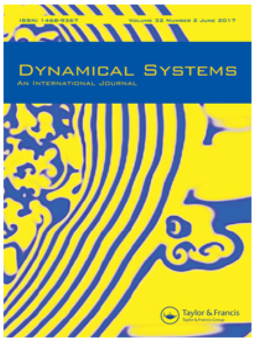

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On spectral decomposition of Smale–Vietoris axiom A diffeomorphisms

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ABSTRACT
We introduce Smale–Vietoris diffeomorphisms that include the classical DE mappings with Smale solenoids. We describe the correspondence between basic sets of axiom A Smale–Vietoris diffeomorphisms and basic sets of non-singular axiom A-endomorphisms. For Smale–Vietoris diffeomorphisms of 3-manifolds, we prove the uniqueness of non-trivial solenoidal basic set. We construct a bifurcation between different types of solenoidal basic sets which can be considered as a destruction (or birth) of Smale solenoid.

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1. Introduction

Stephen Smale, in his celebrated paper [1], introduced the so-called DE maps which arise from expanding maps (the abbreviation DE is formed by first letters of *Derived* from *Expanding* map). Let T be a closed manifold of dimension at least 1, and N an n-disk of dimension $n \geq 2$. Omitting details, one can say that a DE map is the skew map:

$$f : T \times N \to T \times N, \quad (x, y) \mapsto \left(g_1(x),\ g_2(x, y)\right), \tag{1}$$

where $g_1: T \to T$ is an expanding map of degree $d \geq 2$, and

$$g_2|_{\{x\}\times N} : \{x\} \times N \to \{g_1(x)\} \times N,$$

an uniformly attracting map of n-disk $\{x\} \times N$ into n-disk $\{g_1(x)\} \times N$ for every $x \in T$. In addition, f must be a diffeomorphism onto its image $T \times N \to f(T \times N)$. In the particular case, when $T = S^1$ is a circle and $N = D^2$ is a 2-disk with the uniformly attracting g_2, one gets a classical Smale solenoid $\cap_{l\geq 0} f(\mathbb{T} \times D^2) = \mathfrak{S}$ (see Figure 1), which is a topological solenoid.

Recall that a topological solenoid was introduced by Vietoris [2] in 1927 (independently, a solenoid was introduced by van Danzig [3] in 1930, see review in [4]). Smale [1] proved that $\mathfrak{S}(f)$ is a hyperbolic expanding attractor. This construction was generalized by Williams [5,6] who defined g_1 to be expansion mappings of branch manifolds (this allows to Williams to classify interior dynamics of expanding attractors) and by Block [7] who

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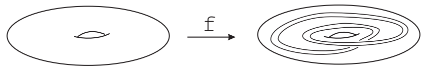

Figure 1. DE map by S. Smale.

defined g_1 to be an Axiom A-endomorphism. The last paper concerns to the Ω-stability and the proving of decomposition of non-wandering set into the so-called basic sets (Spectral Decomposition Theorem for A-endomorphisms). Ideologically, our paper is a continuation of Block,[7] where it was proved the following result (Theorem A). Let f: $M^n \to M^n$ be a Smale–Vietoris diffeomorphism of closed n-manifold M^n and $\mathfrak{B} \subset M^n$ the support of Smale skew-mapping $f|_{\mathfrak{B}}$ (see the notations below). Then $f|_{\mathfrak{B}}$ satisfies axiom A on $\mathfrak{B}$ if and only if g does on T.

Let us mention that in the frame of Smale–Williams construction, the interesting examples of expanding attractors was obtained in [8–12]. Bothe [13] classified the purely Smale solenoids on 3-manifolds. He was the first who also proved that a DE map $S^1 \times D^2 \to S^1 \times D^2$ can be extended to a diffeomorphism of some closed 3-manifold $M^3 \supset S^1 \times D^2$ (see also [14–16]). Ya. Zeldovich and others (see [17]) conjectured that Smale-type mappings could be responsible for the so-called fast dynamos. Therefore, it is natural to consider various generalizations of classical Smale mapping.

In a spirit of Smale construction of DE maps, we here introduce diffeomorphisms called Smale–Vietoris that are derived from non-singular endomorphisms. A non-wandering set of Smale–Vietoris diffeomorphisms belongs to an attractive invariant set of solenoidal type. In the classical case, the invariant set coincides with the non-wandering set consisting of a unique basic set. In general, the non-wandering set does not coincide with the invariant set, and divides into basic sets provided the non-singular endomorphism is an A-endomorphism.

Let N be $(n-k)$-dimensional compact Riemannian manifold with a non-empty boundary where $n-k \geq 1$. For a subset $N_1 \subset N$, we define the diameter $\operatorname{diam} N_1 = \max_{a,b \in N_1} \{\rho_N(a,b)\}$ of N_1 where ρ_N is the metric on N. Denote by $\mathbb{T}^k = \underbrace{S^1 \times \cdots \times S^1}_{k}$ the k-dimensional torus, $k \in \mathbb{N}$. A surjective mapping $g : \mathbb{T}^k \to \mathbb{T}^k$ is called a *d-cover* if g is a preserving orientation local homeomorphism of degree d. A good example is the preserving orientation linear expanding mapping $E_d : \mathbb{T}^k \to \mathbb{T}^k$ defined by an integer $k \times k$ matrix with the determinant equals d. Certainly, E_d is a d-cover.

A skew-mapping

$$F : \mathbb{T}^k \times N \to \mathbb{T}^k \times N, \quad (t,z) \longmapsto \big((g(t), \ \omega(t,z)\big) \tag{2}$$

is called a *Smale skew-mapping*, if the following conditions hold:

- $F : \mathbb{T}^k \times N \to F\left(\mathbb{T}^k \times N\right)$ is a diffeomorphism on its image;
- $g : \mathbb{T}^k \to \mathbb{T}^k$ is a d-cover, $d \geq 2$;
- Given any $t \in \mathbb{T}^k$, the restriction $w|_{\{t\}\times N} : \{t\} \times N \to \mathbb{T}^k \times N$ is the uniformly attracting embedding,

$$\{t\} \times N \to \operatorname{int}\left(\{g(t)\} \times N\right), \tag{3}$$

i.e. there are $0 < \lambda < 1$, $C > 0$, such that

$$\mathrm{diam}\,(F^n(\{t\} \times N)) \leq C\lambda^n \mathrm{diam}\,(\{t\} \times N), \ \ \forall\, n \in \mathbb{N}. \tag{4}$$

When $g = E_d$, Smale skew-mapping is a DE mapping (1) introduced by Smale [1].

A-diffeomorphism f: $M^n \to M^n$ is called a *Smale–Vietoris* diffeomorphism if there is the n-submanifold $\mathbb{T}^k \times N \subset M^n$ such that the restriction $f|_{\mathbb{T}^k \times N} \overset{\text{def}}{=} F$ is a Smale skew-mapping. The sub-manifold $\mathbb{T}^k \times N \subset M^n$ is called a *support of Smale skew-mapping*.

Put by definition,

$$\cap_{l \geq 0} F^l(\mathbb{T}^k \times N) \overset{\text{def}}{=} \mathfrak{S}(f).$$

One can easy to see that the set $\mathfrak{S}(f) = \mathfrak{S}$ is attractive, invariant and closed, so that the restriction

$$f|_{\mathfrak{S}} : \mathfrak{S} \to \mathfrak{S}$$

is a homeomorphism.

The following theorem shows that there is an intimate correspondence between basic sets of $f|_{\mathfrak{B}}$ and basic sets of the A-endomorphism g.

Theorem 1.1: *Let f: $M^n \to M^n$ be a Smale–Vietoris A-diffeomorphism of closed n-manifold M^n and $\mathbb{T}^k \times N = \mathfrak{B} \subset M^n$ be a support of the Smale skew-mapping $f|_{\mathfrak{B}} = F$ (see (2)). Let Ω be a basic set of $g : \mathbb{T}^k \to \mathbb{T}^k$ and $\mathfrak{S} = \cap_{l \geq 0} F^l(\mathbb{T}^k \times N)$. Then $\mathfrak{S} \cap p_1^{-1}(\Omega)$ contains a unique basic set $\Omega_{\mathfrak{S}}$ of f. Here, $p_1 : \mathbb{T}^k \times N \to \mathbb{T}^k$ is the natural projection on the first factor. Moreover,*

(1) *if Ω is a trivial basic set (isolated periodic orbit) of g, then $\Omega_{\mathfrak{S}}$ also is a trivial basic set;*
(2) *if Ω is a non-trivial basic set of g, then $\Omega_{\mathfrak{S}}$ also is a non-trivial basic set;*
(3) *if Ω is a backward g-invariant basic set of g, $\Omega = g^{-1}(\Omega)$, (hence, Ω is non-trivial), then $\Omega_{\mathfrak{S}} = \mathfrak{S} \cap p_1^{-1}(\Omega)$.*

For $k = 1$, when $\mathbb{T}^1 = S^1$ is a circle, the following result says that $NW(F)$ contains a unique non-trivial basic set that is either Smale (one-dimensional) solenoid or a non-trivial zero-dimensional basic set.

Theorem 1.2: *Let f: $M^n \to M^n$ be a Smale–Vietoris A-diffeomorphism of closed n-manifold M^n and $\mathbb{T}^1 \times N = \mathfrak{B} \subset M^n$ the support of Smale skew-mapping $f|_{\mathfrak{B}} = F$. Then, the non-wandering set $NW(F)$ of F belongs to $\mathfrak{S} = \cap_{l \geq 0} F^l(\mathbb{T}^1 \times N)$, and $NW(F)$ contains a unique non-trivial basic set $\Lambda(f)$ that is either*

- *a one-dimensional expanding attractor, and $\Lambda(f) = \mathfrak{S}$, or*
- *a zero-dimensional basic set, and $NW(F)$ consists of $\Lambda(f)$ and finitely many (non-zero) isolated attracting periodic points plus finitely many (possibly, zero) saddle-type isolated periodic points of co-dimension one stable Morse index.*

The both possibilities hold.

It is natural to consider bifurcations from one type of dynamics to another which can be thought of as a destruction (or, a birth) of Smale solenoid. For simplicity, we represent two such bifurcations for $n = 3$ and $M^3 = S^3$ a 3-sphere. Recall that a diffeomorphism f:

$M \to M$ is Ω*-stable* if there is a neighbourhood $U(f)$ of f in the space of C^1 diffeomorphisms $\text{Diff}^1(M)$ such that $f|_{NW(f)}$ conjugate to every $g|_{NW(g)}$ provided $g \in U(f)$.

Theorem 1.3: *There is the family of Ω-stable Smale– Vietoris diffeomorphisms f_μ: $S^3 \to S^3$, $0 \le \mu \le 1$, continuously depending on the parameter μ such that the non-wandering set $NW(f_\mu)$ of f_μ is the following:*

- *$NW(f_0)$ consists of a one-dimensional expanding attractor (Smale solenoid attractor) and one-dimensional contracting repeller (Smale solenoid repeller);*
- *For $\mu > 0$, $NW(f_\mu)$ consists of two non-trivial zero-dimensional basic sets and finitely many isolated periodic orbits.*

Remark 1.1: Theorem 1.3 is also true for every lens space (see the proof of Theorem 1.3).

2. Definitions

A mapping $F: M \times N \to M \times N$ of the type $F(x, y) = (g(x), h(x, y))$ is called a *skew-mapping*. One says also a *skew product transformation* over g or simply, a *skew product*. Denote by End (M) the space of C^1 endomorphisms $M \to M$, i.e. the C^1 maps of M onto itself. An endomorphism g is *non-singular* if the Jacobian $|Dg| \neq 0$. This means that g is a local diffeomorphism. In particular, g is a d-cover mapping. In this paper, we consider non-singular $g \in$ End (M), so that $Dg \neq 0$ and g is not a diffeomorphism.

Fix $g \in$ End (M). A point $x \in M$ is said to be *non-wandering* if given any neighbourhood $U(x) = U$ of x, there is $m \in \mathbb{N}$ such that $g^m(U) \cap U \neq \emptyset$. Denote by $NW(g)$ the set of non-wandering points. Clearly, $NW(g)$ is a closed set and $g(NW(g)) \subset NW(g)$, i.e. $NW(g)$ is a forward g-invariant set. The set $\{x_i\}_{-\infty}^{\infty}$ denoted by $O(x_0)$ is called a *g-orbit* of x_0 if $g(x_i) = x_{i+1}$ for every integer i. A subset $\{x_j, x_{j+1}, \dots, x_{j+r}\} \subset O(x_0)$ consisting of a finitely many points of $O(x_0)$ is called a *compact part* of $O(x_0)$. A g-orbit $\{x_i\}_{-\infty}^{\infty}$ is *periodic* if there is an integer $p \ge 0$ such that $g^p(x_i) = x_{i+p}$ for each $i \in \mathbb{Z}$. Certainly, $NW(g)$ contains all periodic g-orbits.

The orbit $O(x_0)$ is said to be *hyperbolic* if there is a continuous splitting of the tangent bundle

$$\mathbb{T}_{O(x_0)}M = \bigcup_{i=-\infty}^{\infty} \mathbb{T}_{x_i}M = \mathbb{E}^s \bigoplus \mathbb{E}^u = \bigcup_{i=-\infty}^{\infty} \mathbb{E}^s_{x_i} \bigoplus \mathbb{E}^u_{x_i}$$

which is preserved by the derivative Dg such that

$$||Dg^m(v)|| \le c\mu^m ||v||, \ ||Dg^m(w)|| \ge c^{-1}\mu^{-m}||w|| \quad \text{for } v \in \mathbb{E}^s, \ w \in \mathbb{E}^u, \ \forall m \in \mathbb{N}$$

for some constants $c > 0$, $0 < \mu < 1$ and a Riemannian metric on $\mathbb{T}M$. Note that $\mathbb{E}^u(x_0)$ depends on the negative semi-orbit $\{x_i\}_{i=-\infty}^{0}$. It may happen that $\mathbb{E}^u(x_0) \neq \mathbb{E}^u(y_0)$ though $x_0 = y_0$ but $O(x_0) \neq O(y_0)$. Such a phenomenon is impossible for $\mathbb{E}^s(x_0)$, it depends only on x_0 [18].

We say that a non-singular $g \in$ End (M) *satisfies axiom A*, in short, f is an A-endomorphism if

- the periodic g-orbits are dense in $NW(g)$ (it follows that $g(NW(g)) = NW(g)$);

- all g-orbits of $NW(g)$ are hyperbolic, and the corresponding splitting of the tangent bundle $\mathbb{T}_{NW(g)}$ depends continuously on the compact parts of the g-orbits.

Recall that Smale's Spectral Decomposition Theorem says that for every Axiom A-diffeomorphisms the non-wandering set partitions into finitely many non-empty-closed invariant sets each of which is transitive. Similar theorem for A-endomorphisms was proved in [7, Theorem C] and [18, Theorem 3.11 and Proposition 3.13]). Thus, if g is a non-singular A-endomorphism, then the non-wandering set $NW(g)$ is the disjoint union $\Omega_1 \cup \cdots \cup \Omega_k$ such that each Ω_i is closed and invariant, $g(\Omega_i) = \Omega_i$, and Ω_i contains a point whose g-orbit is dense in Ω_i. Each Ω_i is called a *basic sets.*

Following Williams [5,6], we introduce an inverse limit for g: $T \to T$ as follows. Put by definition, $\prod_g = \{ (t_0, t_1, \ldots, t_i, \ldots) \in T^{\mathbb{N}} \ : \ g(t_{i+1}) = t_i, \ i \geq 0 \}$. This set is endowed by the product topology of countable factors. This topology has a basis generating by (ε, r)-neighbourhoods

$$U = \left\{ \{x_i\}_0^\infty \in \prod_g \ : \ x_i \in U_\varepsilon(t_i), \ 0 \leq i \leq r \ \text{ for some } \ \varepsilon > 0, \ r \in \mathbb{N} \right\}, \tag{5}$$

where $\{t_0, t_1, \ldots, t_i, \ldots\} \in \prod_g$. Define the shift map

$$\hat{g} : \prod_g \to \prod_g, \quad \hat{g}(t_0, t_1, \ldots, t_i, \ldots) = \big(g(t_0), t_0, t_1, \ldots, t_i, \ldots\big), \quad (t_0, t_1, \ldots, t_i, \ldots) \in \prod_g .$$

This map $\hat{g} : \prod_g \to \prod_g$ is called the *inverse limit* of g. Indeed, g is a homeomorphism.[6,19]

3. Proofs of main results

We denote by $p_1 : \mathbb{T}^k \times N \to \mathbb{T}^k$, $p_2 : \mathbb{T}^k \times N \to N$ the natural projections $p_1(t, z) = t$ and $p_2(t, z) = z$. A fibre$\{t\} \times N \stackrel{\text{def}}{=} N_t$ of the trivial fibre bundle p_1 is called a t-leaf. It follows from (2) that $F = f|_{\mathfrak{B}}$ takes a t-leaf into $g(t)$-leaf.

Let $t \in \mathbb{T}^k$ and $\varepsilon > 0$. We denote by $U_\varepsilon(t)$ the ε-neighbourhood of the point t, i.e. $U_\varepsilon(t) = \{x \in \mathbb{T}^k \ : \ \varrho(x, t) < \varepsilon\}$ where ϱ is a metric on $\mathbb{T}^k$.

The following technical lemma describes the symbolic model of the restriction $f|_{\mathfrak{S}}$. This lemma is a generalization of the similar classical result by Williams [5,6].

Lemma 3.1: *Let f: $M^n \to M^n$ be a Smale–Vietoris diffeomorphism of closed n-manifold M^n and $\mathbb{T}^k \times N = \mathfrak{B} \subset M^n$ the support of Smale skew-mapping $f|_{\mathfrak{B}} = F$. Then the restriction $f|_{\mathfrak{S}}$ is conjugate to the inverse limit of the mapping $g : \mathbb{T}^k \to \mathbb{T}^k$, where $\mathfrak{S} = \cap_{l \geq 0} F^l(\mathbb{T}^k \times N)$.*

Proof: Recall that given any point $t_0 \in \mathbb{T}^k$, $g^{-1}(t_0)$ consists of d points, one says $t_0^1, t_0^2, \ldots,$ $t_0^d \in \mathbb{T}^k$. Since F is a diffeomorphism on its image, the sets $F(N_{t_0^1}), \ldots, F(N_{t_0^d})$ are pairwise disjoint,

$$F(N_{t_0^i}) \cap F(N_{t_0^j}) = \emptyset, \quad i \neq j, \quad 1 \leq i, j \leq d, \tag{6}$$

Now, for the sake of simplicity, we divide the proof into several steps. The end of each step of the proof will be denoted by □

Step 1: Given any point $p \in \mathfrak{S}$, there is a unique sequence of points $\{t_i\}_{i=0}^{\infty}$, $t_i \in \mathbb{T}^k$, and the corresponding sequence of the leaves $\{N_{t_i}\}_{i=0}^{\infty}$, such that

- $p \in \cdots \subset F^i(N_{t_i}) \subset F^{i-1}(N_{t_{i-1}}) \cdots \subset F(N_{t_1}) \subset N_{t_0}$, $p = \cap_{i\geq 0} F^i(N_{t_i})$;
- $t_i = g(t_{i+1})$, $i \geq 0$.

Proof of Step 1: Put $t_0 = p_1(p) \in \mathbb{T}^k$. Let $g^{-1}(t_0) = \{t_0^1, t_0^2, \ldots, t_0^d\}$. By (6), there is a unique t_0^j such that $p \in F(N_{t_0^j})$. Put by definition $t_0^j = t_1$. Note that $F(N_{t_1}) \subset N_{t_0}$. Similarly, $g^{-1}(t_1)$ consists of d points $t_1^1, t_1^2, \ldots, t_1^d$. By (6), the sets $F(N_{t_1^1}), \ldots, F(N_{t_1^d})$ are pairwise disjoint. Since $p \in F^2(\mathbb{T}^k \times N)$, there is a unique t_1^i such that $p \in F^2(N_{t_1^i})$. Put by definition $t_1^i = t_2$. Note that $p \in F^2(N_{t_2}) \subset F(N_{t_1}) \subset N_{t_0}$. Continuing by this way, one gets the sequences $\{t_i\}_{i=0}^{\infty}$, $\{N_{t_i}\}_{i=0}^{\infty}$ desired. It follows from (4) that $\operatorname{diam} F^i(N_{t_i}) = \operatorname{diam}(F^i(\{t_i\} \times N)) \to 0$ as $i \to \infty$. Hence, $p = \cap_{i\geq 0} F^i(N_{t_i})$.

Let $\hat{g} : \prod_g \to \prod_g$ be the inverse limit of $g : \mathbb{T}^k \to \mathbb{T}^k$ where $\prod_g = \{ (t_0, t_1, \ldots, t_i, \ldots) \in \mathbb{T}^{\mathbb{N}} \ : \ g(t_{i+1}) = t_i, \ \ i \geq 0 \}$. For a point $p \in \mathfrak{S}$, denote by $P(t_0, t_1, \ldots, t_i, \ldots)$, $t_i \in \mathbb{T}^k$, the sequence due to Step 1. Define the mapping

$$\theta : \ \mathfrak{S} \to \prod_g, \quad p \longmapsto P(t_0, t_1, \ldots, t_i, \ldots), \quad p \in \mathfrak{S}.$$

Step 2: The mapping θ is a homeomorphism.

Proof of Step 2: It follows from (4) that θ is injective. Since the intersection of nested sequence of closed subsets is non-empty, θ is surjective. One remains to prove that θ and θ^{-1} are continuous. Take a neighbourhood U of $\theta(p)$, $p \in \mathfrak{S}$. We can assume that U is an (ε, r)-neighbourhood (5), where $\theta(p) = \{t_0, t_1, \ldots, t_i, \ldots\} \in \prod_g$. Moreover, one can assume that $g^{-1}(U_\varepsilon(t_i))$ consists of d pairwise disjoint domains for every $0 \leq i \leq r$. Recall that $t_i = g(t_{i+1})$, $i \geq 0$. Therefore, $t_{r-j} = g^j(t_r)$ for all $1 \leq j \leq r$. Similarly, $x_{r-j} = g^j(x_r)$, $1 \leq j \leq r$. Since g is continuous, there exists $0 < \delta \leq \varepsilon$ such that the inclusion $x_r \in U_\delta(t_r)$ implies $x_i \in U_\varepsilon(t_i)$ for all $i = 0, \ldots, r$. The restriction $F|_{\mathfrak{S}} : \mathfrak{S} \to \mathfrak{S}$ is a diffeomorphism. Therefore, there is a (relative) neighbourhood $U(p)$ of p in $\mathfrak{S}$ such that $p_1(F^{-i}(U(p))) \subset U_\delta(t_i)$ forall $0 \leq i \leq r$. Taking in mind that $g^{-1}(U_\varepsilon(t_i))$ consists of d pairwise disjoint domains, $0 \leq i \leq r$, we see that $\theta(U(p)) \subset U$. Thus, θ is continuous. Since $\prod_g$ is compact, θ^{-1} is also continuous.

Step 3: One holds $\theta \circ F|_{\mathfrak{S}} = \hat{g} \circ \theta|_{\mathfrak{S}}$.

Proof of Step 3 : Take $p \in \mathfrak{S}$ and $\theta(p) = \{t_0, t_1, \ldots, t_i, \ldots\}$ where $t_i = g(t_{i+1})$, $i \geq 0$. By definition of $\hat{g} : \prod_g \to \prod_g$, one holds $\hat{g} \circ \theta(p) = \hat{g}\left(\theta(p)\right) = \hat{g}(\{t_0, t_1, \ldots, t_i, \ldots\}) = \{g(t_0), t_0, t_1, \ldots, t_i \ldots\}$. It follows from (3) that $F(p) \in F(\{t_0\} \times N) \subset N_{g(t_0)}$. Hence, by Step 1, the sequence of points $\{g(t_0), t_0, t_1, \ldots, t_i \ldots\}$ corresponds to $\theta(F(p))$, since

$$\begin{aligned} F(p) &= F\left(\cap_{i\geq 0} F^i(\{t_i\} \times N)\right) = \cap_{i\geq 0} F^{i+1}(\{t_i\} \times N) = \cap_{i\geq 0} F^{i+1}(\{t_i\} \times N) \cap N_{g(t_0)} \\ &= N_{g(t_0)} \cap F(N_{t_0}) \cap F^2(N_{t_1}) \cap \cdots \cap F^{i+1}(N_{t_i}) \cap \cdots . \end{aligned}$$

It follows from Steps 2 and 3 that the mapping θ is a conjugacy between $F|_{\mathfrak{S}}$ and $\hat{g}$. Lemma 3.1 is proved. □

To prove Theorem 1.1, we need some previous results.

Lemma 3.2: *Let* $\bar{t} = \{t_0, t_1, \ldots, t_i, \ldots,\} \in \prod_g, g(t_{i+1}) = t_i, i \geq 0$. *Suppose that* $t_i \in NW(g)$ *for all* $i \geq 0$. *Then,* $\bar{t} \in NW(\hat{g})$ *and* $\theta^{-1}(\bar{t}) \in NW(F)$.

Proof: Since $\bar{t} = \{t_0, t_1, \ldots, t_i, \ldots,\} = \{g^r(t_r), g^{r-1}(t_r), \ldots, t_r, \ldots\}$, we can take the (ε, r)-neighbourhood $V\bar{t}$ as follows:

$$V = \{\{g^r(x_r), g^{r-1}(x_r), \ldots, x_r, \ldots\} \; : \; g^i(x_r) \in U_\varepsilon\left(g^i(t_r)\right), \; 0 \leq i \leq r\}.$$

Since $g, g^2, \ldots, g^r$ are uniformly continuous, there is $0 < \delta \leq \varepsilon$ such that $x \in U_\delta(y)$ implies that $g^i(x) \in U_\varepsilon(g^i(y))$ for all $0 \leq i \leq r$. By condition, $t_r \in NW(g)$. Hence, there exists $n_0 \in \mathbb{N}$ such that $g^{n_0}(V_\delta(t_r)) \cap V_\delta(t_r) \neq \emptyset$. It follows that there is a point $x_0 \in V_\delta(t_r)$ such that $g^{n_0}(x_0) \in V_\delta(t_r)$.

Take $\bar{x}_0 = \{g^r(x_0), g^{r-1}(x_0), \ldots, x_0, \ldots\} \in \prod_g$. Since $x_0 \in V_\delta(t_r)$, $g^i(x_0) \in U_\varepsilon(g^i(t_r))$ for all $0 \leq i \leq r$. Therefore, $\bar{x}_0 \in V$. Since $g^{n_0}(x_0) \in V_\delta(t_r)$, $g^{n_0+i}(x_0) \in U_\varepsilon\left(g^i(t_r)\right)$ for all $0 \leq i \leq r$. Therefore,

$$\hat{g}^{n_0}(\bar{x}_0) = \{g^{n_0+r}(x_0), g^{n_0+r-1}(x_0), \ldots, g^{n_0}(x_0), \ldots\} \in V.$$

As a consequence, $\hat{g}^{n_0}(V) \cap V \neq \emptyset$ and $\bar{t} \in NW(\bar{g})$. A conjugacy map takes a non-wandering set onto a non-wandering set. By Lemma 3.1, $\theta^{-1}(\bar{t}) \in NW(F)$. □

Corollary 3.1: *The following qualities hold* $p_1\left[NW(f_{\mathfrak{B}})\right] = p_1[NW(F)] = NW(g)$.

Proof: Since the projection p_1 is continuous, $p_1[NW(F)] \subset NW(g)$. Take a point $t_0 \in NW(g)$. Since g is an A-endomorphism, $g[NW(g)] = NW(g)$.[7,18] Therefore, there is a sequence $\{t_i | i = 0, 1, \ldots\} \subset NW(g)$ such that $g(t_{i+1}) = t_i$ for every $i \geq 0$. It follows from Lemma 3.2 that $\bar{t} = \{t_0, t_1, \ldots, t_i, \ldots,\} \in NW(\hat{g})$ and $\theta^{-1}(\bar{t}) \in NW(F)$. By definition of the mapping θ, $\theta^{-1}(\bar{t}) \in p_1^{-1}(t_0)$. Hence, $NW(g) \subset p_1[NW(F)]$. □

Lemma 3.3: *Let* $(t_0, z_0) \in \mathfrak{S}$ *be a non-wandering point of f, and* $\theta(t_0, z_0) = \{t_i\}_{i \geq 0}$. *Then,* $t_i \in NW(g)$ *for all* $i \geq 0$.

Proof: According to Corollary 3.1, $p_1\left[NW(f_{\mathfrak{B}})\right] = p_1[NW(F)] = NW(g)$. Therefore, $t_0 \in NW(g)$. Since $F_{\mathfrak{S}} : \mathfrak{S} \to \mathfrak{S}$ is a diffeomorphism, $F^{-1}(NW(F)) = NW(F)$ and $F^{-1}(t_0, z_0) = (t_1, z_1) \in NW(F) = NW(f_{\mathfrak{B}})$. Hence, $t_1 \in NW(g)$ by Step 1. Continuing this way, one gets that $t_i \in NW(g)$ for all $i \geq 0$. □

Corollary 3.2: *Let* $(t_0, z_0) \in \mathfrak{S}$ *be a non-wandering point of f, and* $\theta(t_0, z_0) = \{t_i\}_{i \geq 0}$. *Suppose that* t_0 *belongs to a basic set* Ω *of g. Then* $t_i \in \Omega$ *for all* $i \geq 0$.

Proof: By Lemma 3.3, $t_i \in NW(g)$ for all $i \geq 0$. Since Ω is forward g-invariant, $t_i \in \Omega$ for all $i \geq 0$. □

Lemma 3.4: *Let* Ω *be a non-trivial basic set of g, and* $t_0 \in \Omega$. *Suppose that two points* (t_0, z_1), $(t_0, z_2) \in \mathfrak{S}$ *are non-wandering under f. Then, both* (t_0, z_1) *and* (t_0, z_2) *belong to the same basic set of f.*

Proof: Denote by Ω_j the basic set of F containing the point (t_0, z_j), $j = 1, 2$. Clearly, $\Omega_j \subset \mathfrak{S}$. We have to prove that $\Omega_1 = \Omega_2$. It is sufficient to show that there is a non-wandering point $q \in NW(F)$ such that each point (t_0, z_1) and (t_0, z_2) belongs to the ω-limit set of q.

Let $\bar{t}_j = \theta(t_0, z_j) = \{t_0, t_1^{(j)}, \ldots, t_i^{(j)}, \ldots\}$, $j = 1, 2$. By Corollary 3.2, $t_i^{(j)} \in \Omega$ for all $i \geq 0$, $j = 1, 2$. Since the basic set Ω is transitive, there is a point $x_0 \in \Omega$ such that its positive semi-orbit $O_g^+(x_0)$ is dense in Ω, clos $(O_g^+(x_0)) = \Omega$.

It follows from Corollary 3.1 that there is a point $\bar{x}_0 = \{x_0, x_1, \ldots, x_i, \ldots\} \in \prod_g$ such that $x_i \in \Omega$ for all $i \geq 0$. Take arbitrary (ε, r)-neighbourhood $U(\bar{t}_1)$ of $\bar{t}_1$. Since $g, g^2, \ldots, g^r$ are uniformly continuous, there exists $\delta > 0$ such that the inequality $x \in U_\delta(y)$ implies $g^i(x) \in U_\varepsilon(y)$ for all $0 \leq i \leq r$. Because of the semi-orbit $O_g^+(x_0)$ is dense in Ω, there is $n_0 \in \mathbb{N}$ such that $g^{n_0}(x_0) \in U_\delta(t^{(1)})$. Hence, $\hat{g}^{n_0}(\bar{x}_0) \in U(\bar{t}_1)$. Therefore, $\bar{t}_1 = \theta(t_0, z_1)$ belongs to the ω-limit set of $\bar{x}_0$. Similarly, one can prove that $\bar{t}_2 = \theta(t_0, z_2)$ belongs to the ω-limit set of $\bar{x}_0$ as well. Since θ is a conjugacy mapping, the points $(t_0, z_1) = \theta^{-1}(\bar{t}_1)$ and $(t_0, z_2) = \theta^{-1}(\bar{t}_2)$ belongs to the ω-limit set of the point $q = \theta^{-1}(\bar{x}_0) \in NW(F)$. □

Proof of Theorem 1.1: We know that $p_1[NW(F)] = NW(g)$. Hence, $\mathfrak{S} \cap p_1^{-1}(\Omega)$ contains basic sets of *f*. Suppose that Ω is trivial, i.e. Ω is an isolated periodic orbit:

$$\Omega = Orb_g(q) = \{q, g(q), \ldots, g^{p-1}(q), g^p(q) = q\}, \quad \text{where} \quad q \in \mathbb{T}^k \quad \text{and} \quad p \in \mathbb{N} \quad \text{are a period of} \quad q.$$

By definition of Smale skew-mapping, the restriction of $F = f|_{\mathfrak{B}}$ on the second factor N is the uniformly attracting embedding. Therefore,

$$N_q \supset f^p(N_q) \supset \cdots \supset f^{mp}(N_q) \supset \cdots \text{ and the intersection } \bigcap_{m \geq 0} f^{mp}(N_q) \text{ is a unique point, say } Q.$$

Similarly, $\cap_{m \geq 0} f^{mp}(N_{g^i(q)})$ is a unique point $f^i(Q)$ for every $0 \leq i \leq p-1$. It follows from (2) that $\{Q, f(Q), \ldots, f^{p-1}(Q), f^p(Q) = Q\}$ is an isolated periodic orbit $\mathrm{Orb}_f(Q)$ such that $NW(F) \cap p_1^{-1}(\Omega) = \mathrm{Orb}_f(Q)$. Therefore, $\mathrm{Orb}_f(Q) = \Omega_{\mathfrak{S}}$ is a unique basic set of F that belongs to $\mathfrak{S} \cap p_1^{-1}(\Omega)$.

Let Ω be a non-trivial basic set. It follows from Lemma 3.4 that all basic sets of F that is contained in $\mathfrak{S} \cap p_1^{-1}(\Omega)$ are coincide. Hence, $\Omega_{\mathfrak{S}}$ is a unique non-trivial basic set of *f* contained in $\mathfrak{S} \cap p_1^{-1}(\Omega)$.

Now let Ω be a backward g-invariant basic set of g. Note that the equality $\Omega = g^{-1}(\Omega)$ implies that Ω cannot be a trivial basic set, since g is a d-cover, $d \geq 2$. It follows from Lemma 3.2 that every point of $\mathfrak{S} \cap p_1^{-1}(\Omega)$ is a non-wandering point of *f*. By Lemma 3.4, $\mathfrak{S} \cap p_1^{-1}(\Omega)$ is a unique basic set. Theorem 1.1 is proved. □

Example: Let us consider three endomorphisms $g_i : \mathbb{T}^2 \to \mathbb{T}^2$, $i = 1, 2, 3$, that are 2-covers. g_1 is defined by the matrix $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$. Clearly, g_1 is an expanding A-endomorphism, and $\mathbb{T}^2$ is a unique basic set of g_1. The corresponding diffeomorphism *f* has a unique basic set, say Ω_1, that is locally homeomorphic to the product of $\mathbb{R}^2$ and Cantor set. Thus, Ω_1 is two-dimensional.

Now, let us consider the case when $\mathbb{T}^1 = S^1$ is a circle, and d-cover $g : \mathbb{T}^1 \to \mathbb{T}^1$ is a non-singular endomorphism of S^1. The crucial step of the proof of Theorem 1.2 is the following result.

Lemma 3.5: *Let $g : \mathbb{T}^1 \to \mathbb{T}^1$ be a non-singular A-endomorphism, and $NW(g)$ a non-wandering set of g. Then $NW(g)$ is either $\mathbb{T}^1$ or $NW(g)$ is the union of the Cantor type set Σ and finitely many (non-zero) isolated attracting periodic orbits plus finitely many (possibly, zero) repelling isolated periodic orbits. Moreover, in the last case, Σ is backward g-invariant.*

Proof: Suppose that $NW(g) \neq \mathbb{T}^1$. By [20], g is semi-conjugate to the expanding linear mapping E_d, $E_d(t) = dt \ mod\, 1$, i.e. there is a continuous map $h : \mathbb{T}^1 \to \mathbb{T}^1$ such that $g \circ h = h \circ E_d$. Moreover, h is monotone.[21] As a consequence, given any point $t \in \mathbb{T}^1$, $h^{-1}(t)$ is either a point or a closed segment. Since $NW(g) \neq \mathbb{T}^1$, h is not a homeomorphism. Hence, there are points $t \in \mathbb{T}^1$ for which $h^{-1}(t)$ is a (non-trivial) closed segment. Denote the set of such points by χ. The set χ is countable and invariant under E_d, $E_d(\chi) = E_d^{-1}(\chi) = \chi$.[21,22] Therefore, $h^{-1}(\chi)$ is also invariant under g.

Let as prove that Σ is totally discontinuous. Since h is a semi-conjugacy and χ is invariant, Σ is an invariant set under g. Moreover, h is monotone and χ contains every dense orbits. It follows that Σ is totally discontinuous. As a consequence, $\Sigma = \mathbb{T}^1 \setminus \text{clos}\,(h^{-1}(\chi))$ is the Cantor set consisting on non-wandering points of g. Moreover, Σ is invariant under g (in particular, backward g-invariant). It follows from [23] that the part of $NW(g)$ that different from Σ consists of finitely many (non-zero) isolated attracting periodic orbits and finitely many (possibly, zero) repelling isolated periodic orbits. □

Now, Theorem 1.2 except the realisation part immediately follows from Theorem 1.1 and Lemma 3.5. It remains to construct a Smale–Vietoris A-diffeomorphism whose non-wandering set consists of a non-trivial zero-dimensional basic set and a finitely many (non-zero) isolated periodic orbits. It follows from [13,14] for $n = 3$ and [5,7] for $n \geq 4$ that it is sufficient to construct Smale skew-mapping F: $S^1 \times D^{n-1} \to S^1 \times D^{n-1}$ with the non-wandering set desired because of Smale skew-mapping can be extended to a diffeomorphism of some closed n-manifolds. Moreover, according to Robinson–Williams[12] construction of classical Smale solenoid, we can suppose $n = 3$.

Let g: $S^1 \to S^1$ be a C^∞ non-singular A-endomorphism that is a d-cover ($d \geq 2$) with the non-wandering set $NW(g)$ consisting of a unique attracting fixed point x_0 and a Cantor set Ω. Moreover, one can assume that $Dg|_\Omega = 2d - 1$, $Dg(x_0) = \lambda < 1$ where λ will be specified below. Such endomorphism was constructed by Shub [20]. Hirsch [24] has noticed that such endomorphism can be smoothed to be analytical. Now, the circle S^1 is endowed with the parameter inducing by the natural projection $[0, 1] \to [0, 1]/(0 \sim 1) = S^1$. We can assume that the restriction $g|[0, \frac{1}{2}]$ is a diffeomorphism $[0, \frac{1}{2}] \to [0, \frac{1}{2}]$ with the attracting fixed point $x_0 = \frac{1}{4}$ and two repelling fixed points $0, \frac{1}{2}$. Without loss of generality, one can also assume that $g|_{[\frac{1}{2},1]}(x) = (2d - 1)x \bmod 1$. By construction, $\cup_{n\geq 0} g_d^{-n}\left(0, \frac{1}{2}\right)$ is the stable manifold $W^s(x_0)$ of x_0, and $\Omega = S^1 \setminus W^s(x_0)$ is Cantor set belonging to $NW(g)$. Clearly, given any $y \in S^1$, $\min_{t_k,t_j}\{|t_k - t_j|\} = \frac{1}{2d-1}$ where $t_k \neq t_j$ and $g(t_k) = g(t_j) = y$. We take $0 < \lambda < \frac{1}{4}\sin\frac{\pi}{2d-1}$. After this specification, we denote g by g_d. Put by definition,

$$F(t, z) = \left(g_d(t),\ \lambda z + \frac{1}{2}\exp 2\pi i t\right),\quad F : \mathcal{B} = S^1 \times D^2 \to \mathcal{B}, \tag{7}$$

where $D^2 \subset \mathbb{R}^2$ is the unit disk, and $z = x + iy$, and $\mathcal{B}$ is a support of Smale skew-mapping. Since $\lambda < \frac{1}{4}$, $F(\mathcal{B}) \subset int\, \mathcal{B}$. The Jacobian of F equals

$$DF(t,z) = \begin{pmatrix} Dg_d(t) & 0 \\ \pi i \exp 2\pi it & \lambda Id_2 \end{pmatrix}, \tag{8}$$

where Id_2 is the identity matrix on $\mathbb{C}$ or $\mathbb{R}^2$. Since $Dg_d > 0$ and $\lambda > 0$, F is a local diffeomorphism. It follows from $\lambda < \frac{1}{4}\sin\frac{\pi}{2d-1}$ that F is a (global) diffeomorphism on its image.

Since g_d is an A-endomorphism, the periodic points of g_d are dense in $NW(g_d)$. By Lemma 3.2, the periodic points of F are dense in $NW(F)$. Thus, it remains to prove the $NW(F)$ has a hyperbolic structure. We follow [19, Proposition 8.7.5]. Clearly, the tangent bundle $T(\mathcal{B}) = T(S^1 \times D^2)$ is the sum $T(\mathcal{B}) = T(S^1) \oplus T(D^2)$, and the fibre $T_{(t,z)}(\mathcal{B})$ at each point $(t,z) \in \mathcal{B}$ is the sum of one-dimensional and two-dimensional tangent spaces $T_t(S^1) = \mathbb{E}^1 \cong \mathbb{R}$, $T_z(T^2) = \mathbb{E}^2 \cong \mathbb{R}^2$, respectively. It follows from (8) that $\mathbb{E}^2$ is invariant under DF:

$$\mathrm{DF}_p\begin{pmatrix} \vec{0} \\ \vec{v}_{23} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \lambda\vec{v}_{23} \end{pmatrix}, \quad \vec{v}_{23} \in \mathbb{E}^2.$$

Moreover, since $|\lambda| < 1$, $\mathbb{E}^2$ is the stable bundle, $E^s = \mathbb{E}^2$.

Take $q = (t,z) \in NW(F)$. Then $p_1(q) = t \in NW(g_d)$. If $t = x_0$, then q is a hyperbolic (attractive) fixed point of F. For $t \in \Omega$, we consider the cones

$$C_q^u = \left\{ \begin{pmatrix} \vec{v}_1 \\ \vec{v}_{23} \end{pmatrix} : \vec{v}_1 \in T_t(S^1),\ \vec{v}_{23} \in \mathbb{E}_z^2, \quad |\vec{v}_1| \geq \frac{2d-1}{4}|\vec{v}_{23}| \right\} \subset T(\mathcal{B}) = \mathbb{E}^1 \oplus \mathbb{E}^2.$$

For $\binom{\vec{v}_1}{\vec{v}_{23}} \in C^u$, it follows from (8) that

$$\mathrm{DF}\begin{pmatrix} \vec{v}_1 \\ \vec{v}_{23} \end{pmatrix} = \begin{pmatrix} \vec{v}_1' \\ \vec{v}_{23}' \end{pmatrix} = \begin{pmatrix} (2d-1)\vec{v}_1 \\ \pi i\vec{v}_1 \exp 2\pi it + \lambda\vec{v}_{23} \end{pmatrix}.$$

Hence, $|\vec{v}_{23}| \leq |\pi i \exp 2\pi it\vec{v}_1| + \lambda|\vec{v}_{23}| = \pi|\vec{v}_1| + \lambda|\vec{v}_{23}|$. Taking in mind $\lambda \leq \frac{1}{4}$, one gets

$$\begin{aligned} |\vec{v}_1'| &= (2d-1)|\vec{v}_1| = \frac{2d-1}{4}(4|\vec{v}_1|) \geq \frac{2d-1}{4}\left(\pi|\vec{v}_1| + \frac{1}{2}|\vec{v}_1|\right) \\ &\geq \frac{2d-1}{4}\left(\pi|\vec{v}_1| + \frac{2d-1}{8}|\vec{v}_{23}|\right) \geq \frac{2d-1}{4}\left(\pi|\vec{v}_1| + \lambda|\vec{v}_{23}|\right) \geq \frac{2d-1}{4}|\vec{v}_{23}'|, \end{aligned}$$

since $\frac{2d-1}{8} \geq \frac{1}{4}$. Therefore, $\binom{\vec{v}_1'}{\vec{v}_{23}'} \in C_{F(q)}^u$ and $\mathrm{DF}(C_q^u) \subset C_{F(q)}^u$. As a consequence,

$$\mathrm{DF}^k(C_{F^{-k}(q)}^u) \subset \mathrm{DF}^{k-1}(C_{F^{-k+1}(q)}^u) \subset \cdots \subset \mathrm{DF}(C_{F^{-1}(q)}^u) \subset C_q^u \quad \text{for any} \quad k \in \mathbb{N}.$$

To prove that the intersection of this nested cones is a line, take

$$\begin{pmatrix} \vec{v}_1 \\ \vec{v}_{23} \end{pmatrix}, \begin{pmatrix} \vec{w}_1 \\ \vec{w}_{23} \end{pmatrix} \in C_{F^{-k}(q)}^u, \quad \begin{pmatrix} \vec{v}_1^k \\ \vec{v}_{23}^k \end{pmatrix} = \mathrm{DF}^k\begin{pmatrix} \vec{v}_1 \\ \vec{v}_{23} \end{pmatrix}, \begin{pmatrix} \vec{w}_1^k \\ \vec{w}_{23}^k \end{pmatrix} = \mathrm{DF}^k\begin{pmatrix} \vec{w}_1 \\ \vec{w}_{23} \end{pmatrix}.$$

Put by definition, $|\vec{v}_1^j| = v_1^j$, $|\vec{w}_1^j| = w_1^j$, $\vec{v}_1 = (v_1, 0)$, $\vec{w}_1 = (w_1, 0)$, $v_1 > 0$, $w_1 > 0$. Then,

$$\left|\frac{\vec{v}_{23}^1}{v_1^1} - \frac{\vec{w}_{23}^1}{w_1^1}\right| = \left|\frac{\pi i \vec{v}_1 \exp 2\pi it + \lambda\vec{v}_{23}}{(2d-1)v_1} - \frac{\pi i \vec{w}_1 \exp 2\pi it + \lambda\vec{w}_{23}1}{w_1}\right|$$
$$= \left|\frac{\pi i \exp 2\pi it(w_1\vec{v}_1 - v_1\vec{w}_1)}{(2d-1)v_1w_1} + \frac{\lambda}{2d-1}\left(\frac{\vec{v}_{23}}{v_1} - \frac{\vec{w}_{23}}{w_1}\right)\right| = \frac{\lambda}{2d-1}\left|\frac{\vec{v}_{23}}{v_1} - \frac{\vec{w}_{23}}{w_1}\right|,$$

since $w_1\vec{v}_1 - v_1\vec{w}_1 = |\vec{w}_1|\vec{v}_1 - |\vec{v}_1|\vec{w}_1 = 0$. Therefore,

$$\left|\frac{\vec{v}_{23}^k}{v_1^k} - \frac{\vec{w}_{23}^k}{w_1^k}\right| = \left(\frac{\lambda}{2d-1}\right)^k\left|\frac{\vec{v}_{23}}{v_1} - \frac{\vec{w}_{23}}{w_1}\right|,$$

which goes to 0 as k goes to ∞. Since the difference of slopes goes to 0, the cones converge to a line, say $\mathbb{E}^u$. The calculation gives that the restriction of the derivative DF on $\mathbb{E}^u$ is an expansion.

Taking in mind the realization part of the proof of Theorem 1.2, we see that it is sufficient to construct the corresponding family of d-endomorphisms $S^1 \to S^1$, $d \geq 2$. First, we represent the two-parameter family of circle endomorphisms $f_{\varepsilon,\delta}$ continuously depending on the parameters $\varepsilon \in (0, 1)$ and $\delta \in [0, \frac{1}{4})$.

Let $U_\delta(x)$ be the bump function such that

- $U_\delta(x) = 1$ for $x \in \left[-\frac{\delta}{2}, +\frac{\delta}{2}\right]$, $0 < \delta \leq \frac{1}{4}$;
- $U_\delta(x) = 0$ for $|x| \geq \delta$;
- $U_\delta'(x) \geq 0$ for $x \in \left[-\delta, -\frac{\delta}{2}\right]$, and $U_\delta'(x) \leq 0$ for $x \in \left[\frac{\delta}{2}, \delta\right]$.

Lemma 3.6: *Let*

$$f_{\varepsilon,\delta}(x) = \begin{cases} dx + (-d+\varepsilon)xU_\delta(x) \ \ mod\ 1 & for\ \ \varepsilon \in (0,1),\ \ \delta \in (0, \frac{1}{4}) \\ dx\ \ mod\ 1 & for\ \ \varepsilon = 0,\ \ \delta = 0 \end{cases}$$

Then $f_{\varepsilon,\delta}$ is a structurally stable non-singular circle d-endomorphism such that the non-wandering set $NW(f_{\varepsilon,\delta})$ is the union of a unique hyperbolic attracting point $x = 0$ and a Cantor set provided $\varepsilon \neq 0$ and $\delta \neq 0$. Moreover, $NW(f_{0,0}) = S^1$. In addition, $f_{\varepsilon,\delta} \to E_d$ as $\varepsilon \neq 0$ is fixed and $\delta \to 0$ in the C^0 topology.

Proof: For $\varepsilon \neq 0$ and $\delta \neq 0$, we see

$$f_{\varepsilon,\delta}'(x) = d + (-d+\varepsilon)\left[xU_\delta(x)' + U_\delta(x)\right] = d + (-d+\varepsilon)xU_\delta(x)' + (-d+\varepsilon)U_\delta(x).$$

Clearly, $d + (-d+\varepsilon)U_\delta(x) \geq \varepsilon$. Since $xU_\delta(x)' \leq 0$, $f_{\varepsilon,\delta}'(x) \geq \varepsilon$. Because of outside of the δ-neighbourhood $V_\delta(0)$ of $x_0 = 0$, the mapping $f_{\varepsilon,\delta}$ coincides with the linear d-endomorphism $E_d(x) = dx\ mod\ 1$, $f_{\varepsilon,\delta}$ is a non-singular d-endomorphism. Since $f_{\varepsilon,\delta}'(0) = \varepsilon \in (0,1)$, $x = 0$ is a hyperbolic attracting point. Solving the equation $dx + (-d+\varepsilon)xU_\delta(x) = x$, one gets two fixed points $\pm x_* \in V_\delta(0)$ such that $U_\delta(\pm x_*) = \frac{d-1}{d-\varepsilon}$, where $\frac{\delta}{2} < x_* < \delta$. Moreover, the ω-limit set of any point from $(-x_*, x_*)$ is $x_0 = 0$. Hence, $NW(f_{\varepsilon,\delta})$ equals

$$NW(f_{\varepsilon,\delta}) = \{x_0\} \bigcup \left(S^1 \setminus \cup_{k\geq 0} f_{\varepsilon,\delta}^{-k}(-x_*, x_*)\right),$$

where $C = S^1 \setminus \cup_{k\geq 0} f_{\varepsilon,\delta}^{-k}(-x_*, x_*)$ is Cantor set. For any $x \in C$, one can prove that

$$\begin{aligned} f'_{\varepsilon,\delta}(x) &= d + (-d+\varepsilon)xU'_\delta(x) + (-d+\varepsilon)U_\delta(x) \geq d + (-d+\varepsilon)U_\delta(x_*) \\ &\quad + (-d+\varepsilon)xU'_\delta(x) \\ &= 1 + (-d+\varepsilon)xU'_\delta(x) > 1. \end{aligned}$$

It follows from [23] that $f_{\varepsilon,\delta}$ is structurally stable. At last, for $x \in V_\delta(0)$, one gets

$$\left| f_{\varepsilon,\delta}(x) - E_d(x) \right| = |(-d+\varepsilon)xU_\delta(x)| \leq \delta d \to 0 \quad \text{as} \quad \delta \to 0.$$

As a consequence, $f_{\varepsilon,\delta} \to E_d$ as $\delta \to 0$ in the C^0 topology. □

Recall that original Smale solenoid map is built by a skew map f: $S^1 \times D^2 \mapsto S^1 \times D^2$ by $f(x,y) = (g_1(x), g_2(y))$ so that g_1 is an expanding map on S^1 and g_2 is uniformly attracting. In [13], it was prove that f: $S^1 \times D^2 \mapsto S^1 \times D^2$ can be extended to a diffeomorphism of some lens space $L_{p,q}$ (including S^3). In [14], it was proved that for any given $L_{p,q}$ there is a diffeomorphism of $L_{p,q}$ with one Smale solenoid attractor and one Smale solenoid repeller. The analysis of [13,14] shows that the constructions above can be applied to non-singular endomorrphism g_1 as well. Thus, taking in mind Lemma 3.6 and the technics developed in [13,14] (see also [5,7,15]), one can prove Theorem 1.3.

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