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On the number of maximal independent sets in complete q-ary trees

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Abstract: The paper is concerned with the asymptotic behaviour of the number $\mathrm{mi}(T_{q,n})$ of maximal independent sets in a complete q-ary tree of height n. For some constants α_2 and β_2 the asymptotic formula $\mathrm{mi}(T_{2,n}) \sim \alpha_2 \cdot (\beta_2)^{2^n}$ is shown to hold as $n \to \infty$. It is also proved that $\mathrm{mi}(T_{q,3k}) \sim \alpha_q^{(1)} \cdot (\beta_q)^{q^{3k}}$, $\mathrm{mi}(T_{q,3k+1}) \sim \alpha_q^{(2)} \cdot (\beta_q)^{q^{3k+1}}$, $\mathrm{mi}(T_{q,3k+2}) \sim \alpha_q^{(3)} \cdot (\beta_q)^{q^{3k+2}}$ as $k \to \infty$ for any sufficiently large q, some three pairwise distinct constants $\alpha_q^{(1)}, \alpha_q^{(2)}, \alpha_q^{(3)}$ and a constant b_q.

Keywords: maximal independent set, complete q-ary tree

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1 Introduction

An *independent set* in a graph is an arbitrary set of its pairwise nonadjacent vertices. An independent set in a graph is *maximal* if it is maximal under inclusion. We shall write "m.i.s." to abbreviate the phrase "maximal independent set". The number of independent sets (respectively, maximal independent sets) in a graph G is usually denoted by $i(G)$ (respectively, $\mathrm{mi}(G)$).

The asymptotic behaviour of independent sets in graphs from parametrically defined classes (as a function of the class parameters) was extensively studied. Korshunov and Sapozhenko [3] found the asymptotic behaviour of the number of independent sets in the n-dimensional cube. Kalkin and Wilf obtained the weak asymptotic formula for the number of independent sets in a complete grid graph [4]. Voronin and Demakova [1] found the asymptotic behaviour of the number of independent sets in complete binary trees. The case of complete q-ary trees was considered by Kirschenhofer, Prodinger, and Tichy. Let $T_{q,n}$ denote the complete q-ary tree of height n. Kirschenhofer, Prodinger, and Tichy [5] showed that there exist constants $\beta'_q, \alpha'_q, \alpha'_{q,1}, \alpha'_{q,2}$ ($\alpha'_{q,1} \neq \alpha'_{q,2}$) such that, for any $q \in \{2, 3, 4\}$, $i(T_{q,n}) \sim \alpha'_q \cdot (\beta'_q)^{q^n}$ as $n \to \infty$ and for any $q \geq 5$

$$i(T_{q,2k}) \sim \alpha'_{q,1} \cdot (\beta'_q)^{q^{2k}} \text{ and } i(T_{q,2k+1}) \sim \alpha'_{q,2} \cdot (\beta'_q)^{q^{2k+1}} \text{ as } k \to \infty.$$

The purpose of the present paper is to examine the behaviour of $\mathrm{mi}(T_{q,n})$ as $n \to \infty$ as a function of $q \geq 2$. The main results are as follows.

Theorem 1. *There exist constants α_2 and β_2 such that $\mathrm{mi}(T_{2,n}) \sim \alpha_2 \cdot (\beta_2)^{2^n}$ as $n \to \infty$.*

Theorem 2. *For any sufficiently large q there exist three pairwise distinct constants $\alpha_q^{(1)}, \alpha_q^{(2)}, \alpha_q^{(3)}$ and a constant b_q such that*

$$\mathrm{mi}(T_{q,3k}) \sim \alpha_q^{(1)} \cdot (\beta_q)^{q^{3k}}, \quad \mathrm{mi}(T_{q,3k+1}) \sim \alpha_q^{(2)} \cdot (\beta_q)^{q^{3k+1}}, \quad \mathrm{mi}(T_{q,3k+2}) \sim \alpha_q^{(3)} \cdot (\beta_q)^{q^{3k+2}}$$

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as $k \to \infty$.

2 The asymptotic formula for the number of m.i.s. in the trees $T_{q,n}$

In this section we shall prove Theorems 1 and 2. The proofs are not partitioned into lemmas and theorems, but will be rather given as subsections, each of which constitutes a separate conceptual part of the general argument.

2.1 The derivation of the recurrence relation for the number of m.i.s. in the trees $T_{q,n}$

We set $\mathrm{mi}(q,n) = \mathrm{mi}(T_{q,n})$. Next, we denote by $\mathrm{mi}_+(q,n)$ the number of m.i.s. in the tree $T_{q,n}$ of which each set contains its root r and denote by $\mathrm{mi}_-(q,n)$ the number of m.i.s. in the tree $T_{q,n}$ of which each does not contain the vertex r. It is clear that $\mathrm{mi}(q,n) = \mathrm{mi}_+(q,n) + \mathrm{mi}_-(q,n)$.

Let MIS be some m.i.s. of the tree $T_{q,n}$. By removing the root r of the tree $T_{q,n}$ and removing all the descendants of r we get a set of q^2 subtrees of the tree $T_{q,n}$, in which each subtree is isomorphic to $T_{q,n-2}$. Hence, if $r \in MIS$, then the set $MIS\setminus\{r\}$ is a disjoint union of q^2 sets of which each is an m.i.s. of its subtree $T_{q,n-2}$. Conversely, if in each of given q^2 subtrees we take m.i.s. and augment the union of these sets by the vertex r, then we obtain some m.i.s. of the tree $T_{q,n}$ that contains the vertex r. Hence we have $\mathrm{mi}_+(q,n) = (\mathrm{mi}(q,n-2))^{q^2}$.

Removing the root r from the tree $T_{q,n}$ we get the set of q subtrees of which each is isomorphic to $T_{q,n-1}$. If $r \notin MIS$, then MIS contains one root of some subtree from this set of q subtrees, because MIS is maximal under inclusion.

Conversely, if in each of these q subtrees one takes an m.i.s., assuming that at least one of these sets contains the root of its subtree, then the union of these sets gives an m.i.s. of the tree $T_{q,n}$ that does not contain the vertex r. Hence $\mathrm{mi}_-(q,n)$ is the difference between the number of ways to choose a family of q sets of which each is an m.i.s. of the tree isomorphic to $T_{q,n-1}$ and the number of ways to choose a family of q sets of which each is an m.i.s. of the tree isomorphic to $T_{q,n-1}$ and which does not contain its root. Hence $\mathrm{mi}_-(q,n) = (\mathrm{mi}(q,n-1))^q - (\mathrm{mi}_-(q,n-1))^q$. This equality (which depends on the above equality $\mathrm{mi}_+(q,n-1) = (\mathrm{mi}(q,n-3))^{q^2}$) may be rewritten as follows:

$$\begin{aligned}\mathrm{mi}_-(q,n) &= (\mathrm{mi}(q,n-1))^q - (\mathrm{mi}_-(q,n-1))^q = \\ &= (\mathrm{mi}(q,n-1))^q - (\mathrm{mi}(q,n-1) - \mathrm{mi}_+(q,n-1))^q = \\ &= (\mathrm{mi}(q,n-1))^q - (\mathrm{mi}(q,n-1) - (\mathrm{mi}(q,n-3))^{q^2})^q.\end{aligned}$$

Combining the above relations for $\mathrm{mi}_+(q,n)$ and $\mathrm{mi}_-(q,n)$, we find that

$$\mathrm{mi}(q,n) = (\mathrm{mi}(q,n-2))^{q^2} + (\mathrm{mi}(q,n-1))^q - (\mathrm{mi}(q,n-1) - (\mathrm{mi}(q,n-3))^{q^2})^q. \tag{1}$$

It is easily checked that $\mathrm{mi}(q,0) = 1, \mathrm{mi}(q,1) = 2, \mathrm{mi}(q,2) = 2^q$.

2.2 Particular solution of the resulting recurrence relation

In order to partially solve equation 1) with given initial conditions, we consider the quantity $a(q,n) \triangleq \frac{\mathrm{mi}(q,n)}{(\mathrm{mi}(q,n-1))^q}$.

It is clear that

$$\frac{(\mathrm{mi}(q,n-2))^{q^2}}{(\mathrm{mi}(q,n-1))^q} = \frac{1}{(a(q,n-1))^q} \text{ and } \frac{(\mathrm{mi}(q,n-3))^{q^2}}{\mathrm{mi}(q,n-1)} = \frac{1}{a(q,n-1)\cdot(a(q,n-2))^q}.$$

Hence equality (1) and its initial conditions may be rewritten as follows:

$$\begin{aligned} a(q,n) &= \frac{1}{(a(q,n-1))^q} + 1 - \left(1 - \frac{1}{a(q,n-1)\cdot(a(q,n-2))^q}\right)^q, \\ a(q,1) &= 2, \\ a(q,2) &= 1. \end{aligned} \tag{2}$$

It is easily verified (for example, using induction on n) that for any q the sequence $\{a(q,n)\}$ is bounded from above and below.

Since $a(q,n) = \frac{\mathrm{mi}(q,n)}{(\mathrm{mi}(q,n-1))^q}$, we have $\ln(a(q,n)) = \ln(\mathrm{mi}(q,n)) - q\cdot\ln(\mathrm{mi}(q,n-1))$. Hence, for any $n \geq 3$ and any $i \in \{0,\dots,n-3\}$ we have

$$q^i \cdot \ln(a(q,n-i)) = q^i \cdot \ln(\mathrm{mi}(q,n-i)) - q^{i+1}\cdot\ln(\mathrm{mi}(q,n-i-1)).$$

As a result, $\ln(\mathrm{mi}(q,n)) - q^{n-2}\cdot\ln(\mathrm{mi}(q,2)) = \sum\limits_{i=0}^{n-3}(\ln(a(q,n-i))\cdot q^i)$ for any $n \geq 3$. In other words,

$$\begin{aligned} \ln(\mathrm{mi}(q,n)) &= \sum_{i=3}^{n}(\ln(a(q,i))\cdot q^{n-i}) = q^n \cdot \sum_{i=3}^{n}(\ln(a(q,i))\cdot q^{-i}) = \\ &= q^n \cdot \left(\sum_{i=3}^{\infty}(\ln(a(q,i))\cdot q^{-i}) - \sum_{i=n+1}^{\infty}(\ln(a(q,i))\cdot q^{-i})\right) = \\ &= q^n \cdot \left(\sum_{i=3}^{\infty}(\ln(a(q,i))\cdot q^{-i}) - \sum_{i=n+1}^{\infty}(\ln(a(q,i))\cdot q^{n-i})\right). \end{aligned}$$

Since for any q the sequence $\{a(q,n)\}$ is bounded from above and below by some positive constants (this fact is clear from the inequalities $(\mathrm{mi}(q,n-1))^q \leq \mathrm{mi}(q,n) \leq 2\cdot(\mathrm{mi}(q,n-1))^q$), it follows that the sum $\sum\limits_{i=3}^{\infty}(\ln(a(q,i))\cdot q^{-i})$ of the convergent series is well defined; we denote this sum by $\ln(\beta_q)$. Hence $\ln(\mathrm{mi}(q,n)) = q^n\cdot\ln(\beta_q) + \ln(\alpha_{q,n})$ for some number $\alpha_{q,n}$, and so

$$\mathrm{mi}(q,n) = \alpha_{q,n}\cdot(\beta_q)^{q^n}. \tag{3}$$

Our next purpose is to prove the convergence of the sequence $\{\alpha_{2,n}\}$ and justify the convergence of the sequence $\{\alpha_{q,3k+r}\}$ for any $r \in \{0,1,2\}$ and any sufficiently large q. We have $\ln(\alpha_{q,n}) = -\sum\limits_{i=n+1}^{\infty}(\ln(a(q,i))\cdot q^{n-i})$, and hence to prove the above two facts it suffices to check the convergence of the sequence $\{a(2,n)\}$ and of the sequences $\{a(q,3k)\}$, $\{a(q,3k+1)\}$, $\{a(q,3k+2)\}$ for large q.

Thus, we have $\mathrm{mi}(T_{2,n}) \sim \alpha_2\cdot(\beta_2)^{2^n}$ as $n\to\infty$ for some constant α_2.

For any sufficiently large q there exist pairwise distinct constants $\alpha_q^{(1)}, \alpha_q^{(2)}, \alpha_q^{(3)}$ such that

$$\mathrm{mi}(T_{q,3k}) \sim \alpha_q^{(1)}\cdot(\beta_q)^{q^{3k}}, \quad \mathrm{mi}(T_{q,3k+1}) \sim \alpha_q^{(2)}\cdot(\beta_q)^{q^{3k+1}},$$
$$\mathrm{mi}(T_{q,3k+2}) \sim \alpha_q^{(3)}\cdot(\beta_q)^{q^{3k+2}} \text{ as } k\to\infty.$$

2.3 The case $q = 2$

In this subsection we shall show that the sequence $\{a(2,n)\}$ has limit. We set $g(t_1,t_2) \triangleq 1 + \frac{1}{t_1^2} - (1 - \frac{1}{t_1 t_2^2})^2$. On the halfaxis $[1,+\infty)$ there exists a unique l such that $l = g(l,l)$. This is indeed so, because the function $h(t) \triangleq t - g(t,t)$ has the derivative $h'_t = 1 + \frac{2}{t^3} + 2\cdot(1-\frac{1}{t^3})\cdot\frac{3}{t^4}$, which is positive and continuous at each point of this halfaxis, and besides, $h(1) = -1$ and $h(2) = \frac{97}{64}$. Since $h(1.29) = -0.025...$ and $h(1.3) = 0.005...$ it may be shown that $l = 1.29...$

The function $g(t_1, t_2)$ has the partial derivatives

$$g'_{t_1}(t_1, t_2) = -\frac{2}{t_1^3} - 2 \cdot \left(1 - \frac{1}{t_1 t_2^2}\right) \cdot \frac{1}{t_1^2 t_2^2} \text{ and } g'_{t_2}(t_1, t_2) = -2 \cdot \left(1 - \frac{1}{t_1 t_2^2}\right) \cdot \frac{2}{t_1 t_2^3}.$$

Consider the numbers $A \triangleq g'_{t_1}(l, l) = -1.296...$ and $B \triangleq g'_{t_2}(l, l) = -0.764...$ Clearly, $a(2, n) = g(a(2, n-1), a(2, n-2))$ for any $n \geq 3$. We set $\epsilon_n \triangleq a(2, n) - l$. By Taylor's formula $\epsilon_n = A \cdot \epsilon_{n-1} + B \cdot \epsilon_{n-2} + O(\epsilon_{n-1}^2 + \epsilon_{n-2}^2)$. Thus

$$\epsilon_{n+1} = A \cdot \epsilon_n + B \cdot \epsilon_{n-1} + O(\epsilon_n^2 + \epsilon_{n-1}^2) = (A^2 + B) \cdot \epsilon_{n-1} + AB \cdot \epsilon_{n-2} + O(\epsilon_{n-1}^2 + \epsilon_{n-2}^2)$$

and

$$\begin{aligned}\epsilon_{n+2} &= A \cdot \epsilon_{n+1} + B \cdot \epsilon_n + O(\epsilon_{n+1}^2 + \epsilon_n^2) = (A^2 + B) \cdot \epsilon_n + AB \cdot \epsilon_{n-1} + O(\epsilon_n^2 + \epsilon_{n-1}^2) = \\ &= (A^3 + 2AB) \cdot \epsilon_{n-1} + (A^2 B + B^2) \cdot \epsilon_{n-2} + O(\epsilon_{n-1}^2 + \epsilon_{n-2}^2).\end{aligned}$$

Hence, we have

$$|\epsilon_{n+2}| \leq (|A^3 + 2AB| + |A^2 B + B^2|) \cdot \max(|\epsilon_{n-1}|, |\epsilon_{n-2}|) + O(\epsilon_{n-1}^2 + \epsilon_{n-2}^2). \tag{4}$$

The number $|A^3 + 2AB| + |A^2 B + B^2| = 0.896...$ is smaller than 1. The constant C^*, which is implicitly involved in the O-symbol, may be estimated in terms of the maximum of the absolute values of the second derivatives of the function $g(t_1, t_2)$ on the square $[1, 2]^2$ and in terms of the numbers A, B. Calculating the first few terms of the subsequence $\{a(2, n)\}$ (see Table 2 in section 3) one can verify that there exists n^* for which the remainder $C^* \cdot (\epsilon_{n^*-1}^2 + \epsilon_{n^*-2}^2)$ in formula (4) is majorized by $\frac{1}{10} \cdot \max(|\epsilon_{n^*-1}|, |\epsilon_{n^*-2}|)$. As a result,

$$|\epsilon_{n^*+2}| \leq (|A^3 + 2AB| + |A^2 B + B^2| + \frac{1}{10}) \cdot \max(|\epsilon_{n^*-1}|, |\epsilon_{n^*-2}|).$$

Hence, there exists a number $0 < w < 1$ such that the inequality $|\epsilon_{n+2}| \leq w \cdot \max(|\epsilon_{n-1}|, |\epsilon_{n-2}|)$ holds for any $n \geq n^*$. From this inequality we have $\epsilon_n = O(w^{-n})$. Hence, the sequence $\{a(2, n)\}$ converges to l with exponential rate in n. This proves Theorem 1.

2.4 Solvability of one system of nonlinear equations

In this subsection we shall be concerned with the system of nonlinear equations

$$\begin{cases} x = f(z, y), \\ y = f(x, z), \\ z = f(y, x), \end{cases} \tag{5}$$

where $f(t_1, t_2) \triangleq \frac{1}{t_1^q} + 1 - \left(1 - \frac{1}{t_1 t_2^q}\right)^q$. We shall show that this system has a solution (x_q^*, y_q^*, z_q^*) for any sufficiently large q. In the notation for this function and its derivatives we shall not indicate explicitly its argument q.

Consider the following system of equations, which is a consequence of system (5):

$$\begin{cases} x = f(f(y, x), y), \\ y = f(x, f(y, x)). \end{cases} \tag{6}$$

We set

$$f_1(t_1, t_2) \triangleq t_1 - f(f(t_2, t_1), t_2), \quad f_2(t_1, t_2) \triangleq t_2 - f(t_1, f(t_2, t_1)),$$
$$\mathrm{Tr} \triangleq \left\{(t_1, t_2) : 1 \leq t_1 \leq 1 + \left(\frac{3}{2}\right)^{-q}, 1 \leq t_2 \leq 1 + \left(\frac{3}{2}\right)^{-q}, t_1 \leq t_2\right\}.$$

We claim that the interior of the triangle Tr contains a solution of system (6). Clearly, we have

$$\begin{aligned} f_1(t_1,t_2) &= t_1 - \frac{1}{f^q(t_2,t_1)} - 1 + \left(1 - \frac{1}{f(t_2,t_1)t_2^q}\right)^q, \\ f_2(t_1,t_2) &= t_2 - \frac{1}{t_1^q} - 1 + \left(1 - \frac{1}{t_1 f^q(t_2,t_1)}\right)^q. \end{aligned} \tag{7}$$

Next, it is easily verified that the asymptotic formula

$$f(t_1,t_2) \sim 2, \quad t_1^q \sim 1, \quad t_2^q \sim 1 \tag{8}$$

holds as $q \to \infty$ for any $(t_1,t_2) \in \mathrm{Tr}$.

We note that

$$\begin{aligned} f'_{t_1}(t_1,t_2) &= -\frac{q}{t_1^{q+1}} - q \cdot \left(1 - \frac{1}{t_1 t_2^q}\right)^{q-1} \cdot \frac{1}{t_1^2 t_2^q}, \\ f'_{t_2}(t_1,t_2) &= -q \cdot \left(1 - \frac{1}{t_1 t_2^q}\right)^{q-1} \cdot \frac{q}{t_1 t_2^{q+1}}, \\ \frac{df(t,t)}{dt} &= -\frac{q}{t^{q+1}} - q \cdot \left(1 - \frac{1}{t^{q+1}}\right)^{q-1} \cdot \frac{q+1}{t^{q+2}}. \end{aligned} \tag{9}$$

From equalities (7), (9) and the asymptotic formula (8) it follows that for any point $(t_1,t_2) \in \mathrm{Tr}$ the inequalities

$$\frac{\partial f_1}{\partial t_1}(t_1,t_2) > 0, \quad \frac{\partial f_2}{\partial t_1}(t_1,t_2) > 0, \quad \frac{\partial f_2}{\partial t_2}(t_1,t_2) > 0 \tag{10}$$

hold simultaneously for any sufficiently large q.

We claim that for any $t \in [1, 1 + (\frac{3}{2})^{-q}]$

$$\frac{df_3}{dt}(t) < 0, \tag{11}$$

where $f_3(t) \triangleq f_1(1,t) = -\frac{1}{f^q(t,1)} + (1 - \frac{1}{f(t,1)t^q})^q$. Indeed,

$$\begin{aligned} \frac{df_3}{dt}(t) &= q \cdot \frac{f'_t(t,1)}{f^{q+1}(t,1)} + q \cdot \left(1 - \frac{1}{f(t,1)t^q}\right)^{q-1} \cdot \frac{(f(t,1)t^q)'_t}{f^2(t,1)t^{2q}} = \\ &= q^2 \cdot \frac{-\frac{1}{t^{q+1}} - (1-\frac{1}{t})^{q-1} \cdot \frac{1}{t^2}}{f^{q+1}(t,1)} + q^2 \cdot (1 - \frac{1}{f(t,1)t^q})^{q-1} \cdot \frac{t^{q-1} - (t-1)^{q-1}}{f^2(t,1)t^{2q}} = \\ &= \frac{q^2}{f^2(t,1)t^{q+1}} \cdot \left(\left(1 - \frac{1}{f(t,1)t^q}\right)^{q-1} - \frac{1}{f^{q-1}(t,1)}\right) - q^2 \cdot \frac{(1-\frac{1}{t})^{q-1} \cdot \frac{1}{t^2}}{f^{q+1}(t,1)} - -q^2 \cdot \left(1 - \frac{1}{f(t,1)t^q}\right)^{q-1} \cdot \frac{(t-1)^q}{f^2(t,1)t^{2q}}. \end{aligned} \tag{12}$$

Clearly, $1 - \frac{1}{f(t,1)t^q} < \frac{1}{f(t,1)}$ for any $t \in [1, 1 + (\frac{3}{2})^{-q}]$. Hence from formula (12) we have $\frac{df_3}{dt}(t) < 0$ for any $t \in [1, 1 + (\frac{3}{2})^{-q}]$.

Consider the functions $f_1(t,t)$ and $f_2(t,t)$. From (7), (8) and (9) it follows that for any sufficiently large q both functions are monotone increasing on the interval $[1, 1 + (\frac{3}{2})^{-q}]$. Let us estimate the values of $f_1(t_1,t_2)$ and $f_2(t_1,t_2)$ at the vertices of the triangle Tr. We have $f_1(P_1) = 0$ and $f_2(P_1) = \left(1 - \frac{1}{2^q}\right)^q - 1 < 0$, where $P_1 \triangleq (1,1)$. The value of $f_1(t_1,t_2)$ at $P_2 \triangleq (1, 1 + (\frac{3}{2})^{-q})$ is equal to

$$\left(1 - \frac{1}{f(1 + (\frac{3}{2})^{-q}, 1) \cdot (1 + (\frac{3}{2})^{-q})^q}\right)^q - \frac{1}{f^q(1 + (\frac{3}{2})^{-q}, 1)}.$$

This number is negative, because $f(1 + (\frac{3}{2})^{-q}, 1) < 1 + \frac{1}{1+(\frac{3}{2})^{-q}}$. The value of $f_2(t_1,t_2)$ at P_2 is equal to $(\frac{3}{2})^{-q} + \left(1 - \frac{1}{f^q(1+(\frac{3}{2})^{-q},1)}\right)^q - 1$. The asymptotic formula (8) implies that $f_2(P_2) > 0$ for any sufficiently large q. Similarly, one may show that $f_1(P_3) > 0$ and $f_2(P_3) > 0$ for any sufficiently large q, where $P_3 \triangleq (1 + (\frac{3}{2})^{-q}, 1 + (\frac{3}{2})^{-q})$.

The mapping $F(t_1, t_2) \triangleq (f_1(t_1, t_2), f_2(t_1, t_2))$ sends the cathetus P_1P_2 of the triangle Tr into some curve S_1 connecting the points $F(P_1)$ and $F(P_2)$. Inequalities (10) and (11) show that the abscissa decreases and the ordinate increases as the curve S_1 is traversed from $F(P_1)$ to $F(P_2)$. The cathetus P_2P_3 is mapped into the curve S_2, along which from $F(P_2)$ to $F(P_3)$ both the abscissa and the ordinate are increasing (by the same inequalities). Since $\frac{df_1(t,t)}{dt} > 0$ and $\frac{df_2(t,t)}{dt} > 0$ for any $t \in [1, 1 + (\frac{3}{2})^{-q}]$, the side P_1P_3 is transformed by F into the curve S_3 along which from $F(P_1)$ to $F(P_3)$ both the abscissa and the ordinate are increasing.

The point $F(P_1)$ lies in the lower half-plane on the axis of ordinates, the point $F(P_2)$ lies in the second quadrant, and the point $F(P_3)$ in the first one. Hence the curvilinear triangle Tr', as bounded by the curves S_1, S_2 and S_3, contains the origin in its interior. The mapping F is continuous on Tr, and hence it maps Tr into Tr'. Hence system (6) has a solution (x_q^*, y_q^*, z_q^*) which is an interior point of Tr. Therefore, $1 < x_q^* < y_q^* < 1 + (\frac{3}{2})^{-q}$ and $z_q^* = 2 - O((\frac{3}{2})^{-q})$.

2.5 The case of large q

In this section we shall show that for any sufficiently large q the limit relations

$$a(q, 3k+1) \longrightarrow z_q^*,$$
$$a(q, 3k+2) \longrightarrow x_q^*,$$
$$a(q, 3k+3) \longrightarrow y_q^*$$

hold as $k \to \infty$. From (2) and the definition of the function $f(t_1, t_2)$ it follows that $a(q, n) = f(a(q, n-1), a(q, n-2))$. We set

$$\zeta_{k,q} \triangleq a(q, 3k+1) - z_q^*,$$
$$\eta_{k,q} \triangleq a(q, 3k+2) - x_q^*,$$
$$\theta_{k,q} \triangleq a(q, 3k+3) - y_q^*.$$

From (2) it follows that $\max(|\zeta_{0,q}|, |\eta_{0,q}|, |\theta_{0,q}|) \longrightarrow 0$ as $q \to \infty$ and

$$\begin{aligned} \zeta_{k+1,q} + z_q^* &= f(\theta_{k,q} + y_q^*, \eta_{k,q} + x_q^*), \\ \theta_{k,q} + y_q^* &= f(\eta_{k,q} + x_q^*, \zeta_{k,q} + z_q^*), \\ \eta_{k,q} + x_q^* &= f(\zeta_{k,q} + z_q^*, \theta_{k-1,q} + y_q^*). \end{aligned} \tag{13}$$

Using (13), the equalities $z_q^* = f(y_q^*, x_q^*)$, $y_q^* = f(x_q^*, z_q^*)$, $x_q^* = f(z_q^*, y_q^*)$, and Taylor's formula, we find that

$$\begin{aligned} \zeta_{k+1,q} &= f'_{t_1}(y_q^*, x_q^*) \cdot \theta_{k,q} + f'_{t_2}(y_q^*, x_q^*) \cdot \eta_{k,q} + O(\theta_{k,q}^2 + \eta_{k,q}^2), \\ \theta_{k,q} &= f'_{t_1}(x_q^*, z_q^*) \cdot \eta_{k,q} + f'_{t_2}(x_q^*, z_q^*) \cdot \zeta_{k,q} + O(\eta_{k,q}^2 + \zeta_{k,q}^2), \\ \eta_{k,q} &= f'_{t_1}(z_q^*, y_q^*) \cdot \zeta_{k,q} + f'_{t_2}(z_q^*, y_q^*) \cdot \theta_{k-1,q} + O(\zeta_{k,q}^2 + \theta_{k-1,q}^2). \end{aligned} \tag{14}$$

Next, by (2.5) we have

$$\begin{aligned} \eta_{k,q} &= f'_{t_1}(z_q^*, y_q^*) \cdot \zeta_{k,q} + f'_{t_2}(z_q^*, y_q^*) \cdot \theta_{k-1,q} + O(\zeta_{k,q}^2 + \theta_{k-1,q}^2), \\ \theta_{k,q} &= (f'_{t_1}(x_q^*, z_q^*) \cdot f'_{t_1}(z_q^*, y_q^*) + f'_{t_2}(x_q^*, z_q^*)) \cdot \zeta_{k,q} + \\ &\quad + f'_{t_1}(x_q^*, z_q^*) \cdot f'_{t_2}(z_q^*, y_q^*) \cdot \theta_{k-1,q} + O(\zeta_{k,q}^2 + \theta_{k-1,q}^2), \\ \zeta_{k+1,q} &= (f'_{t_1}(y_q^*, x_q^*) \cdot f'_{t_1}(x_q^*, z_q^*) \cdot f'_{t_1}(z_q^*, y_q^*) + f'_{t_1}(y_q^*, x_q^*) \cdot f'_{t_2}(x_q^*, z_q^*) + \\ &\quad + f'_{t_1}(z_q^*, y_q^*) \cdot f'_{t_2}(y_q^*, x_q^*)) \cdot \zeta_{k,q} + (f'_{t_1}(y_q^*, x_q^*) \cdot f'_{t_1}(x_q^*, z_q^*) \cdot f'_{t_2}(z_q^*, y_q^*) + \\ &\quad + f'_{t_2}(y_q^*, x_q^*) \cdot f'_{t_2}(z_q^*, y_q^*)) \cdot \theta_{k-1,q} + O(\zeta_{k,q}^2 + \theta_{k-1,q}^2). \end{aligned} \tag{15}$$

We have $x_q^* = 1 + O((\frac{3}{2})^{-q})$, $y_q^* = 1 + O((\frac{3}{2})^{-q})$ and $z_q^* = 2 - O((\frac{3}{2})^{-q})$, and hence by (9)

$$|f'_{t_1}(z_q^*, y_q^*)| = O(\frac{q}{2^q}), \quad |f'_{t_2}(z_q^*, y_q^*)| = O(\frac{q^2}{2^q}), \quad |f'_{t_1}(x_q^*, z_q^*)| = O(q),$$
$$|f'_{t_2}(x_q^*, z_q^*)| = O(\frac{q^2}{2^q}), \quad |f'_{t_1}(y_q^*, x_q^*)| = O(q), \quad |f'_{t_2}(y_q^*, x_q^*)| = O(q^2).$$

This shows that the coefficients multiplying $\zeta_{k,q}$ and $\theta_{k-1,q}$ in formulas (2.5) are exponentially decreasing in q. We recall that $\lim_{q\to\infty} \max(|\zeta_{0,q}|, |\eta_{0,q}|, |\theta_{0,q}|) = 0$. Hence, for any sufficiently large q, there exists $0 < w_q < 1$ such that $\max(|\eta_{k,q}|, |\theta_{k,q}|, |\zeta_{k,q}|) = O((w_q)^k)$. Therefore,

$$\lim_{k\to\infty} a(q, 3k+1) = z_q^*,$$
$$\lim_{k\to\infty} a(q, 3k+2) = x_q^*,$$
$$\lim_{k\to\infty} a(q, 3k+3) = y_q^*$$

for any sufficiently large q.

We recall that $\alpha_{q,n} = \exp\left\{-\sum_{i=1}^{\infty}(\ln(a(q, n+i)) \cdot q^{-i})\right\}$ and that $1 < x_q^* < y_q^* < 1 + (\frac{3}{2})^{-q}$ and $z_q^* = 2 - O((\frac{3}{2})^{-q})$ for large q. If $n + 1 \equiv 0 \pmod 3$ and n, q are sufficiently large, then

$$\sum_{i=1}^{\infty}(\ln(a(q, n+i)) \cdot q^{-i}) \approx \left(\frac{y_q^*}{q} + \frac{y_q^*}{q^4} + \frac{y_q^*}{q^7} + \dots\right) + \left(\frac{z_q^*}{q^2} + \frac{z_q^*}{q^5} + \frac{z_q^*}{q^8} + \dots\right) + \left(\frac{x_q^*}{q^3} + \frac{x_q^*}{q^6} + \frac{x_q^*}{q^9} + \dots\right).$$

The last sum is equal to $y_q^* \cdot \frac{q^2}{q^3-1} + z_q^* \cdot \frac{q}{q^3-1} + x_q^* \cdot \frac{1}{q^3-1}$. Similarly, if $n + 1 \equiv 1 \pmod 3$ and if n, q are sufficiently large, then $\sum_{i=1}^{\infty}(\ln(a(q, n+i)) \cdot q^{-i})$ is close to $z_q^* \cdot \frac{q^2}{q^3-1} + x_q^* \cdot \frac{q}{q^3-1} + y_q^* \cdot \frac{1}{q^3-1}$. Finally, if $n + 1 \equiv 2 \pmod 3$ and if n, q are sufficiently large, then $\sum_{i=1}^{\infty}(\ln(a(q, n+i)) \cdot q^{-i})$ is close to

$$x_q^* \cdot \frac{q^2}{q^3-1} + y_q^* \cdot \frac{q}{q^3-1} + z_q^* \cdot \frac{1}{q^3-1}.$$

Hence, for large q the three above sums are close to

$$\frac{q^2 + 2 \cdot q + 1}{q^3 - 1}, \quad \frac{2 \cdot q^2 + q + 1}{q^3 - 1}, \quad \frac{q^2 + q + 2}{q^3 - 1},$$

respectively. Hence, for $q \to \infty$, the subsequences $\{\alpha_{q,3k}\}$, $\{\alpha_{q,3k+1}\}$, $\{\alpha_{q,3k+2}\}$ converge to three pairwise different limits. This proves Theorem 2.

3 Some remarks

It would be interesting to know from what value of the parameter q the relations in the theorem become approximate equalities with different constants. In this regard a numerical experiment was carried out, which gave the following results (in the table we give the first three significant figures in the fractional parts of a number).

The first table shows that for $3 \le q \le 10$ it is highly improbable that the sequence $\{a(q, n)\}$ splits into three convergent subsequences whose terms have numbers correspond to the residue classes mod 3. This observation is supported by the results of numerical experiments with larger n and the same q (not given in the tables). At the same time, Tables 2–4 show that the sequences $\{a(q, 3k)\}$, $\{a(q, 3k+1)\}$, and $\{a(q, 3k+2)\}$ converge for $q \in \{11, 12, 13\}$. Numerical experiments also show the same phenomenon for larger q and k. This supports the conjecture that the conclusion of Theorem 2 (pertaining to the splitting into three convergent sequences) also holds for any $q > 10$ and fails to hold for $3 \le q \le 10$.

q	n									
	10	20	30	40	50	600	700	800	900	1000
2	1.178	1.284	1.303	1.300	1.298	1.298	1.298	1.298	1.298	1.298
3	1.045	1.194	1.445	1.510	1.290	1.106	1.329	1.411	1.118	1.252
4	1.008	1.226	1.805	1.374	1.028	1.037	1.038	1.040	1.042	1.044
5	1.004	1.466	1.566	1.021	1.001	1.765	1.000	1.790	1.019	1.363
6	1.008	1.790	1.108	1.000	1.309	1.039	1.000	1.000	1.001	1.082
7	1.036	1.691	1.000	1.193	1.410	1.213	1.000	1.694	1.000	1.960
8	1.222	1.113	1.000	1.896	1.000	1.000	1.000	1.000	1.000	1.075
9	1.593	1.018	1.333	1.034	1.025	1.000	1.018	1.996	1.025	1.000
10	1.818	1.000	1.995	1.000	1.053	1.000	1.087	1.000	1.038	1.004

Table 1: The values of some terms of the subsequence $\{a(q,n)\}$

q	k									
	1	2	3	4	5	60	70	80	90	100
11	1.942	1.922	1.913	1.909	1.906	1.904	1.904	1.904	1.904	1.904
12	1.966	1.958	1.956	1.955	1.955	1.955	1.955	1.955	1.955	1.955
13	1.979	1.976	1.976	1.976	1.976	1.976	1.976	1.976	1.976	1.976

Table 2: The values of some terms of the subsequence $\{a(q,3k+1)\}$

q	k									
	1	2	3	4	5	60	70	80	90	100
11	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.0008	1.000
12	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
13	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 3: The values of some terms of the subsequence $\{a(q,3k+2)\}$

q	k									
	1	2	3	4	5	60	70	80	90	100
11	1.005	1.007	1.008	1.008	1.008	1.009	1.009	1.009	1.009	1.009
12	1.003	1.004	1.004	1.004	1.004	1.004	1.004	1.004	1.004	1.004
13	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001	1.001

Table 4: The values of some terms of the subsequence $\{a(q,3k)\}$

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