Dmitriy S. Malyshev*

Complexity classification of the edge coloring problem for a family of graph classes

DOI: 10.1515/dma-2017-0011
Received January 11, 2016

Abstract: A class of graphs is called monotone if it is closed under deletion of vertices and edges. Any such class may be defined in terms of forbidden subgraphs. The chromatic index of a graph is the smallest number of colors required for its edge-coloring such that any two adjacent edges have different colors. We obtain a complete classification of the complexity of the chromatic index problem for all monotone classes defined in terms of forbidden subgraphs having at most 6 edges or at most 7 vertices.

Keywords: computational complexity, chromatic index problem, efficient algorithm

Note: Originally published in *Diskretnaya Matematika* (2016) **28**, №2, 44–50 (in Russian).

1 Introduction

The paper is concerned with *ordinary graphs* (that is, with unlabeled undirected graphs without loops and multiple edges). A graph H is called an *induced subgraph* of a graph G if H is obtained by removal from G some set of its vertices (possibly empty). By the removal of a vertex we shall mean the removal of all edges that are incident with it. A graph H is called a *subgraph* of a graph G if H is obtained from G by removing some (or none) vertices and edges.

A class of graphs is called *hereditarily* if it is closed under deletion of vertices. Any hereditarily class (and only hereditarily classes) of graphs $\mathcal{X}$ may be defined by the set of its forbidden induced subgraphs $\mathcal{S}$. This is usually written $\mathcal{X} = \mathrm{Free}(\mathcal{S})$. A *monotone* class of graphs is a hereditarily class which is closed under deletion of edges. Any monotone class $\mathcal{X}$ may be defined in terms of its forbidden subgraphs $\mathcal{S}$; this is written $\mathcal{X} = \mathrm{Free}_s(\mathcal{S})$.

By a *vertex coloring* of a graph G we shall mean any mapping $c : V(G) \longrightarrow \mathbb{N}$ such that $c(u) \neq c(v)$ for any adjacent vertices u and v. An *edge-coloring* of a graph G is any mapping $c : E(G) \longrightarrow \mathbb{N}$ such that $c(e_1) \neq c(e_2)$ for any consecutive edges e_1 and e_2. The smallest number of colors in vertex- and edge-colorings of a graph G are called, respectively, the *chromatic number* and the *chromatic index* of G; these are denoted by $\chi(G)$ and $\chi'(G)$, respectively. Given a graph G, the *k-vertex-coloring problem* or simply the k-VC-problem (respectively, the *k-edge-coloring problem* or the k-EC-problem) is the assignment of k different colors to the vertices (edges) of the graph G. For a given graph G, the *chromatic number problem* and the *chromatic indexes problem* (briefly CN- and CI-problems) is to calculate $\chi(G)$ and $\chi'(G)$, respectively. For an NP-complete problem Π on graphs, the class with polynomially solvable problem Π will be called *Π-simple*; the class with NP-complete problem Π will be called *Π-complex*.

The CN- and CI-problems are related via the concept of an edge graph. A graph H is an *edge graph* of a graph G (written $H = L(G)$) if $V(H) = E(G)$ and two vertices of H are adjacent if and only if the corresponding edges in G are adjacent. Not any graph is an edge graph, for example, the graph $K_{1,3}$, which is a tree with 4 vertices of which one has degree 3, is not not an edge graph. In other words, any edge graph lies in $\mathrm{Free}(\{K_{1,3}\})$. It is plain that $\chi(L(G)) = \chi'(G)$ for any graph G. If an edge graph H is connected and nontriangle, then there exists a unique graph G such that $H = L(G)$; besides the graph G may be calculated

***Corresponding Author: Dmitriy S. Malyshev:** National Research University Higher School of Economics, Lobachevsky State University of Nizhny Novgorod, e-mail: dsmalyshev@rambler.ru

Authenticated | dsmalyshev@rambler.ru author's copy
Download Date | 10/14/17 8:42 AM

from the graph H in a linear time [12]. It follows that for any monotone class $\mathcal{X}$ the CI-problem in the class $\mathcal{X}$ is polynomially equivalent to the CN-problem in the class $L(\mathcal{X})$. According to the well-known Vising theorem [1], any graph G satisfies the inequality $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximal vertex degree of G.

The computational complexities of the 3-VC and 4-VC-problems in the families $\{\text{Free}(\{G\}) : |V(G)| \leq 6\}$ and $\{\text{Free}(\{G\}) : |V(G)| \leq 5\}$ respectively, were calculated in [4, 6]. Moreover, the complete complexity dichotomies for the 3-VC- and 3-EC-problems are known for the families $\{\text{Free}(\{G_1, G_2\}) : \max(|V(G_1)|, |V(G_2)|) \leq 5\}$ and $\{\text{Free}(\mathcal{S})$: any graph from $\mathcal{S}$ contains at most 6 vertices and $\mathcal{S}$ contains at most 2 graphs with 6 vertices$\}$ [2, 10]. In the present paper we shall be concerned with the family of monotone classes defined by forbidding the subgraphs having at most 6 edges or at most 7 vertices. We shall show that such a class is CI-simple if one of the forbidden subgraphs is a subcubic forest; otherwise it is CI-complex. A graph is called *subcubic* if the degree of any of its vertices is at most 3.

We shall use the following notation: $\deg(x)$ is the degree of a vertex x; $N(x)$ is the neighborhood of a vertex x; $G \setminus \{x\}$ is the graph obtained from G by deleting a vertex x; $G_1 + G_2$ is the disjoint union of graphs G_1 and G_2 with disjoint sets of vertices; K_n and O_n are, respectively, the complete and empty n-vertex graphs; a bridge B_k is a graph obtained by joining two vertices of two simple three-vertex paths by a simple path of length k; B_1^+ is the graph obtained by subdividing the edges of the graph B_1 that are incident with vertices of degrees 2 and 3; the butterfly is the graph obtained by identifying two vertices of two triangles, each of which lies its own triangle.

2 Auxiliary results

2.1 Simplest graphs and their relevance

A connected graph G will be called *simple* if any vertex of G has at least two neighbors of degree $\Delta(G)$. This concept is relevant because for any class $\mathcal{X}$ the CI-problem is polynomially reducible to the same problem for the part of $\mathcal{X}$ which consists of all simple graphs from $\mathcal{X}$. This clearly follows from one result of [13]: a graph G containing a vertex x, for which at most one neighbor has the degree $\Delta(G)$, is edge-colorable by $\Delta(G)$ colors if and only if the graph $G \setminus \{x\}$ is $\Delta(G)$ — colorable.

Lemma 1. *If a monotone class $\mathcal{X}$ does not contain at least one of the three graphs $B_1 + K_2, B_1^+, B_2$, then the 3-EC-problem is polynomially solvable in the class $\mathcal{X}$.*

Proof. We shall first show that the 3-EC-problem in the class $\mathcal{X}$ is polynomially reducible to the same problem in the class $\mathcal{X} \cap \text{Free}_s(\{B_1\})$. Let G be an arbitrary simple graph from $\mathcal{X}$ containing B_1 as a subgraph. Next, let $V(B_1) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and $E(B_1) = \{x_1x_2, x_2x_3, y_1y_2, y_2y_3, x_2y_2\}$. It may be assumed that $\Delta(G) \leq 3$, for otherwise G is not 3-edge-colorable. If $\mathcal{X} \subseteq \text{Free}_s(\{B_1^+\})$, then $|V(G)| \leq 6$, because G is connected and since $\Delta(G) \leq 3$. The same argument shows that $|V(G)| \leq 14$ if $G \in \text{Free}_s(\{B_1 + K_2\})$ (no vertex G may be at distance 3 or more from the central edge x_2y_2 of the subgraph B_1). By the *distance* between a vertex $v \in V(G)$ and the edge x_2y_2 we shall mean the smallest distance between v and x_2 and between v and y_2.

Let $\mathcal{X} \subseteq \text{Free}_s(\{B_2\})$. Assume that there exists a vertex G lying at distance 4 from x_2y_2. Then there exist vertices a, b, c which generate in G a simple 3-vertex path and such that the distances between x_2y_2 and the vertices a, b, c are, respectively, 2, 3, 4. Without loss of generality it may be assumed that the vertices x_1 and a are adjacent. Clearly, the vertices x_1 and x_3 are not adjacent, for otherwise the edges ax_1, x_1x_3, x_1x_2, x_2y_2, y_1y_2, y_2y_3 would form B_2. Similarly, the vertex a is not adjacent to any of the vertices y_1 and y_3. Since G is simple, at least one of the vertices x_1 and a is of degree 3.

Let $\deg(x_1) = 3$. If a neighbor x of the vertex x_1 is distinct from a and x_2, then x must coincide with y_1 or with y_3, for otherwise the edges ax_1, xx_1, x_1x_2, x_2y_2, y_1y_2, y_2y_3 would form B_2. We shall assume that $x = y_1$. It is plain that $\deg(b) = 2$, for otherwise there would exist a vertex from $N(b) \setminus \{a, c\}$ not lying in $V(B_1)$, and hence G would contain the subgraph B_2. Since G is simple, we have $\deg(a) = 3$. Clearly, the

Authenticated | dsmalyshev@rambler.ru author's copy
Download Date | 10/14/17 8:42 AM

vertices a and x_3 are adjacent, for otherwise G would contain the subgraph B_2. The degree of the vertex c is at most 2, for otherwise the edges ab, ax_1, ax_3 and the 3 edges that are incident with c would form B_2. This, however, contradicts the simplicity of G, inasmuch as $\deg(b) = \deg(c) = 2$.

Assume now that $\deg(x_1) = 2$. Then $\deg(a) = 3$ and the vertex a is a neighbor of x_3. Again using the fact that G is simple, we see that there exist vertices $b' \in N(b) \setminus \{a, c\}$ and $x' \in N(x_3) \setminus \{a, x_2\}$. We recall that $x' \neq x_1$. Next, it is plain that $x' \in \{y_1, y_3\}$, for otherwise the edges $x'x_3, ax_3, x_3x_2, x_2y_2, y_3y_2, y_2y_1$ would form a subgraph B_2. But then the edges $x'x_3, x_2x_3, ax_3, ab, bc, bb'$ form the subgraph B_2 of the graph G, a contradiction.

The reducibility claimed in the first paragraph of the present section does indeed hold. By the above, any simple graph from $\mathcal{X} \setminus \text{Free}_s(\{B_1\})$ has at most $2 + 4 \cdot (1 + 2 + 4) = 30$ vertices, because any vertex of such a graph is at the distance at most 3 from the central edge of the subgraph B_1. Hence, the number of such graphs is finite.

It is clear that the 3-EC-problem in the class $\mathcal{X} \cap \text{Free}_s(\{B_1\})$ is polynomially equivalent to the 3-VC-problem in the class $L(\mathcal{X} \cap \text{Free}_s(\{B_1\}))$. Next, it is plain that $L(\mathcal{X} \cap \text{Free}_s(\{B_1\})) \subseteq \text{Free}(\{K_{1,3}, \text{butterfly}\})$. The 3-VC-problem is polynomially solvable in the class $\text{Free}(\{K_{1,3}, \text{butterfly}\})$ [11]. Hence, the class $\mathcal{X}$ is 3-EC-simple. □

A simple graph G will be called *simplest* if $\Delta(G) \geq 4$. From Vising theorem and Lemma 1 it follows that for any class of graphs $\mathcal{X}$ the CI-problem is polynomially reducible to the same problem for the part of $\mathcal{X}$ consisting of all possible simplest graphs from $\mathcal{X}$.

Lemma 2. *Any simplest graph from* $\text{Free}_s(\{B_1^+\}) \cup \text{Free}_s(\{B_1 + K_2\}) \cup \text{Free}_s(\{B_2\})$ *contains at most* 62 *vertices.*

Proof. Let G be a simplest graph from $\text{Free}_s(\{B_1^+\}) \cup \text{Free}_s(\{B_1 + K_2\}) \cup \text{Free}_s(\{B_2\})$. Being simple, the graph G contains two adjacent vertices x and y, each of which is of degree $\Delta(G)$.

Let $\Delta(G) \geq 7$ and let z be an arbitrary element of the set $N(x) \setminus \{y\}$ of degree $\Delta(G)$. Such a vertex z always exists, inasmuch as G is a simplest graph. If $G \in \text{Free}_s(\{B_1^+\}) \cup \text{Free}_s(\{B_1 + K_2\})$, then the vertex z cannot have a neighbor outside the set $N(x) \cup N(y)$. If such a vertex z' exists, then the edges $z'z, xz, xz_1, xz_2, xy, yz_3, yz_4$ (where z_1 and z_2 are arbitrary vertices from $N(x) \setminus \{y, z\}$ and z_3 and z_4 are arbitrary vertices from $N(y) \setminus \{x, z, z_1, z_2\}$) form the subgraph G' of the graph G. Clearly, $B_1 + K_2$ and B_1^+ are subgraphs of the graph G'. If $G \in \text{Free}_s(\{B_2\})$, then the vertex z cannot have two neighbors z^*, z^{**} outside the set $N(x) \cup N(y)$, for otherwise the edges $zz^*, zz^{**}, zx, xy, yy^*, yy^{**}$ would form B_2, where y^* and y^{**} are arbitrary vertices from $N(y) \setminus \{x, z\}$. We have $\deg(z) = \Delta(G) \geq 7$, and hence, the set $(N(x) \cup N(y)) \setminus \{x, y\}$ contains at least five neighbors of the vertex z. Let a_1 and a_2 be arbitrary neighbors of the vertex z from this set. The edges $za_1, xz_1, xz_2, xy, yz_3, yz_4$ (where z_1 and z_2 are arbitrary vertices from $N(x) \setminus \{y, z, a_1\}$ and z_3 and z_4 are arbitrary vertices from $N(y) \setminus \{x, z, a_1, z_1, z_2\}$) form the subgraph $B_1 + K_2$ of the graph G. The edges $za_1, za_2, xz, xy, yz', yz''$, where z' and z'' are arbitrary vertices from $N(y) \setminus \{x, z, a_1, a_2\}$, form the subgraph B_2 of the graph G. Besides, the edges $za_1, xz, xz^*, xy, yz^{**}, yz^{***}$ (where z^* is an arbitrary vertex from $N(x) \setminus \{y, z, a_1\}$ and z^{**} and z^{***} are arbitrary vertices from $N(y) \setminus \{x, z, a_1, z^*\}$) form the subgraph B_1^+ of the graph G. This contradiction shows that $4 \leq \Delta(G) \leq 6$.

Let $4 \leq \Delta(G) \leq 6$. It is easily seen that $|(N(x) \cup N(y)) \setminus \{x, y\}| \geq 3$ and that if $|(N(x) \cup N(y)) \setminus \{x, y\}| \geq 4$, then G contains the subgraph B_1. If $|(N(x) \cup N(y)) \setminus \{x, y\}| = 3$, then each vertex from $(N(x) \cup N(y)) \setminus \{x, y\}$ is adjacent to both x and y. If $|V(G)| \geq 6$ and $|(N(x) \cup N(y)) \setminus \{x, y\}| = 3$, then some vertex of G does not lie in $N(x) \cup N(y)$ and is one of the neighbors of the three vertices from $(N(x) \cup N(y)) \setminus \{x, y\}$. Hence, G also contains the subgraph B_1. If $G \in \text{Free}_s(\{B_1 + K_2\}) \cup \text{Free}_s(\{B_1^+\})$, then the distance of any vertex of G from the central edge B_1 is smaller than 3. Hence, G contains at most $2 \cdot (1 + 5 + 5^2) = 62$ vertices. Let us assume that $G \in \text{Free}_s(\{B_2\})$. We shall also assume that there exist a vertex at the distance 3 from the edge xy. Then there exist vertices a, b, c that generate in G a simple 3-path such that the distances between xy and the vertices a, b, c are, respectively, $1, 2, 3$. We shall assume that the vertices x and a are adjacent. It is easy to see that the vertex b is the only neighbor of a outside the set $N(x) \cup N(y)$, for otherwise the three edges that are incident with a (of which one is ax), the edge xy and some two edges incident with y for the subgraph B_2. It is clear that $\deg(b) = 2$, for otherwise, for any vertex

Authenticated | dsmalyshev@rambler.ru author's copy
Download Date | 10/14/17 8:42 AM

$z \in N(b) \setminus \{a, c\}$, the edges bc, bz, ba, xa, xy, xd would form the subgraph B_2, where d is an arbitrary neighbor of x which is different from a, y, z. Since G is simple, we have $\deg(c) \geq 3$ and $\deg(a) \geq 3$. Hence, there is a neighbor e of the vertex a which is distinct from b and x. Clearly, we have $e \in N(x) \cup N(y)$. It is further clear that any three edges incident with c (of which one is the edge cb) and the edges ab, ax, ae form the subgraph B_2. This being so, any vertex of G is at the distance at most 2 from xy. Hence, the number of vertices in G is at most $2 \cdot (1 + 5 + 5^2) = 62$. □

2.2 The clique-width of graphs and its relevance

The clique-width is an important parameter of a graph, because for any constant C many problems on graphs are polynomially solvable in the class of all graphs for which the clique-width is majorized by C (see [5]). The definition of the clique-width of a graph may be found in [5]. Let $\mathcal{T}$ be the class of all forests in each of which any connected component is a tree with at most three leaves. The next lemma (see [3]) gives a sufficient conditions for the clique-width to be uniformly bounded.

Lemma 3. *For any monotone class $\mathcal{X}$ not including the class $\mathcal{T}$ there exists a constant $C(\mathcal{X})$ such that the clique-width of any graph from $\mathcal{X}$ is at most $C(\mathcal{X})$.*

For the CI-problem, it is worth noting that if a monotone class does not contain $\mathcal{T}$, then it is simple.

Lemma 4. *If $\mathcal{X}$ is a monotone class and $\mathcal{T} \nsubseteq \mathcal{X}$, then $\mathcal{X}$ is CI-simple.*

Proof. The CI-problem in the class $\mathcal{X}$ is polynomially equivalent to the CN-problem in the class $L(\mathcal{X})$. By the previous lemma, the clique-width of graphs from the class $\mathcal{X}$ is uniformly bounded. Hence, the clique-width of graphs from the class $L(\mathcal{X})$ is also uniformly bounded [7]. The CN-problem is polynomially solvable in any class of graphs with uniformly bounded clique-width [8]. Hence, the CI-problem is polynomially solvable in the class $\mathcal{X}$. □

3 The main result

Lemma 5. *If H is a graph from $\mathrm{Free}_s(\{G + K_1\})$, then $H \in \mathrm{Free}_s(\{G\})$ or $|V(H)| = |V(G)|$.*

Proof. If the graph H does not contain a subgraph of G, then $H \in \mathrm{Free}_s(\{G\})$. If the graph H contains a subgraph of G, then H cannot contain vertices that fail to lie in this subgraph, because $H \in \mathrm{Free}_s(\{G+K_1\})$. Hence, in this case $|V(H)| = |V(G)|$. □

The main result of the present paper is as follows.

Theorem 1. *Let $\mathcal{S}$ be an arbitrary set of graphs, each of which has at most 6 edges or at most 7 vertices. Then the class $\mathcal{X} = \mathrm{Free}_s(\mathcal{S})$ is CI-simple if $\mathcal{S}$ contains a subcubic forest. Otherwise, $\mathcal{X}$ is CI-complex.*

Proof. Let $\mathcal{Y}_{3,k}$ denote the set of all possible subcubic graphs not containing cycles of length at most k inclusively. If $\mathcal{S}$ does not contain subcubic forests, then $\mathcal{Y}_{3,p} \subseteq \mathcal{X}$ for some p. According to [9] for any k the class $\mathcal{Y}_{3,k}$ is CI-complex. Hence so is the class $\mathcal{X}$.

Any subcubic forest with at most 6 edges or with at most 7 vertices either lies in $\mathcal{T}$ or is of the following 4 types: $B_1 + O_s$, $B_1 + K_2 + O_s$, $B_1^+ + O_s$, $B_2 + O_s$ for some s. Hence, by Lemmas 2, 4, 5 the class $\mathcal{X}$ is CI-simple. □

Acknowledgment: This research was carried out with the financial support of the Russian Foundation for Basic Research (grant no. 16-31-60008-mol_a_dk, the Council on Grants of the President of the Russian Federation (grant no. MK-4819.2016.1), and the LATNA laboratory at the National Research University Higher School of Economics.

Authenticated | dsmalyshev@rambler.ru author's copy
Download Date | 10/14/17 8:42 AM

References

[1] Vizing V. G., “On the estimate of the chromatic class of the p-graph”, *Diskretnyi analiz*, **3** (1964), 25–30 (in Russian).
[2] Malyshev D. S., “The complexity of the edge 3-colorability problem for graphs without two induced fragments each on at most six vertices”, *Sib. elektr. matem. izv.*, **11** (2014), 811–822 (in Russian).
[3] Boliac R., Lozin V. V., “On the clique-width of graphs in hereditary classes”, *Lect. Notes Comput. Sci.*, **2518** (2002), 44–54.
[4] Broersma H. J., Golovach P. A., Paulusma D., Song J., “Updating the complexity status of coloring graphs without a fixed induced linear forest”, *Theor. comput. sci.*, **414**:1 (2012), 9–19.
[5] Courcelle B., Makowsky J., Rotics U., “Linear time solvable optimization problems on graphs of bounded clique-width”, *Theory of Comput. Syst.*, **33**:2 (2000), 125–150.
[6] Golovach P. A., Paulusma D., Song J., “4-coloring H-free graphs when H is small”, *Discrete Appl. Math.*, **161**:1–2 (2013), 140–150.
[7] Gursk, F., Wanke E., “Line graphs of bounded clique-width”, *Discrete Mathematics*, **307**:22 (2007), 2734–2754.
[8] Kobler D., Rotics D., “Edge dominating set and colorings on graphs with fixed clique-width”, *Discrete Appl. Math.*, **126**:2–3 (2003), 197–223.
[9] Lozin V. V., Kaminski M., “Coloring edges and vertices of graphs without short or long cycles”, *Contrib. to Discr. Math.*, **2**:1 (2007), 61–66.
[10] Malyshev D. S., “The complexity of the 3-colorability problem in the absence of a pair of small forbidden induced subgraphs”, *Discrete Math.*, **338**:11 (2015), 1860–1865.
[11] Randerath B., Schiermeyer I., Tewes M., “Three-colourability and forbidden subgraphs. II: polynomial algorithms”, *Discrete Math.*, **251**:1–3 (2002), 137–153.
[12] Roussopoulos N., “A max$\{m, n\}$ algorithm for determining the graph H from its line graph G”, *Inf. Process. Lett.*, **2**:4 (1973), 108–112.
[13] Schrijver A., *Combinatorial optimization — polyhedra and efficiency*, Springer, Berlin, 2003, 1882 pp.

Authenticated | dsmalyshev@rambler.ru author's copy
Download Date | 10/14/17 8:42 AM