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ROUGH DIFFEOMORPHISMS WITH BASIC SETS OF CODIMENSION ONE

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Abstract. The review is devoted to the exposition of results (including those of the authors of the review) obtained from the 2000s until the present, on topological classification of structurally stable cascades defined on a smooth closed manifold M^n ($n \geq 3$) assuming that their nonwandering sets either contain an orientable expanding (contracting) attractor (repeller) of codimension one or completely consist of basic sets of codimension one. The results presented here are a natural continuation of the topological classification of Anosov diffeomorphisms of codimension one. The review also reflects progress related to construction of the global Lyapunov function and the energy function for dynamical systems on manifolds (in particular, a construction of the energy function for structurally stable 3-cascades with a nonwandering set containing a two-dimensional expanding attractor is described).

To the dear memory of D. V. Anosov

Introduction

The present review has to do with traditional research fields of the Nizhniy Novgorod nonlinear oscillations school founded by Academician A. A. Andronov. The historical paper by Andronov and Pontryagin [1] gave an impetus to the development of an important field of qualitative theory of dynamical systems in Nizhniy Novgorod — namely, of topological classification of rough systems and systems with hyperbolic structure of the nonwandering set. The first foundational results in this direction belonged to members of this school: Andronov, Leontovich, Maier, etc. Under a complete topological classification of a certain class G of dynamical systems we understand the solution of the following problems:

- finding topological invariants of the dynamical systems from the class G;
- proof of completeness of the set of the found invariants, i.e., proof of the fact that coincidence of the sets of topological invariants is a necessary and sufficient condition of topological equivalence (conjugacy) of two dynamical systems from G;
- realization, i.e., construction of a standard representative of G with the given set of topological invariants.

Solutions of the problem of topological classification in this exact canonical setting are known only for some classes of structurally stable systems. We restrict ourselves to considering dynamical systems with discrete time (cascades and diffeomorphisms that generate them) on closed manifolds. The equivalence class of rough flows on the circle is uniquely determined by the number of its fixed points. For structurally stable cascades on the circle the complete topological invariant was obtained by Maier [42] in 1939 and consists of a set of three numbers: the number of periodical orbits, their period, and the so-called ordinal number.

The early 1960s were distinguished by a revolutionary discovery related to Smale [60] and Anosov [2]. It was found that structurally stable mappings of a surface can possess a countable set of saddle-type hyperbolic periodic orbits. The dynamics of such systems is chaotic and, unlike regular one, it means the existence of a dense subset of a nontrivial basic set where trajectories of arbitrary close points have different asymptotic behavior. It became clear that the study of such systems

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requires new approaches and methods; their topological invariants are not restricted to combinatory objects but are characterized by algebraic invariants including automorphisms of fundamental groups of supports of the basic sets. For cascades on manifolds with dimension greater than one, existence of homoclinic intersections of invariant manifolds of saddle-type periodic motions becomes possible, which results in the existence of a countable set of periodic trajectories. The first person to discover the complicated structure of the set of trajectories belonging to a neighborhood of a homoclinic trajectory was Poincaré [54]. Then Birkhoff [6] studied two-dimensional area-preserving mappings and showed that the presence of homoclinic intersections implies the existence of infinitely many periodic orbits. The principal advance in this direction was a paper by Shilnikov, where he gave a complete description of the set of all trajectories remaining in a certain neighborhood of a transversal homoclinic trajectory of a flow on a manifold of dimension greater than two. This description, in particular, implies the presence of a countable set of periodic trajectories in the chosen neighborhood [57, 58].

An essential role in understanding the principal difference of structurally stable cascades on manifolds of dimension greater than one and structurally stable flows on surfaces belongs to the example of a structurally stable diffeomorphism of a two-dimensional sphere with infinitely many periodic orbits, constructed by Smale in 1961 [62] and called the "Smale horseshoe." Another very important discovery was made by Anosov in 1962. He established the structural stability of the geodesic flow on a Riemann manifold of negative curvature [2]. In the same paper he introduced an important class of structurally stable flows, and later also diffeomorphisms, which he called U-systems and which later were given the name of Anosov flows and diffeomorphisms. Smale generalized this concept and introduced a class of systems with hyperbolic structure of the nonwandering set, namely, the closure of the set of periodic points [62] (diffeomorphisms with these properties were called A-diffeomorphisms). The nonwandering set of systems from this class admits a decomposition in a finite number of closed invariant basic sets such that on each of these sets the system acts transitively. Dynamics on a nontrivial basic set (which is not a periodic orbit) has properties very similar to those of the diffeomorphism on the nonwandering set in the example called the "Smale horseshoe."

A topological classification of one-dimensional basic sets of A-diffeomorphisms of surfaces was obtained by Plykin, Grines, Zhirov, and Kalay. Moreover, necessary and sufficient conditions of topological conjugacy for structurally stable diffeomorphisms on surfaces were found in the papers by Grines, Bonatti, and Langevin.

It follows from papers [10, 41, 63] that the assumption of the existence of a zero-dimensional or one-dimensional basic set of an A-diffeomorphism $f : M^3 \to M^3$ does not imply restrictions on the topology of the ambient manifold. But if the dimension of the basic set is 2 or 3, this is not the case. Indeed, if the nonwandering set of the diffeomorphism f contains a basic set of dimension three, then f is an Anosov diffeomorphism, the manifold M^3 is a three-dimensional torus $\mathbb{T}^3$, and the topological classification of such diffeomorphisms was obtained by Franks [15] and Newhouse [48].

According to [52], a basic set is an attractor (repeller) if and only if it contains unstable (stable) manifolds of its points. But in general the dimension of the basic set can differ from that of the unstable (stable) manifolds of its points. In the case where the dimension of the attractor (repeller) coincides with that of the unstable (stable) manifolds of its points, the attractor (repeller) is called expanding (contracting).

The dynamics of diffeomorphisms of 3-manifolds with a nonwandering set containing one-dimensional expanding attractors (contracting repellers) is studied in the papers by Bothe [9, 10], Williams [64], Zhuzhoma and Isaenkova [66], etc. Note that the basic sets considered in these papers did not lie on invariant surfaces (i.e., they were not surface sets). Moreover, all examples of diffeomorphisms of a three-dimensional manifold with one-dimensional expanding attractors (contracting repellers) from the aforementioned papers were not structurally stable. The problem of the existence of a structurally stable diffeomorphism with a basic set of this type is open.

In the paper of Bonatti and Gelman [8] a family of structurally stable partially hyperbolic diffeomorphisms was constructed such that their nonwandering sets consisted of exactly one one-dimensional attractor and one one-dimensional repeller. Moreover, the attractor and the repeller belonged to surfaces that were not closed.

In this review we consider A-diffeomorphisms with basic sets Λ of codimension one. Codimension one means that the topological dimension of the basic set is less than the dimension of the supporting manifold by one, $\dim \Lambda = \dim M^n - 1$. Since further we assume that $n \geq 3$, the basic sets under consideration are at least two-dimensional and hence *nontrivial* (distinct from a periodic orbit). It is known that basic sets of codimension one are necessarily either *attractors* or *repellers* whose study is important for applications. An important characteristic of the basic set is its Morse index $u(\Lambda)$, which by definition equals the dimension of the unstable manifold of each periodic orbit from Λ. Theoretically, the Morse index for nontrivial basic sets can take any value from 1 to $n-1$. For the extremal values of the Morse index, there has been recent progress in the understanding of the structure of the ambient manifold, the topological classification, and the existence of an energy function.

The construction of energy functions is related to the "Fundamental theorem of dynamical systems." This theorem proven by Conley [14] in 1978 states that any continuous dynamical system (flow or cascade) has a continuous *Lyapunov function*, i.e., a function decreasing along the trajectories of the system outside of the chain recurrent set and constant on the chain components. From many points of view, it is more meaningful to find information on the existence of an *energy function* of a smooth dynamical system, i.e., a smooth Lyapunov function such that the set of its critical points coincides with the chain recurrent set of the system. The existence of an energy function for any flow follows from a paper of Wilson and Yorke [65]. Cascades, even those with regular dynamics, generally do not have an energy function. Such examples were constructed in a paper of Pixton [49] as well as in papers of Bonatti, Grines, Laudenbach, Pochinka [7, 21, 22]; in the last paper, sufficient conditions for the existence of a Morse energy function for three-dimensional Morse–Smale cascades were also found. More surprising is the fact of the existence of an energy function for some discrete dynamical systems with chaotic behavior.

The structure of the paper is as follows. In Sec. 1, we give basic definitions and construct model examples of basic sets of codimension one (for simplicity, we restrict ourselves to low-dimensional examples, but the idea of construction is retained for high-dimensional ones). In Sec. 2, we give some classical as well as relatively recent results related to the subject under consideration. In Sec. 3, we consider problems of topological classification. Finally, we construct in Sec. 4 an energy function for rough 3-diffeomorphisms with a two-dimensional expanding attractor.

1. Preliminary Information and Model Examples

Let $f \in Diff^1(M^n)$ be a C^1-smooth diffeomorphism of a closed n-dimensional ($n \geq 2$) manifold M^n supplied with some Riemann metric d. A set $\Lambda \subset M^n$ invariant w.r.t. f is called *hyperbolic* if the restriction $T_\Lambda M^n$ of the tangent foliation TM^n of the manifold M^n on Λ can be represented as a Whitney sum $E^s_\Lambda \oplus E^u_\Lambda$ of df-invariant subfoliations E^s_Λ, E^u_Λ ($\dim E^s_x + \dim E^u_x = n$, $x \in \Lambda$), and there exist constants $C_s > 0$, $C_u > 0$, and $0 < \lambda < 1$ such that

$$\|df^m(v)\| \leq C_s \lambda^m \|v\| \quad \text{for } v \in E^s_\Lambda, \qquad \|df^{-m}(v)\| \leq C_u \lambda^m \|v\| \quad \text{for } v \in E^u_\Lambda, \qquad m > 0.$$

The hyperbolic structure generates the existence of the so-called *stable* and *unstable* manifolds that combine points with similar asymptotic behavior w.r.t. positive (resp., negative) iterations [37, 61]. For any point $x \in \Lambda$ there exists an injective immersion $J^s_x : \mathbb{R}^s \to M$, whose range $W^s(x) = J^s_x(\mathbb{R}^s)$ is called the *stable manifold of point* x, such that the following properties hold:

(1) $T_x W^s(x) = E^s_\Lambda$.
(2) Points $x, y \in M$ belong to the same manifold $W^s(x)$ if and only if $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$.
(3) $f(W^s(x)) = W^s(f(x))$.
(4) If $x, y \in \Lambda$, then either $W^s(x) = W^s(y)$, or $W^s(x) \cap W^s(y) = \varnothing$.

(5) If points $x, y \in \Lambda$ are close in M, then $W^s(x)$, $W^s(y)$ are C^1-close on compact sets. This property is usually called *the theorem on continuous dependence of stable manifolds on the initial conditions*.

The *unstable manifold* $W^u(x)$ of a point $x \in \Lambda$ is defined as the stable manifold w.r.t. the diffeomorphism f^{-1}. Unstable manifolds have similar properties. With reference to property (3), stable and unstable manifolds are called *invariant manifolds*.

A point $x \in M^n$ is called *nonwandering* if for each its neighborhood $U(x)$ and each natural number N there exists a $n_0 \in \mathbb{Z}$, $|n_0| \geq N$, such that $f^{n_0}(x) \in U(x)$. The set of nonwandering points of a diffeomorphism f will be denoted by $NW(f)$. A diffeomorphism f *satisfies axiom A* (or, which is the same, is an *A-diffeomorphism*) if the set $NW(f)$ is hyperbolic, and periodic points are dense in $NW(f)$.

Smale [62] proved the following statement known as the *spectral decomposition theorem*. Let a diffeomorphism $f \in Diff^1(M^n)$ satisfy axiom A. Then the set $NW(f)$ can be represented as a finite union of pairwise disjoint closed invariant sets $\Lambda_1, \dots, \Lambda_k$, called *basic sets*, each of them containing a dense orbit. In this case [36], the manifold M^n can be represented as

$$M^n = \bigcup_{i=1}^{k} W^s(\Lambda_i) = \bigcup_{i=1}^{k} W^u(\Lambda_i),$$

where $W^s(\Lambda_i) = \bigcup\limits_{x \in \Lambda_i} W^s(x)$ and $W^u(\Lambda_i) = \bigcup\limits_{x \in \Lambda_i} W^u(x)$. A basic set is called *nontrivial* if it is not a periodic orbit (in particular, not a fixed point).

By transitivity of f on each basic set Λ_i, restrictions of foliations $E^s_{\Lambda_i}$, $E^u_{\Lambda_i}$ on Λ_i have a constant dimension at all points $x \in \Lambda_i$. *The type of a basic set* Λ_i is a pair of numbers (a_i, b_i), where $a_i = \dim E^u_x, b_i = \dim E^s_x$, and x is any point from Λ_i. Here the number a_i is called the *Morse index* of the basic set Λ_i and is denoted $u(\Lambda_i)$. Then $b_i = n - u(\Lambda_i)$.

Results of [4, 11] imply the following specification of the structure of a basic set. Each basic set Λ_i is represented as a finite union of disjoint compact sets $\Lambda_{i1}, \dots, \Lambda_{ih}$, which cyclically pass into each other under the action of f. Moreover, the stable and unstable manifold of each point $x \in \Lambda_{ij}$ contains a dense set in Λ_{ij}. Each Λ_{ij} is called a *C-dense* (*or periodic*) *component* of the basic set Λ_i. A basic set is called *C-dense* if it has exactly one periodic component and coincides with it.

A compact f-invariant set $A \subset M$ is called an *attractor* of diffeomorphism f if it has a compact neighborhood U_A such that $f(U_A) \subset int\, U_A$ and $A = \bigcap\limits_{k \geq 0} f^k(U_A)$. A *repeller* is defined as an attractor for f^{-1}.

By [52], a basic set Λ of a diffeomorphism f is an attractor (repeller) if and only if $\Lambda = \bigcup\limits_{x \in \Lambda} W^u(x)$ $(\Lambda = \bigcup\limits_{x \in \Lambda} W^u(x))$.

An attractor Λ of an A-diffeomorphism f is called an *expanding attractor* if its topological dimension $\dim \Lambda$ equals the dimension of the unstable manifold $W^u_x, x \in \Lambda$. A repeller of the diffeomorphism f is called *contracting* if it is an expanding attractor for f^{-1}.

By [32], a basic set Λ of an A-diffeomorphism $f : M^3 \to M^3$ is called *surface* if it belongs to an f-invariant closed surface M^2_Λ (not necessarily connected) topologically embedded into the manifold M^3 and called the *support* of the set Λ.

Two diffeomorphisms f, $g \in Diff^1(M^n)$ are called *topologically conjugate* if there exists a homeomorphism $\varphi : M^n \to M^n$ such that $\varphi \circ f = g \circ \varphi$. A diffeomorphism $f \in Diff^1(M^n)$ is called *structurally stable* if there exists its neighborhood $U(f) \subset Diff^1(M^n)$ such that each diffeomorphism $g \in U(f)$ is conjugate to f. If we require that the conjugating homeomorphism should be close to the identical one in the C^0-topology, then we come to the definition of a *rough* diffeomorphism. Now it is known that the concepts of "roughness" and "structural stability" are equivalent, though the proof of this fact is quite nontrivial (see [5], where different definitions and respective results are discussed).

In order to formulate conditions of structural stability, one needs the so-called strong transversality condition. Let $W_1, W_2 \subset M^n$ be two immersed manifolds with a nonempty intersection. By definition, W_1, W_2 *intersect transversally* if for any point $x \in W_1 \cap W_2$ the tangent space $T_x M$ is generated by the tangent subspaces $T_x W_1$ and $T_x W_2$. In particular, if W_1, W_2 intersect transversally, then $\dim T_x W_1 + \dim T_x W_2 \geq \dim T_x M^n$.

An A-diffeomorphism is said to *satisfy the strong transversality condition* if for any two points x, $y \in NW(f)$ the manifolds $W^s(x)$, $W^u(y)$ intersect only transversally. It is known [43, 55] that a diffeomorphism is structurally stable if and only if it is an A-diffeomorphism and satisfies the strong transversality condition. Necessity was proved by Mane [43], sufficiency by Robinson [55].

An important and well enough studied class of structurally stable dynamical systems is the class of Anosov diffeomorphisms of codimension one [3]. Recall that an *Anosov diffeomorphism* is a diffeomorphism such that its whole supporting manifold is hyperbolic. An Anosov diffeomorphism $f : M^n \to M^n$ is called a diffeomorphism *of codimension one* if dim $E^s_{M^n} = 1$ or dim $E^u_{M^n} = 1$. It is known that each such Anosov diffeomorphism of codimension one is topologically conjugate to a hyperbolic automorphism of a torus. Moreover, two such diffeomorphisms are topologically conjugate if and only if they are π_1-conjugate [15, 48] (the latter means that they induce conjugate isomorphisms of the fundamental group of the torus). In this case the n-dimensional torus $\mathbb{T}^n, n \geq 2$, is the only basic set of such a diffeomorphism.

Recall that an *algebraic automorphism* of the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is a diffeomorphism $\widehat{C}$ given by the matrix $C = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$ from the set $GL(n, \mathbb{Z})$ of integer matrices with determinant ± 1, i.e., $\widehat{C}(x_1, \dots, x_n) = (a_{11}x_1 + \dots + a_{1n}x_n, \dots, a_{n1}x_1 + \dots + a_{nn}x_n)$ $(mod\ 1)$. An algebraic automorphism $\widehat{C}$ is called *hyperbolic* if the absolute values of the eigenvalues $\lambda_1, \dots, \lambda_n$ of the matrix C are distinct from one. In this case the matrix C is also called *hyperbolic*. A hyperbolic automorphism is called a *hyperbolic automorphism of codimension one* if it has exactly one eigenvalue whose absolute value is either less than one or greater than one, while the other eigenvalues lie respectively either outside the unit circle of the complex plane or inside it.

There are several definitions of orientability of a basic set, two of which are used most widely. One definition is expressed by orientability of the respective subfoliations of the tangent foliation (see, e.g., [40, 50]). The other one introduced by Grines [16–18] uses the intersection index of invariant manifolds. We will say that a basic set Λ is *orientable* if for each point $x \in \Lambda$ and each fixed numbers $\alpha > 0$ and $\beta > 0$, the intersection index $W^s_\alpha(x) \cap W^u_\beta(x)$ is the same ($+1$ or -1) for all intersection points. Otherwise the basic set Λ is called *nonorientable*. Below we will understand orientability of a basic set in the latter sense.

We pass to construction of model examples. Anosov diffeomorphisms are a base for construction of expanding attractors of codimension one. Following [62], one can construct a structurally stable diffeomorphism of the torus $\mathbb{T}^n$ such that its nonwandering set consists of a fixed sink and expanding attractor of codimension one, with the help of the so-called Smale surgery from an Anosov diffeomorphism of codimension one of the n-torus $\mathbb{T}^n$. Such a diffeomorphism is called a *DA*-diffeomorphism. We give a construction for the case $n = 2$.

Let $f_{L_A} : \mathbb{T}^2 \to \mathbb{T}^2$ be the algebraic automorphism of the torus induced by the linear mapping $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, and let p_0 be a fixed saddle point corresponding to the origin in $\mathbb{R}^2$ with the eigenvalues $\lambda^u = \frac{3+\sqrt{5}}{2}$ and $\lambda^s = \frac{3-\sqrt{5}}{2}$. We introduce local coordinates x_1, x_2 in some neighborhood U of the point p_0, ensuring a diagonal form of the matrix of the linear mapping L, i.e., $f_L(x_1, x_2) = (\lambda^u x_1, \lambda^s x_2)$ on U. We choose a value $r_0 \in (0, 1/2)$ such that the 2-ball $B_{r_0}(p_0)$ of

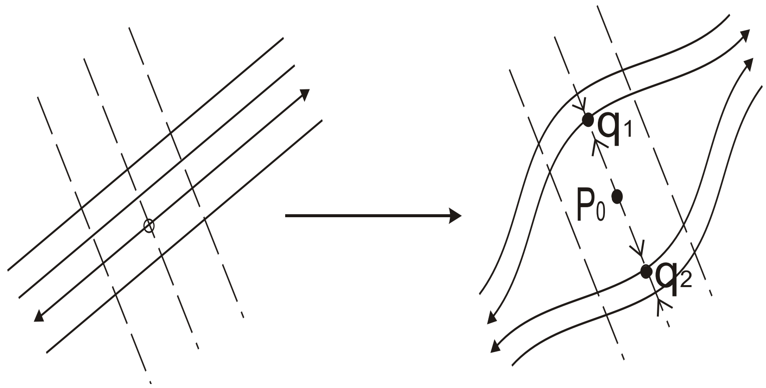

Fig. 1. Smale surgery.

radius r_0 centered at p_0 is contained in U. Let $\delta(r)$ be a function of one variable such that $0 \le \delta(r) \le 1$ for all r, $\delta'(r) < 0$ for $r_0/2 < r < r_0$, and $\delta(r) = \begin{cases} 0, & r \ge r_0, \\ 1, & r \le r_0/2. \end{cases}$

Consider the system of differential equations $\dot{x}_1 = 0$, $\dot{x}_2 = x_2\delta(\| x \|)$. Let φ^t be the flow of this system, $\varphi^t(x_1, x_2) = (x_1, \varphi_2^t(x_1, x_2))$. Then $\varphi^t = id$ outside the ball $B_{r_0}(p_0)$ and $D\varphi_p^t = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}$. Put $f = \varphi^\tau f_{L_A}$ for some $\tau > 0$ such that $e^\tau \lambda^s > 1$. Note that $Df_{p_0} = \begin{pmatrix} \lambda^u & 0 \\ 0 & e^\tau \lambda^s \end{pmatrix}$, so that p_0 is a hyperbolic source. By construction, the diffeomorphism f retains the stable foliation of the Anosov diffeomorphism, and the coordinate axes are f-invariant. Since the diffeomorphisms φ^τ and f_L have opposite directions of motion on the axis Ox_2, the diffeomorphism f has two fixed points q_1 and q_2 symmetric w.r.t. p_0 on the axis Ox_2, being hyperbolic saddle points (see Fig. 1). The following statement takes place (see, e.g., [56]).

Theorem 1.1. *For the diffeomorphism f the set $\Lambda = \mathbb{T}^2 \setminus W^u_{p_0}$ is a one-dimensional attractor, and its spectral decomposition has the form $\{p_0, \Lambda\}$.*

The one-dimensional attractor constructed this way is expanding, of type $(1, 1)$, and orientable.

We give an example of a two-dimensional surface basic set in the three-dimensional space.

It is easy to construct an A-diffeomorphism $f : M^3 \to M^3$ with a basic set homeomorphic to the two-dimensional torus which is a closed two-dimensional submanifold of M^3 (Fig. 2). For this purpose, it suffices to consider a diffeomorphism $f : \mathbb{T}^2 \times \mathbb{S}^1 \to \mathbb{T}^2 \times \mathbb{S}^1$ given by the formula $f(t, s) = (f_A(t), f_{NS}(s))$, where $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov diffeomorphism and $f_{NS} : \mathbb{S}^1 \to \mathbb{S}^1$ is a diffeomorphism of the form "North Pole — South Pole" (a diffeomorphism whose nonwandering set consists of a hyperbolic sink and source). Then the diffeomorphism f has a two-dimensional basic set Λ of type $(1, 2)$, which is an attractor. Moreover, Λ is diffeomorphic to $\mathbb{T}^2$, and the diffeomorphism $f|_\Lambda$ is topologically conjugate to the Anosov diffeomorphism.

2. Williams and Brown Theorems

Using the concept of inverse limit, Williams [64] described the internal dynamics of a restriction of a diffeomorphism to an expanding attractor. We will briefly explain the approach of Williams and mention the development of this approach obtained relatively recently by Brown [13].

Let N be a compact neighborhood of an expanding attractor Λ. Following Williams, we put $x \sim y$ if (and only if) points x, $y \in N$ belong to the same connected component of the intersection $N \cap W^s(z)$ for some point $z \in \Lambda$. Williams proved that the neighborhood N can be chosen to satisfy the following conditions:

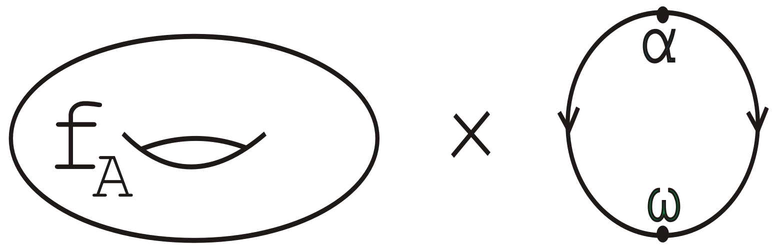

Fig. 2. Construction of a diffeomorphism with a two-dimensional surface basic set.

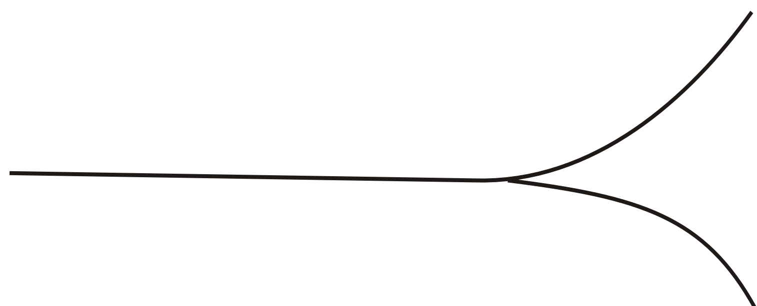

Fig. 3. Branched manifold.

- the factor space $N/_{\sim} \stackrel{\text{def}}{=} K$ is a branched manifold;
- the commutative diagram

$$\begin{array}{ccc} f(N) & \stackrel{f}{\longleftarrow} & N \\ \downarrow \subset & & \\ N & & \downarrow q \\ \downarrow q & & \\ K & \stackrel{g}{\longleftarrow} & K \end{array}$$

holds, where $q : N \to N/_{\sim}$ is the projection onto the factor space. Recall that a branched manifold is a smooth manifold with an exception of finitely many singularities, which for a one-dimensional manifold (this is the case we will basically need) have the form shown in Fig. 3.

Recall that the *inverse limit* w.r.t. mapping $g : K \to K$,

$$\Sigma_g = \varprojlim(K, g) = \varprojlim \left\{ K \stackrel{g}{\longleftarrow} K \stackrel{g}{\longleftarrow} \cdots \stackrel{g}{\longleftarrow} K \stackrel{g}{\longleftarrow} \cdots \right\},$$

is defined as the set of unilateral sequences $(x_0, \ldots, x_i, \ldots)$, where $x_i = g(x_{i+1})$. On Σ_g a *shift* is defined:

$$h : \Sigma_g \to \Sigma_g, \qquad h(x_0, x_1, \ldots) = (g(x_0), x_0, x_1, \ldots).$$

Note that any unilateral sequence $(x_0, \ldots, x_i, \ldots)$ can be considered as a point of the infinite product $\prod\limits_{i=0}^{\infty} K_i$, $K_i = K$, supplied with the Tikhonov topology (of course, K is assumed to have a topological structure). Thus the inverse limit is a subset of a topological space. Further, while saying that some basic set is an inverse limit, we mean that the basic set is homeomorphic to one.

Let K be a branched manifold. The definition of a branched manifold implies that such a manifold has a tangent foliation denoted by $T(K)$. Following Williams [64], we formulate the concept of an expansion of a branched manifold. A C^r-mapping $g : K \to K$, $r \geq 1$, is called an *expansion* if there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$|Dg^m(v)| \geq C\lambda^m |v| \quad \text{for all} \quad m \in \mathbb{N}, \ v \in T(K).$$

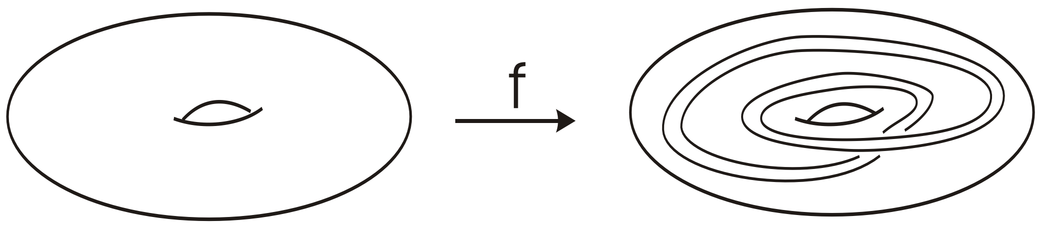

Fig. 4. Smale solenoid

If K is a branched m-manifold and the following conditions hold:

(1) $NW(g) = K$;
(2) g is an expansion;
(3) for any point $z \in K$, there exists a neighborhood U of the point z and a number $j \in \mathbb{N}$ such that $g^j(U)$ is an m-ball,

then Σ_g is called a *generalized m-solenoid*. Williams [64] proved the following theorem.

Theorem 2.1. *Let Λ be an m-dimensional expanding attractor of a diffeomorphism f. Then the restriction $f|_\Lambda$ of the diffeomorphism f to Λ is conjugate to a shift h of some generalized m-solenoid. Conversely, given a shift $h : \Sigma \to \Sigma$ of a generalized m-solenoid Σ, there exist a manifold M and a diffeomorphism $f : M \to M$ such that f has an m-dimensional expanding attractor Λ and its restriction $f|_\Lambda$ is conjugate to h.*

In the special case where K is the circle S^1, and $g = E_2 : x \to 2x \ mod \ 1$ is an expanding endomorphism of degree 2, we obtain the well-known Smale solenoid [62]. Namely, Smale constructed a diffeomorphism of a solid torus onto itself with a one-dimensional expanding attractor that is a topological solenoid. Schematically, the Smale example can be imagined as the combination of an expansion of the solid torus along its internal axis and a subsequent contraction in the direction perpendicular to that axis. Then the obtained (intermediate) solid torus is embedded into the original one so that the axis of the intermediate solid torus spins no less than twice along the axis of the original solid torus and the disk structure is retained (see Fig. 4).

In the case where a basic set Λ is a C-dense attractor with a unit Morse index, Brown [13] showed that Λ is always an inverse limit, and if Λ is not expanding (i.e., its topological dimension $k = \dim \Lambda$ is not less than two), then Λ is a special inverse limit w.r.t. a linear endomorphism or diffeomorphism of the k-dimensional torus $\mathbb{T}^k$. We give a more precise formulation of the Brown theorem.

Theorem 2.2. *Let $f : M^n \to M^n$ be an A-diffeomorphism of a closed manifold M^n, and let Λ be a C-dense basic set of the diffeomorphism f. Assume that the Morse index of the set Λ equals one, i.e., $\dim E^u_x = 1$ for any point $x \in \Lambda$. Then either Λ is a one-dimensional expanding attractor, and in this case Λ is an inverse limit*

$$\Sigma_g = \varprojlim (K, g) = \varprojlim \left\{ K \overset{g}{\longleftarrow} K \overset{g}{\longleftarrow} \cdots \overset{g}{\longleftarrow} K \overset{g}{\longleftarrow} \cdots \right\}$$

w.r.t. expansion $g : K \to K$ of a branched one-dimensional manifold K, or Λ is an inverse limit

$$\Sigma_A = \varprojlim (\mathbb{T}^k, A) = \varprojlim \left\{ \mathbb{T}^k \overset{A}{\longleftarrow} \mathbb{T}^k \overset{A}{\longleftarrow} \cdots \overset{A}{\longleftarrow} \mathbb{T}^k \overset{A}{\longleftarrow} \cdots \right\}$$

w.r.t. the linear endomorphism $A : \mathbb{T}^k \to \mathbb{T}^k$, $k \geq 2$. Moreover, if Λ is locally connected, then Λ is homeomorphic to the k-dimensional torus $\mathbb{T}^k$, and the restriction $f|_\Lambda$ is conjugate to the Anosov automorphism.

The following question by Smale [62, P. 785] remained open until recently: Does there exist a two-dimensional basic set of a diffeomorphism $f : M^3 \to M^3$ that is not a compact submanifold and does not have the local structure of a direct product of $\mathbb{R}^2$ by a Cantor set? In 2010, Brown gave a negative answer to this question. It follows from [13] that any two-dimensional basic set of a diffeomorphism $f : M^3 \to M^3$ is either an expanding attractor (contracting repeller), or a surface attractor (surface repeller). Moreover, it became possible to give a description of basic sets of diffeomorphisms of three-dimensional manifolds.

Theorem 2.3. *Let $f : M^3 \to M^3$ be an A-diffeomorphism of a closed three-dimensional manifold M^3, and let Λ be a C-dense basic set being an attractor of the diffeomorphism f. Then*:

(1) *If $\dim \Lambda = 0$, then Λ is an isolated attracting fixed point.*
(2) *If $\dim \Lambda = 1$, then Λ is a generalized one-dimensional solenoid locally homeomorphic to the product of a segment by a Cantor set in the two-dimensional plane.*
(3) *If $\dim \Lambda = 2$, then*
   - *either $\dim E^u|_\Lambda = 1$, and in this case Λ is homeomorphic to the two-dimensional torus $\mathbb{T}^2$, and the restriction $f|_\Lambda$ is conjugate to the Anosov automorphism of the two-dimensional torus,*
   - *or $\dim E^u|_\Lambda = 2$, and in this case Λ is an expanding attractor locally homeomorphic to the product of the two-dimensional plane by a Cantor set on the line.*
(4) *If $\dim \Lambda = 3$, then $\Lambda = M^3 = \mathbb{T}^3$ is a three-dimensional torus, and the diffeomorphism f is conjugate to the Anosov automorphism of the three-dimensional torus.*

Note the paper [33], where it is proven that any inverse limit

$$\Sigma_A = \varprojlim(\mathbb{T}^k, A) = \varprojlim \left\{ \mathbb{T}^k \xleftarrow{A} \mathbb{T}^k \xleftarrow{A} \cdots \xleftarrow{A} \mathbb{T}^k \xleftarrow{A} \cdots \right\}$$

w.r.t. a linear endomorphism or diffeomorphism $A : \mathbb{T}^k \to \mathbb{T}^k$ (in other words, the determinant of the matrix A must be distinct from zero) is homeomorphic to an attractor of a diffeomorphism of some analytic manifold.

3. Classification of Basic Sets of Codimension One

Let Λ be a basic set of codimension one of an A-diffeomorphism f of a closed n-dimensional manifold M^n, i.e., the topological dimension of the basic set Λ is $n-1$. According to [51, 64], Λ is an attractor or a repeller. Generally, unless otherwise stated, we will consider Λ as an attractor for the sake of definiteness. As soon as the dimension n of the supporting manifold equals 2, the topological classification of basic sets is obtained in [17–19, 52] (see [27] for an extensive bibliography). Therefore, we consider below the case $n \geq 3$. We restrict ourselves to attractors of codimension one whose Morse index is 1 or $n-1$. Since numbers are representatives in the group $\mathbb{Z}_n$ for $\pm 1 (mod\ n)$, we will identify the index $n-1$ with the index -1. The Morse index is called the *Morse index with unit module* if it equals 1 or $n-1$. Note that for $n = 3$, a basic set of codimension one always has a Morse index with unit module.

In [38], the inverse limit

$$\Sigma = \varprojlim(\mathbb{T}^k) = \varprojlim \left\{ \mathbb{T}^k \xleftarrow{A_1} \mathbb{T}^k \xleftarrow{A_2} \cdots \xleftarrow{A_m} \mathbb{T}^k \xleftarrow{A_{m+1}} \cdots \right\}$$

was considered (in fact, the construction considered in this paper was more general, but the cited one suffices for our purposes), and it was shown that if there is an infinite set of indexes m for which $|\det A_m| \geq 2$ for the determinant of the matrix A_m, then the inverse limit Σ cannot be embedded in any closed $(k+1)$-dimensional manifold. From this and Theorem 2.2 we obtain the following result.

Theorem 3.1. *Let $f : M^n \to M^n$ be an A-diffeomorphism of a closed n-dimensional manifold M^n, $n \geq 3$, and let Λ be an attractor of codimension one of the diffeomorphism f with the Morse index with unit module. Then exactly one of the following possibilities takes place*:

- *either* Λ *is an expanding attractor locally homeomorphic to the product of* $\mathbb{R}^{n-1}$ *by a Cantor set on the line*;
- *or* Λ *is homeomorphic to the* $(n-1)$*-dimensional torus* $\mathbb{T}^{n-1}$, *and the restriction of the diffeomorphism* $f|_\Lambda$ *is conjugate to an Anosov diffeomorphism of codimension one.*

Further we consider classification results in the class of structurally stable diffeomorphisms that have an attractor of codimension one with the Morse index with unit module. By the alternative given above, we first consider expanding attractors, and then those homeomorphic to tori (here we restrict ourselves to the case $n = 3$, where a complete classification is obtained).

3.1. Expanding attractors. A topological classification of structurally stable diffeomorphisms with orientable expanding attractors of codimension one on closed n-dimensional manifolds for $n \geq 3$ is obtained in [28–31].

Let $f : M^n \to M^n$ be a structurally stable diffeomorphism such that its nonwandering set contains an expanding orientable attractor Λ of topological dimension $(n-1)$. Then dim $W^s(x) = 1$ for any point $x \in \Lambda$, which allows one to introduce the notation $(y, z)^s$ ($[y, z]^s$) for an open (closed) arc of the stable manifold $W^s(x)$ bounded by points $y, z \in W^s(x)$.

The set $W^s(x) \setminus x$ consists of two connected components. At least one of these components has a nonempty intersection with the set Λ. A point $x \in \Lambda$ is called a *boundary point* if one of the connected components of the set $W^s(x) \setminus x$ does not intersect with Λ, we will denote this component by $W^{s\varnothing}(x)$. The set Γ_Λ of all boundary points of the set Λ is nonempty and consists of a finite number of periodic points that are divided into *associated* couples (p, q) of points of the same period so that the 2-*bunch* $B_{pq} = W^u(p) \cup W^u(q)$ is a boundary achievable from inside[1] of the connected component of the set $M \setminus \Lambda$.

For each couple (p, q) of associated boundary points of the set Λ, we construct the so-called *characteristic sphere*.

Let B_{pq} be a 2-bunch of the attractor Λ, consisting of two unstable manifolds $W^u(p)$ and $W^u(q)$ of associated boundary points p and q respectively, and let m_{pq} be the period of the points p, q. Then for any point $x \in W^u(p) \setminus p$ there exists a unique point $y \in (W^u(q) \cap W^s(x))$ such that the arc $(x, y)^s$ does not intersect with the set Λ. We define the mapping

$$\xi_{pq} : B_{pq} \setminus \{p, q\} \to B_{pq} \setminus \{p, q\}$$

by putting $\xi_{pq}(x) = y$ and $\xi_{pq}(y) = x$. Then $\xi_{pq}(W^u(p) \setminus p) = W^u(q) \setminus q$ and $\xi_{pq}(W^u(q) \setminus q) = W^u(p) \setminus p$, i.e., the mapping ξ_{pq} takes the pierced unstable manifolds of the 2-bunch into each other and is an involution ($\xi^2_{pq} = id$). By the theorem on continuous dependence of invariant manifolds on compact sets, the mapping ξ_{pq} is a homeomorphism.

The restriction $f^{m_{pq}}|_{W^u(p)}$ has exactly one hyperbolic repelling fixed point p; hence there exists a smooth closed $(n-1)$-disk $D_p \subset W^u(p)$ such that $p \in D_p \subset int(f^{m_{pq}}(D_p))$. Then the set $C_{pq} = \bigcup\limits_{x \in \partial D_p} (x, \xi_{pq}(x))^s$ is homeomorphic to a closed cylinder $\mathbb{S}^{n-2} \times [0, 1]$. The set C_{pq} is called a *connecting cylinder*. The circle $\xi_{pq}(\partial D_p)$ bounds in $W^u(q)$ a two-dimensional $(n-1)$-disk D_q such that $q \in D_q \subset int(f^{m_{pq}}(D_q))$. The set $S_{pq} = D_p \cup C_{pq} \cup D_q$ is homeomorphic to a $(n-1)$-dimensional sphere called the *characteristic sphere* corresponding to the bunch B_{pq} (see Fig. 5).

Put $T(f) = NW(f) \setminus \Lambda$ and formulate the main dynamic properties of the diffeomorphism $f \in G$ as a theorem.

Theorem 3.2. *Let* $f : M^n \to M^n$ *be a structurally stable diffeomorphism such that its nonwandering set contains an expanding orientable attractor* Λ *of topological dimension* $(n-1)$. *Then the following facts take place*:

[1]Let $G \subset M$ be an open set with boundary ∂G ($\partial G = cl(G) \setminus int(G)$). A subset $\delta G \subset \partial G$ is called *achievable from inside* of the domain G, if for any point $x \in \delta G$ there exists an open arc completely lying in G and such that x is one of its endpoints.

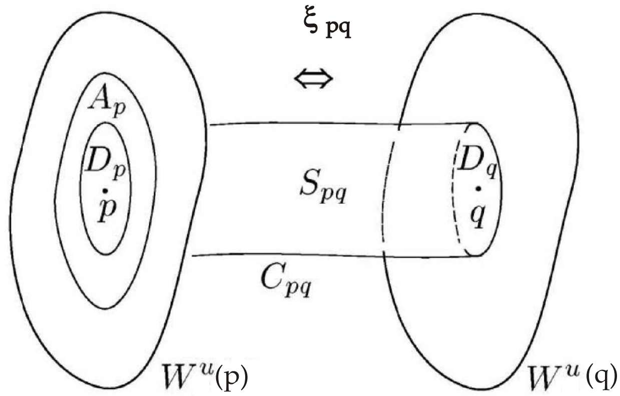

Fig. 5. Characteristic sphere.

(1) *the ambient manifold M^n is homeomorphic to the n-dimensional torus $\mathbb{T}^n$;*
(2) *each characteristic sphere S_{pq} bounds an n-ball Q_{pq} such that $T(f) \subset \bigcup\limits_{(p,q)\subset\Gamma_\Lambda} Q_{pq}$;*
(3) *for each associated couple (p,q) of boundary points, there exists a natural number k_{pq} such that $T(f) \cap Q_{pq}$ consists of k_{pq} periodic sources $\alpha_1, \dots, \alpha_{k_{pq}}$ and $k_{pq}-1$ periodic saddle points $P_1, \dots, P_{k_{pq}-1}$ alternate on the simple arc $l_{pq} = W^{s\varnothing}(p) \cup \bigcup\limits_{i=1}^{k_{pq}-1} W^s(P_i) \cup \bigcup\limits_{i=1}^{k} W^s(\alpha_i) \cup W^{s\varnothing}(q)$ (see Fig. 6).*

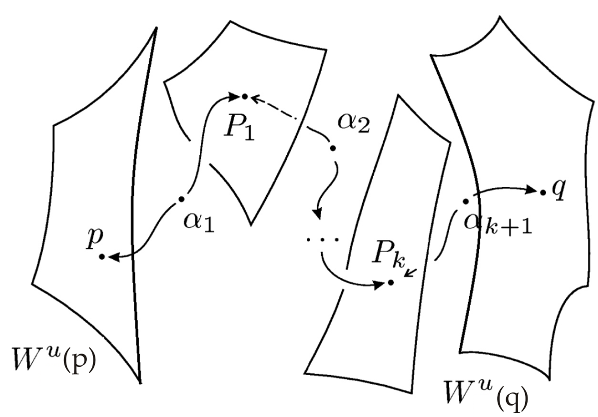

Fig. 6. Arc l_{pq}

After Theorem 3.2, a natural step in classification of structurally stable diffeomorphisms with basic sets of codimension one is classification of structurally stable diffeomorphisms with expanding attractors or contracting repellers of codimension one on the torus T^n, $n \geq 3$. Let $f : T^n \to T^n$ be such diffeomorphism. Assume for definiteness that f has an expanding attractor Λ of codimension one. Denote by

$$f_* : H_1(T^n, \mathbb{R}^n) \to H_1(T^n, \mathbb{R}^n)$$

the automorphism of the one-dimensional group of homologies $H_1(T^n, \mathbb{R}^n) \simeq \mathbb{R}^n$ of the torus T^n induced by the diffeomorphism f.

Theorem 3.3. *Let $f : T^n \to T^n$ be an A-diffeomorphism of the n-dimensional torus T^n, $n \geq 3$, possessing an orientable expanding attractor Λ of codimension one. Then f_* is a hyperbolic automorphism of codimension one.*

Following Franks [15], we will call a diffeomorphism $f : M \to M$ a π_1-*diffeomorphism* if for any homeomorphism $g : K \to K$ of a compact CW-complex K onto itself and a continuous mapping $h : K \to M$ such that the relation $f_* \circ h_* = h_* \circ g_*$ holds, there exists a unique mapping $h' : K \to M$ taking a basis point on K to a basis point on M, homotopic to h, and such that $f \circ h' = h' \circ g$.

By Theorem 3.3, there exists an algebraic automorphism $A(f) : T^n \to T^n$ with $f_* = A(f)_*$, which is hyperbolic. By [15, Proposition 2.1], a hyperbolic automorphism of the torus is a π_1-diffeomorphism. Hence there exists a continuous mapping $h : T^n \to T^n$, homotopic to an identical one and such that $h \circ f = A(f) \circ h$. We assume that

$$P(f, h) = \{x \in T^n | h^{-1}(x) \text{ contains more than one point}\}.$$

Lemma 3.1. *Let $f : T^n \to T^n$ be an A-diffeomorphism of the n-dimensional torus T^n, $n \geq 3$, possessing an orientable expanding attractor Λ of codimension one, and let $h : T^n \to T^n$ be a continuous mapping homotopic to the identity and such that $h \circ f = A(f) \circ h$. Then h satisfies the following conditions*:

- $h(\Lambda) = T^n$.
- *If $\{p_i, q_i\}_{i=1}^k$ is a family of couples of associated boundary periodic points of the diffeomorphism f, then $h(p_i) = h(q_i)$ is a periodic point of the automorphism $A(f)$ for each $i = 1, \ldots, k$.*
- $h(W^u(p_i)) = h(W^u(q_i))$, $i = 1, \ldots, k$.
- $P(f, h) = \bigcup_{i=1}^{k} h(W^u(p_i))$.
- *Let K_i be a component of the set $T^n \setminus \Lambda$; then $h(K_i)$ is the unstable manifold $W^u(h(p_i)) = W^u(h(q_i))$ of the automorphism $A(f)$, where p_i, q_i are associated boundary periodic points such that $\delta(K_i) = W^u(p_i) \cup W^u(p_i)$. Moreover, $h(K_i \cup \delta(K_i)) = W^u(h(p_i))$.*
- *Let $\breve{\Lambda} \subset \Lambda$ be the union of unstable manifolds that do not contain boundary periodic points*; *then the restriction $h|\breve{\Lambda}$ is a homeomorphism onto its range.*

Let Λ be an orientable expanding attractor of codimension one of the structurally stable diffeomorphism f, and let S_{pq} be the characteristic sphere that corresponds to the 2-bunch $B_{pq} = W^u(p) \cup W^u(q)$ of the attractor Λ, where p, $q \in \Lambda$ are associated boundary periodic points. One can show that inside the sphere S_{pq} there are $d \geq 1$ periodic points of index n and $d - 1 \geq 0$ periodic points of index $n-1$. Put $d(p, q) = d$. If Λ is a repeller, then we denote by $d(p, q)$ the number of periodic points of index 0 inside S_{pq}. In both cases the number $d(p, q)$ is well-defined, since it does not depend on the choice of the characteristic sphere S_{pq}. It is easy to see that points $f^j(p)$, $f^j(q)$ are associated boundary and periodic, and the number of periodic points of the same index inside the spheres S_{pq} and $f^j(S_{pq}) = S_{f^j(p), f^j(q)}$ is the same for any $j \in \mathbb{Z}$. Therefore, we can assign the number $d(O(p, q)) \stackrel{\text{def}}{=} d(p, q)$ to the union $O(p, q) = O(p) \cup O(q)$ of the orbits of points p, q.

Boundary periodic points are divided into couples $\{O(p_i, q_i)\}_{i=1}^k$ of orbits of associated boundary points. Let $\{d(O(p_i, q_i))\}_{i=1}^k$ be the corresponding numbers defined above that indicate the number of periodic points of index n (if Λ is an attractor) or of index 0 (if Λ is a repeller) in the corresponding characteristic spheres. By Lemma 3.1, $h(O(p_i)) = h(O(q_i))$ is a periodic orbit of the automorphism $A(f)$. Assign to each orbit $h(O(p_i)) = h(O(q_i))$ the number $d(O(p_i, q_i))$. The family $\{h(O(p_i)), d(O(p_i, q_i))\}_{i=1}^{k}$ is called the *d-signature of the diffeomorphism* f and denoted by $\mathcal{D}(f, h)$.

Let A be a hyperbolic automorphism of codimension one of the torus T^n, and let $\{O_j\}_{j=1}^{r}$ be a finite set of periodic orbits O_j of the automorphism A. To each orbit O_j we assign a natural number $d_j \in \mathbb{N}$ in an arbitrary way. The family $\{O_j, d_j\}_{j=1}^{r}$ is called an *admissible d-signature of the automorphism A*. By Theorem 3.2, the d-signature of the structurally stable diffeomorphism f is admissible.

Let $\{O_j^1, d_j^1\}_{j=1}^{r_1}$, $\{O_j^2, d_j^2\}_{j=1}^{r_2}$ be admissible d-signatures of hyperbolic automorphisms A_1 and A_2, respectively. These signatures are called *equivalent* if there exists a linear diffeomorphism (i.e., a composition of an automorphism and a shift) $\psi : T^n \to T^n$ such that $\psi(\bigcup_{j=1}^{r_1} (O_j^1)) = \bigcup_{j=1}^{r_2} (O_j^2)$, $d(\psi(O_j)) = d(O_j)$ for all $1 \le j \le r_1$, and there holds the relation $\psi \circ A_1 = A_2 \circ \psi$. It follows immediately from the definition that $r_1 = r_2$.

The next theorem solves the problem of topological conjugacy in the class of structurally stable diffeomorphisms on the torus T^n ($n \ge 3$) that have orientable expanding attractors or contracting repellers of codimension one. It shows that the d-signature is a complete invariant of conjugacy in this class of diffeomorphisms.

Theorem 3.4. *Let $f_1, f_2 : T^n \to T^n$ be structurally stable diffeomorphisms that have orientable expanding attractors Λ_1 and Λ_2 of codimension one, respectively. Then the diffeomorphisms f_1, f_2 are conjugate if and only if their d-signatures $\mathcal{D}(f_1, h_1)$, $\mathcal{D}(f_2, h_2)$ are equivalent, where $h_i : T^n \to T^n$ $(i = 1, 2)$ are continuous mappings homotopic to the identity and such that $h_i \circ f_i = A(f_i) \circ h_i$.*

The next theorem solves the implementation problem in the class of structurally stable diffeomorphisms on the torus T^n ($n \ge 3$) that have orientable basic sets of codimension one (expanding attractors or contracting repellers). Namely, given an admissible topological invariant (d-signature), a structurally stable diffeomorphism with this invariant is constructed.

Theorem 3.5. *Let $A : T^n \to T^n$ be a hyperbolic automorphism with an unstable foliation of codimension one in each leaf of the tangent foliation of the torus T^n, $n \ge 3$ (which means that the stable manifolds of all points are one-dimensional). For each admissible d-signature $\{O_j, d_j\}_{j=1}^{r}$ of the automorphism A there exists a structurally stable diffeomorphism $f : T^n \to T^n$ possessing an orientable expanding attractor of codimension one such that $\mathcal{D}(f, h) = \{O_j, d_j\}_{j=1}^{r}$, where $f_* = A_*$ and $h : T^n \to T^n$ is a continuous mapping homotopic to the identity and satisfying the relation $h \circ f = A \circ h$.*

As for nonorientable basic sets of codimension one, the following result holds (see [44, 67]).

Theorem 3.6. *Let $f : M^{2m+1} \to M^{2m+1}$ be a structurally stable diffeomorphism of a closed $(2m+1)$-dimensional manifold M^{2m+1}, $2m + 1 \ge 3$. Then the spectral decomposition of the diffeomorphism f does not contain nonorientable expanding attractors and contracting repellers of codimension one.*

The first example of a nonorientable basic set of codimension one that is an expanding attractor or a contracting repeller was constructed by Plykin [52] on the two-dimensional sphere S^2 (hence in this example the basic set of codimension one is one-dimensional). Note also that the diffeomorphism in Plykin's example is structurally stable. This example shows that Theorem 3.6 does not hold true in the dimension $n = 2$. For dimensions $2m \ge 4$, the question of existence of nonorientable expanding attractors and contracting repellers of codimension one for structurally stable diffeomorphisms remains open. Nevertheless, there exist Ω-stable diffeomorphisms with such basic sets [44, 67].

3.2. Surface basic sets. Let a diffeomorphism $f : M^3 \to M^3$ defined on a smooth closed orientable 3-manifold M^3 satisfy Smale's axiom A, and let the nonwandering set $NW(f)$ of the diffeomorphism f contain a two-dimensional surface basic set $\mathcal{B}$. Then according to Plykin, $\mathcal{B}$ is either an attractor or a repeller.

The following statements are proved in [32].

Theorem 3.7. *For any two-dimensional surface attractor (repeller) $\mathcal{B}$ of an A-diffeomorphism $f : M^3 \to M^3$ the following holds:*

- *$\mathcal{B}$ has type $(2,1)$ $((1,2))$ and hence is not an expanding attractor (contracting repeller);*
- *$\mathcal{B}$ coincides with its support, which is a union of finitely many manifolds tamely embedded[2] into M^3 and homeomorphic to the two-dimensional torus;*
- *the restriction of some power of the diffeomorphism f to any connected component of the support is conjugate to a hyperbolic automorphism of the torus.*

We will consider the class G of all A-diffeomorphisms $f : M^3 \to M^3$ whose nonwandering sets $NW(f)$ consist only of two-dimensional surface basic sets.

Let $f \in G$. Denote by $\mathcal{A}$ $(\mathcal{R})$ the union of all attractors (repellers) belonging to $NW(f)$. The next statement specifies the topology of the manifold M^3 (see [25]).

Lemma 3.2. *For each diffeomorphism $f \in G$, the sets $\mathcal{A}$ and $\mathcal{R}$ are nonempty, and the boundary of each connected component V of the set $M^3 \setminus (\mathcal{A} \cup \mathcal{R})$ consists of exactly one periodic component $A \subset \mathcal{A}$ and one periodic component $R \subset \mathcal{R}$. In this case the closure cl V is homeomorphic to the manifold $\mathbb{T}^2 \times [0,1]$.*

Thus the supporting manifold M^3 is homeomorphic to the factor space M_τ obtained from $\mathbb{T}^2 \times [0,1]$ by identification of points $(z,1)$ and $(\tau(z),0)$, where $\tau : \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism. Thus M_τ is a locally trivial bundle over the circle with fiber torus.

The next lemma is a well-known topological fact.

Lemma 3.3. *The manifold M_τ is homeomorphic to the manifold $M_{\widehat{J}}$, where $J \in GL(2,\mathbb{Z})$ is the matrix defined by the action of the automorphism $\tau_* : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2)$.*

We represent the manifold $M_{\widehat{J}}$ as the space of orbits $M_{\widehat{J}} = (\mathbb{T}^2 \times \mathbb{R})/\Gamma$, where $\Gamma = \{\gamma^k, k \in \mathbb{Z}\}$ is the group of powers of the diffeomorphism $\gamma : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ given by the formula $\gamma(z,r) = (\widehat{J}(z), r-1)$. Denote the natural projection by $p_{\widehat{J}} : \mathbb{T}^2 \times \mathbb{R} \to M_{\widehat{J}}$.

Denote by $\mathcal{C}$ the set of hyperbolic matrices from $GL(2,\mathbb{Z})$. For $C \in \mathcal{C}$, denote by $Z(\widehat{C})$ the centralizer $\widehat{C}$, i.e., $Z(\widehat{C}) = \{\widehat{J} : J \in GL(2,\mathbb{Z}), \widehat{C}\widehat{J} = \widehat{J}\widehat{C}\}$.

The next result is proved in [53].

Lemma 3.4. *The group $Z(\widehat{C})$, $C \in \mathcal{C}$, is isomorphic to the group $\mathbb{Z} \oplus \mathbb{Z}_2$.*

Put $Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $-Id = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathcal{J} = \mathcal{C} \cup Id \cup (-Id)$. Since $\widehat{C}$ and $-\widehat{C}$ belong to $Z(\widehat{C})$, Lemma 3.4 entails the following fact.

Lemma 3.5. *If $\widehat{J} \in Z(\widehat{C})$ for $C \in \mathcal{C}$, then $J \in \mathcal{J}$. Moreover, C and J have the same form in the following sense: $C = (-Id)^{j_C}\xi^{k_C}$ and $J = (-Id)^{j_J}\xi^{k_J}$, where $\xi \in \mathcal{C}$, $k_C, k_J \in \mathbb{Z}$, $j_C, j_J \in \{0,1\}$.*

The next theorem, proved in [25], distinguishes the set of all manifolds that admit diffeomorphisms from the class G.

---

[2] One should stress that the support of a two-dimensional surface set f may be nonsmooth at any point (a corresponding example is given in [39]), but it is not wild at any point. Recall that a C^0-mapping $g : B \to X$ is called a *topological embedding* of a topological manifold B into a manifold X if it homeomorphically maps B onto a subspace $g(B)$ with topology induced from X. In this case the image $A = g(B)$ is called a *topologically embedded manifold*. Note that a topologically embedded manifold is generally not a topological submanifold. If A is a submanifold, then it is called *tame or tamely embedded*; otherwise A is called *wild or wildly embedded* and the points where the conditions that define a topological submanifold do not hold are called *wildness points*. By the results of [47], a topological embedding of an orientable surface into a 3-manifold is tame if and only if it is cylindrical. Recall that a two-dimensional surface $S_g \subset W$ is called *cylindrically embedded into the 3-manifold W* if there exists a topological embedding $h : \mathbb{S}_g \times [-1,1] \to W$ such that $h(\mathbb{S}_g \times \{0\}) = S_g$, where $\mathbb{S}_g$ is a standard orientable two-dimensional surface of genus $g \geq 0$.

Theorem 3.8. *Let a manifold M^3 admit a diffeomorphism f from the class G. Then M^3 is diffeomorphic to the manifold $M_{\widehat{J}}$, where $J \in \mathcal{J}$.*

Remark 3.1. In [35] a similar conclusion on the structure of the manifold is obtained under the assumption that the manifold M^3 be irreducible (i.e., each two-dimensional sphere cylindrically embedded into M^3 bounds a three-dimensional ball in it) and admit a diffeomorphism $f : M^3 \to M^3$ with an *invariant Anosov torus* (i.e., diffeomorphisms with a smooth f-invariant submanifold homeomorphic to the torus and such that f induces hyperbolic action on its fundamental group). Note that Theorem 3.8 does not require irreducibility of the manifold M^3.

Let $MS(\mathbb{S}^1)$ be the class of structurally stable transformations of the circle, which coincides, according to Maier [42], with the class of Morse–Smale diffeomorphisms on $\mathbb{S}^1$. We divide $MS(\mathbb{S}^1)$ into two subclasses $MS_+(\mathbb{S}^1)$ and $MS_-(\mathbb{S}^1)$ consisting of orientation-preserving and orientation-changing diffeomorphisms, respectively. We formulate Maier's results on topological classification of structurally stable transformations of the circle.

Theorem 3.9.

(1) *For each diffeomorphism $\varphi \in MS_+(\mathbb{S}^1)$, the nonwandering set $NW(\varphi)$ consists of $2n, n \in \mathbb{N}$, periodic orbits of period k.*

(2) *For each diffeomorphism $\varphi \in MS_-(\mathbb{S}^1)$, the set $NW(\varphi)$ consists of $2q, q \in \mathbb{N}$, periodic points, among which two are fixed, and the rest have the period* 2.

Let $\varphi \in MS_+(\mathbb{S}^1)$. Enumerate the periodic points from $NW(\varphi)$: $p_0, p_1, \dots, p_{2nk-1}, p_{2nk} = p_0$, starting from arbitrary periodic point p_0 clockwise. Then $\varphi(p_0) = p_{2nl}$, where l is an integer such that $l = 0$ for $k = 1$, $l \in \{1, \dots, k-1\}$ for $k > 1$, and (k, l) are coprime[3]. Note that the number l does not depend on the point p_0. For $\varphi \in MS_-(\mathbb{S}^1)$ we set $\nu = -1$ if its fixed points are sources, $\nu = 0$ if they are a sink and a source, and $\nu = +1$ if they are sinks. Note that $\nu = 0$ if q is odd and $\nu = \pm 1$ if q is even.

Theorem 3.10.

(1) *Two diffeomorphisms $\varphi; \varphi' \in MS_+(\mathbb{S}^1)$ with parameters $n, k, l; n', k', l'$ are topologically conjugate if and only if $n = n'$, $k = k'$, and at least one of the following assertions holds*:
   - $l = l'$ *(in this case, if $l \neq 0$, then the conjugating homeomorphism is preserving orientation)*,
   - $l = k' - l'$ *(in this case, the conjugating homeomorphism is reversing orientation)*.

(2) *Two diffeomorphisms $\varphi; \varphi' \in MS_-(\mathbb{S}^1)$ with parameters q, ν; q', ν' are topologically conjugate if and only if $q = q'$ and $\nu = \nu'$.*

For $n, k \in \mathbb{N}$ and an integer l such that $l = 0$ for $k = 1$ and $l \in \{1, \dots, k-1\}$ for $k > 1$, we construct a standard representative φ_+ in $MS_+(\mathbb{S}^1)$ with parameters n, k, l. For $q \in \mathbb{N}$, $\nu \in \{-1, 0, +1\}$ we construct a standard representative φ_- in $MS_-(\mathbb{S}^1)$ with parameter q. Let us introduce the following mappings:

$\psi_m : \mathbb{R} \to \mathbb{R}$ is the shift by a time unit of the flow $\dot{r} = \sin(2\pi m r)$ for $m \in \mathbb{N}$;
$\chi_{k,l} : \mathbb{R} \to \mathbb{R}$ is the diffeomorphism given by the formula $\chi_{k,l}(r) = r - \frac{l}{k}$;
$\chi : \mathbb{R} \to \mathbb{R}$ is the diffeomorphism given by the formula $\chi(r) = -r$;
$\tilde{\varphi}_{n,k,l} = \psi_{n \cdot k} \chi_{k,l} : \mathbb{R} \to \mathbb{R}$;
$\tilde{\varphi}_{q,0} = \psi_q \chi : \mathbb{R} \to \mathbb{R}$ for odd q;
$\tilde{\varphi}_{q,+1} = \psi_q \chi : \mathbb{R} \to \mathbb{R}$ and $\tilde{\varphi}_{q,-1} = \tilde{\varphi}^{-1}_{q,+1} : \mathbb{R} \to \mathbb{R}$ for even q.

Set $\tilde{\Pi}_+ = \{\tilde{\varphi}_+ = \tilde{\varphi}_{n,k,l}\}$ and $\tilde{\Pi}_- = \{\tilde{\varphi}_- = \tilde{\varphi}_{q,\nu}\}$. One can check directly that $\tilde{\varphi}_\sigma(r + \mu) = \tilde{\varphi}_\sigma(r)$ for $\sigma \in \{+, -\}$ and $\mu \in \mathbb{Z}$. Hence, the following diffeomorphisms are well defined: $\varphi_\sigma = \pi \tilde{\varphi}_\sigma \pi^{-1} : \mathbb{S}^1 \to \mathbb{S}^1$. Set $\Pi_+ = \{\varphi_+\}$, $\Pi_- = \{\varphi_-\}$ and $\Pi = \Pi_+ \cup \Pi_-$. Denote by $\tilde{\phi}_\sigma : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ the product of the diffeomorphism $\tilde{\varphi}_\sigma \in \tilde{\Pi}_\sigma$ and automorphism $\widehat{C}$, $C \in \mathcal{C}$, i.e., $\tilde{\phi}_\sigma(z, r) = (\widehat{C}(z), \tilde{\varphi}_\sigma(r))$.

[3]Indeed, A. G. Mayer used the number r_1 instead of l such that $l \cdot r_1 \equiv 1 (mod\ k)$ and called it the *ordering number*.

Lemma 3.6. *The diffeomorphism $\tilde{\phi}_\sigma$ can be projected onto the diffeomorphism $\phi_\sigma : M_{\widehat{J}} \to M_{\widehat{J}}$ by the relation $\phi_\sigma = p_J \tilde{\phi}_\sigma p_J^{-1}$ if and only if*

- *$CJ = JC$ for $\sigma = +$,*
- *$J \in \{Id, -Id\}$ for $\sigma = -$.*

Thus, we get the following description of the models. Let $J_+ \in \mathcal{J}$ and $C_+ \in \mathcal{C}$ be such that $C_+J_+ = J_+C_+$. Let $J_- \in \{Id, -Id\}$ and $C_- \in \mathcal{C}$. Set $\tilde{\phi}_\sigma(z,r) = (\widehat{C}_\sigma(z), \tilde{\varphi}_\sigma(r))$. It is immediately verified that $\tilde{\phi}_\sigma\gamma_\sigma = \gamma_\sigma\tilde{\phi}_\sigma$ where $\gamma_\sigma(z,r) = (J_\sigma(z), r-1)$ is the generator of the group $\Gamma_\sigma = \{\gamma_\sigma^i, i \in \mathbb{Z}\}$. Then the following concept is well defined.

Definition 3.1. We say that the diffeomorphism $\phi_\sigma : M_{\widehat{J}_\sigma} \to M_{\widehat{J}_\sigma}, \sigma \in \{+,-\}$ is the *locally direct product* of $\widehat{C}_\sigma$ and φ_σ if $\phi_\sigma = p_{J_\sigma}\tilde{\phi}_\sigma p_{J_\sigma}^{-1}$ and write $\phi_\sigma = \widehat{C}_\sigma \otimes \varphi_\sigma$.

By Φ_+ (Φ_-) denote the set of all locally direct products ϕ_+ (ϕ_-). Thus, each diffeomorphism $\phi_+ \in \Phi_+$ is uniquely defined by parameters $\{J_+, C_+, n, k, l\}$ and each diffeomorphism $\phi_- \in \Phi_-$ is uniquely defined by parameters $\{J_-, C_-, q, \nu\}$. Set $\Phi = \Phi_+ \cup \Phi_-$. The following result provides algebraic criteria for topological conjugacy of the diffeomorphisms from Φ (see the proof in [24]).

Theorem 3.11.

(1) *Two diffeomorphisms $\phi_+, \phi'_+ \in \Phi_+$ with parameters $\{J_+, C_+, n, k, l\}$ and $\{J'_+, C'_+, n', k', l'\}$ are topologically conjugate if and only if $n = n'$, $k = k'$, there exists a matrix $H \in GL(2,\mathbb{Z})$ such that $C_+H = HC'_+$, and at least one of the following assertions holds*:
   - *$J_+H = HJ'_+$ and $l = l'$,*
   - *$J_+^{-1}H = HJ'_+$ and either $l = l' = 0$ or $l = k' - l'$.*

(2) *Two diffeomorphisms $\phi_-, \phi'_- \in \Phi_-$ with parameters $\{J_-, C_-, q, \nu\}$ and $\{J'_-, C'_-, q', \nu'\}$ are topologically conjugate if and only if $J_- = J'_-$, $q = q'$, $\nu = \nu'$, and there exists a matrix $H \in GL(2,\mathbb{Z})$ such that $C_-H = HC'_-$.*

(3) *There are no topologically conjugate diffeomorphisms $\phi_+ \in \Phi_+$ and $\phi_- \in \Phi_-$.*

Recall that (see, e.g., [12, 34]) a diffeomorphism g on M^3 is called[4] *partially hyperbolic* if there exists a continuous splitting of the tangent bundle $T_{M^3} = E^s \oplus E^c \oplus E^u$ invariant under the derivative Dg, where $dim\ E^s = dim\ E^c = dim\ E^u = 1$ and the strong expansion of the unstable bundle E^u and the strong contraction of the stable bundle E^s dominate any expansion or contraction on the center E^c. Herewith g is *dynamically coherent* if there are g-invariant foliations tangent to $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$ (and consequently there is g-invariant foliation tangent to E^c). Note that the constructed model is dynamically coherent if we can replace $\dot{r} = \sin(2\pi mr)$ by the vector field $\dot{r} = ln(\mu) \cdot \sin(2\pi mr)$ in the construction of $\phi_\sigma \in \Phi_\sigma$ above, where $\mu < |\lambda|$ and absolute values of eigenvalues of C_σ are $|\lambda|$ and $\frac{1}{|\lambda|}$. Thus, by Theorem 3.11, we get the following result.

Corollary 3.1. *Any diffeomorphism ϕ from the class Φ is topologically conjugate to some dynamically coherent diffeomorphism.*

Recall that two diffeomorphisms $f : M^3 \to M^3$ and $f' : M'^3 \to M'^3$ are called *ambient Ω-conjugate* if there exists a homeomorphism $h : M^3 \to M'^3$ such that $h(NW(f)) = NW(f')$ and $hf|_{NW(f)} = f'h|_{NW(f)}$. Next theorems are proved in [24].

Theorem 3.12. *Any diffeomorphism from the class G is ambient Ω-conjugate to some diffeomorphism from the class Φ.*

Theorem 3.13. *Any structurally stable diffeomorphism from the class G is topologically conjugate to some diffeomorphism from the class Φ.*

[4] More exactly, a diffeomorphism f is *partially hyperbolic* if there is $N \in \mathbb{N}$ and a Dg-invariant continuous splitting $TM^3 = E^s \oplus E^c \oplus E^u$ into one-dimensional subbundles such that $||Dg^N|_{E^s_x}|| < ||Dg^N|_{E^c_x}|| < ||Dg^N|_{E^u_x}||$ and $||Dg^N|_{E^s_x}|| < 1 < ||Dg^N|_{E^u_x}||$ for every $x \in M^3$.

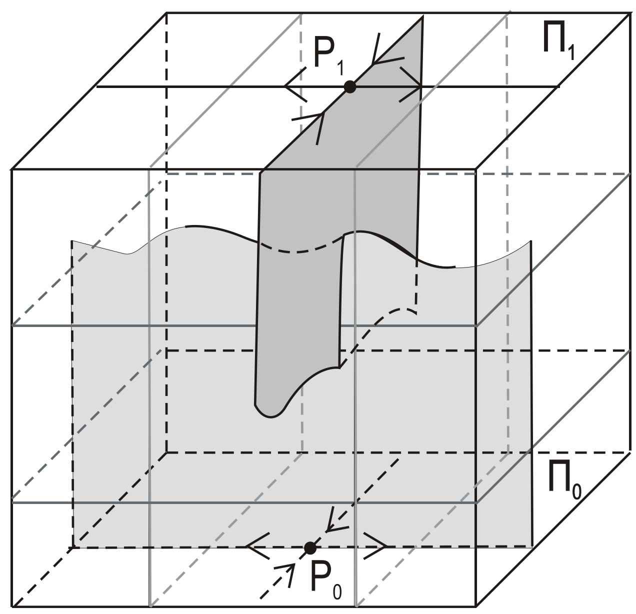

Fig. 7. Domains B^u and B^s.

Note that there exist diffeomorphisms in the class G that are not topologically conjugate to any diffeomorphism from the class Φ. The proof of the main classification Theorem 3.13 reduces to the proof of the existence of a one-dimensional foliation $\mathcal{I}_f$ for a structurally stable diffeomorphism f from the class G. Since this is the most nontrivial part of the theorem, we give an idea of its proof.

By Theorem 3.12, f is ambiently Ω-conjugate to some diffeomorphism $\phi : M_{\widehat{J}} \to M_{\widehat{J}}$ from the class Φ by means of a homeomorphism $h : M^3 \to M_{\widehat{J}}, J \in \mathcal{J}$. For our purposes it suffices to assume that ϕ belongs to Φ_+ and is determined by parameters $C \in \mathcal{C}$, $n \in \mathbb{N}, k = 1$, and $l = 0$ (otherwise one can choose an adequate power of f). Put $\psi = hfh^{-1} : M_{\widehat{J}} \to M_{\widehat{J}}$. In connection with the homeomorphism ψ, we may use the concepts and notations of stable and unstable manifolds of nonwandering points, understanding them as preimages w.r.t. h of the respective objects of the diffeomorphism f. By construction, ψ and ϕ coincide on the nonwandering set, and by Theorem 3.12 there exists a lifting $\tilde{\psi} : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ of the homeomorphism ψ, which coincides with $\tilde{\phi}$ on the set $\mathbb{T}^2 \times \left(\bigcup\limits_{i \in \mathbb{Z}} \frac{i}{2n} \right)$.

Denote by $p : \mathbb{R}^2 \to \mathbb{T}^2$ a universal covering such that $p(x, y) = (x(mod\ 1), y(mod\ 1))$, and by $\eta : \mathbb{R}^3 \to \mathbb{T}^2 \times \mathbb{R}$ the covering $\eta(x, y, z) = (p(x, y), z)$. Put $\eta_{\widehat{J}} = p_{\widehat{J}}\eta : \mathbb{R}^3 \to M_{\widehat{J}}$. Denote by $\check{\psi} : \mathbb{R}^3 \to \mathbb{R}^3$ the lifting $\tilde{\psi}$ w.r.t. η. For the stable (unstable) manifold $W^s(x)$ $(W^u(x))$ of the nonwandering point $x \in NW(\psi)$, denote by $w^s(\check{x})$ $(w^u(\check{x}))$ the connected component of the set $\eta_{\widehat{J}}^{-1}(W^s(x))$ $\eta_{\widehat{J}}^{-1}(W^u(x))$, passing through the point $\check{x} \in \eta_{\widehat{J}}^{-1}(x)$. Since each lifting $\check{C} : \mathbb{R}^2 \to \mathbb{R}^2$ of the diffeomorphism $\widehat{C}$ has the form $\check{C}(x, y) = (ax + by + \alpha, cx + dy + \beta)$ for some $\alpha, \beta \in \mathbb{Z}$, the homeomorphism $\check{\psi}$ has exactly one fixed saddle point P_i on the plane $\Pi_i = \mathbb{R}^2 \times \left\{ \frac{i}{2n} \right\}$ for each $i \in \mathbb{Z}$. Note that the homeomorphism $\check{\psi}|_{\Pi_i}$ possesses two transversal one-dimensional $\check{\psi}$-invariant layerings F_i^s, F_i^u on Π_i consisting of parallel lines with different irrational slopes μ_s and μ_u.

Put $N_\gamma^u(P_0) = \bigcup\limits_{\check{x} \in L_0^u(P_0)} w_\gamma^u(\check{x})$ $(N_\gamma^s(P_1) = \bigcup\limits_{\check{x} \in L_1^s(P_1)} w_\gamma^s(\check{x}))$ for some fixed $\gamma > 0$. Based on the continuous dependence of invariant manifolds, one can prove the existence of numbers $b_1^u, b_2^u, b_1^s, b_2^s$ such that the closed domain B^u (B^s) bounded by the planes $\Pi_{-1}, \Pi_1, Q_1^u = \{(x, y, z) \in \mathbb{R}^3 : y = \mu_u x + b_1^u\}$, and $Q_2^u = \{(x, y, z) \in \mathbb{R}^3 : y = \mu_u x + b_2^u\}$ $(\Pi_0, \Pi_2, Q_1^s = \{(x, y, z) \in \mathbb{R}^3 : y = \mu_s x + b_1^s\},\ Q_2^s =$

$\{(x,y,z) \in \mathbb{R}^3 : y = \mu_s x + b_2^s\}$) contains $N_\gamma^u(P_0)$ ($N_\gamma^s(P_1)$) in its interior. Then one can prove that $w^u(P_0) \cap w^s(P_1) \neq \varnothing$ (see Fig. 7) and, moreover, $(cl\ w^u(P_0)) \cap \Pi_1 = w^u(P_1)$ and $(cl\ w^s(P_1)) \cap \Pi_0 = w^s(P_0)$. By density of periodic points in the basic set, we make sure that for any point $x \in \Pi_0$ there exists a point $y \in \Pi_1$ such that $(cl\ w^u(x)) \cap \Pi_1 = w^u(y)$ and, conversely, for each point $y \in \Pi_1$ there exists a point $x \in \Pi_0$ such that $(cl\ w^s(y)) \cap \Pi_0 = w^s(x)$.

On the set $\mathbb{R}^2 \times [0, 1/2n)$ there exists a $\check{\psi}$-invariant two-dimensional foliation R_0 (R_1) such that each its leaf G_0 (G_1) is homeomorphic to a half-plane and coincides with $w^u(x) \cap (\mathbb{R}^2 \times [0, 1/2n))$ ($w^s(x) \cap (\mathbb{R}^2 \times [0, 1/2n))$) for some point $x \in \Pi_0$ ($x \in \Pi_1$). Since $cl\ (G_1) \cap \Pi_0 = \{(x,y,z) \in \Pi_0 : y = \mu_s x + b_{G_1}\}$ for some $b_{G_1} \in \mathbb{R}$, the intersection $Y = G_0 \cap G_1$ is nonempty for any leaves $G_0 \in R_0, G_1 \in R_1$ and $cl\ (Y) \setminus Y$ consists of two points $P^0_{G_0,G_1} \in \Pi_0, P^1_{G_0,G_1} \in \Pi_1$. Due to structural stability of the diffeomorphism f, any connected component of Y cannot bound a disk in G_0, since otherwise such a disk would be foliated by traces of intersections with leaves of the foliation R_1 and this foliation would necessarily have singularities, which would violate the strong transversality condition. Thus Y consists of a single curve z such that $cl\ z \cap (\Pi_0 \cup \Pi_1) = P^0_{G_0,G_1} \cup P^1_{G_0,G_1}$. This completes the proof.

4. Energy Function for Rough Diffeomorphisms with a Two-Dimensional Expanding Attractor

The most complete results in the field of construction of energy functions are obtained for *Morse–Smale systems*, which are structurally stable systems whose chain recurrent set consists of finitely many hyperbolic fixed points and periodic orbits. In 1961, Smale [59] proved the existence of an energy function being a Morse function of a *gradient-like flow* (Morse–Smale flow without closed trajectories). In 1968, Meyer [45] generalized this result and constructed an energy function that was a Morse–Bott function for an arbitrary Morse–Smale flow. Recall that a point $p \in M^n$ is called a *critical point* of a C^r-smooth ($r \geq 2$) function $\psi : M^n \to \mathbb{R}$ if in some local coordinates $x_1, \dots, x_n$ ($x_j(p) = 0$ for all $j = \overline{1,n}$) one has $\frac{\partial\psi}{\partial x_1}(p) = \cdots = \frac{\partial\psi}{\partial x_n}(p) = 0$ ($\operatorname{grad}\psi(p) = 0$). A critical point p is called *nondegenerate* if the matrix of second derivatives $\frac{\partial^2\psi}{\partial x_i \partial x_j}(p)$ (the Hessian matrix) is not degenerate; otherwise the point p is called *degenerate*. A function $\psi : M^n \to \mathbb{R}$ is called a *Morse function* if all its critical points are nondegenerate and is called a *Morse–Bott function* if the Hessian at each critical point is not degenerate in the normal direction to the critical level set.

In 1977, Pixton [49] established the existence of an energy function, which was a Morse function, for Morse–Smale diffeomorphisms on surfaces. Moreover, he constructed a diffeomorphism on a 3-sphere, having no energy function and showed that this effect was related to the wild embedding of separatrices of saddle points. Conditions for the existence of an energy function for Morse–Smale cascades on 3-manifolds were studied in [21, 22] . These studies made clear that many Morse–Smale cascades on 3-manifolds have no energy function.

In the next section, we give the results of [21] (see also [27]) concerning the criterion for the existence of an energy function in the basin of a one-dimensional attractor of a gradient-like 3-diffeomorphism.

4.1. Existence of an energy function in the basin of a one-dimensional attractor of a gradient-like 3-diffeomorphism. Let g be a Morse–Smale diffeomorphism on a manifold N, and a Morse function $\varphi : N \to \mathbb{R}$ be a Lyapunov function for g. By [49] (see also [27]), any periodic point β is the maximum of the restriction of φ to the unstable manifold W^u_β and the minimum of the restriction of φ to the stable manifold W^s_β. If these extrema are nondegenerate, then the invariant manifolds of the point β are transversal to all regular level sets φ in some neighborhood U_β of the point β. A Lyapunov function $\varphi : N \to \mathbb{R}$ for a Morse–Smale diffeomorphism $f : N \to N$ is called a *Morse–Lyapunov function* if any periodic point β is a nondegenerate maximum (minimum) of the restriction of φ to the unstable (stable) manifold $W^u(\beta)$ ($W^s(\beta)$).

Among the Lyapunov functions for a Morse–Smale diffeomorphism g, the Morse–Lyapunov functions form an open set dense in the C^∞-topology.

If β is a critical point of a Morse function $\varphi : N \to \mathbb{R}$, then by the Morse Lemma (see, e.g., [46]), in some neighborhood $V(\beta)$ of the point β there exists a local coordinate system $x_1, \dots, x_n$ called the *Morse coordinates* such that $x_j(p) = 0$ for each $j = \overline{1,n}$ and φ has the form $\varphi(x) = \varphi(\beta) - x_1^2 - \dots - x_b^2 + x_{b+1}^2 + \dots + x_n^2$, where b is the index[5] of the point β. If φ is a Lyapunov function for a Morse–Smale diffeomorphism $f : N \to N$, then by [49], one has $b = dim\ W^u(\beta)$ for each periodic point $\beta \in Per(g)$.

If φ is a Lyapunov function for a Morse–Smale diffeomorphism g, then any periodic point of the diffeomorphism g is a critical point of the function φ. The converse is generally not true: a Lyapunov function can have critical points that are not periodic points for g. It is proven by Grines, Laudenbach, and Pochinka [20] that the Lyapunov function in the Pixton example (see Fig. 8) has no less than six critical points.

Recall that a Morse–Smale diffeomorphism $g : N \to N$ is called *gradient-like* if for any couple of periodic points β, γ $(\beta \neq \gamma)$, the condition $W^u(\beta) \cap W^s(\gamma) \neq \varnothing$ implies that $\dim W^s(\beta) < \dim W^s(\gamma)$. The next definition distinguishes for gradient-like diffeomorphisms a class of Morse–Lyapunov functions with additional properties similar to those ones of functions introduced by Smale [59] for gradient-like vector fields.

A Morse–Lyapunov function φ for a gradient-like diffeomorphism g is called a *self-indexing energy function* if the following conditions hold:

(1) the set of critical points of the function φ coincides with the set $Per(g)$ of periodic points of the diffeomorphism g;
(2) $\varphi(\beta) = dim\ W^u(\beta)$ for each point $\beta \in Per(g)$.

Note that the concept of a Lyapunov function is well-defined on any g-invariant subset of the manifold N.

The following considerations deal only with three-dimensional manifolds.

Let $g : N \to N$ be a gradient-like diffeomorphism. Let $\Sigma^+(\Omega^+)$ be a subset of the set of saddle points with one-dimensional unstable invariant manifolds (sink points) and the set $A^+ = W^u(\Sigma^+) \cup \Omega^+$ be closed and g-invariant. Then A^+ is an attractor of the diffeomorphism g. The set

$$W^s(A^+) = \bigcup_{\beta^+ \in (\Sigma^+ \cup \Omega^+)} W^s(\beta^+)$$

is g-invariant and is called the *basin of the one-dimensional attractor* A^+. Denote by c^+ the number of connected components of the attractor A^+, by r^+ the number of saddle points and by s^+ the number of sink points in A^+. Put $\delta(A^+) = c^+ + r^+ - s^+$. The attractor A^+ is called *tightly embedded* if it has a neighborhood P^+ with the following properties:

(1) $g(P^+) \subset int\ P^+$;
(2) P^+ is a disjoint union of c^+ handlebodies[6] such that the sum of their genera equals $\delta(A^+)$;
(3) for each saddle point $\sigma^+ \in \Sigma^+$, the intersection $W^s_{\sigma^+} \cap P^+$ consists of one two-dimensional disk.

Theorem 4.1. *A self-indexing energy function φ_{A^+} of a diffeomorphism g exists in the basin $W^s(A^+)$ of the attractor A^+ if and only if this attractor is tightly embedded.*

A *tightly embedded repeller* A^- of a gradient-like diffeomorphism $g : N \to N$ and its basin are defined as a tightly embedded attractor A^+ and its basin for the diffeomorphism g^{-1}. In this case the function $\varphi_{A^-}(x) = 3 - \varphi_{A^+}(x)$ is a self-indexing function of the diffeomorphism g in the basin of the repeller A^-.

[5] *The index of a critical point* β is the number of negative eigenvalues of the matrix $\frac{\partial^2 \varphi}{\partial x_i \partial x_j}(\beta)$.

[6] *A handlebody of genus* $\delta \geq 0$ is a compact three-dimensional manifold with an edge obtained from a 3-ball by pairwise identification of 2δ two-dimensional pairwise disjoint disks on the boundary of the ball by means of an orientation-changing mapping.

In the aforementioned Pixton example the nonwandering set $g : \mathbb{S}^3 \to \mathbb{S}^3$ consists of exactly four fixed points: one source α, two sinks ω_1, ω_2, and one saddle σ. The one-dimensional attractor A^+ of this diffeomorphism coincides with the closure of the stable manifold of the saddle σ and $\delta(A^+) = 0$. In this case any three-dimensional ball containing the attractor A^+ in its interior intersects $W^s(\sigma)$ no less than by three connected components (see Fig. 8). Thus the attractor A^+ is not tightly embedded and by Proposition 4.1, there exists no energy function in the basin of the one-dimensional Pixton attractor.

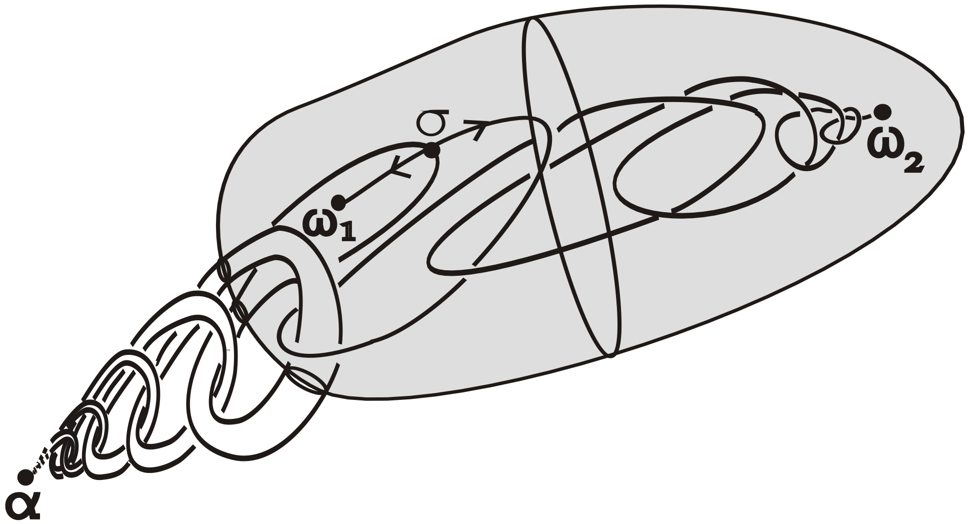

Fig. 8. The Pixton example.

4.2. Construction scheme. In this subsection, we give an idea of the proof of the following theorem (the detailed proof can be found in [26]).

Theorem 4.2. *For any structurally stable diffeomorphism $f : M^3 \to M^3$ such that its nonwandering set contains a two-dimensional expanding attractor Λ, there exists an energy function that is a Morse function outside of Λ.*

The proof of Theorem 4.2 is based on Theorems 3.2 and 4.1. Here we give an idea of the proof.

Let (p, q) be a couple of associated boundary points of period m_{pq} of the basic set Ω. Put $A^-_{pq} = \bigcup\limits_{j=0}^{m_{pq}-1} f^j(\bigcup\limits_{i=1}^{k_{pq}-1} W^s(P_i) \cup \bigcup\limits_{i=1}^{k_{pq}-1} W^s(\alpha_i))$. By construction, the set A^-_{pq} is a repeller of the diffeomorphism f and $\delta(A^-_{pq}) = 0$. Show that it is tightly embedded. For this purpose, it suffices to show that there exists a 3-ball P^-_{pq} such that $f^{-m_{pq}}(P^-_{pq}) \subset int\ P^-_{pq}$ and the intersection $P^-_{pq} \cap W^u(P_j)$ consists of exactly one two-dimensional disk for each saddle $P_j, j \in \{1, \ldots, k_{pq} - 1\}$.

Due to structural stability of the diffeomorphism f, each arc $(x, \varphi_{pq}(x))^s, x \in D_p \setminus p$, intersects $W^u(P_j)$ exactly at one point for all $j = 1, \ldots, k_{pq} - 1$. Indeed, assuming the converse, we find a point where the stable manifold of this point touches the unstable manifold $W^u(P_j)$, which contradicts the strong transversality condition. Thus the 3-ball Q_{pq} intersects the two-dimensional unstable manifold of the saddle $P_j, j \in \{1, \ldots, k_{pq} - 1\}$ at exactly one two-dimensional disk. The required 3-ball P^-_{pq} is obtained from Q_{pq} by pressing the disks D_p, D_q inside and smoothing the angles (see Fig. 9).

By Lemma 4.1, in the basin $W^u(A^-_{pq})$ of the repeller A^-_{pq} there exists a self-indexing energy function $\varphi_{A^-_{pq}}$ of the diffeomorphism f. Put $b_{pq} = inf\{\varphi_{A^-_{pq}}(z), z \in W^u_{A^-_{pq}}\}$. Define a function $g_{pq} : (b_{pq}, 3] \to (0, 3]$ in the following way: if $b_{pq} > -\infty$, then we set $g_{pq}(x) = 2^{\frac{(2-b_{pq})(3-x)}{x-b_{pq}}} 3^{\frac{(3-b_{pq})(x-2)}{x-b_{pq}}}$ and, if $b_{pq} =$

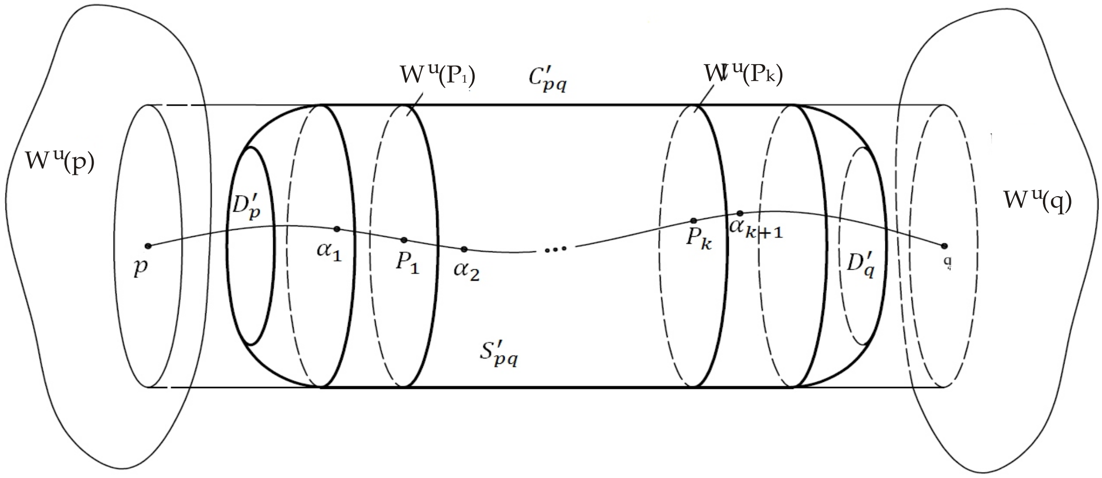

Fig. 9. Neighborhood P^-_{pq}

$-\infty$, then $g_{pq}(x) = 2^{3-x}3^{x-2}$. By construction, the function g_{pq} is infinitely smooth, has a positive derivative, $g_{pq}(2) = 2$, $g_{pq}(3) = 3$, and $\lim\limits_{x\to b_{pq}} g_{pq}(x) = 0$. Consider the superposition $\varphi_{pq} = g_{pq}\varphi_{A^-_{pq}}$. Since $\operatorname{grad}\varphi_{pq} = g'_{pq} \cdot \operatorname{grad}\varphi_{A^-_{pq}}$ and the Hessians $\Delta\ \varphi_{pq}$ and $\Delta\ \varphi_{A^-_p q}$ are connected by the relation $\Delta\ \varphi_{pq} = g''_{pq} \cdot (\operatorname{grad}\varphi_{A^-_{pq}}) \cdot (\operatorname{grad}\varphi_{A^-_{pq}})^{\mathrm{T}} + g'_{pq} \cdot \Delta\ \varphi_{A^-_{pq}}$, the function φ_{pq} is a Morse energy function for f in the basin $W^u_{A^-_{pq}}$.

Set $A^- = \bigcup\limits_{(p,q)\subset\Gamma_\Lambda} A^-_{pq}$, $W^u(A^-) = \bigcup\limits_{(p,q)\subset\Gamma_\Lambda} W^u(A^-_{pq})$ and denote by φ_{A^-} the function composed of the functions φ_{pq}, $(p,q) \subset \Gamma_\Lambda$. Define a function φ on the manifold M^3 by the formula

$$\varphi(z) = \begin{cases} \varphi_{A^-}(z), \text{ if } z \in W^u(A^-); \\ 0, \text{ if } z \in \Lambda. \end{cases}$$

By construction, the function φ is a Lyapunov function for the diffeomorphism f; moreover, it is a Morse function on $M^3 \setminus \Lambda$. The required energy function is the superposition $\psi = g\varphi$, where $g : [0,3] \to [0,3]$ is a C^2-smooth function constructed as follows.

Let d be a Riemannian metric on the manifold M^3, and the distance between sets be defined as the infimum of distances between the elements of these sets, i.e., $\forall X, Y \subset M:\ d(X,Y) = \inf\{d(x,y) : x \in X, y \in Y\}$. For $c \in (0,3]$, put $\alpha(c) = \min\{1, d^2(\varphi^{-1}(c), \Lambda)\}$ and $\beta(c) = \max\{1, \max\limits_{x\in\varphi^{-1}([c,3])} |\operatorname{grad}\varphi(x)|\}$. By construction, the functions $\alpha(c)$ and $\beta(c)$ are continuous, $\alpha(c)$ is nondecreasing on $(0,3]$ and there exists a value $c^* \in (0,3]$ such that $\alpha(c)$ monotonically increases on $(0,c^*]$, and $\beta(c)$ is nonincreasing. Then the function $\frac{\alpha(c)}{\beta(c)}$ is nondecreasing on the half-interval $(0,3]$ and $\lim\limits_{c\to 0} \frac{\alpha(c)}{\beta(c)} = 0$.

Using a partition of unity, we construct a C^2-smooth function $g : [0,3] \to [0,3]$ such that

(1) $g'(c) > 0$ for any $c \in (0,3]$;
(2) $g(c) \le \frac{\alpha(c)}{\beta(c)}$ for any $c \in (0,1/2]$;
(3) $g'(c) \le \frac{\alpha(c)}{\beta(c)}$ for any $c \in (0,1/2]$;
(4) $g(2) = 2$ and $g(3) = 3$.

Since $\operatorname{grad} \psi = g' \cdot \operatorname{grad} \varphi$ and the Hessians $\Delta\, \psi$ and $\Delta\, \varphi$ are related by $\Delta\, \psi = g'' \cdot (\operatorname{grad} \varphi) \cdot (\operatorname{grad} \varphi)^{\mathrm{T}} + g' \cdot \Delta\, \varphi$, the function ψ is a Morse energy function for f on the set $M^3 \setminus \Lambda$. By construction, ψ is smooth on the whole manifold M^3.

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