studied in already classical works by P. Blanchard, J. Franks, R. Bowen, S. Batterson, J. Smillie, W. Jaco, P. Shalen, C. Narasimhan and other. Description of periodic data of gradient-like diffeomorphisms of surfaces was given by A. Bezdenezhnykh and V. Grines [1], using J. Nielsen's classification of periodic surface transformations. In the paper by V. Grines, O. Pochinka, S. Van Strien [6] it was shown that the study of periodic data of arbitrary Morse-Smale diffeomorphisms on surfaces is reduced by filtration to the problem of computing periodic data of diffeomorphisms with a unique saddle periodic orbit. The present paper is devoted to a solution of this problem.

As first we will recall some basic definitions and notations.

Let S_g be a closed orientable surface of genus $g \geq 0$ with a metric d and $f : S_g \to S_g$ be an orientation preserving diffeomorphism. A point $x \in S_g$ is called *wandering* for f, if there exists an open neighborhood U_x of x such that $f^n(U_x) \cap U_x = \varnothing$ for all $n \in \mathbb{N}$. Otherwise, x is *non-wandering*.

The set of all non-wandering points of f is denoted by Ω_f and called the *non-wandering set* of f.

When Ω_f is finite then every point $p \in \Omega_f$ is periodic, and its period will be denoted by $m_p \in \mathbb{N}$. Moreover, p is called *hyperbolic* if the Jacobian matrix $\left(\frac{\partial f^{m_p}}{\partial x}\right)|_p$ has eigenvalues whose absolute values are distinct from 1. If the absolute values of all eigenvalues are less (resp. greater) than 1, then p is called a *sink* (resp. *source*). Sink and source points are also called *nodes*, while hyperbolic periodic points being not *nodes* are called *saddles.*

Hyperbolicity of a periodic point p leads to the existence of the *stable*, W^s_p, and *unstable*, W^u_p, manifolds, which are defined as follows:

$$W^s_p = \{x \in S_g : \lim_{k\to+\infty} d(f^{k\cdot per(p)}(x), p) = 0\},$$
$$W^u_p = \{x \in S_g : \lim_{k\to+\infty} d(f^{-k\cdot per(p)}(x), p) = 0\}.$$

The stable and unstable manifolds are called *invariant manifolds.* A connected component of the set $W^u_p \setminus p$ (resp. $W^s_p \setminus p$) is called *unstable* (resp. *stable*) *separatrix.*

A diffeomorphism $f : S_g \to S_g$ is *Morse-Smale* if the set of its wandering points consists of finitely many periodic points and there is no separatrixes connecting saddles.

To the orbit $\mathcal{O}_p$ of a periodic point p of a Morse-Smale diffeomorphism f one can associate the following numbers (m_p, q_p, ν_p) called the *periodic data* of p, where

- m_p is the period of p,
- $q_p = \dim W^u_p$, and
- ν_p is the *orientation type of* p which equals $+1$ (resp. -1) whenever $f^{m_p}|_{W^u_p}$ preserves (resp. reverses) orientation.

Denote by $G(S_g)$ the set of Morse-Smale diffeomorphisms $f : S_g \to S_g$ having a unique saddle periodic orbit $\mathcal{O}_\sigma$ and satisfying $\nu_\sigma = +1$. The case $\nu_\sigma = -1$ was investigated in [9].

Let $f \in G(S_g)$. It is well known that the Euler characteristic for an orientable surface of genus g is expressed by the formula: $\chi(S_g) = 2 - 2g$, (see, for example, [5], [2]). On the other hand, by [13], a Morse-Smale diffeomorphism induces a cellular decomposition of S_g whose open cells are unstable submanifolds of periodic points:

$$S_g = \bigcup_{p\in\Omega_f} W^u_p.$$

Then

$$c_2 - c_1 + c_0 = 2 - 2g, \tag{1.1}$$

where c_2 is be the number of the sources of f corresponding to 2-cells, c_1 is the number saddles corresponding to 1-cells, and c_0 is the number of sinks (0-cells).

The following theorem describes the numbers of the periodic orbits of $f \in G(S_g)$.

Theorem 1.1. *The non-wandering set of every diffeomorphism $f \in G(S_g)$ consists of a unique saddle orbit and three node orbits: either one sink orbit and two source orbits or one source orbit and two sink orbits.*

In what follows we assume that a diffeomorphism $f \in G(S_g)$ has a unique sink orbit $\mathcal{O}_\omega$ and two source orbits $\mathcal{O}_{\alpha_1}, \mathcal{O}_{\alpha_2}$ (otherwise, we could just replace f with its inverse f^{-1}).

By assumption f has one saddle periodic orbit preserving the orientation, hence $\nu_\sigma = +1$ and $q_\sigma = 1$. The orbits of $\mathcal{O}_{\alpha_1}, \mathcal{O}_{\alpha_2}$ consist of sources, so $q_{\alpha_1} = q_{\alpha_2} = dim\, W^u_{\alpha_i} = 2$. Moreover, since f preserves the orientation of open connected invariant subsets, we get that $\nu_{\alpha_i} = +1$. The orbit of $\mathcal{O}_\omega$ consists of sinks, whence $q_\omega = \dim W^u_\omega = 0$ and $\nu_\omega = +1$. Thus, a part of the periodic data is already known and our task is to find the periods $m_\omega, m_\sigma, m_{\alpha_1}, m_{\alpha_2}$. Notice that from (1.1) we have

$$m_{\alpha_1} + m_{\alpha_2} - m_\sigma + m_\omega = 2 - 2g. \tag{1.2}$$

Below (a, b) means the greatest common divisor of the natural numbers a, b, also we assume $(0, b) = b$.

Theorem 1.2. *Every diffeomorphism $f \in G(S_g)$ has the following periodic data:*

$$\begin{aligned} m_\omega = m, \qquad m_\sigma = km, \\ m_{\alpha_1} = (k, j+1)\left(\frac{k}{(k, j+1)}, m\right), \\ m_{\alpha_2} = (k, j)\left(\frac{k}{(k, j)}, m\right), \end{aligned} \tag{1.3}$$

for some $m \in \mathbb{N}$, $k \in \mathbb{N}$, $j \in \{0, \ldots, k-1\}$.

Furthermore, for every collection of the natural numbers $k \in \mathbb{N}$, $m \in \mathbb{N}$, $j \in \{0, \ldots, k-1\}$ there exists a diffeomorphism $f \in G(S_g)$ with a periodic data of the form (1.3) on a surface of genus

$$g = 1 + \frac{1}{2}\left((k-1)m - (k, j+1)\left(\frac{k}{(k, j+1)}, m\right) - (k, j)\left(\frac{k}{(k, j)}, m\right)\right).$$

Corollary 1.3. *Due to theorem above every orientable surface of genus g admits a diffeomorphism from the class $G(S_g)$, with the following periodic data: $m_\sigma = 2g + 1$, $m_\omega = m_{\alpha_1} = m_{\alpha_2} = 1$, see Figure 1.1.*

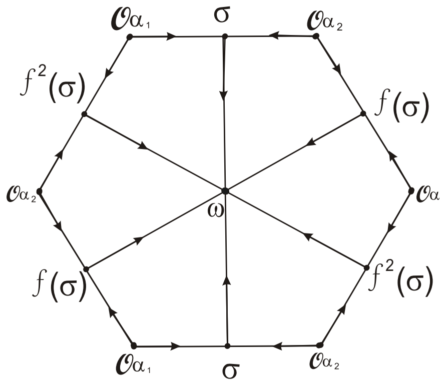

FIGURE 1.1. Diffeomorphisms from $G(S_1)$ with $m_\sigma = 3$, $m_\omega = m_{\alpha_1} = m_{\alpha_2} = 1$

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2. STRUCTURE OF THE NON-WANDERING SET

In this section we prove Theorem 1.1. Detailed proofs of auxiliary statements given in this section can be found in [5].

Proof. Let us show that the non-wandering set of every diffeomorphism $f \in G(S_g)$ consists of a unique saddle orbit and three node orbits (either one sink and two source, or one source and two sink orbits).

For the saddle separatrix ℓ we denote by m_ℓ its *period*, that is, the smallest natural number μ such that $f^\mu(\ell) = \ell$. Also denote by $\mathcal{O}_\ell$ the orbit of the separatrix ℓ. By [5, Proposition 2.3, p. 31], the closure of each unstable saddle separatrix contains a unique sink, while the closure of each stable saddle separatrix contains a unique source. Assume that the unstable separatrix ℓ^u of a saddle point σ contains a sink ω in its closure. Let m be the period of ω. According to [10, Theorem 5.5], f^m is locally conjugate at the point ω with the linear diffeomorphism of $\mathbb{R}^2$ given by the formula

$$L(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right).$$

Let $\mathcal{O}_\omega$ be the orbit of the point ω, $V_\omega = W^s_{\mathcal{O}_\omega} \setminus \mathcal{O}_\omega$, $\hat{V}_\omega = V_\omega/f$ be the orbit space of the action of the group $F = \{f^k, k \in \mathbb{Z}\}$ on V_ω, and $p_\omega : V_\omega \to \hat{V}_\omega$ the natural projection.

Notice that V_ω is diffeomorphic to a disjoint union of m open cylinders $S^1 \times \mathbb{R}$ that are cyclically interchanged by f. Moreover, due to [5, Proposition 2.5, p. 35], the space $\hat{V}_\omega$ is diffeomorphic to a two-dimensional torus. Let $x \in V_\omega$ be any point and $\hat{x} = p_\omega(x)$. Then the natural projection $p_\omega : V_\omega \to \hat{V}_\omega$ is a covering map and we have the following exact sequence

$$0 \to \pi_1(V_\omega, x) \to \pi_1(\hat{V}_\omega, \hat{x}) \xrightarrow{\eta_\omega} \pi_0\mathbb{Z} \to \pi_0 V_\omega \to 1, \tag{2.1}$$

where η_ω is the *boundary* homomorphism. Since the connected component of x in V_ω consists of points $\{f^{km_\omega}(x)\}_{k\in\mathbb{Z}}$, it follows that the image of η_ω is a subgroup $m_\omega\mathbb{Z} \subset \mathbb{Z} \cong \pi_0\mathbb{Z}$.

In other words, we get an epimorphism $\eta_\omega : \pi_1(\hat{V}_\omega) \to m_\omega\mathbb{Z}$ onto the subgroup of $\mathbb{Z}$ consisting of multiples of m_ω. For the convenience of the reader let us recall the definition of η_ω. Let $[\hat{c}] \in \pi_1(\hat{V}_\omega, \hat{x})$ be a loop in $\hat{V}_\omega$ and $c : [0,1] \to V_\omega$ be its lift starting at $c(0) = x \in V_\omega$. Then the end point $c(1) = f^n(x)$, for some $n \in m_\omega\mathbb{Z}$, and $\eta_\omega([\hat{c}]) = n$.

Denote by $a : \mathbb{R}^2 \to \mathbb{R}^2$ the diffeomorphism given by the formula

$$a(x,y) = \left(\frac{x}{2}, 2y\right).$$

Evidently, it has a unique fixed saddle point at the origin O with the stable manifold $W^s_O = Ox$ and the unstable manifold $W^u_O = Oy$. Then the diffeomorphism f^{m_σ} in some neighborhood of the point σ is topologically connected to the diffeomorphism a in a neighborhood of the point O (see, for example, [10, Theorem 5.5]).

Let $\hat{\ell}^u = p_\omega(\ell^u)$ and $j_{\hat{\ell}^u} : \hat{\ell}^u \to \hat{V}_\omega$ be the inclusion map. It follows from [5, Proposition 2.5, p.35] that the set $\hat{\ell}^u$ is a circle smoothly embedded in $\hat{V}_\omega$ and such that $\eta_\omega(j_{\hat{\ell}^u *}(\pi_1(\hat{\ell}^u))) = m_{\ell^u}\mathbb{Z}$. Notice that $p_\omega(\mathcal{O}_{\ell^u}) = \hat{\ell}^u$.

Figure 2.1 depicts the torus $\hat{V}_\omega$ with the projection $\hat{\ell}^u$ of the separatrix ℓ^u such that $\frac{m_{\ell^u}}{m_\omega} = 3$.

Let $\mathcal{N} = \{(x,y) \in \mathbb{R}^2 : |xy| \leq 1\}$. Notice that the set $\mathcal{N}$ is invariant with respect to the diffeomorphism a. A neighborhood of N_σ of the point σ is called *linearizing* if there exists a homeomorphism $\mu_\sigma : N_\sigma \to \mathcal{N}$ conjugating the restriction $f^{m_\sigma}|_{N_\sigma}$ with the diffeomorphism $a|_{\mathcal{N}}$.

In this case the neighborhood

$$N_{\mathcal{O}_\sigma} = \bigcup_{j=0}^{m_\sigma - 1} f^j(N_\sigma)$$

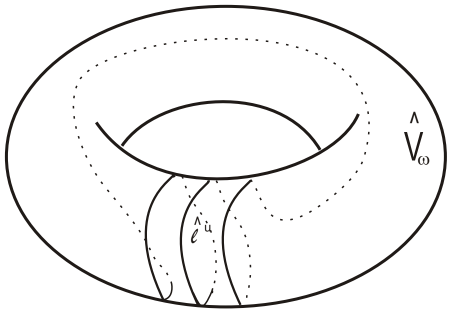

FIGURE 2.1. The projection of the saddle separatrix in the orbit space of the sink basin homeomorphic to the torus

of the orbit $\mathcal{O}_\sigma = \bigcup\limits_{j=0}^{m_\sigma - 1} f^j(\sigma)$ equipped with the map $\mu_{\mathcal{O}_\sigma} : N_{\mathcal{O}_\sigma} \to \mathcal{N}$, defined by

$$\mu_{\mathcal{O}_\sigma}|_{f^j(N_\sigma)} = \mu_\sigma \circ f^{-j} : f^j(N_\sigma) \to \mathcal{N}, \qquad j = 0, \ldots, m_\sigma - 1,$$

is called the *linearizing neighborhood of the orbit* $\mathcal{O}_\sigma$.

Due to [5, Theorem 2.2, p. 29], the saddle point (orbit) of the diffeomorphism f has a linearizing neighborhood.

Let $\mathcal{N}^u = \mathcal{N} \setminus Ox$ and $\hat{\mathcal{N}}^u = \mathcal{N}^u/a$ be the orbit space of the action of the group $\{a^n, n \in \mathbb{Z}\}$ on $\mathcal{N}^u$. Then the natural projection $p_{\hat{\mathcal{N}}^u} : \mathcal{N}^u \to \hat{\mathcal{N}}^u$ is a covering map. Moreover, the fundamental domain of the action of the group $\{a^n, n \in \mathbb{Z}\}$ on $\mathcal{N}^u$ consists of two disjoint curvilinear trapezoids, each of which has equivalent points belonging to the horizontal segments of the boundary. In Figure 2.2 these trapezoids are shaded and it is shown how we can obtain the manifold $\hat{\mathcal{N}}^u$ by identifying their boundaries via the diffeomorphism a. Thus the space $\hat{\mathcal{N}}^u$ is homeomorphic to a pair of two-dimensional annuli K_1, K_2.

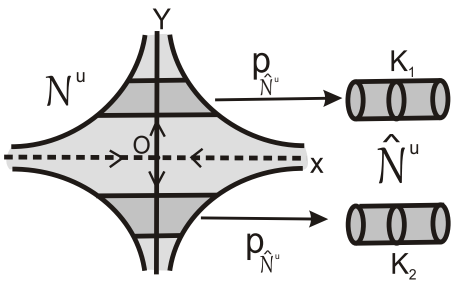

FIGURE 2.2. The orbit space $\hat{\mathcal{N}}^u$

Let

$$N^u_\sigma = N_\sigma \setminus W^s_\sigma, \qquad N^s_\sigma = N_\sigma \setminus W^u_\sigma.$$

Denote by N_{ℓ^u} the connected component of the set N^u_σ containing an unstable separatrix ℓ^u. Let also $\hat{N}_{\ell^u} = p_{_\omega}(N_{\ell^u})$ and $j_{N_{\hat{\ell}^u}} : N_{\hat{\ell}^u} \to \hat{V}_\omega$ be the inclusion map. The set $\hat{N}_{\ell^u}$ is a smoothly embedded annulus in $\hat{V}_\omega$ such that $\eta_\omega(j_{\hat{N}_{\ell^u}*}(\pi_1(\hat{N}_{\ell^u}))) = m_{\ell^u}\mathbb{Z}$.

Denote by A the union of all sink points of the diffeomorphism f. Let also $V_A = W^s_A \setminus A$, $\hat{V}_A = V_A/f$, and $p_A : V_A \to \hat{V}_A$ be the natural projection. Similarly to the above arguments, the orbit space in the sink basin is homeomorphic to the torus, which implies that each connected component of the set $\hat{V}_A$ is homeomorphic to a two-dimensional torus, and the number of connected components coincides with the number of sink orbits.

Let $N^u_{\mathcal{O}_\sigma} = N_{\mathcal{O}_\sigma} \setminus W^s_{\mathcal{O}_\sigma}, N^s_{\mathcal{O}_\sigma} = N_{\mathcal{O}_\sigma} \setminus W^u_{\mathcal{O}_\sigma}$, $\hat{N}^u_{\mathcal{O}_\sigma} = N^u_{\mathcal{O}_\sigma}/f$. It follows from [5, Theorem 2.4, p. 42] that the set $\hat{N}^u_{\mathcal{O}_\sigma}$ is a pair of annuli smoothly embedded in $\hat{V}_A$. Also due to [5, Corollary 2.1, p. 46] the set $\hat{V}_A$ is not empty and, by [5, Corollary 2.2, p. 62], each torus in $\hat{V}_A$ has to contain at least one annulus from the set $\hat{N}^u_{\mathcal{O}_\sigma}$. Thus $\hat{V}_A$ contains one or two connected components.

Similar statements can be formulated for the source point α and for the stable separatrix ℓ^s of the saddle point σ such that $\ell^s \subset W^u_\alpha$.

Denote by R the union of the source points of the diffeomorphism f. Let also $V_R = W^u_R \setminus R$, $\hat{V}_R = V_R/f$ and $p_{_R} : V_R \to \hat{V}_R$ be the natural projection. Similarly to the above arguments, the orbit space in the source basin is homeomorphic to the torus, which implies that each connected component of the set $\hat{V}_R$ is homeomorphic to a two-dimensional torus, and the number of connected components coincides with the number of source orbits. On the other hand, it follows from the equality $V_R = (V_A \setminus N^u_{\mathcal{O}_\sigma}) \cup N^s_{\mathcal{O}_\sigma}$ (see, for instance, [5, Theorem 2.1, p. 28]) that

$$\hat{V}_A = (\hat{V}_R \setminus \hat{N}^u_{\mathcal{O}_\sigma}) \cup \hat{N}^s_{\mathcal{O}_\sigma}.$$

Thus, to get the space $\hat{V}_R$ we have to delete $\hat{N}^u_{\mathcal{O}_\sigma}$ from the torus $\hat{V}_A$ and glue the set $\hat{N}^s_{\mathcal{O}_\sigma}$ to the boundary of the resulting set.

Each of the sets $\hat{N}^u_{\mathcal{O}_\sigma}$, $\hat{N}^s_{\mathcal{O}_\sigma}$ consists of two annuli. Moreover, the annuli $\hat{N}^u_{\mathcal{O}_\sigma}$ are homotopically non-trivially embedded in the torus $\hat{V}_A$. If we assume that $\hat{V}_A$ consists of a unique connected component then $\hat{V}_A \setminus \hat{N}^u_{\mathcal{O}_\sigma}$ consists of two annuli and a gluing $\hat{N}^s_{\mathcal{O}_\sigma}$ to their boundaries gives two two-dimensional tori (see Figure 2.3, where the transition from the sink basins to the sources basins is illustrated by the example of a diffeomorphism

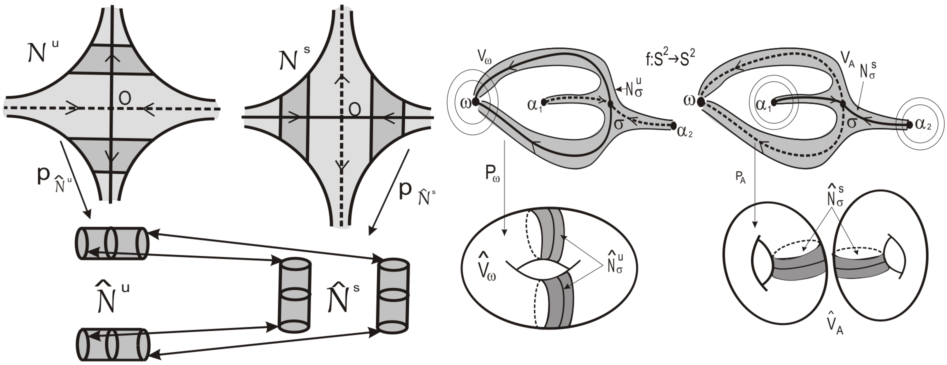

FIGURE 2.3. Regluing along annuli

of the 2-sphere. For convenience, in the above-mentioned basins, fundamental regions are selected, after the identifying of their boundary circles the corresponding tori in the quotient spaces are obtained). This means that there are exactly two source orbits for the diffeomorphism f, that is, $R = \mathcal{O}_{\alpha_1} \cup \mathcal{O}_{\alpha_2}$ for some periodic sources α_1, α_2.

If we assume that $\hat{V}_A$ consists of two connected components then the similar cut and gluing operation implies the existence of the unique source orbit in this case. □

3. Periodic Data

This section is devoted to the proof of Theorem 1.2. Firstly let us show that every diffeomorphism $f \in G(S_g)$ has following periodic data, see (1.3):

$$m_\omega = m, \qquad m_\sigma = km,$$
$$m_{\alpha_1} = (k, j+1)\left(\frac{k}{(k, j+1)}, m\right),$$
$$m_{\alpha_2} = (k, j)\left(\frac{k}{(k, j)}, m\right),$$

where $m \in \mathbb{N}, k \in \mathbb{N}, j \in \{0, \ldots, k-1\}$ are natural numbers.

Let us introduce an abstract model of dynamics in the basin of a periodic sink of *period* m. Let $m \geq 1$ be an integer and $V_m = \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{Z}_m$. Thus V_m is a model for the basin of a periodic sink of *period* m. Let $k \in \mathbb{N}$, $\tau \in \{0, \ldots, k-1\}$ and

$$\gamma_1^\tau = \bigcup_{\tau=0}^{k-1} e^{i\pi\left(\frac{1}{2} - \frac{2\tau}{k}\right)} \times \mathbb{R}^+, \qquad \gamma_2^\tau = \bigcup_{\tau=0}^{k-1} e^{i\pi\left(\frac{1}{2} - \frac{2\tau+1}{k}\right)} \times \mathbb{R}^+,$$

$$\gamma_1=\bigcup_{\tau=0}^{k-1}\gamma_1^\tau\times\mathbb{Z}_m,\qquad\qquad \gamma_2=\bigcup_{\tau=0}^{k-1}\gamma_2^\tau\times\mathbb{Z}_m,$$

Here $\gamma_1\cup\gamma_2$ models the saddle unstable separatrices, see Figure 3.1.

Let $n\geq 0$ be any integer satisfying the following conditions:

- if $k=1$, then $n=0$;
- otherwise, $n\in\{1,\dots,k-1\}$ is such that mn and k are co-prime.

Here mn models the period of periodic unstable separatrices in V_m and $\frac{mn}{k}$ represents their "rotation number", i.e. how the diffeomorphism permutes these separatrices. As a local model for the diffeomorphism on the basin we take the contraction $\phi_{m,k,n}:V_m\to V_m$ given by the formula:

$$\phi_{m,k,n}(z,r,w)=\left(ze^{-\frac{2\pi mn}{k}i},\ \tfrac{r}{2^m},\ w+1 \bmod m\right).$$

Then

$$\phi_{m,k,n}(\gamma_i^\tau\times\{w\})=\gamma_i^{\tau+mn}\times\{w+1 \bmod m\} \tag{3.1}$$

for all $i=1,2$, $\tau=0,\dots,k-1$, $w=0,\dots,m-1$.

Notice that $\hat V_{m,k,n}=V_m/\phi_{m,k,n}$ is a torus. Let $p_{m,k,n}:V_m\to\hat V_{m,k,n}$ the natural projection. Then $\hat\gamma_i=p_{m,k,n}(\gamma_i), i=1,2$ is a knot in $\hat V_{m,k,n}$.

Let $f\in G(S_g)$. For the sink orbit $\mathcal{O}_\omega$ put $V_\omega=W^s_{\mathcal{O}_\omega}\setminus\mathcal{O}_\omega$. Denote by $\hat V_\omega=V_\omega/f$ the orbit space of the action of the group $F=\{f^i, i\in\mathbb{Z}\}$ on V_ω and by $p_{_\omega}:V_\omega\to\hat V_\omega$ the natural projection. The unstable separatrices ℓ_1^u, ℓ_2^u of the saddle point σ have period m_σ and lie in the basin V_ω. Since the group F acts transitively on the connected components of V_ω (the number of such components is m) and on the orbit of each unstable separatrix (the number of the connected components of this orbit is m_σ), it follows that in each connected component of the set V_ω there exists the same number of separatrices from that orbit. Hence the period m_σ is a multiple of the period m.

Thus each connected component of V_ω contains $k:=\frac{m_\sigma}{m}$ separatrices from the orbit of the separatrix ℓ_i^u. Let $\hat\ell_1^u=p_{_\omega}(\ell_1^u)$ and $\hat\ell_2^u=p_{_\omega}(\ell_2^u)$. Then there is a number n and a diffeomorphism $\hat h_\omega:\hat V_\omega\to\hat V_{m,k,n}$ transforming the knots $\hat\ell_1^u$, $\hat\ell_2^u$ to the knots $\hat\gamma_1$, $\hat\gamma_2$. Thus, there is a lift $h_\omega:V_\omega\to V_m$ of $\hat h_\omega$ which sends the separatrices $W^u_{\mathcal{O}_\sigma}\setminus\mathcal{O}_\sigma$ to the frame of rays $\gamma_1\cup\gamma_2$ and conjugates diffeomorphism $f|_{V_\omega}$ with the diffeomorphism $\phi_{m,k,n}$, e.g. [5, Statement 10.35, p. 243]. In this case, we may identify the conjugated objects everywhere below.

Notice that $\gamma_1^0\times\{0\}$ corresponds to one of unstable separatrices, say ℓ_1^u, of the saddle point. Let $\gamma_2^j\times\{\rho\}$ be another separatrix ℓ_2^u, where

$j \in \{0, \ldots, k-1\}$ and $\rho \in \{0, \ldots, m-1\}$. Then

$$(\rho, m) = 1$$

due to connectivity of the ambient surface S_g. Moreover, as h_ω conjugates f with $\phi_{m,k,n}$, it follows from (3.1) that for every $\tau \in \{0, \ldots, k-1\}$

$$\gamma_1^\tau \times \{w\} \qquad \text{and} \qquad \gamma_2^{(\tau+j) \bmod k} \times \{w + \rho \bmod m\}$$

are stable and unstable manifolds of the same saddle point, so the parameters j and ρ determine the are responsible for division of separatrices into stable-unstable submanifolds of the same saddle.

Notice that the choice of j and ρ depends on the order which h_ω maps $\gamma_1^0 \times \{0\}$ and $\gamma_2^j \times \{\rho\}$ to separatrices of the saddle point. It exchange those separatrices, so $\gamma_1^0 \times \{0\}$ will correspond to ℓ_2^u, and $\gamma_2^{j'} \times \{\rho'\}$ to ℓ_1^u, then the pairs (j, ρ) and (j', ρ') are related by the formulas:

$$j + j' + 1 = k, \qquad\qquad \rho + \rho' + 1 = m.$$

This leads to the formulas

$$(k, j' + 1) = (k, j), \qquad\qquad (m, \rho' + 1) = (m, \rho),$$

guaranteeing that in (1.3) the periods $m_{\alpha_1}, m_{\alpha_2}$ do not depend on the order of separatrices.

By Theorem 1.1, the non-wandering set of f contains exactly two source orbits $\mathcal{O}_{\alpha_1}$,$\mathcal{O}_{\alpha_2}$ such that $cl(\ell_1^u) = \ell_1^u \cup \alpha_1$, and $cl(\ell_2^u) = \ell_2^u \cup \alpha_2$. Thus

$$W^s_{\mathcal{O}_\omega} = S_g \setminus cl(W^s_{\mathcal{O}_\sigma}).$$

If we remove from our surface S_g the closures of m_σ stable manifolds, then we get m disks (the basins of the sinks). Since each stable manifold locally separates two such discs on the supporting surface, it follows that every stable manifold will included twice to the boundaries of the disks after cutting. Thus, the boundary of each disk consists of $\frac{2m_\sigma}{m} = 2k$ stable manifolds so that disk can be regarded as $2k$-gon (see Figures 3.1 and 3.2 on the left).

The stable separatrices are called s_1- and s_2-*curves*, the unstable separatrices (they are located on the rays of the frames γ_1 and γ_2) are called *u-curves frames*, and the segments connecting the vertices of the polygon with its center are called *t-curves*. Thus, this (colored) curves divide every polygon into the triangles with s_i-, t-, u-sides. Let us enumerate these triangles as it shown on Figures 3.1 and 3.2 on the left.

As u-sides belonging to the rays

$$\gamma_1^\tau \times \{w\}, \qquad \text{and} \qquad \gamma_2^{(\tau+j) \bmod k} \times \{(w + \rho) \bmod m\}$$

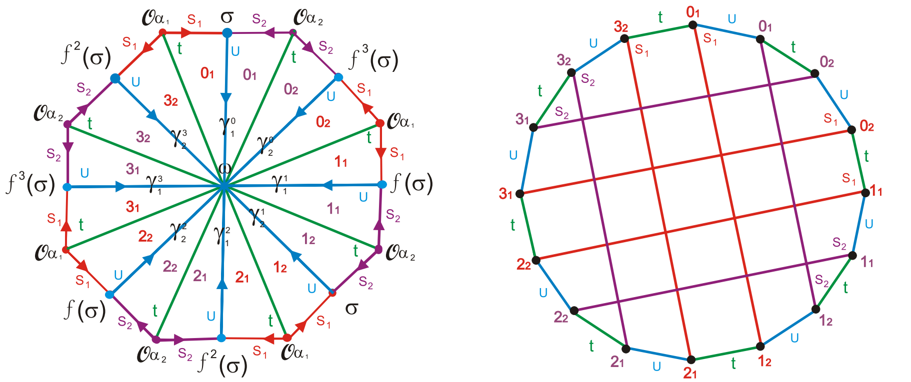

FIGURE 3.1. The octagon Π that is the closure of the sink basin of the diffeomorphism $f \in G$ (on the left) and the four-color graph T_f constructed on it (on the right). Here $m = 1$, $k = 4$, $n = 1$, $j = 1, \rho = 0$

are separatrices of the same saddle point of f, it follows that in order to get the surface S_g from the polygons $\Pi_0, \dots, \Pi_{m-1}$ we have to identify the pairs of those sides of polygons which are transversal to this pair of the separatrices.

To compute periods of source points we associate a four-color graph with the diffeomorphism f in the following way (see for details, for example, [7] and [4]):

1) the vertices of the graph T_f one-to-one correspond to the triangular regions;
2) two vertices of the graph are incident to the edge of color s_1, s_2, t or u if the triangular areas corresponding to these vertices have a common s_1, s_2, t or u side (see Figure 3.1 and 3.2 on the right).

Denote by B_f the set of vertices of the graph T_f and by Δ_f the set of triangles in the partition of the polygon. Let also $\pi_f : \Delta_f \to B_f$ be a one-to-one correspondence between the set of triangular domains of the diffeomorphism f and the set of vertices of the graph T_f. Then f induces an automorphism $P_f = \pi_f \circ f \circ \pi_f^{-1}$ of the set of vertices and edges of the graph T_f. Moreover,

- the set of sink points of the diffeomorphism f is in a one-to-one correspondence with the set of tu-cycles of the graph T_f;
- the set of saddle points of the diffeomorphism f is in a one-to-one correspondence with the set of su -cycles of the graph T_f;

- the set of source points of the diffeomorphism f is in a one-to-one correspondence with the set of ts -cycles of the graph T_f.

Thus, to determine the period m_{α_i} of the point α_i, $i = 1, 2$, we have to calculate the number of $s_i t$-cycles. As every such cycle is an image of an other such cycle by f, we see that all cycles must have the same period. Hence, the length of each such cycle is some even number (as edges s_i and t follow one after other), which we denote by $2\lambda_i$. Notice that the number of s_i- and t-edges in all $s_i t$-cycles equals $2km$ then m_{α_i} is calculated by the formula

$$m_{\alpha_i} = \frac{km}{\lambda_i}. \tag{3.2}$$

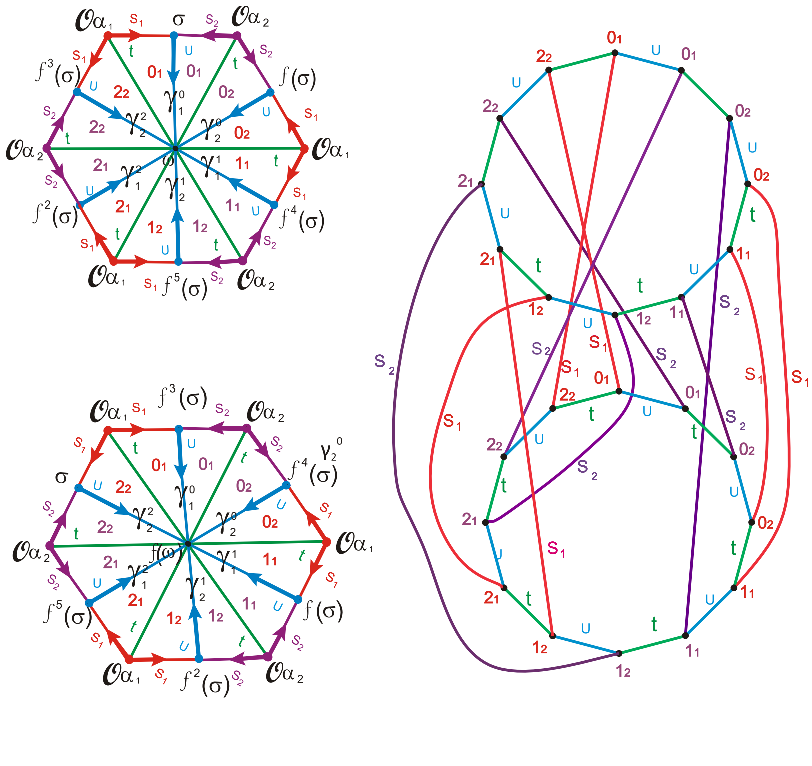

FIGURE 3.2. Hexagons Π and $f(\Pi)$ which are the closures of the sinks basins of the diffeomorphism $f \in G$ (left) and the four-color graph T_f (right) constructed from them. Here $m = 2$, $k = 3$, $n = 1$, $j = 2$, and $\rho = 1$

Now we will calculate the length of s_1t-cycle starting from the s_1-edge $(0_1, j_2)$. We get the following sequence of the vertices

$$0_1 \to j_2 \to ((j+1) \bmod k)_1 \to ((2j+1) \bmod k)_2 \to \\ \to (2(j+1) \bmod k)_1 \to \cdots \to (\lambda_1(j+1) \bmod k)_1,$$

where $(\tau)_i$ correspond to the separatrice γ_i^τ, $i = 1, 2$. Since that sequence constitute a cycle, we obtain that

$$\lambda_1(j+1) \bmod k = 0, \qquad \lambda_1\rho \bmod m = 0,$$

whence

$$\lambda_1(j+1) = lk, \qquad \lambda_1\rho = rm$$

for some l, r.

Let $A = (k, j+1)$. Then $k = pA, j+1 = qA$, where $(p,q) = 1$. Hence,

$$\lambda_1 = \frac{lp}{q} = \frac{rm}{\rho}.$$

As λ_1 is a natural, $(p,q) = 1$, and $(\rho, m) = 1$, it follows that $l = \mu q$ and $r = \nu\rho$. Hence $\lambda_1 = \mu p = \nu m$ and $(\mu, \nu) = 1$, since λ_1 is the minimal number with the property $\lambda_1 = \tilde{\mu} p = \tilde{\nu} m$ for some natural $\tilde{\mu}$, $\tilde{\nu}$.

Let $B = (p, m)$ then $p = xB$, $m = yB$, where $(x, y) = 1$. Therefore $\mu x = \nu y$, $\mu = y, \nu = x$, and $\lambda_1 = yp$. Thus

$$m_{\alpha_1} = \frac{km}{\lambda_1} = \frac{km}{yp} = \frac{pAm}{yp} = AB = (k, j+1)\left(\frac{k}{(k, j+1)}, m\right).$$

A similar construction for α_2 gives $m_{\alpha_2} = (k, j)\left(\frac{k}{(k,j)}, m\right)$. By (1.2),

$$m + (k, j+1)\left(\frac{k}{(k, j+1)}, m\right) + (k, j)\left(\frac{k}{(k, j)}, m\right) - km = 2 - 2g.$$

In an addition, every collection of natural numbers $m \in \mathbb{N}$, $k \in \mathbb{N}$, $j \in \{0, \ldots, k-1\}$ can be realized by an admissible four-color graph, which in turn, due to [7], allows to construct a diffeomorphism $f \in G(S_g)$ with a periodic data of the form (1.3) on a surface of genus

$$g = 1 + \frac{1}{2}\left((k-1)m - (k, j+1)\left(\tfrac{k}{(k,j+1)}, m\right) - (k, j)\left(\tfrac{k}{(k,j)}, m\right)\right).$$

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Elena Nozdrinova
National Research University "Higher School of Economics"
Email: maati@mail.ru
ORCID: orcid.org/0000-0001-5209-377X

Olga Pochinka
National Research University "Higher School of Economics"
Email: olga-pochinka@yandex.ru
ORCID: orcid.org/0000-0002-6587-5305

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On measures of nonplanarity of cubic graphs

Leonid Plachta

Abstract. We study two measures of nonplanarity of cubic graphs G, the genus $\gamma(G)$, and the edge deletion number $ed(G)$. For cubic graphs of small orders these parameters are compared with another measure of nonplanarity, the rectilinear crossing number $\overline{cr}(G)$. We introduce operations of connected sum, specified for cubic graphs G, and show that under certain conditions the parameters $\gamma(G)$ and $ed(G)$ are additive (subadditive) with respect to them.

The minimal genus graphs (i.e. the cubic graphs of minimum order with given value of genus γ) and the minimal edge deletion graphs (i.e. cubic graphs of minimum order with given value of edge deletion number ed) are introduced and studied. We provide upper bounds for the order of minimal genus and minimal edge deletion graphs.

Анотація. З відомої теореми Куратовського випливає, що кубічний граф є непланарним тоді і тільки тоді, коли він не місить підграфів, гомеоморфних $K_{3,3}$. Для непланарних графів існує декілька характеристик графа, які визначають міру його непланарності. Для заданого 3-зв'язного кубічного графу G позначимо через $ed(G)$ найменше число ребер в G, після викидання яких дістанемо планарний підграф, а через $g(G)$ (орієнтовний) рід графа G. Крім того, нехай $\overline{cr}(G)$ позначає мінімальне число внутрішніх перетинів ребер графа G серед усіх прямолінійних імерсій графа в площині. Кубічний граф G називається k-мінімальним відносно параметра $ed(G)$ (відповідно, параметрів $\overline{cr}(G)$, $g(G)$), якщо $ed(G) = k$ (відповідно, $\overline{cr}(G) = k$, $g(G) = k$) і порядок графа G є мінімальним серед усіх 3-зв'язних кубічних графів з даною властивістю.

В роботі досліджуються k-мінімальні відносно параметрів $ed(G)$ і $g(G)$ 3-зв'язних кубічних графи. Описані операції на 2-зв'язних і 3-зв'язних кубічних графах (зв'язна сума, подвійна зв'язна сума), які мають властивість адитивності (субадитивності) відносно параметрів $ed(G)$ і $g(G)$.

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Обчислені характеристики $ed(G)$ і $g(G)$ k-мінімальних відносно параметра $\overline{cr}(G)$ кубічних графів для малих чисел k. Дається порівняння характеристик $ed(G)$, $g(G)$ і $\overline{cr}(G)$ для зв'язних кубічних графів G.

Запропонований конструктивний метод, який дозволяє отримувати 2-зв'язні і циклічно 4-зв'язні кубічні графи G з як завгодно великими характеристиками $ed(G)$ і $g(G)$. Даний метод базується на адитивних властивостях операцій "зв'язна сума" і "подвійна зв'язна сума" на кубічних графах. Отримані також верхні оцінки порядку мінімальних відносно характеристик $ed(G)$ і $g(G)$ графів G, в класі 2-зв'язних і 3-зв'язних кубічних графів G. Сформульована також відкрита проблема, яка стосується нижньої оцінки для порядку k-мінімальних відносно параметрів $ed(G)$ і $g(G)$ кубічних графів.

1. Introduction

We consider finite graphs $G = (V, E)$ without loops and multiple edges. The number of vertices of G is called the *order* of G and denoted by $|G|$. The number of edges $|E|$ of the graph G is called its *size* and denoted by $||G||$. The Kuratowski theorem states that a graph G is planar if and only if it does not contain subgraphs homeomorphic to K_5 and $K_{3,3}$. A regular graph of valence 3 is called *cubic*. For cubic graphs, the only forbidden graphs are those which are not homeomorphic to $K_{3,3}$. There are different measures of nonplanarity of a graph. Let us recall their definitions.

For a given connected graph G denote by $\gamma(G)$ the *(orientable) genus* of G i.e. the minimal genus of an orientable closed connected surface M such that G has an embedding in M. Note that each such embedding is 2-cell. The problem of deciding whether a cubic graph G has the genus $\gamma(G) \leq m$ is known to be NP-complete, [21]. There are some upper and lower bounds of $\gamma(G)$ for different classes of graphs G, [19]. For cubic graphs G, the precise values of the parameter $\gamma(G)$ are known only for special classes of them (for example, for some snarks, etc., see [15, 19]).

Another well known measure of nonplanarity of a graph G is the *crossing number* $cr(G)$ (the *rectilinear* crossing number $\overline{cr}(G)$). This is the minimal number of proper double crossings of edges among all immersions of G in the plane (the minimal number of proper double crossings of edges among all rectilinear immersions of G in the plane, respectively). The computation of the crossing number of a graph is also an NP-complete problem, [7]. Note that, in general, $cr(G)$ and $\overline{cr}(G)$ are distinct numbers, [2]. There are estimations of the parameters $cr(G)$ and $\overline{cr}(G)$ for complete graphs, complete bipartite graphs, and other special classes of graphs (see, for example [10, 20]). The precise values of $cr(G)$ and $\overline{cr}(G)$ are known only for particular nonplanar graphs (for example, for small complete and complete bipartite graphs, [17, 20]).

For a given graph G, denote by $ed(G)$ the minimal number of edges in G such that after their deletion the resulting graph becomes planar. The parameter $ed(G)$ is called *the edge deletion number of* G and the corresponding problem of finding the minimal set of edges to be deleted in a graph G is known as MINED. Even for cubic graphs, the problem MINED is known to be NP-complete, [5]. Algorithms of computing $ed(G)$, in particular, for cubic graphs, are described in [3, 5, 4].

Comparing with the parameters $\gamma(G)$ and $cr(G)$, there are much more fewer results concerning evaluation of the number $ed(G)$. Pegg jr and Exoo [12] introduced the notion of a minimal crossing graph. For a given natural number k a cubic graph G is called *minimal k-crossing graph* (in original, k-crossing graph [12]) if G has a minimal order among all cubic graph H with $\overline{cr}(H) = k$. By this analogy, we introduce minimal k-genus and minimal k-edge deletion graphs. Denote by $\beta(G)$ the *cyclomatic* number of the connected graph G, $\kappa(G)$ the *vertex connectivity* and $\lambda(G)$ the *edge connectivity* of G. For graphs G of maximal degree at most three the number $\lambda(G)$ and $\kappa(G)$ coincide, [6]. By this reason, for cubic graphs G we will abbreviate the terminology and use the term "the connectivity of G".

We say that a connected graph $G = (V, E)$ is *cyclically k-edge connected* if no set of fewer than k edges is cycle-separating in G. The *edge cyclic connectivity* $\zeta(G)$ of the cubic graph G is the largest integer $k \leq \beta(G)$ for which G is cyclically k-edge connected. For any cubic connected graph G we have obviously $\kappa(G) = \lambda(G) \leq \zeta(G)$. Note that $\zeta(G)$ is equal to $\beta(G)$ if and only if G does not have any cycle-separating edge cut. Moreover for cubic graphs G with $\zeta(G) \leq 3$ the values of vertex connectivity, edge connectivity and cyclic k-edge connectivity coincide, [16]. As an example, for the Petersen graph P we have $\kappa(P) = \lambda(P) = 3$ but $\zeta(P) = 5$.

We say that a connected cubic graph G is *cyclically k-vertex connected* if it contains no cycle-separating vertex cut with fewer than k vertices. For exception of few graphs (which are $K_4, K_{3,3}$ and the multigraph Θ_2), the notions of cyclically k-vertex connected graph and cyclically k-edge connected graph coincide, [16, Proposition 3].

In Section 2, we evaluate genus and edge deletion number of minimal k-crossing graphs for small numbers k. These auxiliary results are used in Section 3.

Battle et al. [1] have shown that the genus of any connected graph is equal to the sum of blocks with respect to its block decomposition. This is perhaps the first known result on additivity of the (orientable) genus of graph. The operation of the vertex amalgamation applied to 2-connected cubic graphs gives a separable graph which contains a vertex of degree 4.

Another operation is the edge amalgamation of graphs G_1 and G_2, [13]. Miller, [13], introduced the generalized genus of a graph and showed that it is additive with respect to the edge amalgamation of two graphs. The operation of edge amalgamation does not preserve the class of cubic graphs. In [8], Gross also studied bar-amalgamation of graphs. All these operations, when applied to cubic graphs, produce the graphs which are outside the given class. Moreover they are not compatible with such properties of graphs (including cubic graphs) as the vertex connectivity and the edge connectivity. Therefore they cannot serve as a good tool for the construction of graphs with big numbers of the parameters ed and γ inside the class of cubic graphs with the given connectivity.

In Section 3, we introduce two operations of connected sum which are suitable for cubic graphs. We study additivity properties of genus and edge deletion number with respect to these operations. The first operation, when applied to two 2-connected cubic graphs, results in a 2-connected cubic graph. Similarly, the second operation preserves, in general, the class of 3-connected (or even cyclically 4-edge connected) cubic graphs. Additivity properties of cubic graphs are provided by Theorems 3.1, 3.2 and 3.3 (subject to the parameter γ), and by Theorems 3.4 and 3.5 (subject to the parameter ed). By using using these properties, we provide upper bounds for the order of a 2-connected and 3-connected cubic graphs G, which are minimal with respect to these parameters (Corollaries 3.1-3.4). These are the main results of the paper.

In [18] we use more subtle arguments for obtaining upper bounds of the order of minimal cubic graphs G with prescribed value of the parameter γ. The additivity properties of the parameter γ given by Theorems 3.1, 3.2 and 3.3 are also essential in this relation.

2. Measures of nonplanarity of cubic graphs: small orders

We start by considering the parameters $cr(G)$ and $\overline{cr}(G)$ for small cubic graphs G and compare these numbers with the parameters $\gamma(G)$ and $ed(G)$. Denote by $g(G)$ the girth of the graph G. In our study of the (orientable) genus of cubic graphs we shall use the notion of the rotation system on a graph.

A *rotation system* on a graph $G = (V, E)$ is a family $\Pi = \{\pi_v : v \in V\}$, where π_v is a cyclic permutation of the edges incident with v. With any 2-cell embedding φ of a graph G into an oriented closed surface S it is associated a rotation system Π on G. The pair (G, Π) is called a *rotation graph*. Moreover for a given rotation graph (G, Π) one can construct a system R of (oriented) circuits on G in such a way that each edge e of G is contained twice in the circuits, but with opposite orientations. The circuits

c from R can be thought of as oriented boundaries of 2-dimensional discs D_c. Gluing together the family of discs $D_c, c \in R$, along their oriented boundaries, we shall obtain an oriented closed surface S and this provides a natural 2-cell embedding ψ of G into S. The circuits c from R are called the *facial cycles* of the embedding ψ. The correspondence between the 2-cell embeddings of a graph G into oriented surfaces and rotation systems on G is one-to-one, in a usual sense. For more details see [9], [14].

It is easy to see that we have the following inequalities:

$$\gamma(G) \ \leq \ ed(G) \ \leq \ cr(G) \ \leq \ \overline{cr}(G).$$

It can be shown that for cubic graphs the difference between any two of the parameters $\gamma(G), ed(G), cr(G)$ of G can be arbitrarily large. This can be proven, for example, by using results of Sections 2 and 3. Moreover, there exist graphs G for which the number $cr(G)$ is less than $\overline{cr}(G)$ (more precisely, $cr(G) = 4$ and $\overline{cr}(G) = m$ for any $m > 4$, [2]).

We shall say that a cubic graph G is *minimal l-genus graph* if $\gamma(G) = l$ and it is of minimum order among all 2-connected cubic graphs with this property. Similarly, for a given nonnegative integer l, a cubic graph G is minimal l-edge deletion graph, if $ed(G) = l$ and G is of minimum order among all 2-connected cubic graphs with this property.

In this section, we evaluate or estimate the order of minimal graphs with respect to parameters γ and ed for small numbers l. First count all minimal l-crossing graphs G for small values l. Minimal l-crossing graphs have been described up to value $l \leq 8$ in [12]. Note that for $l = 9$ it is unknown any minimal crossing graph G. At present, for $l \geq 10$, there are known only hypothetically minimal l-crossing graphs. Using minimal l-crossing graphs, we will find some minimal cubic graphs with respect to parameters ed and γ. For cubic graphs of small order we use the notations as in [12].

In the following, we will work in the piece wise linear category PL, [11]. Therefore surfaces are 2-dimensional PL-manifolds, graphs are 1-dimensional polyhedra, the maps (embeddings) of graphs are PL-maps (PL-embeddings) and the images of graphs under such maps are subpolyhedra of PL-manifolds (surfaces). For more detailed information about the category PL see also [14].

1. For $l = 1$ there is a unique minimal crossing graph, the graph $K_{3,3}$. We have obviously

$$ed(K_{3,3}) = \overline{cr}(K_{3,3}) = cr(K_{3,3}) = \gamma(K_{3,3}) = 1.$$

2. For $l = 2$ there are two minimal crossing graphs. These are the Petersen graph P (see Figure 2.1b) and the graph $CNG2B$ (see Figure 2.1a). We have obviously

$$ed(P) = 2, \qquad \gamma(P) = 1, \qquad ed(CNG2B) = \gamma(CNG2B) = 1.$$

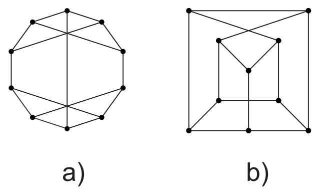

FIGURE 2.1. The minimal 2-crossing graphs

We shall say that a system F of (oriented) circuits in a graph G is *admissible* if it determines a rotation system or can be completed to a rotation system on G by adding some additional circuits in G. In the latter case, we will say that F is *incomplete*. Note that for a connected cubic graph G of order 12, any incomplete admissible system F on G consisting of 4 circuits can be completed to a rotation system R that induces embedding of G in torus or the sphere S^2.

Lemma 2.1. *For any connected cubic graph G of order 12 we have*

$$\gamma(G) \leq 1.$$

Proof. Let G be a connected cubic graph of order 12. We know from [12] that if $|G| \leq 12$, then $\overline{cr}(G) \leq 2$. If $\kappa(G) = 1$, the assertion follows (by using for example results from [1]). If $\kappa(G) = 2$, then after removal two nonincident edges from G we shall obtain two planar subcubic graphs, G_1 and G_2. The only case, when one can worry about, is that G_1 and G_2 are isomorphic to $K_{3,3} - e$ where e is an edge of $K_{3,3}$. But in this case, G is isomorphic to a connected sum of two copies of $K_{3,3}$ and has genus 1 (see Section 3). Therefore we may assume that G is 3-connected. If $\overline{cr}(G) = 1$ we have obviously $\gamma(G) = 1$. If $\overline{cr}(G) = 2$ and the equality reaches via a straight line drawing G in the plane, in which one edge intersects two another edges, the assertion also easily follows.

Assume that there is an immersion of G in the oriented plane P in which we have crossings of two pairs of different edges: e_1 and e_2, and f_1 and f_2. Deleting e_1, e_2, f_1 and f_2 from G, we get a subcubic multigraph H which has a natural embedding φ in P. We may assume without loss of generality that H is connected, otherwise one can use a flip and redraw G in the plane with a fewer number of crossings. Denote by Π the rotation system on H associated with the embedding φ. Now consider all possible configurations of the induced plane embedding of the (multi)graph H and the positions of the deleted edges with respect to it.

a) There is an inner face r of the embedding φ which contains two pairs of crossing edges, e_1 and e_2, and f_1 and f_2. We have three types of configurations describing positions of these edges inside a *regular* face r. In any case, we are able to define an admissible system of circuits in G which, after a suitable completion, generates an embedding of G into torus.

In the first case (see Figure 2.2), we have inside r two internal vertices of G and two crossings of different pairs of edges. The position of external two edges of H and incident vertices is irrelevant and so, is not indicated here. One can replace the facial circuit dr of the rotation system Π with three new circuits c_1, c_2, c_3 as indicated in Figure 2.2. The system of circuits $F = \{c_1, c_2, c_3\}$ in G is admissible and contain each "internal" edge of the immersed graph G twice, but with opposite orientations. The orientation of edges positioned on dr coincide with the one of the cycle dr. We can complete F to a rotation system on G by adding three outer circuits of Π, u_1, u_2 and u_3. The system of circuits $\mathcal{F} = \{c_1, c_2, c_3, u_1, u_2, u_3\}$ defines a rotation system T on G of genus 1.

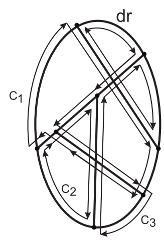

FIGURE 2.2.

In the second case, we have inside r one internal vertex of G and two crossings of edges, as shown in Figure 2.3a). The edges of G positioned inside r are called internal while the edges in the exterior of r are called external (subject to the given immersion of G in P). Let E' be the set of external edges of G. The vertices of G incident to external edges are called external. Denote by H_1 the graph $G - E'$. Then H_1 is immersed via φ in the closure of the face r. Consider in G the system of circuits

$$F_1 = \{c_1, c_2, c_3, c_4, c_5, c_6\}$$

indicated in Figure 2.3b). F_1 defines an embedding of H_1 into a sphere S^2. We have two possibilities:

a) G is obtained from H_1 by gluing two nonincident edges e_1 and e_2 to H_1;
b) G is obtained from H_1 by gluing the graph $K_{1,3}$ along three vertices of degree 1.

Consider the subcase a). Suppose that the end vertices of e_1 (or e_2) are lying on the same circuit c_i. One can replace c_i in the system F_1 with two new circuits, c_i' and c_i'', as shown in Figure 2.3c). Denote by F_2 the new

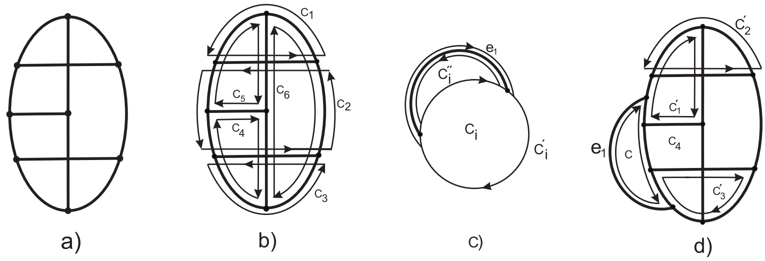

FIGURE 2.3.

family consisting of seven circuits. Obviously F_2 is a rotation system on the graph $H_1 \cup e_1$ ($H_1 \cup e_2$, respectively). The remaining edge e_k from $\{e_1, e_2\}$ joins the vertices of different circuits of F_2, say d_i and d_j. This results in a single circuit d in G containing both d_1 and d_2 and the edge e_k, which is involved twice and with opposite directions. Finally we get a rotation system R on G consisting of six circuits.

Now suppose that both e_1 and e_2 joins the vertices of the same circuits, c_i and c_j of F_1. The vertices of e_1 and e_2 divide c_i into two directed paths, l_1 and l_2, and c_j into two directed paths, m_1 and m_2. Combining the paths l_1, l_2, m_1 and m_2 with the oriented edges e_1 and e_2, we get two circuits in G, d_i and d_j. Each edge $e_i, i = 1, 2$, is contained both in d_i and d_j, but with opposite orientation. Replacing the pair of circuits c_i and c_j with the pair d_i and d_j, we obtain an admissible system F in G, consisting of six circuits. Therefore F defines in G a rotation system of genus one.

In the remaining cases, we can take three circles c_1', c_2' and c_3' of F_1, endow them with a suitable orientation, and add to them a new circuit c of G, which contains e_1 or e_2. This results in an admissible system F' on G consisting of 4 circuits. The choice of c_1', c_2' and c_3' depends on a position of vertices of e_1 or e_2 on dr (see, for example, Figure 2.3d). Completing F' by two new circuits, we get a rotation system R on G of genus one.

The subcase b) is handled in the same way as subcase a). We omit here the details.

The third case of configuration for H_1 is indicated in Figure 2.4a). We also have two possibilities:

a) G is obtained from H_1 by gluing up two nonincident edges e_1 and e_2;
b) G is obtained from H_1 by gluing up the graph $K_{1,3}$ along three vertices of degree 1.

There is no essential difference between the subcases a) and b). Let us consider the subcase a). We can choose two circuits c_1 and c_2 in H_1 and add to them two circuits d_1 and d_2 in G such that d_1 contains the edge e_1

and d_2 contains the edge e_2. As a result, we get an incomplete admissible system F' of circuits in G. The choice of circuits c_1 and c_2 depends on the position of vertices of the edges e_1 and e_2 on dr, see Figure 2.4b) and c). Then F' can be completed to an admissible system F in G, consisting of six circuits. An exceptional case of the pairs of crossing edges e_1, e_2 and f_1, f_2

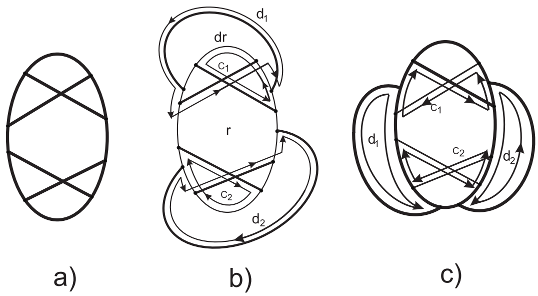

FIGURE 2.4.

inside a *nonregular* face r is shown in Figure 2.5. The following family of circuits in G determines a rotation system R of genus one:

$c_1 = (v_1, v_2, v_3, v_4)$, $c_2 = (v_4, v_3, v_5, v_6, u_6)$,

$c_3 = (v_6, u_2, u_3, u_5, u_6)$, $c_4 = (u_2, u_1, u_4, u_3)$,

$c_5 = (u_1, u_2, v_6, v_5, v_1, v_4, u_6, u_5)$, $c_6 = (u_1, u_5, u_3, u_4, v_2, v_1, v_5, v_3, v_2, u_4)$.

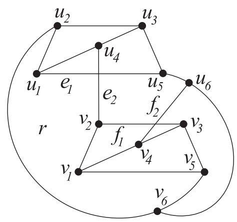

FIGURE 2.5.

b) There are two faces r_1 and r_2 of the embedding φ such that r_1 contains the crossing of e_1 and e_2, and r_2 contains the crossing of f_1 and f_2.

If r_1 and r_2 are disjoint, the existence of a rotation system Π' on G with 6 circuits is obvious. If r_1 and r_2 have a unique edge in common, we have a configuration shown in Figure 2.6. There is a rotation system R on G with 6 facial circuits. We indicate here only a noncomplete admissible system consisting of four circuits, $F = \{c_1, c_2, c_3, c_4\}$. The circuits are the following:

$c_1 = (u_2, u_3, u_4, u_1)$, $c_2 = (u_1, u_4, u_5, u_{11}, u_{12})$,

$$c_3 = (u_{11}, u_5, u_6, u_9, u_{10}), \qquad c_4 = (u_9, u_6, u_7, u_8).$$

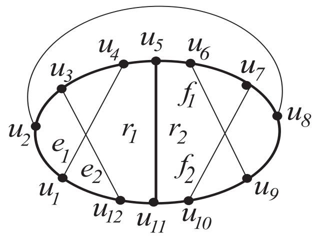

FIGURE 2.6.

Assume now that r_1 and r_2 have two edges in common, see Figure 2.7. In this case, we indicate a rotation system R on G with the following six circuits:

$$c_1 = (u_2, u_1, v_2, v_1), \qquad c_2 = (v_5, v_4, u_5, u_4),$$
$$c_3 = (u_1, u_2, u_3, u_4, u_5, u_6), \qquad c_4 = (v_1, v_2, v_3, v_4, v_5, v_6),$$
$$c_5 = (u_2, v_1, v_6, u_6, u_5, v_4, v_3, u_3), \qquad c_6 = (u_1, u_6, v_6, v_5, u_4, u_3, v_3, v_2).$$

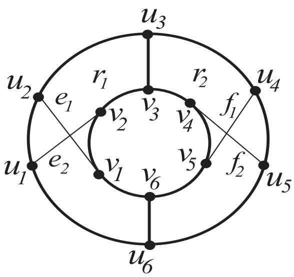

FIGURE 2.7.

c) It can occur that H is a multigraph with three loops and the pairs of crossing edges of G are situated in the outer face p of the embedding φ. We depict in Figure 2.8 such a configuration. In this case, we indicate the following noncomplete admissible system of circuits in G:

$$c_1 = (u_{10}, u_9, u_{12}, u_{11}), \qquad c_2 = (u_{11}, u_{12}, u_3, u_4),$$
$$c_3 = (u_4, u_3, u_2, u_1), \qquad c_4 = (u_5, u_6, u_7).$$

Note that the case when one pair of crossing edges of G is inside p and the other one is inside a region bounded by a loop of H is not admissible by the assumption that the graph G is 2-connected. Lemma 2.1 is completed. □

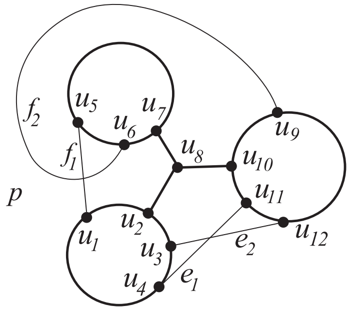

FIGURE 2.8.

3. For $l = 3$ there are eight minimal crossing graphs. We count them according to [12]: $CNG\ 3A$, $CNG\ 3B$, $CNG\ 3D$, $CNG\ 3E$, $CNG\ 3F$, $CNG\ 3H$, $GP(7,2)$, the Heawood graph H (see Figure 2.9).

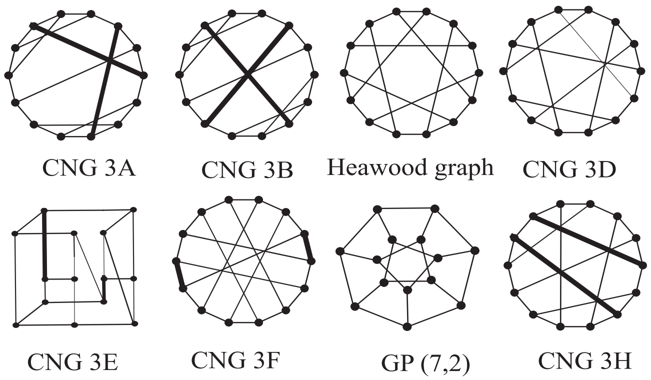

FIGURE 2.9. The minimal 3-crossing graphs

By direct computation, we have

$$\begin{aligned} ed(CNG\ 3A) &= ed(CNG\ 3B) = ed(CNG\ 3E) \\ &= ed(CNG\ 3F) = ed(CNG\ 3H) = 2. \end{aligned}$$

In Figure 2.9, for each of these graphs we indicate by bold line the two edges after removal of which we obtain a planar subgraph. In Figure 2.10a), we indicate 3-crossing drawing of the graph $GP(7,2)$. By bold lines there are indicated two edges in $GP(7,2)$ after removal of which we obtain a planar subgraph. Similarly, in Figure 2.10b) we indicate 3-crossing drawing of the graph $CNG\ 3D$. Removing two bold edges from it also leads to a planar subgraph.

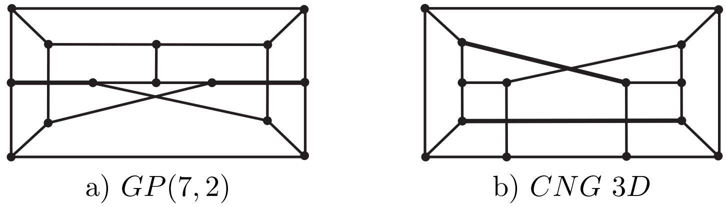

FIGURE 2.10.

Lemma 2.2. *For the Heawood graph H we have $ed(H) = 3$.*

Proof. It is well known that H is symmetric graph. Remove any edge e from H. The resulting graph U can be represented as a sum of two subgraphs H_1 and H_2 and the edge f, see Figure 2.11. The subgraphs H_1 and H_2 have in common a path l of length 5. Denote by $E(l)$ the set of edges of the path l.

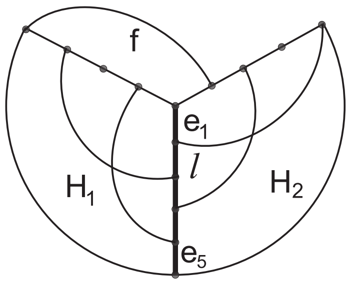

FIGURE 2.11.

Removing the edge f from U, we obtain obviously a non planar graph. Let h be any edge of the graph $H_1 \setminus E(l)$. There is a path $p_1 \subset H_1 \setminus \{E(l) \cup h\}$ such that $H_2 \cup p_1$ contains a graph homeomorphic to $K_{3,3}$. Similarly, for any edge g of the graph $H_2 \setminus E(l)$ there is a path $p_2 \subset H_2 \setminus \{E(l) \cup g\}$ such that $H_1 \cup p_2$ contains a graph homeomorphic to $K_{3,3}$. Therefore removing any edge k from $U \setminus E(l)$ results in a non planar subgraph. On the other hand, it is not difficult to check that removing any edge $e_i \in E(l)$ from U also leads to a graph which contains a subgraph homeomorphic to $K_{3,3}$. As an example consider the subgraph $U \setminus e_5$ shown in Figure 2.12. The subgraph homeomorphic to $K_{3,3}$ is depicted here by bold line. Therefore $ed(H) \geq 3$. Since $cr(H) = 3$ it follows that $ed(H) = 3$. $\square$

It is known that the Heawood graph H is toroidal, [19]. The Heawood graph is also cyclically 4-edge connected. To show this we first note that H is symmetric. Remove an edge e from H as in the proof of Lemma 2.2.

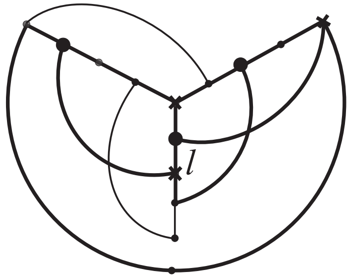

FIGURE 2.12.

Ignoring two vertices of degree 2 in the resulting graph U' we shall obtain a cubic graph U of degree 12, which is homeomorphic to U'. Evidently, U is 3-connected. It follows that U is cyclically 3-edge connected, so it H is cyclically 4-edge connected. The last fact will be used in Section 3.

Note also that $\gamma(CNG\ 3A) = 2$. The proof of this fact will be given in Section 3. It follows that $CNG\ 3A$ is a minimal 2-genus graph. It is not difficult to check that $\zeta(CNG\ 3A) = 3$. By direct computation, the remaining seven 3-crossing graphs have genus equal to one. We omit here the details of this computation.

4. For $l = 4$ there are two minimal crossing graphs: 8-crossed prism graph Pr_8, see Figure 2.13a), and the Möbius-Kantor graph MK, see Figure 2.13b). By direct computation we have $ed(MK) = 3$ and $ed(Pr_8) = 2$. Moreover it is known that the Möbius-Kantor graph MK is toroidal, [12]. It is not difficult to show that the graph Pr_8 is also toroidal.

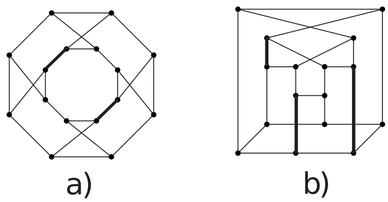

FIGURE 2.13. Graphs Pr_8 and MK

3. Additivity of parameters γ and ed and minimal cubic graphs

In this section we introduce two operations on graphs and establish some additivity properties of parameters ed and γ with respect to them, in the case of cubic graphs. The first operation is the connected sum of graphs and the second one is the double (crossed) connected sum of them. We also provide some upper bounds for the order of minimal edge deletion and minimal genus graphs within the classes of 2-connected and 3-connected cubic graphs.

Let G_1 and G_2 be 2-connected cubic graphs with distinguished edges e in G_1 and f in G_2. Let u_1, u_2 be the vertices of e and v_1, v_2 the vertices of f, respectively. Remove from G_1 the edge e, and from G_2 the edge f. Take the disjoint sum G of resulting graphs, $G = (G_1 - e) \sqcup (G_2 - f)$, and joint in G the pairs of vertices: u_1 with v_1, and u_2 with v_2, respectively. Denote the resulting graph by $G_1 \star G_2$. We shall say that $G_1 \star G_2$ is *the connected sum* of the graphs G_1 and G_2 with respect to the pair of edges e and f. Note $G_1 \star G_2$ is also 2-connected cubic graph.

Let G_1 and G_2 be any two 3-connected graphs. Take in G_1 a pair of nonincident edges (e_1, e_2), and in G_2 a pair of nonincident edges (f_1, f_2). Denote the vertices of e_1 by u_1, u_2, and the vertices of e_2 by v_1, v_2, respectively. Similarly, let s_1, s_2 be the vertices of f_1, and t_1, t_2 the vertices of f_2. Delete in G_1 the edges e_1 and e_2, and in G_2 the edges f_1 and f_2. Then take a disjoint sum $G = (G_1 - e_1 - e_2) \sqcup (G_2 - f_1 - f_2)$ of two graphs and joint in G the following pairs of vertices: u_1 and s_1, u_2 and s_2, v_1 and t_1, and v_2 and t_2, respectively. Denote the resulting 2-connected graph $G_1 * G_2$ and call it *a double connected sum* of G_1 and G_2. The four edges joining the graphs $G_1 - e_1 - e_2$ and $G_2 - f_1 - f_2$ are called the bridge edges of the graph $G_1 * G_2$ and are denoted h_1, h_2, h_3 and h_4, see Figure 3.1.

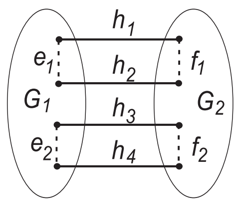

FIGURE 3.1. A double connected sum of graphs G_1 and G_2

If in the above definition we join u_1, u_2 with the vertices incident to different edges f_1 and f_2 (then v_1 and v_2 are also joined with the vertices of different edges f_1 and f_2), the resulting cubic graph is called *the crossed connected sum* of G_1 and G_2 and is denoted by $G_1 \sharp G_2$, see Figure 3.2.

It is clear that the operations of double connected sum and crossed connected sums are not determined uniquely and the result $G_1 * G_2$ depends on the distinguished edges of two graphs.

Let e be an edge of the connected cubic graph G. We shall say that e is *inessential* (subject to the parameter γ) if $\gamma(G) = \gamma(G - e)$. Otherwise e is called *essential*. It is naturally to ask whether the (oriented) genus is additive under taking the operations of connected sum and double connected

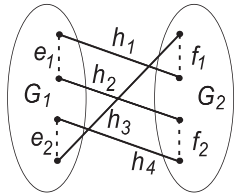

FIGURE 3.2. The crossed connected sum of graphs G_1 and G_2

sum of cubic graphs. In general, the answer is negative. For example, we have $\gamma(K_{3,3}) = 1$ while $\gamma(K_{3,3} \star K_{3,3}) = 1 \neq 2$.

Similarly, the genus is not additive subject to the operation of double connected sum of cubic graphs.

The following assertions show that under certain conditions, the (oriented) genus is subadditive or additive with respect to the operations defined above.

Theorem 3.1. *Let G_1 and G_2 be 2-connected cubic graphs of genus k and l, respectively. Let e and f be distinguished edges of G_1 and G_2, respectively and $G_1 \star G_2$ be the connected sum of G_1 and G_2. Then*

$$\gamma(G_1) + \gamma(G_2) \geq \gamma(G_1 \star G_2) \geq \gamma(G_1) + \gamma(G_2) - 1.$$

Moreover if e is inessential in G_1 or f is inessential in G_2, then

$$\gamma(G_1 \star G_2) = k + l.$$

Proof. Let $\varphi_1\colon G_1 \to M_1$ be a minimal embedding of the graph G_1 in the surface M_1 with $\gamma(G_1) = \gamma(M_1)$ and $\varphi_2\colon G_2 \to M_2$ be a minimal embedding of the graph G_2 in the surface M_2 with $\gamma(G_2) = \gamma(M_2)$. Cut an open disc D_1 in M_1 containing the edge e of G_1 and an open disc D_2 in M_2 containing the edge f of G_2. Then join the resulting surfaces M_1' and M_2' with a tube t, where one connected component of ∂t is identified with ∂D_1 and the other connected component of ∂t is identified with ∂D_2. The resulting surface is denoted by M. Drawing the bridge edges h_1 and h_2 in the tube t, we obtain an embedding of the graph $G_1 \star G_2$ into the connected surface M. The inequality $\gamma(G_1) + \gamma(G_2) \geq \gamma(G_1 \star G_2)$ now follows.

We continue with proving the second assertion. Denote the two bridge edges of $G_1 \star G_2$ by h_1 and h_2. Let $\varphi\colon G_1 \star G_2 \to M$ be a minimal embedding of the graph $G_1 \star G_2$ in a closed orientable surface M. Then φ is a 2-cell embedding. Consider a regular neighborhood N_1 of the polyhedron $\varphi(G_1) - e$ in M. Then N_1 is a compact 2-manifold with the boundary ∂M. The compact 2-manifold $M_1 = \overline{M \setminus N_1}$ is decomposed into several connected components $S_1, \ldots, S_l$. We also have $\partial(N_1) = \partial(M_1)$. The connected graph $\varphi(G_2 - f)$ is contained in one such connected component, say S_1. All other connected components of $M \setminus N_1$ must be open 2-disks. We have $\partial(S_1) = m_1 \sqcup m_2 \sqcup \ldots \sqcup m_k$ where each m_i is a circle. The 2-manifolds N_1 and S_1 have each boundary component m_i in common.

We claim that each m_i intersects at least one bridge edge h_i of $G_1 \star G_2$. Otherwise we can cut the manifold M along the circle m_j, not intersecting the bridge edges, past the two holes by discs to obtain another closed orientable 2-manifold M'. Note that $\gamma(M') = \gamma(M) - 1$ and the graph $G_1 \star G_2$ has an embedding in M'. But this contradicts to the assumption that the embedding φ of the graph $G_1 \star G_2$ is minimal.

Denote by Π the rotation system on $G_1 \star G_2$ induced by φ. Consider the facial circuits of the rotation system Π that contains bridge edges h_1 and h_2. We have the following (alternative) possibilities.

1) There are facial circuits c_1 and c_2 of Π such that c_1 contains h_1(twice) and c_2 contains h_2(twice). The corresponding closed faces r_1 and r_2 bounded by c_1 and c_2, respectively, form two handles in M, H_1 and H_2. Then $\chi(M) \leq 0$. Cutting M along the meridians m_1 and m_2 of H_1 and H_2 and pasting the holes by discs, we shall obtain two disjoint closed orientable surfaces, M_1 and M_2. This induces embeddings of the graph $G_1 - e$ in the surface M_1 and the graph $G_2 - f$ in the surface M_2. We have $\gamma(M) = \gamma(M_1) + \gamma(M_2) + 1$. Since $\gamma(G_1 - e) = \gamma(G_1)$ or $\gamma(G_2 - f) = \gamma(G_2)$, the assertion follows.

2) There are two facial circuits c_1 and c_2 of Π each of which contains both the edges h_1 and h_2. Fix an orientation on M. Let r_1 and r_2 be the faces of the embedding φ bounded by c_1 and c_2, respectively. The closed faces r_1 and r_2 glued along the edges h_1 and h_2 form a handle. Removing from M the (open) faces r_1, r_2 together with the edges h_1 and h_2, we shall obtain two disjoint 2-manifolds, M_1' and M_2' with boundaries $\partial M_1'$ and $\partial M_2'$, respectively. Elimination of the edges h_1 and h_2 in $G_1 \star G_2$ leads to a surgery of the rotation system Π and induces actually the rotation systems Π_1 and Π_2 on the graphs $G_1 - e$ and $G_2 - f$, respectively. More precisely, instead of the facial circuits c_1 and c_2 in Π we have two new circuits, d_1 and d_2, respectively, in Π_1 and Π_2. We thus have $\gamma(G_1 \star G_2) \geq \gamma(G_1 - e) + \gamma(G_2 - f)$. The rotation systems Π_1 and Π_2 generates embeddings of the graphs $G_1 - e$ and $G_2 - f$ in the surfaces M_1 and M_2, respectively. By drawing the edge e in the face D_1 bounded by the circuit d_1 and the edge f in the disc D_2 bounded by the circuit d_2, we obtain embeddings of G_1 into M_1 and G_2 into M_2. Therefore we have $\gamma(M) \geq \gamma(G_1) + \gamma(G_2)$.

3) There is a unique facial circuit c of Π which contains both the edges h_1 and h_2 twice. Now we proceed just as in the case 1). After surgery of the surface M we shall obtain two disjoint surfaces, M_1 and M_2, such that

$\gamma(M) = \gamma(M_1) + \gamma(M_2) + 1$. Moreover, G_1 has embedding in M_1 and G_2 has embedding in M_2. Since $\gamma(G_1 - e) = \gamma(G_1)$ or $\gamma(G_2 - f) = \gamma(G_2)$ we have $\gamma(M) \geq \gamma(G_1) + \gamma(G_2)$ completing the proof of the second assertion.

The inequality $\gamma(G_1 \star G_2) \geq \gamma(G_1) + \gamma(G_2) - 1$ follows directly from the proof of the second assertion through the careful analysis of the cases 1)-3). □

Corollary 3.2. *Let G_1 be a 2-connected cubic graphs with the distinguished edge e. Let e' be a distinguished edge of the graph $K_{3,3}$ and $H = G_1 \star K_{3,3}$ be a connected sum of G_1 and $K_{3,3}$ subject to the edges e and e'. If e is inessential in G_1, then $\gamma(H) = \gamma(G_1) + 1$.* □

Now take in the graph $K_{3,3}$ an edge e and replace it with two parallel edges, e_1 and e_2. The resulting cubic graph is denoted by K_1. It is clear that both e_1 and e_2 are inessential in K_1. Take e_1 as a distinguished edge of K_1 and consider the connected sum $K_2 = K_1 \star K_{3,3}$. By Corollary 3.2, $\gamma(K_2) = 2$. Iterating this process, we obtain a sequence K_l of 2-connected cubic graphs with $\gamma(K_l) = l$. Note that the order of K_l is equal to $8l - 2$.

Corollary 3.3. *If H is a minimal l-genus graph in the class of 2-connected graphs, then $|H| \leq 8l - 2$.* □

Theorem 3.4. *Let G_1 be a 3-connected cubic graph with the pair of distinguished edges e_1 and e_2 and G_2 be a cyclically 4-edge connected cubic graph with the pair of distinguished edges f_1 and f_2. Assume that $\gamma(G_1 - e_1) = \gamma(G_1)$ or $\gamma(G_1 - e_2) = \gamma(G_1)$ and $\gamma(G_2 - \{f_1, f_2\}) \geq \gamma(G_2) - 1$. Then $G_1 * G_2$ is a 3-connected graph and $\gamma(G_1 * G_2) \geq \gamma(G_1) + \gamma(G_2) - 1$.*

Proof. The fact that the graph $G_1 * G_2$ is 3-connected does not depend on topological properties of graphs G_1 and G_2 and actually follows from the proof of Theorem 3.14 (see below).

Let ψ be an embedding of the graph $G_1 * G_2$ in a surface M of minimal genus. Consider a subpolyhedron $P = \psi(G_1 - \{e_1, e_2\})$ in M. Let $N(P)$ be a regular neighborhood of P in M. This is a compact submanifold of M (see, for example [11]) and its boundary $\partial(N(P))$ of $N(P)$ consists of k disjoint circles $c_i, \ldots, c_k$.

Let S be a complementary submanifold of $N(P)$ in M. It consists of several connected components S_i, $S = \sqcup_i S_i$. Since the graph $G_2 - \{f_1, f_2\}$ is connected, it is contained in one such component, say S_j. Denote by M_2 the closure of submanifold S_j in M. We have obviously $\partial M_2 \subset \partial(N(P))$, so ∂M_2 is the disjoint union of several circles c_i, i.e. $\partial M_2 = c_{i_1} \sqcup \ldots \sqcup c_{i_l}$, where $l \leq k$.

As the embedding of $G_1 * G_2$ is genus minimal, all other connected components of $M \setminus N(P)$ must be open 2-disks. Moreover each boundary component c_i of M_2 must intersect at least one bridge edge h_1, h_2, h_3, h_4. We may suggest without loss of generality that each bridge edge h_m, $m = 1, \ldots, 4$, of $G_1 * G_2$ intersects in M a unique circle c_m at one point and each such intersection is transversal (see also the proof of Theorem 3.5). In particular, ∂M_2 consists of at most 4 circles i.e. $l \leq 4$.

Denote by M_1 the submanifold $M \setminus S_j$. Both the submanifolds M_1 and M_2 are connected and by construction we have $\partial M_1 = \partial M_2 = \sqcup_{s=1}^{l} c_{i_s}$. Moreover the graph $G_1 - \{e_1, e_2\}$ is embedded in M_1 and $G_2 - \{f_1, f_2\}$ is embedded in M_2. Glue the boundary components of ∂M_1 and ∂M_2 by discs and denote the obtained surfaces by M_1' and M_2', respectively.

Since $G_1 - \{e_1, e_2\} \subseteq M_1'$, we have that $\gamma(M_1') \geq \gamma(G_1 - \{e_1, e_2\})$. Moreover, from the inequality $\gamma(G_1 - e_1) \leq \gamma(G_1 - \{e_1, e_2\}) + 1$ and assumption $\gamma(G_1 - e_1) = \gamma(G_1)$ we also get that

$$\gamma(M_1') \geq \gamma(G_1 - \{e_1, e_2\}) \geq \gamma(G_1 - e_1) - 1 = \gamma(G_1) - 1.$$

As $G_2 - \{f_1, f_2\} \subseteq M_2'$, by the assumption $\gamma(G_2 - \{f_1, f_2\}) \geq \gamma(G_2) - 1$, we have

$$\gamma(M_2') \geq \gamma(G_2 - \{f_1, f_2\}) \geq \gamma(G_2) - 1.$$

We have to show that $\gamma(M) \geq \gamma(G_1) + \gamma(G_2) - 1$. Suppose

$$\gamma(M) \geq \gamma(G_1) + \gamma(G_2) - 2.$$

By the above reasoning, this is possible whenever we have the following $\gamma(M_1') = \gamma(G_1) - 1$ and $\gamma(M_2') = \gamma(G_2) - 1$. In other words,

$$\gamma(M) = \gamma(M_1) + \gamma(M_2),$$

so M is obtained from M_1 and M_2 by gluing along one boundary component. Therefore $\partial M_1 = \partial M_2$ is a circle c. In particular, M_1' is obtained from M_1 by attaching a disc D, so $\gamma(M_1') = \gamma(M_1)$. It follows that the both ends of e_2 belong to c, so one can extend embedding $G_1 - \{e_1, e_2\} \subset M_1$ to embedding $G_1 - e_1 \subset M_1' = M_1 \cup D$ by drawing e_2 in the closed 2-cell $\bar{D}$. We thus get embedding of $G_1 - e_1$ into surface M_1' of genus $\gamma(G_1) - 1$ contradicting to our assumption. □

Note also that an analogue of Theorem 3.4 holds also for crossed connected sum of cubic graphs.

Let G_1 be a 2-connected cubic graph of genus $k > 0$ which has the pair of distinguished non incident edges $\{e_1 = (u_1, v_1), e_2 = (u_2, v_2)\}$ and let G_2 be a connected cubic graph with the pair of distinguished non incident edges $\{f_1 = (u_1', v_1'), f_2 = (u_2', v_2')\}$. Assume that the following two conditions holds:

(i) at least one of the edges e_1, e_2 in G_1 is inessential;
(ii) $\gamma(G_2) = 1$ and either $\gamma(G_2 - \{f_1, f_2\}) = 1$, or $\gamma(G_2 - \{f_1, f_2\}) = 0$

and for any plain embedding of $G_2 - \{f_1, f_2\}$ there is no facial circuit c' containing the four vertices u'_1, v'_1, u'_2, v'_2 and the only possibility that the two facial circuits c'_1, c'_2 cover all these vertices is that one of them contains the vertices u'_1, u'_2 and the other one contains the vertices v'_1, v'_2.

For a moment, let $G_1 \sharp G_2$ denote the crossed connected sum of cubic graphs G_1 and G_2 in which the vertices of the pair $\{u_1, v_1\}$ are joined to the vertices of the pair $\{u'_1, u'_2\}$ and the vertices of the pair $\{u_2, v_2\}$ to the vertices of the pair $\{v'_1, v'_2\}$.

Theorem 3.5. *Let G_1 and G_2 be cubic graphs that satisfy conditions* (i) *and* (ii). *Assume that G_1 is 3-connected and G_2 is cyclically 4-edge connected. Then $G_1 \sharp G_2$ is 3-connected graph and $\gamma(G_1 \sharp G_2) = k + 1$.*

Proof. The proof of the first assertion follows from the proof of the first part of Theorem 3.14.

It remains to prove the second assertion. Suppose that

$$\gamma(G_1 \sharp G_2) \leq k.$$

Let φ be an embedding of $G_1 \sharp G_2$ into an orientable surface M of genus k, and ψ be the embedding of the subgraph $G_1 - \{e_1, e_2\}$ into M induced by the embedding φ. Let also $N(G_1)$ be an open regular neighborhood of the polyhedron $\psi(G_1 - \{e_1, e_2\})$ in M.

Let s be a connected component of the 2-manifold $M_2 = M \setminus N(G_1)$ containing the image $\varphi(G_2 - \{f_1, f_2\})$. Then s cannot be a disc (i.e. a face of the embedding ψ). Indeed, otherwise the bridge edges of $G_1 \sharp G_2$ would join the four vertices from $G_2 - \{f_1, f_2\}$ to four vertices of $G_1 - \{e_1, e_2\}$ in a disc. But this is impossible by condition (ii). Therefore s contains tubes (i.e. is a submanifold with nontrivial fundamental group). It follows that $\gamma(G_1 \sharp G_2) \geq k$.

It can occur that ∂s consists of one connected component, a circle c. Then $M_1 = \overline{M \setminus s}$ is a 2-manifold with the boundary $\partial M_1 = c$. After gluing a disc D to M_1 along the circle c we shall obtain a surface T of genus $k - 1$. In this case we can draw the edge e_1 (or the edge e_2) in the disc D and obtain an embedding of the graph $G_1 - e_1$ into the surface M_1 contradicting with the equality $\gamma(G_1 - e_1) = k$. We thus exclude this possibility.

Suppose now that s is glued to the rest of the surface M along two or more circles c_i. The number of circles cannot be bigger than two, otherwise the genus of M would be greater than k, contradicting to our assumption.

Assume that s has two boundary components, c_1 and c_2. Then s is a cylinder and $\gamma(\overline{M \setminus s}) = k-1$. There are two tubes t_1 and t_2 inside s which contain four bridge edges of the graph $G_1 \sharp G_2$. A tube t_i, $i = 1, 2$, cannot contain three bridge edges h_l, otherwise one circle c'_i would contain three vertices from the set $L = \{u'_1, v'_1, u'_2, v'_2\}$ and the other circle c'_{3-i} contains the remaining vertex, which is impossible by condition (ii).

Therefore the first tube t_1, bounded by c_1 on one side, contains two bridge edges h_1 and h_2 joining the ends of the edge e_1 to the vertices, say u'_1 and u'_2, positioned on the facial circuit c'_1 of $G_2 - f_1 - f_2$.

Similarly, the second tube t_2, bounded by c_2 on one side, contains the remaining bridge edges h_3 and h_4 which join the ends of the edge e_2 to the vertices v'_1 and v'_2, positioned on the second facial circuit c'_2 of $G_2 - f_1 - f_2$, see Figure 3.3.

In this case we can add the edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ to the subgraph $G_1 - e_1 - e_2$ and draw them in the 2-manifold $N(G_1)$. It follows that the graph G_1 admits embedding in a surface of genus $k - 1$ contradicting to the condition (i). This completes the proof of the second assertion. □

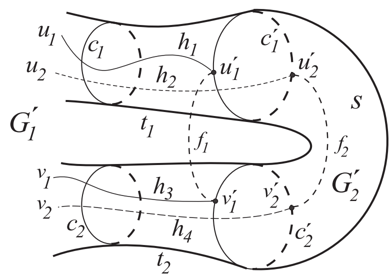

FIGURE 3.3.

Example 3.6. Consider the cubic graph K obtained from $K_{3,3}$ by doubling an edge e. Instead of e, we have in K two edges e_1 and e_2, see Figure 3.4. Take the edges e_1 and e_2 to be distinguished in K. Removing e_1 and e_2 from K we shall obtain a subcubic graph K'. Obviously, K' is homeomorphic to the complete graph K_4 so there is a unique embedding ρ of K' in the sphere S^2. The pairs of vertices $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are positioned on two different faces of ρ and there is no face r of ρ that contains three of these vertices in the boundary. It follows that K satisfies condition (i) (subject to the pair of edges e_1 and e_2). It is also clear that K satisfies the condition (i) as well (subject to the pair of edges e_1 and e_2).

It follows that $\gamma(K\sharp K) = 2$. Note also that $K\sharp K$ is cyclically 4-edge connected cubic graph.

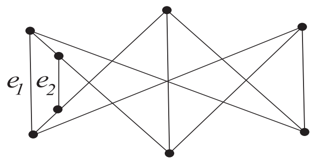

FIGURE 3.4. The cubic graph K

Lemma 3.7. *The genus of cubic graph CNG 3A is equal to* 2.

Proof. Cut the graph CNG $3A$ across four edges as shown in Figure 3.5. We have a decomposition of CNG $3A$ into two planar graphs G_1 and G_2 such that G_1 contains four semiedges e_1, e_2, e_3 and e_4 and G_2 contains four semiedges f_1, f_2, f_3 and f_4.

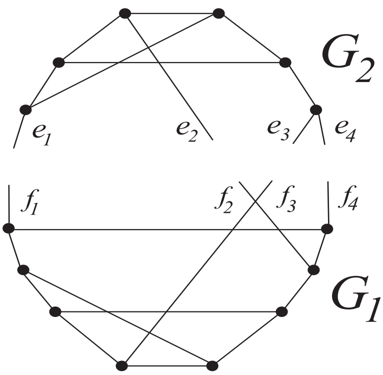

FIGURE 3.5.

Suppose that the graph CNG $3A$ is toroidal. Let φ denote embedding of this graph in the torus T. Then φ induces embeddings φ_1 and φ_2 of the subgraphs G_1 and G_2, respectively, in the torus. Let N_1 and N_2 be open regular neighborhoods of the graphs $\varphi_1(G_1)$ and $\varphi_1(G_2)$, respectively, in T. Then G_1 is contained in one connected component t of the 2-manifold $T \setminus N_2$ and G_2 is contained in one connected component s of the 2-manifold $\overline{T \setminus N_1}$. The component t cannot be a disc since there is no planar embedding of G_1 which contains all semiedges inside the same region r. Similarly the component s is not a disc. Therefore the only possibility to obtain embedding of the graph CNG $3A$ in the torus is as follows. The subgraph

G_1 is embedding into a sphere S_1 with two holes, the subgraph G_1 is embedding into a sphere S_2 with two holes and the spheres S_1 and S_2 are joining by two tubes τ_1 and τ_2 which contain four pairs of glued semiedges: $(e_1, f_1), (e_2, f_2), (e_3, f_3)$ and (e_4, f_4). By careful inspection all possibilities we can easily check that this is impossible. □

Now starting from the graphs $CNG3A$ and K in Example 3.6, we can inductively construct a sequence of 3-connected cubic graphs H_l of order $8l$. Note that at each inductive step l, there is at least two nonincident inessential edges in H_l. By Lemma 3.7 and Theorem 3.5 we have $\gamma(H_l) = l$.

Corollary 3.8. *If H is minimal l-genus graph in the class of 3-connected cubic graphs, then $|H| \leq 8l$.*

Denote by $\chi'(G)$ the chromatic index of the graph G. A cubic graph G is called *colorable* if $\chi'(G) = 3$, otherwise G is called *uncolorable* (i.e. $\chi'(G) = 4$) or a *weak snark*. A weak snark which is cyclically 4-edge connected and whose girth is at least five is called a *snark*, [15].

The Petersen graph is a simplest example of a snark. Using the operation of dot product, see Figure 3.6, one obtains from any two snarks of orders k and l, respectively, a bigger snark of order $k + l - 2$. Note that the dot product $G_1 \cdot G_2$ of two cubic graphs G_1 and G_2 is defined non uniquely.

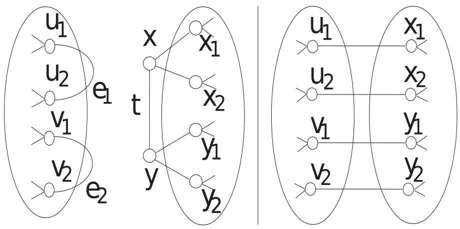

FIGURE 3.6. The dot product of two snarks

In [15] the authors consider different powers P^k of the Petersen graph P and study their genus. A k-th power P^k of the Petersen graph P is defined inductively: $P^k = P \cdot P^{k-1}$, where $\cdot$ denote a dot product of the cubic graphs. Since the dot product of two cubic graphs is defined non uniquely, there are several powers P^n of the snark P for each natural number $n \geq 2$.

In [15] the authors construct for each pair (k, n) of natural numbers k and n, where $k \leq n$ and $k, n \geq 1$, the powers P^n such that $\gamma(P^n) = k$. Note that the order of P^n is equal to $8n + 2$. This is an open problem to evaluate the number $ed(P^n)$ of the powers P^n of P such that $\gamma(P^n) = k$.

In the remaining part of this section, we study additivity properties of the parameter ed subject to operations of connected and double connected sum of graphs within the classes of 2-connected and 3-connected cubic graphs. A simple example shows that this parameter is not additive under the connected sum of cubic graphs. It suffice to consider the graphs $K_{3,3} * K_{3,3}$ and $K_{3,3} \star K_{3,3}$. Indeed we have $ed(K_{3,3}) = 1$ and $ed(K_{3,3} \star K_{3,3}) = 1$. Moreover $ed(K_{3,3} * K_{3,3}) = 1$ for appropriate choice of pairs of the non incident edges in the first and second copies of $K_{3,3}$. However under certain conditions an analogue of additivity property holds also for the parameter ed.

Let G be a cubic graph and e and f are two distinguished edges of G. We shall say that the edge e of G is *inessential* (subject to the characteristic ed) if $ed(G - e) = ed(G)$.

Example 3.9. Let H denote the Heawood graph. Take any two edges e and f in H, subdivide e with the vertex x and f with the vertex y and connect the new vertices with an edge h. Denote the resulting cubic graph by T. We claim that the edge h is inessential in T. To show this note that $ed(T) = 3$. Indeed the equality $ed(T) = 4$ would imply that $cr(T) = 4$. But the only 4-crossing minimal cubic graphs are Pr_8 and MK. As was mentioned in Section 2, the following equalities hold: $ed(Pr_8) = 2$ and $ed(MK) = 3$. Therefore, we have $ed(T) = 3$. Notice that T cannot be the graph MK, since the latter graph is symmetric, so removing any edge from it and ignoring two new vertices of degree 2, we obtain a cubic graph U of degree 14 and with the girth equal to five. But it is well known that H is of girth six. Note however that h is a unique inessential edge in T, since removal any other edge in T leads to a cubic graph of order 14 that is not isomorphic to H, so to a graph L with $ed(L) = 2$.

Example 3.10. Take in the Heawood graph H an edge e and replace it with two parallel edges, e' and e''. Let H' denote the resulting cubic graph of order 16. Then both e' and e'' are obviously inessential edges of H'.

Theorem 3.11. *Let G_1 and G_2 be two 2-connected cubic graphs with the distinguished edges e in G_1 and f in G_2, respectively. If $ed(G_1) = k > 0$ and $ed(G_2) = l > 0$, then $ed(G_1 \star G_2) \geq k + l - 1$. Moreover if e is inessential in G_1 and f is inessential in G_2, then $ed(G_1 \star G_2) = k + l$.*

Proof. Denote the vertices of e in G_1 by u_1 and u_2 and the vertices of f in G_2 by v_1 and v_2. Put $ed(G_1 \star G_2) = m$. Let $E = \{e_1, \ldots, e_m\}$ be the minimal set of edges in $G_1 \star G_2$ such that $G_1 \star G_2 - E$ is planar. By minimality of E, the graph $G_1 \star G_2 - E$ is connected.

Assume that E contains neither $t_1 = (u_1, v_1)$ nor $t_2 = (u_2, v_2)$. Then either there exists a path p_1 joining u_1 to u_2 in $G_1 - \{e, e_1, \ldots, e_m\}$ or a path p_2 in $G_2 - \{f, e_1, \ldots, e_m\}$ joining v_1 to v_2.

Suppose that the first possibility occurs. Then p_1 together with the edges t_1 and t_2 form in $G_1 \star G_2$ a path p that joins v_1 to v_2 and do not intersect $(G_2 - f)$, for exception the end vertices v_1 and v_2. Evidently, p is homeomorphic to the removed edge f in G_2 and replaces actually it. Since $G_2 - \{f, e_1, \dots, e_m\} \cup p$ is planar, it follows that $|E(G_2 - f) \cap E| \geq l$. As $|E(G_1 - e) \cap E| \geq k-1$ and $|E(G_2 - f) \cap E| \geq l$, the inequality $|E| \geq k+l-1$ follows. In the second case, we have $|E(G_1 - e) \cap E| \geq k$ and the assertion also follows.

Assume now that E contains one of the edges t_1 or t_2. Then E contains at least $k-1$ edges of $G_1 - e$ and $l-1$ edges of $G_2 - f$, and the first assertion follows.

The second assertion of the theorem follows directly from the definitions of the connected sum of cubic graphs and the minimal edge deletion set. □

Let G be a connected cubic graph with $ed(G) = l$ and $L \subset E(G)$ be an edge deletion subset of G, so the graph $G - L$ is planar. Note that if the subgraph $G - L$ is disconnected, then $|L| \geq l + 1$. Indeed, suppose contrary that $|L| = l$. We can add some edge r from L to the graph $G - L$ and obtain a planar subgraph U of G. But this contradicts to the assumption that $ed(G) = l$.

Let G be a cyclically 4-edge connected cubic graph with $ed(G) = l$. Let also $e_2 = (u_2, u_2')$ and $f_2 = (v_2, v_2')$ be a pair of non incident distinguished and inessential edges in G. Put $G' = G - \{e_2, f_2\}$. Consider an l-cut L in G' which decomposes the graph G' into two planar components, say G_1' and G_2'. It may occur that in a plane embedding of $G_1' \sqcup G_2'$ the pair of vertices $\{u_2, v_2\}$ ($\{u_2, v_2'\}$, respectively) are in the same facial cycle of G_1' and the pair of vertices $\{u_2', v_2'\}$ ($\{u_2', v_2\}$, respectively) are in the same facial cycle of G_2'. Then L is called a *cut separating* $\{u_2, v_2\}$ from $\{u_2', v_2'\}$ ($\{u_2, v_2'\}$ from $\{u_2', v_2\}$, respectively). We shall say that G *has the property* P (subject to the pair of edges e_2 and f_2) if no such separating l-cut L exists in G'.

Example 3.12. Let H' be a cubic graph as in Example 3.10 and

$$\{e' = (u, u'), \quad e'' = (v, v')\}$$

be a pair of distinguished edges in H'. We assert that H' has the property P subject to the pair of edges $\{e', e''\}$. Indeed, first note that H' is cyclically 4-edge connected and $ed(H') = 3$. Remove the edges e' and e'' from H' and denote the resulting graph G', see Figure 3.7.

The graph G' is also cyclically 4-edge connected and $ed(G') = 2$. Indeed, suppose that G does not have the property P. Let L be a 3-cut of G' that separates $\{u, v\}$ from $\{u', v'\}$ and G_1 and G_2 the corresponding planar components of $G' - L$ such that $u, v \in G_1$ and $u', v' \in G_2$. The only

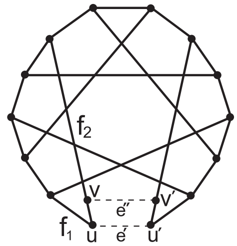

FIGURE 3.7.

possibility for such a cut is that L contains both the edges f_1, f_2 (so G_1 is simply K_2) and some other edge f of G'. Consider the component K of $G' - \{f_1, f_2\}$ that contains the pair of vertices $\{u', v'\}$. K is of order 14 and contains four vertices of degree two, see Figure 3.8.

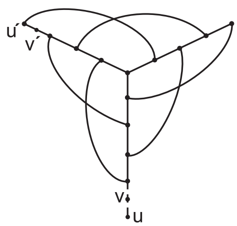

FIGURE 3.8.

We can ignore the vertices of degree two in K and consider the corresponding cubic graph K' of degree 10 which is homeomorphic to K. It is easy to see that $g(K') = 5$. It is known that the only cubic graph of degree 10 and of girth 5 is the Petersen graph P. However $ed(P) = 2$, so removal any edge u from it does not lead to a planar graph. It follows that removal any edge f from K does not lead to a planar graph. This contradicts to our assumption that $G_2 = K - f$ is planar. It is rather obvious fact that there is no 3-cut in G' separating $\{u, v'\}$ from $\{u', v\}$. Therefore H' has the property P subject to the pair of edges $\{e', e''\}$.

Example 3.13. Let MK denote the Möbius-Kantor graph. It is well known that MK is a symmetric graph. Take in the graph MK any edge e remove it from MK. Forgetting two vertices of degree 2 in the resulting graph, we obtain a cubic graph U of order 14 with $g(U) = 5$. Then U is not isomorphic to the Heawood graph, so by the results of Section 2 we have $ed(U) = 2$. It follows that each edge of MK is essential. ow replace the edge e in MK with two parallel edges $\{e' = (u, u'), e'' = (v, v')\}$. Denote the resulting graph by W. It is easy to see that both e' and e'' are inessential edges of W. Let e' and e'' be the distinguished edges of W. By the construction, W is cyclically 4-edge connected and $ed(W) = 3$.

Moreover, we assert that W has the property P subject to the pair of edges $\{e', e''\}$. Indeed, suppose the contrary. Remove the edges e' and e'' from W and denote the resulting graph W', see Figure 3.9. Since W' differs from U by subdivision only, it is cyclically 4-edge connected and we have $ed(W') = 2$.

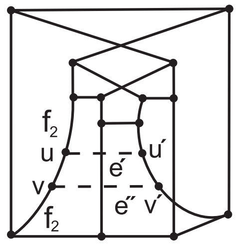

FIGURE 3.9.

Let M be a 3-cut of W' that separates $\{u, v\}$ from $\{u', v'\}$ and W_1 and W_2 the corresponding planar components of $W' - M$ where $u, v \in W_1$ and $u', v' \in W_2$. The only possibility for such a cut is that M contains both the edges f_1, f_2 (so W_1 is simply K_2) and some other edge f of W'. Consider the component R of $W' - \{f_1, f_2\}$ that contains the pair of vertices $\{u', v'\}$ (up to homeomorphism of graphs, to obtain R from MK we simply remove from it a star of vertex i.e. the graph $K_{1,3}$). As before, we can forget the vertices u' and v' of degree two in R and consider the corresponding cubic graph R' of degree 12 that is homeomorphic to R. It is easy to see that $g(R') = 5$, see Figure 3.10. It is not difficult to check that $ed(R') = 2$. It follows that removing an edge f from R does not lead to a planar graph. This contradicts to our assumption that $W_2 = R - f$ is planar. It is rather an obvious fact that there is no 3-cut in W' separating $\{u, v'\}$ from $\{u', v\}$. Therefore W has the property P subject to the pair of edges $\{e', e''\}$.

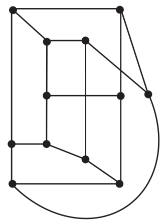

FIGURE 3.10.

Theorem 3.14. *Let G_1 be a 3-connected cubic graph with $ed(G_1) = k > 0$ and G_2 a cyclically 4-edge connected cubic graph with $ed(G_2) = l > 0$. Let $\{e_1, f_1\}$ be a pair of distinguished non incident edges in G_1 and $\{e_2, f_2\}$ a pair of non incident distinguished edges in G_2. Assume that in both the pairs each edge is inessential and G_1 has the property P with respect to the pair $\{e_1, f_1\}$ and G_2 has the property P with respect to the pair $\{e_2, f_2\}$. Then $G_1 * G_2$ is a 3-connected cubic graph and $ed(G_1 * G_2) \geq k + l$.*

Proof. Let $e_1 = (u_1, u_1'), f_1 = (v_1, v_1'), e_2 = (u_2, u_2')$ and $f_2 = (v_2, v_2')$. The bridge edges in the graph $H = G_1 * G_2$ are the following:

$$h_1 = (u_1, u_2), \qquad h_2 = (u_1', u_2'), \qquad h_3 = (v_1, v_2), \qquad h_4 = (v_1', v_2').$$

The subgraph of H formed by the bridge edges h_1, h_2, h_3, h_4 is denoted by B. Put $G_1' = G_1 - \{e_1, f_1\}$ and $G_2' = G_2 - \{e_2, f_2\}$.

Note that if G is cyclically 4-edge connected cubic graph, then any 3-edge cut of G is of kind $K_{1,3}$ (see, for example, [22]). We claim that the graph H is 3-connected. Suppose contrary that $\lambda(H) = 2$. Let $A = \{a_1, a_2\}$ be a cut of H consisting of two edges a_1 and a_2. Depending on the positions of edges a_1 and a_2 in H, the following situations can occur.

1) $a_1 \in E(G_1')$ and $a_2 \in E(G_2')$. Under this assumption we have

$$\lambda(G_1') = \lambda(G_2') = 1,$$

so the graph H is decomposed into two components by removing the edges a_1 and a_2 as shown in the Figure 3.11. It follows that $E_1 = \{e_1, f_1, a_1\}$ is a 3-cut of the graph G_1 and $E_2 = \{e_2, f_2, a_2\}$ is a 3-cut of the graph G_2. However since G_2 is cyclically 4-edge connected, this is impossible because the graph formed by the set of edges E_2 is not isomorphic to $K_{1,3}$.

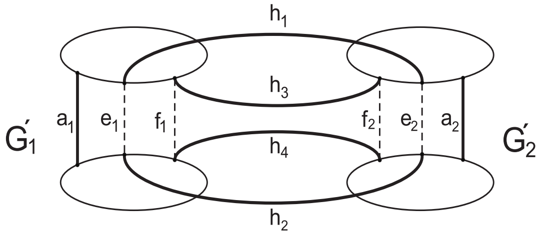

FIGURE 3.11.

2) Both a_1 and a_2 are the edges of G_1'. In this case, there are paths p_1 and p_2 in $H - E(G_1')$ that join the vertices u_1, u_1' and v_1, v_1', respectively. This means that U decomposes G_1' into two subgraphs, say T_1 and T_2, so that one such T_i contains all the vertices u_1, u_1', v_1, v_1'. This would imply that $\lambda(G_1) = 2$ contradicting to our assumption.

3) Both a_1 and a_2 are the edges of G'_2. This case can be handled just in the same way as the case 2).

Finally, it is obviously that H does not possesses a 2-cut which contains at least one bridge edge. Therefore H is 3-connected. It remains to show that $ed(G_1 * G_2) \geq k + l$.

Let $R = \{r_1, \ldots r_s\}$ be a minimal edge deletion set in H. Denote by q the number $|R \cap \{h_1, h_2, h_3, h_4\}|$. The planar graph $H - R$ is connected. The proof of the inequality is actually reduced to analyzing the following three cases:

Case 1) $q \geq 2$. By assumption, we have

$$ed(G'_1) \geq k - 1 \qquad \text{and} \qquad ed(G'_2) \geq l - 1.$$

Since the graphs G'_1 and G'_2 are planar, it follows that

$$|E(G'_1) \cap R| \geq ed(G'_1) \geq k - 1, \qquad |E(G'_2) \cap R| \geq ed(G'_2) \geq l - 1.$$

As the sets $E(G'_1) \cap R$ and $E(G'_1) \cap R$ are disjoint, we have

$$|R| \geq k - 1 + l - 1 + q \geq k + l,$$

so $ed(H) \geq k + l$.

Case 2) $q = 0$, i.e. R does not contain any bridge edge h_i. Consider a planar drawing g of the connected graph $H - R$. Let g_1 and g_2 be the planar embeddings of the subgraphs $G'_1 - R$ and $G'_2 - R$, respectively, induced by g. To prove the assertion in this case we have to inspect the following three subcases.

(i) *Both the subgraphs $G'_1 - R$ and $G'_2 - R$ are connected.* Since $G'_1 - R$ is connected, the plane subgraph $D_2 = (G'_2 \cup B) - R$ is contained in a face μ of the plane embedding g_1 of the graph $G'_1 - R$. This means that the vertices u_1, u'_1, v_1, v'_1 of $G'_1 - R$ are situated on the same facial circuit c, the circuit that bounds the face μ. We can draw the edge e_1 (or the edge f_1) in the face μ and obtain a planar embedding of the subgraph $G_1 - (f_1 \cup R)$, see Figure 3.12. Since the edge f_1 is inessential in G_1, we have $|E(G'_1) \cap R| \geq k$. In the same way we can prove that $|E(G'_2) \cap R| \geq l$. It follows that $|R| \geq k + l$.

(ii) *Both the subgraphs $G'_1 - R$ and $G'_2 - R$ are disconnected.* Then $|E(G'_2) \cap R| \geq l$ and $|E(G'_1) \cap R| \geq k$, so $|R| = k + l$;

(iii) *One of the subgraphs $G'_1 - R$ and $G'_2 - R$ is connected and the other is disconnected.* Suppose for instance that $G'_1 - R$ is connected and $G'_2 - R$ is disconnected. Then $G'_2 - R$ consists of two connected components, say U_1 and U_2. Since $G'_2 - R$ is disconnected, we have $|E(G'_2) \cap R| = l$. If $|E(G'_1) \cap R| = k$ we have $|R| = k + l$. Suppose that $|E(G'_1) \cap R| = k - 1$. Then in any plane embedding of $W = G'_1 - R$, the pair of vertices u_1 and u'_1 and the pair of vertices v_1, v'_1 cannot be neighboring (positioned on the

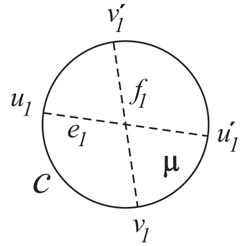

FIGURE 3.12.

same face). For the given plane subgraphs $G_2' - R$ and $G_1' - R$, the only possibility to draw the graph H in the plane is to take a face c_1 of $G_1' - R$ containing u_1 and v_1, a face c_2 containing u_1' and v_1' (if such exist), insert a connected component U_i into the face c_1, the second connected component U_{3-i} into the face c_2, then connect the vertices u_1 with u_2 and the vertices v_1 with v_2 with bridge edges h_1 and h_3 inside c_1, and the vertices u_1' with u_2' and the vertices v_1' with v_2' inside the face c_2 with bridge edges h_2 and h_4, see Figure 3.13. This is possible only if u_2 and v_2 are vertices of the same face of the plane graph U_i and u_2' and v_2' are vertices of the same face of the plane graph U_{3-i}. However the latter is excluded by property P. Therefore $|E(G_1') \cap R| = k$ and $|R| = k + l$.

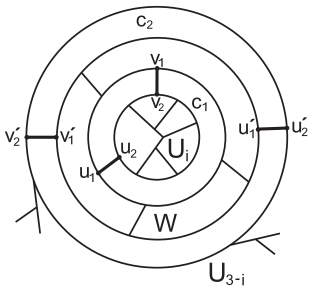

FIGURE 3.13.

Case 3) $q = 1$. If $|E(G_2') \cap R| \geq l$ or $|E(G_1') \cap R| \geq k$, the assertion follows. Suppose that $|E(G_2') \cap R| = l - 1$ and $|E(G_1') \cap R| = k - 1$. By the same arguments as before we conclude that in this case both the graphs $G_1' - R$ and $G_2' - R$ are connected. Let g be a planar embedding of the connected graph $H - R$. The embedding g induces planar embeddings g_1 and g_2 of the subgraphs $G_1' - R$ and $G_1' - R$, respectively. Since $G_1' - R$ is connected, the plane subgraph $D_2 = (G_2' \cup B) - R$ is contained in a face γ of the plane graph $G_1' - R$. This means that three vertices of $G_1' - R$ from the set $\{u_1, u_1', v_1, v_1'\}$ are situated in the same facial circuit c, the circuit that bounds the face γ. Therefore we can draw the edge e_1 or the edge f_1 in the face γ which gives in planar embedding of the subgraph $G_1-(f_1\cup R)$ ($G_1-(e_1\cup R)$, respectively). Since the edges e_1 and f_1 are inessential in G_1 we have $|E(G_1')\cap R|\geq k$, contradicting to our assumption.

We conclude that in any case $|R|\geq k+l$. We thus have proved that $ed(H)\geq k+l$. □

Theorem 3.14 can be used in constructing 3-connected and even cyclically 4-edge connected cubic graphs G of order $16n$ such that $ed(G)\geq 3n$ for any natural number $n>0$. We can start from the graph H' as in Example 3.10. Let e' and e'' be two parallel edges in H' obtained by doubling an edge e in the Heawood graph H. As was noted before, H is cyclically 4-connected and $\zeta(H')=4$. Taking two copies of H', the graphs G' and G_1, and applying to them the operation of double connected sum (just as it was described in Theorem 3.14), we shall obtain a 3-connected cubic graph H_2 of order 32. By Theorem 3.14, since H' has the property P, we get that $ed(H_2)=6$. It is not difficult to check that $\zeta(H_2)=4$. The thorough analysis of the proof of Theorem 3.14 shows that each bridge edge h_i in H_2 is also inessential.

Now we can take three copies of the cubic graph H', the graphs U_1, U_2 and U_3 with the pairs of distinguished edges e_1, f_1, and e_2, f_2 and e_3, f_3, respectively, remove all them and join the resulting graphs G_1, G_2 and G_3 by six bridge edges h_i as shown in Figure 3.14. The resulting cubic graph H_3 is 3-connected and we have $ed(H_3)=9$. The proof of this assertion actually follows from the proof of Theorem 3.14 and uses in an essential way the facts that H' is cyclically 4-edge connected and H has the property P. We omit here the details.

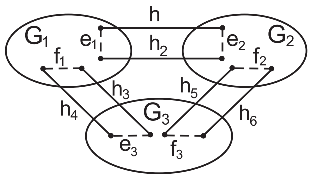

FIGURE 3.14. The cubic graph H_3

Iterating the process of joining the several copies of the graph H' in a cycle in the way as before, we obtain a sequence of cyclically 4-edge connected cubic graphs H_n of order $16n$ with $ed(H_n)=3n$.

Corollary 3.15. *If H is an $3l$-edge deletion minimal graph in the class of 3-connected cubic graphs, then $|H|\leq 16l$.*

Question. We provided some upper bounds for the order of minimal edge deletion (cubic) graphs and minimal genus (cubic) graphs. What about nontrivial lower bounds for these graph parameters?

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Leonid Plachta
AGH University of Science and Technology (Al. Mickiewicza 30, 30-059 Cracow, Poland), & Institute of Applied Problems of Mechanics and Mathematics of NAS of Ukraine (3-b Naukova St., 79060, Lviv, Ukraine)
Email: dept25@gmail.com

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Moyal and Rankin-Cohen deformations of algebras

Volodymyr Lyubashenko

Abstract. It is proven that Rankin-Cohen brackets form an associative deformation of the algebra of polynomials whose coefficients are holomorphic functions on the upper half-plane.

Анотація. Будується вкладення добутку верхньої напівплощини на комплексну площину з виколотим нулем в комплексний простір $\mathbb{C}^2$ з координатами (q, p). Уздовж нього переноситься бівекторне поле

$$M = \partial_p \otimes \partial_q - \partial_q \otimes \partial_p,$$

двоїсте до стандартної симпліціальної форми на $\mathbb{C}^2$. Подібним чином з деформації Муаяля алгебри поліномів від двох змінних отримано асоціативну деформацію алгебри поліномів, коефіцієнти яких є голоморфними функціями на верхній напівплощині. Мета і основний результат: доведено, що n-й член цієї деформації збігається з n-ю дужкою Ранкіна-Коена. Це пов'язує деформацію з модулярними формами. В доведенні використовується результат Ель Градекі про єдиність з точністю до числового множника $SL(2, \mathbb{R})$-еквіваріантного бідиференціального оператора на просторах голоморфних функцій на верхній напівплощині з певною дією групи $SL(2, \mathbb{R})$. Як члени згаданої деформації, так і дужки Ранкіна-Коена задовольняють ці умови, отже, є пропорційними. Перевіряється, що коефіцієнт пропорційності дорівнює 1. Наведене доведення основного твердження коротше за існуючі: П. Коен, Маніна і Загіра; Конна і Московічі; В. Овсієнка.

Аннотация. Доказано, что скобки Ранкина-Коэна образуют ассоциативную деформацию алгебры многочленов, коэффициенты которых являются голоморфными функциями на верхней полуплоскости.

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1. Rankin-Cohen brackets

Denote $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$, $\mathbb{C}^\times = \mathbb{C} - \{0\}$. Then the left action

$$SL(2,\mathbb{R}) \times (\mathbb{H} \times \mathbb{C}^\times) \to \mathbb{H} \times \mathbb{C}^\times, \qquad \gamma.(z,X) = \Big(\frac{az+b}{cz+d}, \frac{X}{cz+d}\Big),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, induces the right action of $SL(2,\mathbb{R})$ on the algebra of holomorphic functions $\mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times)$. The restriction of this action to the subspace $X^k\mathcal{H}ol(\mathbb{H})$, $k \in \mathbb{Z}$, equips $\mathcal{H}ol(\mathbb{H})$ with the right action $|_k$ of $SL(2,\mathbb{R})$. Recall that a modular (automorphic) form of weight k [10, Definition 2.1] is, in particular, a fixed point of this action. Rankin described in [8] $SL(2,\mathbb{R})$-invariant polydifferential operators on $\mathbb{H}$. Earlier Gordan proposed bidifferential $SL(2,\mathbb{R})$-invariant operators and called them transvectants [6]. In modular form theory these operators are known as Rankin-Cohen brackets. They were defined by H. Cohen as follows in [1]. Let $k, l \in \mathbb{Z}$, $n \in \mathbb{N} = \mathbb{Z}_{\geqslant 0}$, $f, g \in \mathcal{H}ol(\mathbb{H})$. Then

$$[X^k f, X^l g]_n^{RC} = X^{k+l+2n} \sum_{r+s=n}^{r,s\geqslant 0} (-1)^s \binom{k+n-1}{s} \binom{l+n-1}{r} f^{(r)} g^{(s)},$$

where

$$f^{(r)} = \partial_z^r f, \qquad \partial_z f = (\partial_x f - i\partial_y f)/2.$$

H. Cohen proved in [1, Theorem 7.1] that the operation $[\_,\_]_n^{RC}$ on the algebra $\mathcal{H}ol(\mathbb{H})[X^{-1}, X]$ is $SL(2,\mathbb{R})$-equivariant.

2. Deformations

A *deformation* of an associative $\mathbb{C}$-algebra $(A, m : A \otimes A \to A)$ is a $\mathbb{C}$-bilinear map

$$\mu : A \times A \to A[[\hbar]], \qquad \mu(a,b) = \sum_{n=0}^{\infty} \hbar^n \mu_n(a,b)$$

such that

$$\mu_0(a,b) = m(a \otimes b) = ab, \qquad \mu(1,a) = a = \mu(a,1),$$

and the $\mathbb{C}[[\hbar]]$-bilinear map $\tilde{\mu} : A[[\hbar]] \times A[[\hbar]] \to A[[\hbar]]$, which extends μ by $\mathbb{C}[\hbar]$-bilinearity and $\hbar$-adic continuity, is associative.

Let $\mathfrak{a} \subset \mathrm{Der} A$ be an abelian $\mathbb{C}$-subalgebra of the Lie $\mathbb{C}$-algebra of derivations of A. View an element

$$P = \sum_{i=1}^{l} \xi_i \otimes \eta_i \in \mathfrak{a} \otimes_{\mathbb{C}} \mathfrak{a}$$

as a $\mathbb{C}$-linear operator $A^{\otimes 2} \to A^{\otimes 2}$. It is well-known that

$$a \star_P b \equiv \mu_P(a,b) = m \exp(\hbar P).(a \otimes b) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} m P^n.(a \otimes b)$$

is a deformation of A. An example is provided by the Moyal deformation of $A = \mathcal{H}ol(\mathbb{C}^2)$, $\mathbb{C}^2 = \{(q,p)\}$, $P = M = \partial_p \otimes \partial_q - \partial_q \otimes \partial_p$. The point of this note is to show that the Rankin-Cohen brackets give a deformation of the algebra $\mathcal{H}ol(\mathbb{H})[X]$, related to the Moyal deformation.

Following a remark of Eholzer, Zagier noticed in [12] that many identities involving operations $[,]_n^{RC}$ follow from the statement that these form a deformation of the algebra of modular forms. This was further elaborated by P. Cohen, Manin, Zagier [2], where they show that the deformed algebra of modular forms is isomorphic to certain algebra of invariant pseudodifferential operators.

Connes and Moscovici discovered certain Hopf algebra $\mathcal{H}_1$ during their work on foliations [3]. They described also its action [4] on the algebra of modular forms from which a deformation of the latter is obtained. Moreover the Rankin-Cohen deformation generalizes to a universal deformation formula of $\mathcal{H}_1$ as shown by Tang and Yao [11]. These results are reviewed in [9].

Our proof of the main result is concise. It is most close to the proof by V. Ovsienko [7] with the difference that he looks for a symplectomorphism whereas we find an open embedding of $\mathbb{H} \times \mathbb{C}^\times$ into $\mathbb{C}^2$, compatible with the Poisson structure.

3. Rankin-Cohen brackets as deformation

Consider the open embedding

$$\Psi = \big(\mathbb{H} \times \mathbb{C}^\times \hookrightarrow \mathbb{C} \times \mathbb{C}^\times \xrightarrow[\cong]{\psi} \mathbb{C} \times \mathbb{C}^\times \hookrightarrow \mathbb{C}^2\big),$$

where $\psi(z,X) = (zX^{-1}, X^{-1}) = (q,p)$. Vector field ∂_p (resp. ∂_q) on $\mathbb{C}^2$ is lifted along Ψ to vector field $\xi = -X^2\partial_X - zX\partial_z$ (resp. $\eta = X\partial_z$) on $\mathbb{H} \times \mathbb{C}^\times$. Thus,

$$M = \partial_p \otimes \partial_q - \partial_q \otimes \partial_p$$

is lifted to

$$P = RC = \xi \otimes \eta - \eta \otimes \xi$$

and ξ and η commute. Hence,

$$F \star_{RC} G = m \exp(\hbar RC).(F \otimes G) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} m (RC)^n.(F \otimes G)$$

is a deformation of $\mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times)$.

Theorem 3.1. *Let $k, l \in \mathbb{Z}$, $n \in \mathbb{N}$. Assume that one of the following conditions holds:*

- $k > 0$;
- $l > 0$;
- $k, l \leqslant 0$ *and* $n < \max\{1-k, 1-l\}$;
- $k, l \leqslant 0$, $n > 1 - k - l$.

Then for all $f, g \in \mathcal{H}ol(\mathbb{H})$ and $F = X^k f$, $G = X^l g$ we have

$$(n!)^{-1} m(RC)^n.(F \otimes G) = [F, G]_n^{RC}.$$

Proof. The embedding $\Psi : \mathbb{H} \times \mathbb{C}^\times \hookrightarrow \mathbb{C}^2$ is $SL(2,\mathbb{R})$-equivariant, where the left action on $\mathbb{C}^2 = \{(q,p)\}$ is the standard action of matrices on vectors. The mapping

$$M = \partial_p \otimes \partial_q - \partial_q \otimes \partial_p \;\in\; \mathrm{End}_{\mathbb{C}}(\mathcal{H}ol(\mathbb{C}^2) \otimes \mathcal{H}ol(\mathbb{C}^2))$$

commutes with the action of $SL(2,\mathbb{R})$, or, equivalently, of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Hence, the Moyal deformation is $SL(2,\mathbb{R})$-equivariant. This implies and can be checked directly that

$$RC = \xi \otimes \eta - \eta \otimes \xi \;\in\; \mathrm{End}_{\mathbb{C}}(\mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times) \otimes \mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times))$$

commutes with the action of $\mathfrak{sl}(2,\mathbb{R})$ and $SL(2,\mathbb{R})$. Thus,

$$\star_{RC} : \mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times) \otimes \mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times) \to \mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times)[[\hbar]]$$

is a homomorphism of $SL(2,\mathbb{R})$-modules. Accordingly to El Gradechi [5, Proposition 3.11] if the hypotheses on (k,l,n) hold, then there is only 1-dimensional vector space of $SL(2,\mathbb{R})$-equivariant bidifferential operators

$$(\mathcal{H}ol(\mathbb{H}), |_k) \otimes (\mathcal{H}ol(\mathbb{H}), |_l) \to (\mathcal{H}ol(\mathbb{H}), |_{k+l+2n}).$$

Both maps

$$f \otimes g \mapsto (n!)^{-1} m(RC)^n.(X^k f \otimes X^l g), \qquad f \otimes g \mapsto [X^k f, X^l g]_n^{RC}$$

belong to this space, therefore, they are proportional. One checks that the proportionality constant is 1. In fact, evaluate both operators on the following $f \otimes g$.

If $k = l = n = 0$, take $f = g = 1$.

If $k > 0$ or $k < 0$, $k \leqslant l \leqslant 0$, $n < 1 - k$, take $f = 1$, $g = z^n$.

If $l > 0$ or $l < 0$, $l \leqslant k \leqslant 0$, $n < 1 - l$, take $f = z^n$, $g = 1$.

If $k, l \leqslant 0$ and $n > 1-k-l$ choose $r, s \in \mathbb{N}$, such that $r+s = n$, $r+k > 0$ and $s + l > 0$, and take $f(z) = z^r$ and $g(z) = z^s$. (Hint: use coordinates (q,p) to make the computations easier). □

Corollary 3.2. *The Rankin-Cohen deformation* $\star_{RC}$ *restricted to* $A^{\otimes 2}$, $A = \mathcal{H}ol(\mathbb{H})[X]$ *coincides with the map*

$$A \otimes A \to A[[\hbar]], \qquad X^k f \otimes X^l g \mapsto \sum_{n=0}^{\infty} \hbar^n [X^k f, X^l g]_n^{RC}.$$

Proof. For all $(k,l,n) \in \mathbb{N}^3$ the statement follows from Theorem 3.1 except for $(k,l,n) = (0,0,1)$. In the latter case $[f,g]_1 = 0 = m(RC)(f \otimes g)$ for all $f, g \in \mathcal{H}ol(\mathbb{H})$. □

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Volodymyr Lyubashenko
Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska st., Kyiv, 01004, Ukraine
Email: lub@imath.kiev.ua
ORCID: orcid.org/0000-0002-3480-1514

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Bypassing dynamical systems: a simple way to get the box-counting dimension of the graph of the Weierstrass function

Claire David

Abstract. In the following, bypassing dynamical systems tools, we propose a simple means of computing the box dimension of the graph of the classical Weierstrass function defined, for any real number x, by

$$\mathcal{W}(x) = \sum_{n=0}^{+\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right),$$

where λ and N_b are two real numbers such that $0 < \lambda < 1$, $N_b \in \mathbb{N}$ and $\lambda\, N_b > 1$, using a sequence a graphs that approximate the studied one.

Анотація. В роботі пропонується простий спосіб обчислення що box-counting розмірність (або розмірності Мінковського) графіка класичної неперервної ніде не диференційовної функції Вейєрштраса

$$\mathcal{W}(x) = \sum_{n=0}^{+\infty} \lambda^n \cos(2\pi N_b^n x),$$

де λ та N_b — дійсні числа такі, що $0 < \lambda < 1$, $N_b \in \mathbb{N}$, $\lambda N_b > 1$.

Відмітимо, що попередні роботи, в яких обчислювалась box-counting розмірність графіка функції Вейєрштраса, базувались на таких поняттях теорії динамічних систем як: розмірність Ляпунова еквівалентних притягуючих торів, вкладення графа в атрактор динамічної системи, b-відображення пекаря одиничного квадрату, стійкі та нестійкі підмноговиди та ін.

В даній статті ми «обходимо» техніку теорії динамічних систем за допомогою послідовностей графіків $(\Gamma_{\mathcal{W}_m})_{m\in\mathbb{N}}$, що збігаються до графіка функції Вейєрштраса. Основний інструмент — це ітерована система функцій, тобто сім'я C^∞ стискаючих відображень $T_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 0, \ldots, N_b - 1$, визначених за формулами

$$T_i(x,y) = \left(\tfrac{x+i}{N_b}, \lambda y + \cos(2\pi \cdot \tfrac{x+i}{N_b})\right),$$

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Нехай P_i — нерухома точка T_i і $V_0 = \{P_0, \ldots, P_{N_b-1}\}$. Далі для кожного натурального m покладемо $V_m = \cup_{i=0}^{N_b-1} T_i(V_{m-1})$. З'єднавши послідовні точки в множині V_m у напрямку зростання абсциси отримаємо граф $\Gamma_{\mathcal{W}_m}$. Обчислення розмірності графіка базується на оцінках виду:

$$C_{\inf}|x_1 - x_2|^{2-D_{\mathcal{W}}} \leq |\mathcal{W}(x_1) - \mathcal{W}(x_2)| \leq C_{\sup}|x_1 - x_2|^{2-D_{\mathcal{W}}}$$

де $[x_1, x_2] \subset [0,1]$ — довільний досить малий інтервалу, $D_{\mathcal{W}} = 2 + \frac{\ln\lambda}{\ln b}$, а $C_{\inf}$ і $C_{\sup}$ строго додатні числа. Їх доведення займає лише дві сторінки і не вимагає спеціальних знань, що робить доступним роботу для широкої аудиторії.

Introduction

The determination of the box and Hausdorff dimension of the graph of the Weierstrass function has, since long been, a topic of interest. In the following, we show that the box-counting dimension (or Minskowski dimension) can be obtained directly, without using dynamical systems tools.

Let us recall that, given $\lambda \in (0,1)$, and b such that $\lambda b > 1 + \frac{3\pi}{2}$, the Weierstrass function

$$x \in \mathbb{R} \mapsto \mathcal{W}(x) = \sum_{n=0}^{+\infty} \lambda^n \cos(\pi b^n x)$$

is continuous everywhere, while nowhere differentiable. The original proof, by K. Weierstrass [18], can also be found in [17]. It has been completed by the one, now a classical one, in the case where $\lambda b > 1$, by G. Hardy [5].

After the works of A. S. Besicovitch and H. D. Ursell [2], it is Benoît Mandelbrot [12] who particularly highlighted the fractal properties of the graph of the Weierstrass function. He also conjectured that the Hausdorff dimension of the graph is $D_{\mathcal{W}} = 2 + \frac{\ln\lambda}{\ln b}$. Interesting discussions in relation to this question have been given in the book of K. Falconer [4]. A series of results for the box dimension can be found in the works of J. L. Kaplan et al. [8] (where the authors show that it is equal to the Lyapunov dimension of the equivalent attracting torus), in the one of F. Przytycki and M. Urbański [14], and in those by T. Y. Hu and K-S. Lau [6]. As for the Hausdorff dimension, a proof was given by B. Hunt [7] in 1998 in the case where arbitrary phases are included in each cosinusoidal term of the summation. Recently, K. Barańsky, B. Bárány and J. Romanowska [1] proved that, for any value of the real number b, there exists a threshold value λ_b belonging to the interval $(\frac{1}{b}, 1)$ such that the aforementioned dimension is equal to $D_{\mathcal{W}}$ for every b in $(\lambda_b, 1)$. Results by W. Shen [16] go further than the ones of [1]. In [9], G. Keller proposes what appears as a much simpler and very original proof.

May one wish to understand the proofs mentioned above, it requires theoretical background in dynamic systems theory. For instance, in the work of J. L. Kaplan et al. [8], the authors call for results that cannot be understood without knowledge on the Lyapunov dimension. One may also note that their proof, which enables one to obtain the box-counting dimension of the aforementioned graph, involves sequences revolving around the Gamma Function, Fourier coefficients, integration in the complex plane, definition of a specific measure, the solving of several equations, thus, a lot of technical manipulations (on eleven pages), to yield the result.

Following those results, F. Przytycki and M. Urbański [14] give a general method leading to the value of this box-counting dimension. It was initially devoted to the calculation of the Hausdorff dimension of the graph. It appears simpler than the one by Kaplan et al., calling for Frostman's lemma [15], [13]. The authors deal with continuous functions f satisfying conditions of the form:

$$\sup\left\{|f(a_1) - f(a_2)| \,:\, x_1 \leq a_1 \leq a_2 \leq x_2\right\} \geq C\,|x_1 - x_2|^{\alpha} \qquad (\star)$$

for all $(x_1, x_2) \in [0,1]^2$, where C and $\alpha < 1$ denote strictly positive real constants.

In order to apply the results by F. Przytycki and M. Urbański, one thus requires the estimate $(\star)$, which is not that easy to prove. The same kind of estimate is required to obtain the Hausdorff dimension of the graph. As evoked above, existing works in the literature all call for the theory of dynamical systems.

Until now, the simplest calculation is the one by G. Keller [9], where the author bypasses the Ledrappier-Young theory on hyperbolic measures [10], [11], embedding the graph into an attractor of a dynamical system. The proof requires b-baker maps, acting on the unit square. It also requires results on stable and unstable manifolds, as well as results on related fibers.

In our work [3], where we build a Laplacian on the graph of the Weierstrass function $\mathcal{W}$, we came across a simpler means of computing the box dimension of the graph, using a sequence a graphs that approximate the studied one, bypassing all the aforementioned tools. The main computation, which, for any small interval $[x_1, x_2] \subset [0,1]$, leads to an estimate of the form:

$$C_{inf}|x_1 - x_2|^{2-D_{\mathcal{W}}} \leq |\mathcal{W}(x_1) - \mathcal{W}(x_2)| \leq C_{sup}|x_1 - x_2|^{2-D_{\mathcal{W}}}$$

where $D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln b}$, and C_{inf} and C_{sup} denote strictly positive constants, is done in barely two pages, and does not require specific know-ledge, putting the result at the disposal of a wider audience. The key results are exposed in the sequel.

1. Framework of the study

In this section, we recall results that are developed in [3]. We consider the case when the real number b is an integer, that we thus choose to denote by N_b.

Notation. We will denote by $\mathbb{N}$ the set of natural integers. In the following, λ and N_b are two real numbers such that:

$$0 < \lambda < 1 \qquad N_b \in \mathbb{N}, \qquad \text{and} \qquad \lambda\, N_b > 1.$$

We will consider the Weierstrass function $\mathcal{W}$, defined, for any $x \in \mathbb{R}$, by:

$$\mathcal{W}(x) = \sum_{n=0}^{+\infty} \lambda^n \cos(2\pi N_b^n x).$$

Periodic properties of the Weierstrass function. Notice, that for any real number x:

$$\mathcal{W}(x+1) = \sum_{n=0}^{+\infty} \lambda^n \cos(2\pi N_b^n x + 2\pi N_b^n) = \sum_{n=0}^{+\infty} \lambda^n \cos(2\pi N_b^n x) = \mathcal{W}(x).$$

Hence the study of the Weierstrass function can be restricted to the interval $[0,1)$.

In the sequel, we place ourselves in the Euclidean plane of dimension 2, referred to a direct orthonormal frame. The usual Cartesian coordinates are (x,y).

The restriction $\Gamma_{\mathcal{W}}$ to $[0,1)\times\mathbb{R}$, of the graph of the Weierstrass function, is approximated by means of a sequence of graphs, built through an iterative process. For this purpose, we introduce the iterated function system, i.e. the family of C^∞ contractions from $\mathbb{R}^2$ to $\mathbb{R}^2$:

$$\{T_0, \dots, T_{N_b-1}\}$$

where, for any integer i belonging to $\{0,\dots,N_b-1\}$, and any (x,y) in $\mathbb{R}^2$:

$$T_i(x,y) = \left(\tfrac{x+i}{N_b},\, \lambda y + \cos\!\left(2\pi \cdot \tfrac{x+i}{N_b}\right)\right).$$

We will denote by:

$$D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln N_b}$$

the Hausdorff dimension of $\Gamma_{\mathcal{W}}$ (see [1], [9]).

Proposition 1.1. $\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}})$.

Definition 1.2. For any integer i belonging to $\{0, \ldots, N_b - 1\}$, let us denote by:

$$P_i = (x_i, y_i) = \left(\frac{i}{N_b - 1}, \frac{1}{1-\lambda} \cos \frac{2\pi i}{N_b - 1}\right)$$

the fixed point of the contraction T_i.

We will denote by V_0 the ordered set (according to increasing abscissae), of the points:

$$\{P_0, \ldots, P_{N_b-1}\}.$$

The set of points V_0, where, for any i of $\{0, \ldots, N_b - 2\}$, the point P_i is linked to the point P_{i+1}, constitutes an oriented graph (according to increasing abscissa), that we will denote by $\Gamma_{\mathcal{W}_0}$. We will call V_0 the *set of vertices* of the graph $\Gamma_{\mathcal{W}_0}$.

For any natural integer m, we set:

$$V_m = \bigcup_{i=0}^{N_b-1} T_i(V_{m-1}).$$

The set of points V_m, where two consecutive points are linked, is an oriented graph (according to increasing abscissa), which we will denote by $\Gamma_{\mathcal{W}_m}$. We will call V_m the set of vertices of the graph $\Gamma_{\mathcal{W}_m}$. We will also denote, in the sequel, by

$$\mathcal{N}_m^{\mathcal{S}} = 2\, N_b^m + N_b - 2$$

the number of vertices of the graph $\Gamma_{\mathcal{W}_m}$, and write:

$$V_m = \left\{P_0^m, P_1^m, \ldots, P_{\mathcal{N}_m^{\mathcal{S}}-1}^m\right\}.$$

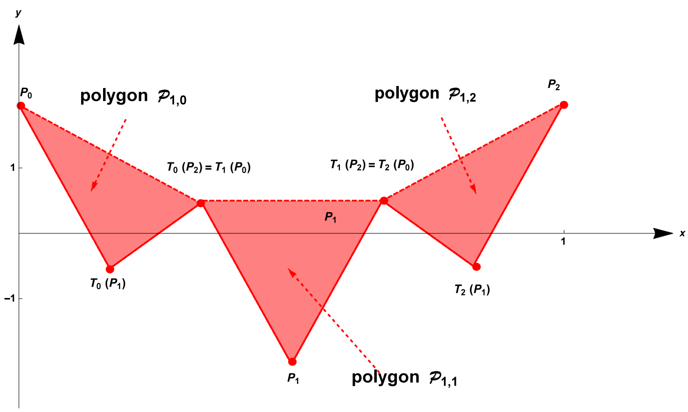

FIGURE 1.1. The polygons $\mathcal{P}_{1,0}$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, in the case where $\lambda = \frac{1}{2}$ and $N_b = 3$.

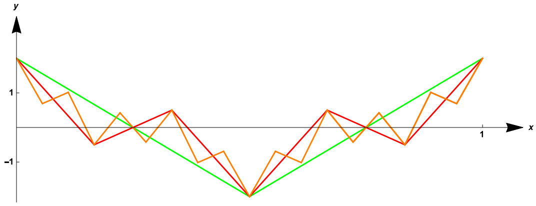

FIGURE 1.2. The graphs $\Gamma_{\mathcal{W}_0}$, $\Gamma_{\mathcal{W}_1}$ (in red), $\Gamma_{\mathcal{W}_2}$, $\Gamma_{\mathcal{W}}$, in the case where $\lambda = \frac{1}{2}$ and $N_b = 3$.

Definition 1.3 (Consecutive vertices on the graph $\Gamma_{\mathcal{W}}$)**.** Two points X and Y of $\Gamma_{\mathcal{W}}$ will be called *consecutive vertices* of the graph $\Gamma_{\mathcal{W}}$ if there exists a natural integer m, and an integer j of $\{0, \ldots, N_b - 2\}$, such that:

$$\begin{cases} X = (T_{i_1} \circ \cdots \circ T_{i_m})(P_j), \\ Y = (T_{i_1} \circ \cdots \circ T_{i_m})(P_{j+1}), \end{cases}$$

or:

$$\begin{cases} X = (T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_m})(P_{N_b-1}), \\ Y = (T_{i_1+1} \circ T_{i_2} \cdots \circ T_{i_m})(P_0), \end{cases}$$

where $(i_1, \ldots, i_m) \in \{0, \cdots, N_b - 1\}^m$.

Definition 1.4. For every $m \in \mathbb{N}$ the $\mathcal{N}_m^{\mathcal{S}}$ consecutive vertices of the graph $\Gamma_{\mathcal{W}_m}$ are, also, the vertices of N_b^m simple polygons $\mathcal{P}_{m,j}$, $0 \leq j \leq N_b^m - 1$, with N_b sides. For any integer j such that $0 \leq j \leq N_b^m - 1$, one obtains each polygon by linking the point number j to the point number $j+1$ if $j = i \,\text{mod} N_b$, $0 \leq i \leq N_b - 2$, and the point number j to the point number $j - N_b + 1$ if $j = -1 \,\text{mod} N_b$. These polygons generate a Borel set of $\mathbb{R}^2$.

Definition 1.5 (Word, on the graph $\Gamma_{\mathcal{W}}$)**.** Let m be a strictly positive integer. We will call *number-letter* any integer $\mathcal{M}_i$ of $\{0, \ldots, N_b - 1\}$, and *word of length* $|\mathcal{M}| = m$, on the graph $\Gamma_{\mathcal{W}}$, any set of number-letters of the form:

$$\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_m).$$

We will write:

$$T_{\mathcal{M}} = T_{\mathcal{M}_1} \circ \cdots \circ T_{\mathcal{M}_m}.$$

Definition 1.6 (Edge relation, on the graph $\Gamma_{\mathcal{W}}$)**.** Given a natural integer m, two points X and Y of $\Gamma_{\mathcal{W}_m}$ will be called *adjacent* if and only if X and Y are two consecutive vertices of $\Gamma_{\mathcal{W}_m}$. We will write:

$$X \underset{m}{\sim} Y.$$

This edge relation ensures the existence of a word $\mathcal{M} = (\mathcal{M}_1, \dots, \mathcal{M}_m)$ of length m, such that X and Y both belong to the iterate:

$$T_{\mathcal{M}} V_0 = (T_{\mathcal{M}_1} \circ \dots \circ T_{\mathcal{M}_m}) V_0.$$

Given two points X and Y of the graph $\Gamma_{\mathcal{W}}$, we will say that X and Y are *adjacent* if and only if there exists a natural integer m such that:

$$X \underset{m}{\sim} Y.$$

Proposition 1.7 (Addresses, on the graph of the Weierstrass function). *Given a strictly positive integer m, and a word $\mathcal{M} = (\mathcal{M}_1, \dots, \mathcal{M}_m)$ of length $m \in \mathbb{N}^\star$, on the graph $\Gamma_{\mathcal{W}_m}$, for any integer j of $\{1, \dots, N_b - 2\}$, any $X = T_{\mathcal{M}}(P_j)$ of $V_m \setminus V_0$, i.e. distinct from one of the N_b fixed point P_i, $0 \leq i \leq N_b - 1$, has exactly two adjacent vertices, given by:*

$$T_{\mathcal{M}}(P_{j+1}) \qquad \textit{and} \qquad T_{\mathcal{M}}(P_{j-1}),$$

where:

$$T_{\mathcal{M}} = T_{\mathcal{M}_1} \circ \dots \circ T_{\mathcal{M}_m}.$$

By convention, the adjacent vertices of $T_{\mathcal{M}}(P_0)$ are $T_{\mathcal{M}}(P_1)$ and $T_{\mathcal{M}}(P_{N_b-1})$, those of $T_{\mathcal{M}}(P_{N_b-1})$, $T_{\mathcal{M}}(P_{N_b-2})$ and $T_{\mathcal{M}}(P_0)$.

Notation. For any integer j belonging to $\{0, \dots, N_b - 1\}$, any natural integer m, and any word $\mathcal{M}$ of length m, we set:

$$\begin{aligned} T_{\mathcal{M}}(P_j) &= \left(x\left(T_{\mathcal{M}}(P_j)\right), y\left(T_{\mathcal{M}}(P_j)\right)\right), \\ T_{\mathcal{M}}(P_{j+1}) &= \left(x\left(T_{\mathcal{M}}(P_{j+1})\right), y\left(T_{\mathcal{M}}(P_{j+1})\right)\right), \\ L_m &= x\left(T_{\mathcal{M}}(P_{j+1})\right) - x\left(T_{\mathcal{M}}(P_j)\right) = \frac{1}{(N_b - 1) N_b^m}, \\ h_{j,m} &= y\left(T_{\mathcal{M}}(P_{j+1})\right) - y\left(T_{\mathcal{M}}(P_j)\right). \end{aligned}$$

2. Main results

Theorem 2.1 (An upper bound and a lower bound, for the box-dimension of the graph $\Gamma_{\mathcal{W}}$). *For any integer j belonging to $\{0, 1, \dots, N_b - 2\}$, each natural integer m, and each word $\mathcal{M}$ of length m, let us consider the rectangle, whose sides are parallel to the horizontal and vertical axes, of width:*

$$L_m = x\left(T_{\mathcal{M}}(P_{j+1})\right) - x\left(T_{\mathcal{M}}(P_j)\right) = \frac{1}{(N_b - 1) N_b^m}$$

and height $|h_{j,m}|$, such that the points $T_{\mathcal{M}}(P_j)$ and $T_{\mathcal{M}}(P_{j+1})$ are two vertices of this rectangle.

(i) *If the integer N_b is odd, then:*

$$L_m^{2-D_{\mathcal{W}}}\,(N_b-1)^{2-D_{\mathcal{W}}}\times$$
$$\times\left\{\tfrac{2}{1-\lambda}\,\sin\tfrac{\pi}{N_b-1}\min_{0\le j\le N_b-1}\left|\sin\tfrac{\pi(2j+1)}{N_b-1}\right|-\tfrac{2\,\pi}{N_b\,(N_b-1)}\,\tfrac{1}{\lambda\,N_b-1}\right\}\le|h_{j,m}|$$

(ii) *If the integer N_b is even, then*

$$L_m^{2-D_{\mathcal{W}}}(N_b-1)^{2-D_{\mathcal{W}}}\times$$
$$\times\max\Big\{\tfrac{2}{1-\lambda}\,\sin\tfrac{\pi}{N_b-1}\min_{0\le j\le N_b-1}\left|\sin\tfrac{\pi\,(2j+1)}{N_b-1}\right|-$$
$$-\tfrac{2\,\pi}{N_b\,(N_b-1)}\,\tfrac{1}{\lambda\,N_b-1},\,\tfrac{4}{N_b^2}\,\tfrac{1-N_b^{-2}}{N_b^2-1}\Big\}\le|h_{j,m}|$$

Also:

$$|h_{j,m}|\le\eta_{\mathcal{W}}\,L_m^{2-D_{\mathcal{W}}}\,(N_b-1)^{2-D_{\mathcal{W}}}$$

where the real constant $\eta_{\mathcal{W}}$ is given by:

$$\eta_{\mathcal{W}}=2\pi^2\left\{\frac{(2N_b-1)\lambda(N_b^2-1)}{(N_b-1)^2(1-\lambda)(\lambda N_b^2-1)}+\frac{2N_b}{(\lambda N_b^2-1)\,(\lambda N_b^3-1)}\right\}.$$

Corollary 2.2. *The box-dimension of the graph $\Gamma_{\mathcal{W}}$ is exactly $D_{\mathcal{W}}$.*

Proof. By definition of the box-counting dimension $D_{\mathcal{W}}$ (we refer, for instance, to [4]), the smallest number of squares, the side length of which is at most equal to L_m, that can cover the graph $\Gamma_{\mathcal{W}}$ on $[0,1)$, obeys, approximately, a power law of the form:

$$c\,L_m^{-D_{\mathcal{W}}},\quad c>0.$$

Let us set

$$C=\max\Big\{\Big\{\tfrac{2}{1-\lambda}\,\sin\tfrac{\pi}{N_b-1}\min_{0\le j\le N_b-1}\left|\sin\tfrac{\pi(2j+1)}{N_b-1}\right|-\tfrac{2\,\pi}{N_b\,(N_b-1)}\,\tfrac{1}{\lambda\,N_b-1}\Big\},$$
$$\tfrac{4}{N_b^2}\,\tfrac{1-N_b^{-2}}{N_b^2-1},\,\eta_{\mathcal{W}}\Big\}$$

and consider the subdivision of the interval $[0,1)$ into:

$$N_m=\frac{1}{L_m}=(N_b-1)\,N_b^m$$

sub-intervals of length L_m. One has to determine a natural integer $\tilde N_m$ such that the graph of $\Gamma_{\mathcal{W}}$ on $[0,1)$ can be covered by $N_m\times\tilde N_m$ squares of side L_m. By considering, the vertical amplitude of the graph, one gets:

$$\tilde N_m=\left\lfloor\frac{C\,L_m^{2-D_{\mathcal{W}}}}{L_m}\right\rfloor+1,\qquad\text{i.e.}\qquad\tilde N_m=\left\lfloor C\,L_m^{1-D_{\mathcal{W}}}\right\rfloor+1.$$

Thus,

$$N_m \times \tilde{N}_m = \frac{\tilde{N}_m}{L_m} = \frac{1}{L_m} \left\lfloor C\, L_m^{1-D_{\mathcal{W}}} \right\rfloor + \frac{1}{L_m}.$$

The integer $N_m \times \tilde{N}_m$ then obeys a power law of the form

$$N_m \times \tilde{N}_m \approx c\, L_m^{-D_{\mathcal{W}}}$$

where c denotes a strictly positive constant. □

Proof of Theorem 2.1. Preliminary computations. For any pair of integers (i_m, j) of $\{0, \ldots, N_b - 2\}^2$

$$T_{i_m}(P_j) = \left(\frac{x_j + i_m}{N_b},\ \lambda y_j + \cos\left(2\pi\, \frac{x_j + i_m}{N_b}\right)\right).$$

For any triple of integers (i_m, i_{m-1}, j) of $\{0, \ldots, N_b - 2\}^3$

$$\begin{aligned} T_{i_{m-1}}(T_{i_m}(P_j)) &= \\ &= \left(\frac{\frac{x_j + i_m}{N_b} + i_{m-1}}{N_b}, \lambda^2\, y_j + \lambda\, \cos\left(2\pi\, \frac{x_j + i_m}{N_b}\right) + \cos\left(2\pi\, \frac{\frac{x_j + i_m}{N_b} + i_{m-1}}{N_b}\right)\right) \\ &= \left(\frac{x_j + i_m}{N_b^2} + \frac{i_{m-1}}{N_b}, \lambda^2 y_j + \lambda \cos\left(2\pi\, \frac{x_j + i_m}{N_b}\right) + \cos\left(2\pi\, \left(\frac{x_j + i_m}{N_b^2} + \frac{i_{m-1}}{N_b}\right)\right)\right). \end{aligned}$$

For any quadruple of integers $(i_m, i_{m-1}, i_{m-2}, j)$ of $\{0, \ldots, N_b - 2\}^4$

$$\begin{aligned} T_{i_{m-2}}(T_{i_{m-1}}(T_{i_m}(P_j))) &= \left(\frac{x_j + i_m}{N_b^3} + \frac{i_{m-1}}{N_b^2} + \frac{i_{m-2}}{N_b},\ \lambda^3 y_j + \lambda^2 \cos\left(2\pi \frac{x_j + i_m}{N_b}\right) + \right. \\ &\left. + \lambda \cos\left(2\pi\left(\frac{x_j + i_m}{N_b^2} + \frac{i_{m-1}}{N_b}\right)\right) + \cos\left(2\pi\left(\frac{x_j + i_m}{N_b^3} + \frac{i_{m-1}}{N_b^2} + \frac{i_{m-2}}{N_b}\right)\right)\right). \end{aligned}$$

Given a strictly positive integer m, and two points X and Y of V_m such that:

$$X \underset{m}{\sim} Y$$

there exists a word $\mathcal{M}$ of length $|\mathcal{M}| = m$, on the graph $\Gamma_{\mathcal{W}}$, and an integer j of $\{0, \ldots, N_b - 2\}^2$, such that:

$$X = T_{\mathcal{M}}(P_j), \qquad\qquad Y = T_{\mathcal{M}}(P_{j+1}).$$

Let us write $T_{\mathcal{M}}$ under the form:

$$T_{\mathcal{M}} = T_{i_m} \circ T_{i_{m-1}} \circ \ldots \circ T_{i_1},$$

where $(i_1, \ldots, i_m) \in \{0, \ldots, N_b - 1\}^m$.

One has then:

$$x(T_{\mathcal{M}}(P_j)) = \frac{x_j}{N_b^m} + \sum_{k=1}^{m} \frac{i_k}{N_b^k}, \qquad x(T_{\mathcal{M}}(P_{j+1})) = \frac{x_{j+1}}{N_b^m} + \sum_{k=1}^{m} \frac{i_k}{N_b^k},$$

and:

$$\begin{cases} y(T_{\mathcal{M}}(P_j)) = \lambda^m\, y_j + \sum\limits_{k=1}^{m} \lambda^{m-k}\, \cos\Big(2\pi\big(\frac{x_j}{N_b^k} + \sum\limits_{\ell=0}^{k} \frac{i_{m-\ell}}{N_b^{k-\ell}}\big)\Big) \\ y(T_{\mathcal{M}}(P_{j+1})) = \lambda^m\, y_{j+1} + \sum\limits_{k=1}^{m} \lambda^{m-k}\, \cos\Big(2\pi\big(\frac{x_{j+1}}{N_b^k} + \sum\limits_{\ell=0}^{k} \frac{i_{m-\ell}}{N_b^{k-\ell}}\big)\Big) \end{cases}$$

Determination of a lower bound. Let us note that:

$$\begin{aligned} &h_{j,m} - \lambda^m (y_{j+1} - y_j) = \\ &\quad = \sum_{k=1}^{m} \lambda^{m-k} \Big\{ \cos\Big(2\pi\big(\tfrac{x_{j+1}}{N_b^k} + \sum_{\ell=0}^{k} \tfrac{i_{m-\ell}}{N_b^{k-\ell}}\big)\Big) - \cos\Big(2\pi\big(\tfrac{x_j}{N_b^k} - \sum_{\ell=0}^{k} \tfrac{i_{m-\ell}}{N_b^{k-\ell}}\big)\Big)\Big\} \\ &\quad = -2 \sum_{k=1}^{m} \lambda^{m-k} \sin\big(\pi \tfrac{x_{j+1}-x_j}{N_b^k}\big) \sin\Big(2\pi\big(\tfrac{x_{j+1}+x_j}{2\,N_b^k} + \sum_{\ell=0}^{k} \tfrac{i_{m-\ell}}{N_b^{k-\ell}}\big)\Big). \end{aligned}$$

Taking into account:

$$\begin{aligned} \lambda^m (y_{j+1} - y_j) &= \frac{\lambda^m}{1-\lambda} \left(\cos \frac{2\,\pi\,(j+1)}{N_b - 1} - \cos \frac{2\,\pi\, j}{N_b - 1} \right) \\ &= \frac{-2\lambda^m}{1-\lambda}\, \sin \frac{2\pi(j+1-j)}{2(N_b-1)}\, \sin \frac{2\pi(j+1+j)}{2(N_b-1)} \\ &= \frac{-2\lambda^m}{1-\lambda}\, \sin \frac{\pi}{N_b-1}\, \sin \frac{\pi(2j+1)}{N_b-1}, \end{aligned}$$

the triangular inequality leads then to:

$$\begin{aligned} &\big|y(T_{\mathcal{M}}(P_{j+1})) - y(T_{\mathcal{M}}(P_j))\big| = \\ &= \Big|\lambda^m (y_{j+1} - y_j) - 2\sum_{k=1}^{m} \lambda^{m-k} \sin \tfrac{\pi}{N_b^{k+1}(N_b-1)}\, \sin\big(\tfrac{\pi(2j+1)}{N_b^{k+1}(N_b-1)} + 2\pi \sum_{\ell=0}^{k} \tfrac{i_{m-\ell}}{N_b^{k-\ell}}\big)\Big| \\ &\geq \Big|\lambda^m |y_{j+1} - y_j| - 2\sum_{k=1}^{m} \lambda^{m-k} \Big|\sin \tfrac{\pi}{N_b^{k+1}(N_b-1)}\, \sin\big(\tfrac{\pi(2j+1)}{N_b^{k+1}(N_b-1)} + 2\pi \sum_{\ell=0}^{k} \tfrac{i_{m-\ell}}{N_b^{k-\ell}}\big)\Big|\Big| \\ &= \lambda^m \Bigg| \tfrac{2}{1-\lambda} \sin \tfrac{\pi}{N_b-1} \cdot \Big|\sin \tfrac{\pi(2j+1)}{N_b-1}\Big| - \\ &\qquad - 2\sum_{k=1}^{m} \lambda^{-k} \sin \tfrac{\pi}{N_b^{k+1}(N_b-1)} \cdot \Big|\sin\big(\tfrac{\pi(2j+1)}{N_b^{k+1}(N_b-1)} + 2\pi \sum_{\ell=0}^{k} \tfrac{i_{m-\ell}}{N_b^{k-\ell}}\big)\Big|\Bigg|. \end{aligned}$$

One also has:

$$2\sum_{k=1}^{m}\lambda^{-k}\sin\tfrac{\pi}{N_b^{k+1}(N_b-1)}\left|\sin\left(\tfrac{\pi(2j+1)}{N_b^{k+1}(N_b-1)}+2\pi\sum_{\ell=0}^{k}\tfrac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right|\leq$$

$$\begin{aligned}&\leq 2\sum_{k=1}^{m}\lambda^{-k}\sin\tfrac{\pi}{N_b^{k+1}(N_b-1)}\leq 2\sum_{k=1}^{m}\lambda^{-k}\tfrac{\pi}{N_b^{k+1}(N_b-1)}\\ &=\tfrac{2\pi}{N_b(N_b-1)}\sum_{k=1}^{m}\tfrac{1}{\lambda^k N_b^k}=\tfrac{2\pi}{N_b(N_b-1)}\lambda^{-1}N_b^{-1}\tfrac{1-\lambda^{-m}N_b^{-m}}{1-\lambda^{-1}N_b^{-1}}\\ &\leq\tfrac{2\pi}{N_b(N_b-1)}\lambda^{-1}N_b^{-1}\tfrac{1}{1-\lambda^{-1}N_b^{-1}}=\tfrac{2\pi}{N_b(N_b-1)}\tfrac{1}{\lambda N_b-1}\end{aligned}$$

which yields:

$$-2\sum_{k=1}^{m}\lambda^{-k}\sin\tfrac{\pi}{N_b^{k+1}(N_b-1)}\cdot\left|\sin\left(\tfrac{\pi(2j+1)}{N_b^{k+1}(N_b-1)}+2\pi\sum_{\ell=0}^{k}\tfrac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right|\geq$$

$$\geq-\tfrac{2\pi}{N_b(N_b-1)}\tfrac{1}{\lambda N_b-1}.$$

Thus:

$$\frac{2}{1-\lambda}\sin\frac{\pi}{N_b-1}\cdot\left|\sin\frac{\pi(2j+1)}{N_b-1}\right|-$$

$$-2\sum_{k=1}^{m}\lambda^{-k}\sin\tfrac{\pi}{N_b^{k+1}(N_b-1)}\cdot\left|\sin\left(\tfrac{\pi(2j+1)}{N_b^{k+1}(N_b-1)}+2\pi\sum_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right|\geq$$

$$\geq\frac{2}{1-\lambda}\sin\frac{\pi}{N_b-1}\cdot\left|\sin\frac{\pi(2j+1)}{N_b-1}\right|-\frac{2\pi}{N_b(N_b-1)}\frac{1}{\lambda N_b-1}.$$

Lemma 2.2.1. *The following result holds: for $0\leq j\leq N_b-1$:*

$$\sin\frac{\pi(2j+1)}{N_b-1}=0 \textit{ if and only if } N_b \textit{ is even and } j=\frac{N_b}{2}-1.$$

Proof. Since $0\leq j\leq N_b-1$, one has:

$$1\leq 2j+1\leq 2N_b-1 \quad\text{and thus}\quad 0<\frac{2j+1}{N_b-1}\leq 2+\frac{1}{N_b-1}.$$

Then, $\sin\frac{\pi(2j+1)}{N_b-1}=0$ if and only if

$$\frac{2j+1}{N_b-1}=1 \quad\text{or}\quad \frac{2j+1}{N_b-1}=2.$$

The second case has to be rejected, since it would lead to

$$j=N_b-\frac{3}{2}\notin\mathbb{N}$$

The only possibility is thus when N_b is an even number:

$$j = \frac{N_b}{2} - 1.$$

The converse is obvious. □

First case: $\sin \frac{\pi(2j+1)}{N_b-1} \neq 0$. One has then:

$$\left|\sin \frac{\pi(2j+1)}{N_b-1}\right| \geq \min_{0\leq j\leq N_b-1} \left|\sin \frac{\pi(2j+1)}{N_b-1}\right| = \left|\sin \frac{\pi}{N_b-1}\right| \geq \frac{2}{N_b-1}.$$

This leads to:

$$\begin{aligned}
&\frac{2}{1-\lambda} \sin \frac{\pi}{N_b-1} \cdot \left|\sin \frac{\pi(2j+1)}{N_b-1}\right| - \\
&\quad - 2\sum_{k=1}^{m} \lambda^{-k} \sin \tfrac{\pi}{N_b^{k+1}(N_b-1)} \cdot \left|\sin\left(\tfrac{\pi(2j+1)}{N_b^{k+1}(N_b-1)} + 2\pi \sum_{\ell=0}^{k} \tfrac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right| \geq \\
&\qquad \geq \frac{2}{1-\lambda} \sin \frac{\pi}{N_b-1} \sin \frac{\pi}{N_b-1} - \frac{2\pi}{N_b(N_b-1)} \frac{1}{\lambda N_b - 1} \\
&\qquad \geq \frac{2}{1-\lambda} \frac{4}{(N_b-1)^2} - \frac{2\pi}{N_b(N_b-1)} \frac{1}{\lambda N_b - 1} \\
&\qquad = \frac{2}{N_b-1} \left\{ \frac{4}{1-\lambda} \frac{1}{N_b-1} - \frac{\pi}{N_b} \frac{1}{\lambda N_b - 1} \right\} \\
&\qquad = \frac{2\left(4N_b(\lambda N_b - 1) - \pi(1-\lambda)(N_b-1)\right)}{N_b(N_b-1)(1-\lambda)(\lambda N_b - 1)}.
\end{aligned}$$

The function

$$\lambda \longmapsto 4N_b(\lambda N_b - 1) - \pi(1-\lambda)(N_b-1)$$

is affine and strictly increasing in λ, and quadratic and strictly increasing in N_b, for strictly positive values of N_b. This ensures the positivity of:

$$\begin{aligned}
&\tfrac{2}{1-\lambda} \sin \tfrac{\pi}{N_b-1} \cdot \left|\sin \tfrac{\pi(2j+1)}{N_b-1}\right| - \\
&\quad - 2\sum_{k=1}^{m} \lambda^{-k} \sin \tfrac{\pi}{N_b^{k+1}(N_b-1)} \cdot \left|\sin\left(\tfrac{\pi(2j+1)}{N_b^{k+1}(N_b-1)} + 2\pi \sum_{\ell=0}^{k} \tfrac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right|.
\end{aligned}$$

Second case: $\sin \frac{\pi(2j+1)}{N_b-1} = 0$. One has then:

$$\begin{aligned}
&|y(T_{\mathcal{M}}(P_{j+1})) - y(T_{\mathcal{M}}(P_j))| \geq \\
&\qquad \geq 2\lambda^m \left|\sum_{k=1}^{m} \lambda^{-k} \sin \tfrac{\pi}{N_b^{k+1}} \cdot \left|\sin\left(\tfrac{\pi}{N_b^{k+1}} + 2\pi \sum_{\ell=0}^{k} \tfrac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right|\right|.
\end{aligned}$$

Thanks to the periodic properties of the sine function, one may only consider the case when:

$$0 \leq \frac{\pi}{N_b^{k+1}} + 2\pi \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N_b^{k-\ell}} \leq \frac{\pi}{2}.$$

Thus,

$$\begin{aligned}
\left|y(T_{\mathcal{M}}(P_{j+1})) - y(T_{\mathcal{M}}(P_j))\right| &\geq \sum_{k=1}^{m} \lambda^{-k} \frac{2}{N_b^{k+1}} \left\{ \frac{2}{N_b^{k+1}} + 2 \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N_b^{k-\ell}} \right\} \\
&\geq \sum_{k=1}^{m} \lambda^{-k} \frac{2}{N_b^{k+1}} \left\{ \frac{2}{N_b^{k+1}} \right\} \\
&= \frac{4\,\lambda^{-1}}{N_b^4} \frac{1-\lambda^{-m} N_b^{-2m}}{1-\lambda^{-1} N_b^{-2}} \\
&= \frac{4}{N_b^2 (N_b-1)^2} \frac{1-\lambda^{-m} N_b^{-m}}{\lambda N_b - 1} = \frac{4}{N_b^2} \frac{1-N_b^{-2}}{N_b^2-1}.
\end{aligned}$$

<u>General case.</u> The above results enable us to obtain the predominant term of the lower bound of $\left|y(T_{\mathcal{M}}(P_{j+1})) - y(T_{\mathcal{M}}(P_j))\right|$, which is thus:

$$\lambda^m = e^{m\,(D_{\mathcal{W}}-2)\,\ln N_b} = N_b^{m\,(D_{\mathcal{W}}-2)} = L_m^{2-D_{\mathcal{W}}}\,(N_b-1)^{2-D_{\mathcal{W}}}.$$

<u>Determination of an upper bound.</u> One has

$$\begin{aligned}
|h_{j,m}| &\leq \frac{2\lambda^m}{1-\lambda} \frac{\pi^2\,(2j+1)}{(N_b-1)^2} + 2\sum_{k=1}^{m} \lambda^{m-k}\,\pi \left\{ \frac{2j+1}{(N_b-1)\,N_b^k} + 2\sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N_b^{k-\ell}} \right\} \frac{\pi}{(N_b-1)\,N_b^k} \\
&= \frac{2\lambda^m}{1-\lambda} \frac{\pi^2\,(2j+1)}{(N_b-1)^2} + \frac{2\pi^2\lambda^m}{N_b-1} \sum_{k=1}^{m} \left\{ \frac{(2j+1)\,\lambda^{-k}}{(N_b-1)\,N_b^{2k}} + 2\sum_{\ell=0}^{k} \frac{i_{m-\ell}\,\lambda^{-k}}{N_b^{2k-\ell}} \right\} \\
&= \frac{2\lambda^m}{1-\lambda} \frac{\pi^2\,(2j+1)}{(N_b-1)^2} + \\
&\quad + \frac{2\pi^2\lambda^m}{N_b-1} \left\{ \frac{\lambda^{-1} N_b^{-2}\,(2j+1)}{(N_b-1)} \frac{(1-\lambda^{-m} N_b^{-2m})}{1-\lambda^{-1} N_b^{-2}} + 2\sum_{k=1}^{m} \frac{(N_b-1)\,\lambda^{-k}}{N_b^{2k}} \frac{1-N_b^{-k-1}}{1-N_b^{-1}} \right\} \\
&\leq \frac{2\lambda^m}{1-\lambda} \frac{\pi^2\,(2N_b-1)}{(N_b-1)^2} + \frac{2\pi^2\lambda^m}{N_b-1} \frac{(2N_b-1)}{(N_b-1)} \frac{(1-\lambda^{-m} N_b^{-2m})}{\lambda N_b^2 - 1} \\
&\quad + \frac{2\pi^2\lambda^m}{N_b-1}\, 2\, \frac{\lambda^{-1} N_b^{-2}\,(N_b-1)\,(1-\lambda^{-m} N_b^{-2m})}{(1-N_b^{-1})\,(1-\lambda^{-1} N_b^{-2})}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{2\,\pi^2\,\lambda^m}{N_b-1}\, 2\, \frac{\lambda^{-1}\,N_b^{-3}\,(N_b-1)\,(1-\lambda^{-m}\,N_b^{-3m})}{(1-N_b^{-1})\,(1-\lambda^{-1}\,N_b^{-3})} \\
&\leq \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,N_b-1)}{(N_b-1)^2} + \frac{2\,\pi^2\,\lambda^m}{N_b-1}\,\frac{(2\,N_b-1)}{(N_b-1)}\,\frac{1}{\lambda\,N_b^2-1} \\
&+ \frac{4\,\pi^2\,N_b\,\lambda^m}{N_b-1}\left\{\frac{1}{\lambda\,N_b^2-1} - \frac{1}{\lambda\,N_b^3-1}\right\} \\
&= 2\,\pi^2\,\lambda^m \left\{\frac{(2\,N_b-1)\,\lambda\,(N_b^2-1)}{(N_b-1)^2\,(1-\lambda)\,(\lambda\,N_b^2-1)} + \frac{2\,N_b}{(\lambda\,N_b^2-1)\,(\lambda\,N_b^3-1)}\right\}.
\end{aligned}
$$

Since

$$
x(T_{\mathcal{M}}(P_{j+1})) - x(T_{\mathcal{M}}(P_j)) = \frac{1}{(N_b-1)N_b^m}
$$

and

$$
D_{\mathcal{W}} = 2 + \frac{\ln\lambda}{\ln N_b}, \qquad \lambda = e^{(D_{\mathcal{W}}-2)\,\ln N_b} = N_b^{(D_{\mathcal{W}}-2)},
$$

one has thus

$$
\begin{aligned}
|h_{j,m}| \leq\ & 2\pi^2 L_m^{2-D_{\mathcal{W}}}(N_b-1)^{2-D_{\mathcal{W}}} \times \\
& \times \left\{\frac{(2N_b-1)\lambda(N_b^2-1)}{(N_b-1)^2(1-\lambda)(\lambda N_b^2-1)} + \frac{2N_b}{(\lambda N_b^2-1)(\lambda N_b^3-1)}\right\}.
\end{aligned}
$$

Theorem 2.1 is completed. □

Remark 2.3. In [7] B. Hunt uses the fact that the Hausdorff dimension of a fractal set $\mathcal{F}$ can be obtained by means of what is called the t-energy, $t \in \mathbb{R}$, of a Borel measure supported on $\mathcal{F}$ (one may refer to [4], for instance):

$$
I_t(\mu) = \iint \frac{d\mu(x)\,d\mu(x')}{|x-x'|^t}
$$

which enables one to obtain:

$$
\dim \mathcal{F} = \sup\,\{t \in \mathbb{R}, |\ \mu \text{ supported on } \mathcal{F}\,, I_t(\mu) < +\infty\}\,.
$$

A lower bound t_0 of the Hausdorff dimension can thus be obtained by building a measure μ supported on $\mathcal{F}$ such that:

$$
I_{t_0}(\mu) < +\infty.
$$

B. Hunt proceeds as follows: he introduces the measure $\mu_{\mathcal{W}}$ supported on $\Gamma_{\mathcal{W}}$, induced by the Lebesgue measure μ on $[0,1]$. Thus

$$
I_t(\mu_{\mathcal{W}}) = \iint \frac{d\mu_{\mathcal{W}}(x)\,d\mu_{\mathcal{W}}(x')}{\{|x-x'|^2 + |\mathcal{W}(x) - \mathcal{W}(x')|^2\}^{\frac{t}{2}}}.
$$

We could also have used a similar argument since, in our case:

$$|x-x'|^{2-D_{\mathcal{W}}} \lesssim |\mathcal{W}(x)-\mathcal{W}(x')| \lesssim |x-x'|^{2-D_{\mathcal{W}}}.$$

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Claire David
Sorbonne Université, CNRS, Laboratoire Jacques-Louis Lions, 4, place Jussieu 75005, Paris, France
Email: Claire.David@upmc.fr
ORCID: orcid.org/0000-0002-4729-0733

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тел. (0342) 58 04 32, +38 050 540 30 64
email: gsm1502@ukr.net
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