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Stochastic Logistic Model of the Global Financial Leverage

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Abstract:
Debt, as one of basic human relations, has profound effects on economic growth. Debt accumulation in the global economy was modeled by the stochastic logistic equation reflecting causality between leverage and its rate of change. The model, identifying interactions and feedbacks in aggregate behaviour of creditors and borrowers, addressed various issues of macrofinancial stability. Qualitatively diverse patterns, including the Wicksellian (normal) market, the Minsky financial bubbles and the Fisherian debt-deflation, were discerned by appropriate combinations of rates of return, spreads and leverage. The Kolmogorov-Fokker-Plank equation was used to find out the stationary gamma distribution of leverage that was instrumental for the evaluation of appropriate failure and survival functions. Two patterns corresponding to different forms of a stationary gamma distribution were recognized in the long run leverage dynamics and were simulated as scenarios of a possible system evolution. In particular, empirically parameterized asymptotical distribution indicated excessive leverage and unsustainable global debt accumulation. It underlined the necessity of comprehensive reforms aiming to decrease uncertainty, debt and leverage. Assuming these reforms were successfully implemented, global leverage distributions would have converged in the long run to a peaked gamma distribution with the mode identical to the anchor leverage. The latter corresponded to a balanced long run debt demand and supply, hence to fairly evaluated financial assets fully collateralized by real resources. A particular case of macrofinancial Tobin's q-coefficients following the Ornstein-Ulenbeck process was studied to evaluate a reasonable range of squeezing the bloated world finance. The model was verified on data published by the IMF in *Global Financial Stability Reports* for the period 2003–2013.

Keywords: leverage, logistic equation, debt collateral, gamma distribution, hazard function

JEL classification: C2, E4, E5, G1, N2

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Entia non sunt multiplicanda
praeter necessitatem
Lex parsimoniae

1 Introduction

Financial intermediation, intertwined with real markets activity, has for many centuries been one of the major factors of economic development and growth. Historical experience of countries with sophisticated financial markets has convincingly demonstrated their advantages over economies without such structures. According to statistics of the International Monetary Fund (IMF), summarized in Table 1, contemporary world finance is an immensely huge and complex system. The total amount of the world financial assets in the year 2013, about \$283 trillion, was almost four times the world GDP. These staggering numbers reflected the functioning of a bewilderingly complicated network of banks, companies, markets and instruments encompassing actions of innumerable producers, consumers, investors, creditors and borrowers operated on different financial and real markets.

Table 1: Global financial system in 2003–2013 (US dollars, trillion).

Years	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013

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Total Assets, A_t	128.3	144.7	151.8	190.4	229.7	214.4	232.2	250.1	255.9	268.6	282.8
Stocks, e_t	31.2	37.2	37.2	50.8	65.1	33.5	47.2	55.1	47.1	52.5	62.6
Debts, x_t^1	52.0	57.9	58.9	68.7	79.8	83.5	92.1	94.8	98.4	99.1	99.8
Bank assets, x_t^2	40.6	49.6	55.7	70.9	84.8	97.4	93.0	100.1	110.4	117.0	120.4
World GDP, Y_t	36.2	40.9	44.5	48.2	54.5	60.9	57.8	62.9	69.9	72.2	74.6

Source: IMF, Global financial stability report, annual issues (International Monetary Fund 2004–2014).

The global financial system is beyond the scope of any supranational regulation; even the well-coordinated efforts of major central banks were only able to produce very limited results. It has been expanded as if governed by its own laws, exclusively, though, since time immemorial, financial practice, in order to diminish risks, tried to balance financial assets with their collaterals. Being subordinated to the real economy in the long run, a mere "contrivance for sparing of time and labour", in the words of John Stuart Mill (1909, III, 7.8), money and finance has been transformed into a dominant force. It made persistent debt accumulation, probably, the most prominent feature of the world economic system. Global debt, growing by 5.3 percent annually in 2007–2014 according to McKinsey Global Institute (2015) and outpacing the world GDP, pushed the aggregate leverage persistently up, and became one of the major economic concerns.

Debt, as one of the fundamental relations in the economy, has profound effects on all of humanity and has, naturally, been studied extensively. It was treated within the broad philosophical context comprising of different social activities (Anderson 2014). In particular, leverage, as a general measure of financial intermediation and indebtedness, has attracted a lot of attention recently. An excessive leverage had been considered among the major culprits of the credit crunch 2007–2008 (Cassidy 2009), and since then vivid discussions were going on in both the academic and banking communities about its systemic monitoring and management (Holmstrom 2015). Modeling financial leverage has become a noticeable feature of modern academic research (Adrian and Shin 2010; Geanakoplos 2010; Peters 2010; Thurner, Farmer, and Geanakoplos 2012; Aumanns et al. 2015); and logistic models in economics and finance drew attention recently (Mao, Marion, and Renshaw 2000; Solomon and Richmond 2001; Kwasnicki 2013). In our opinion, these works, by revealing important relationships and feedbacks in leveraged financial processes, has stressed the necessity for a general systemic approach to understanding their complicated dynamics. Macrofinance, an integral part of it, has its own domain of research since the behaviour of vast aggregates of borrowers and creditors deviates, and sometimes dramatically, from the activity of any particular market entity.

This paper is an attempt, by means of studying basic relations among leverage, its rate of change, various rates of return and uncertainty, to recognize regular patterns in the behaviour of macrofinancial systems. In particular, consistent analysis of interactions, feedbacks and uncertainty in the global leverage dynamics was facilitated by application of deterministic and stochastic logistic models. Models of leverage were verified on the IMF statistics about the global financial system for the period of 2003–2013 and were illustrated with numerical simulations using *Wolfram Mathematica 10*.

Asymptotics of random leverage was modeled via stationary gamma distribution that helped to identify some pivots in its stable behaviour, including anchor leverage, collateral and debt-to-capital ratios. Survival and failure functions of stationary gamma distribution indicated different scenarios of global debt accumulation, including sustainable in the long term, associated with much lower, relative to contemporary, leverage. Conditions of convergence to a stationary leverage distribution were studied in detail in order to evaluate benchmarks for implementation effective assets/debt management and control. Assuming macrofinancial equity-to-capital coefficients, analogous to Tobin's q, follow the Ornstein-Uhlenbeck process, the "true value" of global financial assets was estimated.

The paper deals with continuous logistic models since enormous size, plentiful instruments and liquid markets made assumptions of continuity (and differentiability in a deterministic case) of macrofinancial processes quite plausible[1]; in this respect the discrete sample of Table 1 merely reflects our restricted ability to measure macrofinancial processes. Hence presently empirical estimations are of tentative character and they serve rather as numerical illustrations intended to make the model more perceptible and convincing.

Continuous logistic models, proposed in economic literature, proved to be very demanding conceptually. Had this requirement been satisfied, these models provided better understanding of nonlinear feedbacks engendered by interactions of aggregate creditors and borrowers in uncertain debt processes (Smirnov 2012, 2014, 2016). Run-of-the-mill assertion should be added, though: as a rule, advantages of macrofinancial logistic modeling came at a cost of abstracting away, in the short run especially, from many realistic features and factors of diversified markets.

2 Logistic Hypothesis Testing

Empirical evidence *pro et contra* for logistic hypothesis was checked in standard statistical tests of a linear relationship between global leverage, l_t, and its relative rate of change, $\Delta l_t / l_t$:

$$\Delta l_t / l_t = \widehat{a} - \widehat{b} l_t + \varepsilon_t \tag{1}$$

where $\widehat{a}, \widehat{b}$ are regression parameters and $\varepsilon_t$is the normally and independently distributed noise factor of the debt accumulation process. Equation (1) reflects cumulative effects, $\Delta l_t / l_t$, as they are captured by empirical data in the subsequent period, of interactions between aggregates of borrowers and creditors that formed debt and leverage, l_t, in the preceding period.

Statistics to be tested are given in columns 2 and 3 of Table 2 that was compiled on the IMF data represented in Table 1. Empirical leverage values, l_t, in Table 2 were considered as observations of a random sample from its general population, and their first differences were calculated as $\Delta l_t = l_{t+1} - l_t$. The sample was kept deliberately small in order to preserve relative homogeneity of general economic conditions: during the period under consideration only one major disruption took place, namely the "credit crunch" of 2008–2009, and it was largely attributed to the excesses of financial leverage.

Table 2: Leverage, its relative rate of change, spreads and capital intensity.

Years	Leverage, l_t	Relative Rate, $\Delta l_t / l_t$	Capital Intensity, w_t	Relative Rate, $\Delta w_t / w_t$	Spread, $a_t = \mu_t - r_t$	Spread, $c_t = \rho_t - r_t$	Parameter, $b_t = a_t^2 / c_t$
2003	3.97	−0.020	0.252	0.02			
2004	3.89	0.049	0.257	−0.047	0.0207	0.0852	0.005
2005	4.08	−0.083	0.245	0.089	−0.0169	−0.066	−0.0043
2006	3.74	−0.056	0.267	0.059	0.0363	0.1474	0.0088
2007	3.53	0.813	0.283	−0.449	0.0273	0.1024	0.0072
2008	6.4	−0.231	0.156	0.301	−0.1656	−0.5844	−0.0468
2009	4.92	−0.077	0.203	0.084	0.0598	0.3858	0.0093
2010	4.54	0.196	0.220	−0.164	0.0242	0.1145	0.0051
2011	5.43	−0.057	0.184	0.059	−0.0481	−0.2165	−0.0107
2012	5.12	−0.117	0.195	0.133	0.0146	0.0797	0.0027
2013	4.52		0.221		0.033	0.1734	0.0063

Empirically given sequence of pairs $\{\Delta l_t / l_t ; l_t\}, t = 1, 2, ..., 10$ suggested that the variate $\Delta l_t / l_t$ might randomly depend upon the variate l_t. Indeed, observations of global leverage and its relative rate of change in 2003–2013 were correlated with a coefficient of correlation, $R = -0.5652$.

A non-parametric test was performed on the bivariate sample with null hypothesis $H_0 : R = -0.5652$ against its alternative $H_a : R \neq -0.5652$. The *CorrelationTest* procedure of *Mathematica10* recovered a p-value, $p = 0.4848$, associated with the hypothesis H_0; it was large enough not to reject the existence of a strongly non-zero correlation between global leverage and its relative rate of change.[2] For comparison, the null hypothesis of a zero correlation between the same observations recovered a small p-value ($p = 0.0133$) and was rejected.

represented in Figure 1.

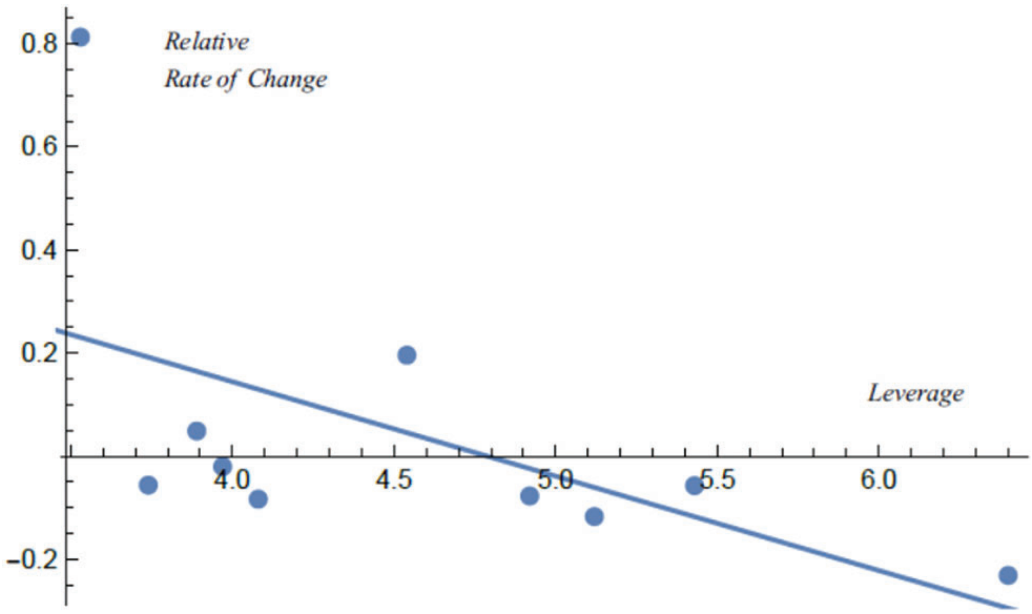

Figure 1: Linear regression of leverage and its relative rate of change.

The linear regression was tested against the null hypothesis, $H_0 : \widehat{b} = 0$, assuming negative feedbacks in the debt market (parameter $\hat{b}$). In a one-sided test the null hypothesis was rejected at a 10 % significance level (the critical value of the *t-statistic* with 8 degrees of freedom was 1.397). Thus the hypothesis of a linear regression between leverage and its relative rates of change did not contradict to the empirical information about the process under consideration. The test results are given in Table 3.

Table 3: Characteristics of the linear model fit.

	Estimate	Standard error	t statistic	p value
$\widehat{a}$	0.877112	0.438707	1.99931	0.080602
$\widehat{b}$	−0.183159	0.0945117	−1.93795	0.0886262

The simple statistical exercise pursued a limited goal. It was shown, on one hand that empirical evidence did not impede exploration of the logistic hypothesis notwithstanding deficiencies engendered by a small sample. However, direct calibration of the model (1) did not contribute to its economic cogency: debt process was explained in too general terms to be convincing. In particular, regression coefficients bore no resemblance to the rates of return nor contained any recipe of their recovering out of the empirical data. Yet the core of financial dynamics is constituted by combinations of leverage and rates of return (or their spreads). Modeling various configurations among leverage and rates of return seems to be the most promising way of providing conclusive explanations with regard to underlying processes.[3] To facilitate analysis along this avenue, the logistic hypothesis has to be consistently reconstructed from scratch in economic terms.

3 Basic Macrofinancial Rates and Equations

Logistic modeling of a continuous macrofinancial system was started with the following assertion of (Goldsmith 1959, 116): "The two main – and related – tools of comparative financial morphology and dynamics are sectoral balance sheets for stocks and sources-and-uses-of-funds statements for each sector for flows". Accordingly, data in Table 1 was structured as a sequence of scalar financial balances:

$$A(t) = x(t) + e(t) \tag{2}$$

including variables of assets, $A(t)$; debt, $x(t) = x^1(t) + x^2(t)$; and capital, $e(t)$. Due to relations: $\lim_{T\to t} A(t,T) = A(t)$ and $\lim_{T\to t} x(t,T) = x(t)$ where T is the time to maturity, aggregate debt and financial assets were considered functions of astronomical time, t, only. Aggregate financial flows, subject to the balance between saving and investment, were represented as:

$$dA(t) = dx(t) + de(t) \tag{3}$$

where d was the operator of taking differentials.

Actual data from Table 1 was used to evaluate three instantaneous (in terms of future value) rates of return: on assets, $\mu = dA(t)/A(t)dt$, on aggregate debt, $r = dx(t)/x(t)dt$, and on equity, $\rho = de(t)/e(t)dt$. These rates, because of ignoring the "time-to-maturity" coordinate and due to the impossibility to fix a "macrofinancial portfolio", are inexact analogues to the widely used yields on assets, on investment, and on equities. Denoted respectfully as *ROA* (red curve), *ROI* (green curve) and *ROE* (blue curve) structural parameters of macrofinancial system are reproduced in Figure 2.

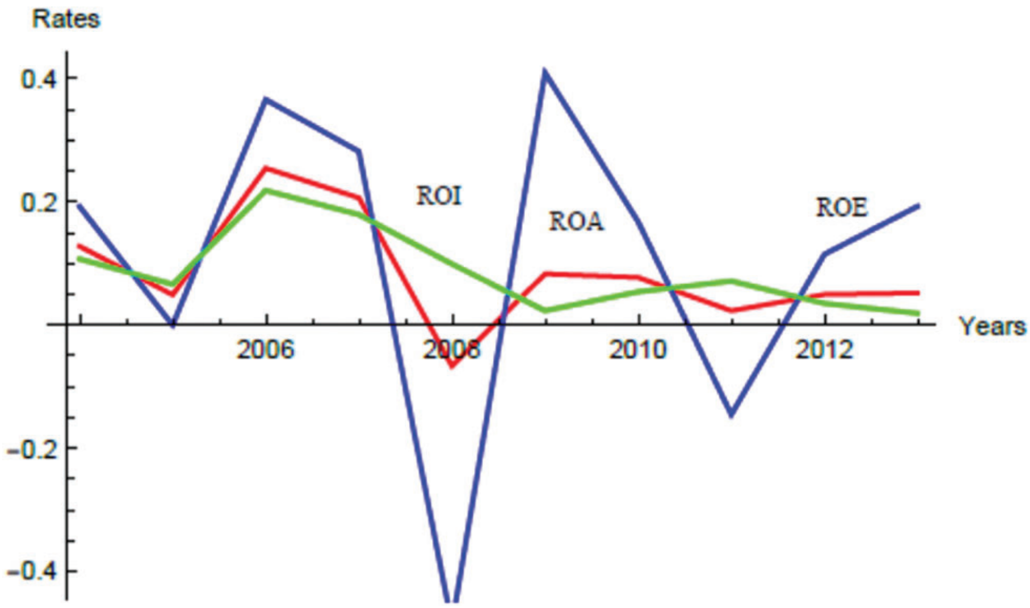

Figure 2: Global rates of return (computed on the *GFSR* data).

As seen, empirical macrofinancial rates could be positive or negative. For example, the "credit crunch" of 2008 took place for both negative *ROA* and *ROE* while negative *ROE* alone corresponded to the market correction in 2011. Parameter *ROI* remained positive though decreasing persistently over the analysed period.

The balance of financial flows (3) being parameterized by rates of return, takes a simple form:

$$\mu A = rx + \rho e \tag{4}$$

Given parameters and a vector of any two known variables, eqs (2) and (3) could be solved simultaneously for any year, as was confirmed by the data in Table 1. Yet it is more appealing to get a scalar representation of system (2–3) by introducing macrofinancial leverage – a ratio of assets to equities:

$$l(t) = A(t)/e(t) \text{ and/or } x(t)/e(t) = l(t) - 1 \tag{5}$$

as a major variable reflecting the process of debt accumulation. Thus, it is easy to see (omitting time for the moment) that the combined actions of aggregates of creditors and borrowers in a balanced financial market are given by the linear equation:

$$\rho = r + (\mu - r)l \tag{6}$$

where $\rho \geq \mu$ for any $1 \leq l < \infty$. Given parameters r, μ, ρ the root of eq. (6)

$$l^* = \left(\frac{\rho - r}{\mu - r}\right) \equiv \frac{c}{a}$$

represents macrofinancial leverage as a ratio of two spreads, or relative expected net revenue of aggregate capital holders, $c = \rho - r$, and of aggregate asset holders, $a = (\mu - r)$. Spreads are negative, simultaneously, in the periods of critical market turbulence when leverage multiplies losses of capital and asset holders. It was, once again painfully, shown in the last credit crunch.[4] Empirical value of financial spreads were given in Table 2.

The microfinancial analogue of eq. (6):

$$\rho = \mu + (\mu - r)[l - 1] \tag{7}$$

is widely used by market participants to boost their *ROE* for any given positive spread $(\mu - r)$. Equation (6) appears in the Modigliani-Miller Theorem, and, implying $r = 0$, it constitutes the widespread DuPont model, $\rho = \mu l$.

If parameter $r \equiv ROI$ is an investment return, then parameters $\mu \equiv ROA$ and $\rho \equiv ROE$are related to the standard rates like yield-to-maturity (YTM), γ, and current yield, δ. The rate return on investment is equal to the difference between *YTM* and current yield, or $dx/x \equiv r = \gamma - \delta$, in the linear debt model: $dx(t) = (\gamma - \delta)x(t)dt$. Comparing the latter to model (6) ties up different rates into the following relation:

$$\gamma - \delta = (l - 1)^{-1}[\mu l - \rho] \tag{8}$$

Relations, established above, incorporated consistently leverage and different rates of return into the structure of a macrofinancial process thus helping to explain its behavior in standard terms of financial analytics and practice. Now, let us show that the root of eq. (6) coincides with the stationary solution to the dynamic logistic model.

4 Logistic Leverage and its Phase Portrait

Persistent interactions of collective borrowers and creditors gave rise to the emergence of feedbacks among assets, debt and capital. In the continuous context their decisions about instantaneous (expected) leverage could be modeled by a simple linear equation:

$$dl(t) = [\mu - \rho]\, l(t)dt \tag{9}$$

that appears after differentiation of leverage, $l(t)$, with respect to time. Equation (9) connects current leverage to its expected instantaneous rate of change, but has a trivial (zero) stationary solution. However, historically, even the Dutch *Wisselbank* (1603) took non-zero deposits, having them kept fully reserved though (Galbraith 1975; Ferguson 2009). In 2003–2013 macrofinancial leverage was no smaller than 3.9. To preserve this important feature, the point corresponded to stationary leverage was translated onto the balanced financial market line:

$$l^* \equiv K = a/b = c/a; \quad l^*/gt1 \tag{10}$$

after substituting an expression (6) for parameter ρ into eq. (9). These operations transformed (8) into a nonlinear two-parametric, or logistic, model of leverage:

$$dl(t)/dt = [a - b\,l(t)]\,l(t)\,; \;\; l(0) = l_0 \tag{11}$$

Precise economic meaning of the model parameters, explained in Section 3, makes ODE (10) much more informative than its analogue (1). Parameter $a = \mu - r$, associated with expected relative revenue of asset holders, defines a tendency of leverage to grow (or decrease) while parameter, $b = (\mu - r)^2/(\rho - r)$, measures intensity of feedbacks between current and future (expected) leverage. Since leverage is a positive number, parameters $\{a, b\}$are either positive or negative, simultaneously. By the model construction one of its steady states is located to the right of the point $l = 1$; the other steady state, $l_1 = 0$, is associated, by definition, with the market collapse.

The model (10) was illustrated by the phase diagram (Figure 3) drawn on empirical data of the "credit crunch" 2007–2008 given in Table 2. In the pre-crisis year 2007 parameters $\{a, b\}$ were positive, and the RHS of eq. (11), viewed as a function of the leverage, $f(l) = 0.0273\,l - 0.0072l^2$, was represented by the upper curve in Figure 3. At the stationary point $l^* = a/b = 3.79$the Jacobian of the dynamical system (10) is negative, $\partial f/\partial l = -0.0273 < 0$, and, in the vicinity of the stationary point, its behaviour is stable. Negative feedbacks forced the leverage to converge to its stationary value though difficulties for borrowers overloaded with large debts were increased and financial frictions were intensified. Thus, arrows along the upper curve were pointed towards stationary leverage.

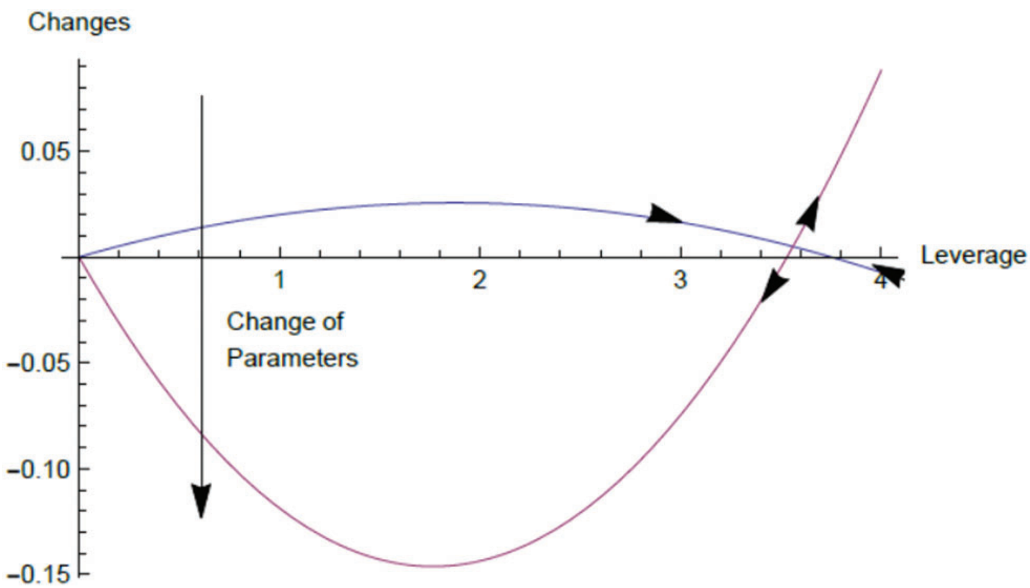

Figure 3: Phase portrait of the global financial system in 2007–2008.

The "credit crunch" of 2008 made parameters of the logistic model negative, transforming the RHS of (10) into function: $f(l) = -0.1656\,l + 0.0469\,l^2$ (the lower curve in (Figure 3). At the stationary point $l^* = 3.53$ the system's Jacobian was positive, $\partial f/\partial l = 0.1656/gt0$, signaling the "irrational exuberance" of the debt market; its instability was depicted by the outward pointed arrows on the lower curve in Figure 3. Hence the "credit crunch" of 2008 could be modeled similar to a "trans-critical" bifurcation that is generic for the logistic dynamical systems. Theoretically, the number of stationary (critical) points in a transcritical bifurcation remains fixed though some of them might change their stability as parameters vary. In our example, due to roughness of empirical information the point of bifurcation appeared to be "blurred", $l^* \in [3.53; 3.79]$, yet the dramatic transformation of a stable stationary leverage into the unstable one was reproduced rather convincingly.

5 Deterministic Leverage Dynamics

The aggregate behaviour of creditors and borrowers, as drawn on the empirical data in Figure 4, could be explained in standard economic terms. The short run dynamics (near stationary leverage) of an aggregate of investors is represented by indicators of debt supply, $(\mu - r)$, and demand for debt, $(\rho - r)/l$, which are balanced at stationary leverage:

$$(\rho - r)/l^* = (\mu - r) \tag{12}$$

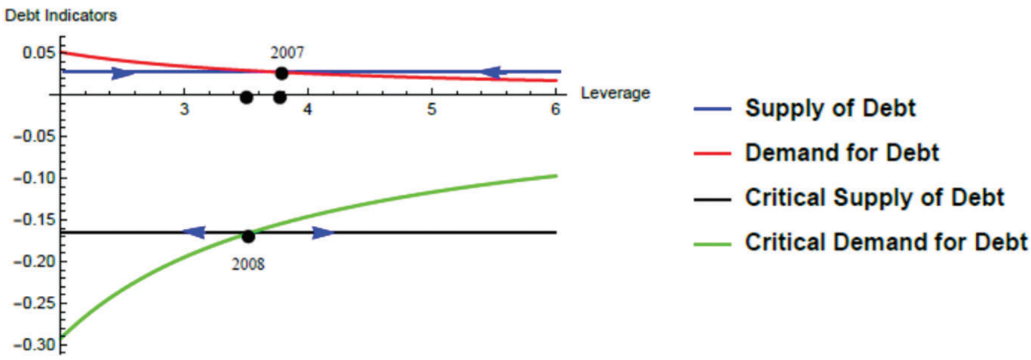

Figure 4: Attractor and repeller of the global finance in 2007–2008.

At a stable stationary leverage (upper part of Figure 4, $l^* = 3.79$) the Jacobian of (10) is negative, $\partial f/\partial l = -a < 0$. In its vicinity for any leverage, $l_1 < l^*$, the demand for debt exceeds its supply, $(\rho - r)\,l_1/gt(\mu - r)$, and investors in aggregate would borrow additional funds thus increasing leverage until it reaches the stationary point. On the other hand, at any l_2/gtl^*, the demand for loans is less than their supply, and investors, by selling their assets, would drive the leverage down. Thus, in the market with positive spreads creditors and borrowers behave in accordance with the analysis originally developed by Knut Wicksell (1898).

Negative parameters $\{a, b\}$ make the Jacobian of (10) positive, $\partial f/\partial l = -a/gt0$, thus the stationary leverage became unstable, as in the lower part in Figure 4($l^* = 3.53$). In the vicinity of a stationary point they apply the same strategies: pursuing positive effects at l/gtl^* they increase leverage, and short their debt positions minimizing net losses at $l < l^*$. But in the disturbed financial market their strategies produce dramatically different consequences: contrary to the “normal” market, purchases or sales do not end up at the stationary point; once started they are going on unboundedly (theoretically, at least). When a typical investor (in $l_1 < l^*$) decreases her/his debt exposure by selling-off spasmodically large chunks of their assets, this process quickly degenerates into a vicious debt deflation, as was convincingly explained by Irving Fisher (1933). On the other hand, the hectic purchases of new assets on borrowed funds in l_2/gtl^* provoke assets overvaluation and blowing financial bubbles, as was described by Hayman Minsky (2008).

All in all, stationary leverage reflects market processes of valuation and payment of excess returns in the global financial system. Market participants use leverage as a measure of valuation, by equating two spreads in either one of the following relations:

$$(\rho - r) = (\mu - r)l^* \text{ or } (\rho - r)l^{*-1} = (\mu - r)$$

.One of these relations is redundant since they define the same leverage; yet, their alternative usage might accentuate differences in market conditions if spreads are treated as a rude measure of market riskiness.

On a sample of given (and different) empirical rates of return stationary leverage (9), as a ratio of two spreads, is evaluated either in the speculative, overvalued (bull) market, or in the shaky, undervalued (bear) financial market. In other words, stationary leverage *per se* does not imply balancing the total debt demand against its supply in the near term, though it would be difficult, if possible, to explain their permanent deviation from each other. Thus, the long run behavior of the financial market is subject to different rules to be studied in Section 9.

Debt market regimes, or patterns, are consistently represented by the family of logistic trajectories:

$$l(t) = K\left\{1 + (\frac{K}{l_0} - 1)\exp[-at]\right\}^{-1} \tag{13}$$

where each trajectory $l(t)$ is specified by a particular initial state, l_0, and parameters, a and b. Alternatively, any particular solution to the logistic eq. (11) can be written as a weighted harmonic average of initial leverage, l_0, and its stationary state, K:

$$l(t) = 1/\left\{K^{-1}(1 - \exp[-at]) + l_0^{-1}\exp[-at]\right\}. \tag{14}$$

The family of trajectories (12) with empirical parameters: $l^* \equiv K = 5.27$; $l_0^1 = 6.45$; $l_0^2 = 4.52$; $a_1 \equiv a_H = 0.0598$; $a_2 = -0.1656$ is represented in Figure 5. Different market patterns[5] are explained as follows: trajectories of financial leverage are stable for positive parameters $\{a, b\}$; financial bubbles appear for $l_0/gtl^*$and negative parameters; and the debt-deflation spirals exist for $l_0 < l^*$ and negative parameters.

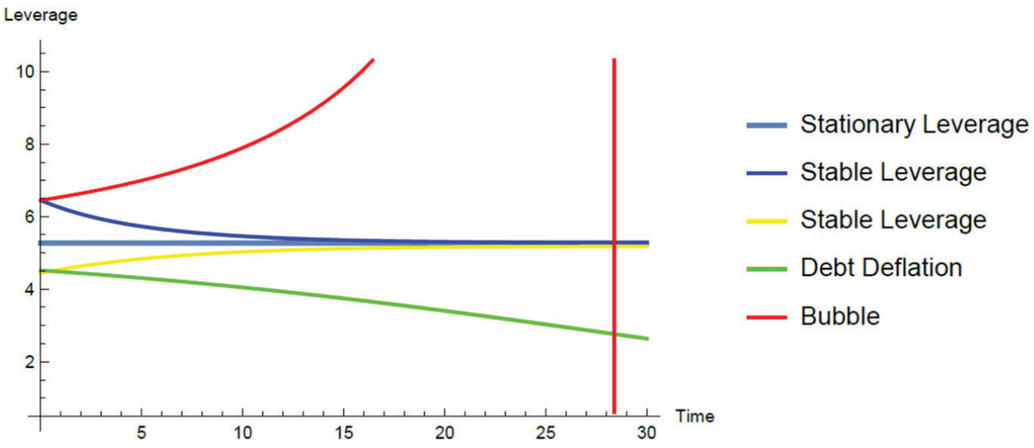

Figure 5: Deterministic logistic leverage trajectories.[6]

In the normal, or Wicksellian, market investors borrow additional funds (thus increasing leverage) and sell *en masse* their assets in the opposite case. The market exists for positive spreads,[7] and leverage trajectories (blue and yellow curves) approach asymptotically their attractor, or stable long run leverage (light blue curve). Accordingly, stationary leverage is unstable (the repeller) on the abnormal market where either assets overvaluation would lead to the Minsky bubble of unboundedly increasing demand for assets (red curve), or falling asset prices would drive investors to collapse along the Fisherian debt deflation trajectory (green curve).

Note, in addition, that reflecting economic realities, the logistic leverage model allows for negative parameters (spreads) – however abnormal, short lived, and highly undesirable – as quite feasible and legitimate characteristics of financial markets. Addressing, via negative spreads, the phenomena of an irrationally exuberant market – bubbles and collapses – makes the financial logistic model very different from its biological or ecological analogues where, as a rule, negative parameters are not feasible (Gabriel, Sausy, and Bersier 2003).

6 Leverage and Capital Intensity

Aggregate debt, as a component of financial liabilities, plays an important but passive role in macrofinancial dynamics for it multiplies the effects of capital investment that are realized in future income. Logistic eq. (11), describing the evolution of aggregate assets per unit of capital, is complementary to the equation of capital dynamics as well. Since the product of the capital intensity variable, $w(t) = e(t)/A(t) = l^{-1}(t)$, and the leverage is equal to one:

$$w(t) * l(t) = 1 \tag{15}$$

their marginal changes (with respect to time), $\dot{w}(t)$ and $\dot{l}(t)$, are subject to the following equation:

$$\dot{w}(t)l^2(t) + \dot{l}(t) = 0 \tag{16}$$

where $l^2(t)$ plays a role of a scale factor since units for the rates of change are different. Equation (16) means that in a riskless world the marginal increase in leverage, $\dot{l}(t)$, is equivalent to a decrease in l^2 units of the marginal intensity of capital, $\dot{w}(t)$, and *vice versa*. Instantaneous changes in capital intensity and leverage in the absolute terms are subject to differential equations:

$$\dot{l}(t) = al(t) - bl^2(t) \tag{17}$$

$$\dot{w}(t) = -aw(t) + b \tag{18}$$

which have different dynamics, scales and stationary points. Figure 6 gives the phase portraits of the leverage (blue curve) and the capital intensity (red line) for stable solutions to both systems (10) and (15), stationary leverage $l^* \equiv K = a/b$, and stationary capital intensity $w^* = b/a$.

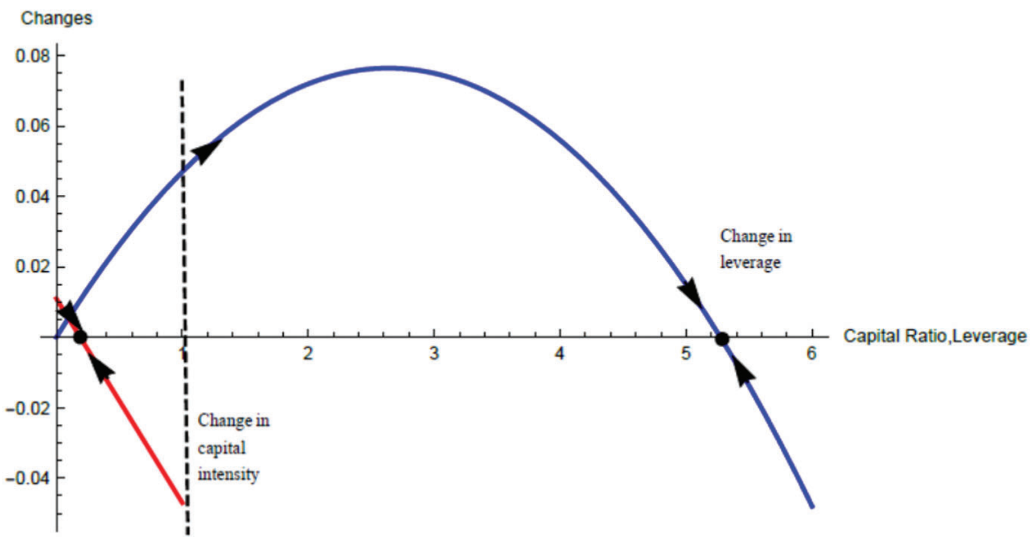

Figure 6: Phase portraits of leverage and capital intensity.

By economic meaning, the rate of change of capital intensity in recessions is negative while the rate of leverage change is positive; in booms signs of these rates would become the opposite. Thus, their relative rates of growth are subject to equality:

$$\frac{\dot{l}(t)}{l(t)} \frac{w(t)}{\dot{w}(t)} = -1 \tag{19}$$

which for random rates is modified into an expectation of their respective product. It follows that the expected rate of leverage change depends upon correlation between the rates of growth of capital intensity and leverage:

$$\mathrm{E}\left[\frac{\dot{l}(t)}{l(t)}\right] = -\frac{1}{\mathrm{E}\left[\frac{\dot{w}(t)}{w(t)}\right]}\left(1 + Cov\left[\frac{\dot{l}(t)}{l(t)}, \frac{\dot{w}(t)}{w(t)}\right]\right) \tag{20}$$

Analogues of eq. (20) were extensively studied within the framework of the CCAPM model containing a “stochastic discounting factor; it was concisely described, for example, in (Campbell, Lo, and MacKinlay 1997 **Economic Sciences Prize Committee 2013**). In our case, an expectation (17), according to data in Table 2, indicated a tendency of leverage to grow at the rate 0.065 annually: $-1/8.83 * (1 - 1.57) = 0.065$, which cast doubt on the stability of the process. This preliminary result will be elaborated in Section 8.

7 Gamma Distributed Stochastic Leverage

The deterministic logistic model revealed important interconnections among leverage, its relative rate of change and rates of return that might be found in a single trajectory with zero volatility. Deterministic model *per se* is a bad predictor since the leverage sample (Table 2) has by no means negligible annualized volatility.[8] Looking from another angle, the same data might be viewed as a sample from an ensemble of up to 11 random realizations of a stochastic leverage process. Stochastic approach provides powerful instruments of the leverage analysis and prediction, namely, the diffusion and the Kolmogorov-Fokker-Planck equations as different representations of the same process. Since both of them contain information about deterministic and random parts of the system all the preceding analysis of leverage is valid and important. More of that, an asymptotic leverage distribution could be returned by solving the stationary KFP equation, and in a particular form of the gamma distribution it is useful in the long term financial management and control.

Continuous random logistic leverage process $l(t)$ is generalized as a diffusive process following a stochastic differential equation, SDE:

$$dl(t) = [a - bl(t)]l(t)dt + \sigma\, l(t)dW(t) \tag{21}$$

where σ is a constant parameter of volatility, and $W(t) = \int_0^t dW_u$ is the standard Brownian motion.[9] The random future leverage change, $dl(t)$, consists of the deterministic drift, $[a - bl(t)]l(t)dt$, and a diffusion component, $\sigma l(t)dW(t)$, linearly dependent on leverage; hence the SDE for the leverage relative rate of change, $dl(t)/l(t)$, contains drift, linearly dependent upon leverage, and independent of leverage noise fluctuations, $\sigma\, dW(t)$). This simple structure resembles eq. (1) tested against empirical data in Section 2. As a result, logistic SDE (18) preserves economic interpretation of the deterministic model (10) and facilitates realistic leverage dynamics in the long run.

The strong solution to (18) is represented by a family of admissible random realizations:

$$l(t) = \frac{l_0\, K \exp[(a - 0.5\sigma^2)t + \sigma W(t)]}{K + a\, l_0 \int_0^t \exp[(a - 0.5\sigma^2)u + \sigma W(u)]du}$$

that coincide with solution of the deterministic model for zero volatility, $\sigma = 0$, (Skiadas 2010). A sample of five leverage realizations is shown in Figure 7 where the red curve represents the trajectory of expected leverage.

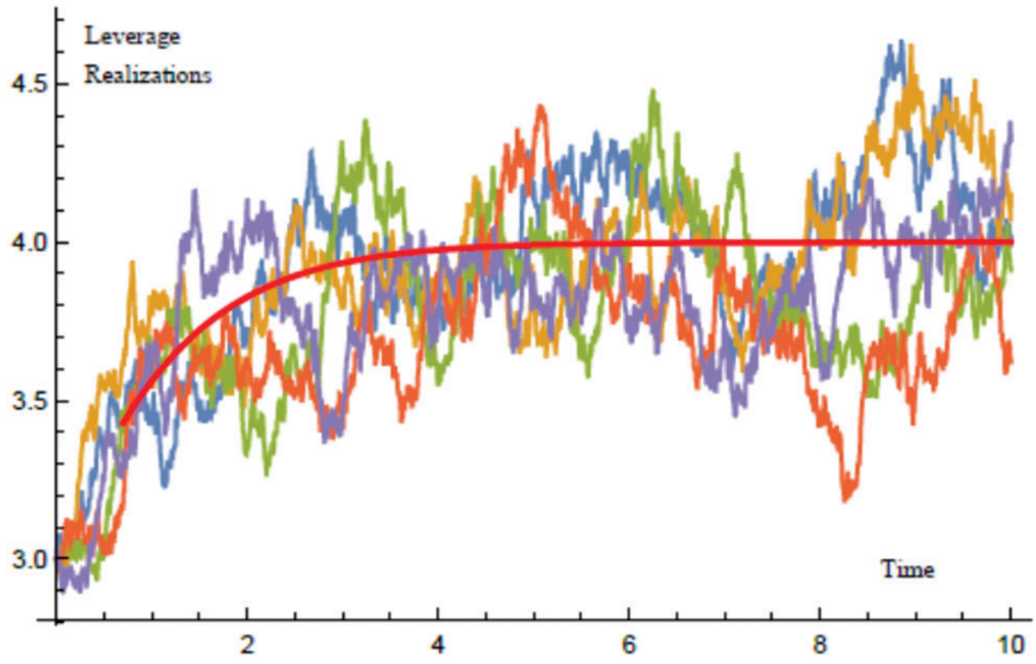

Figure 7: Five realizations of stochastic leverage.

General information about the ensemble of leverage realizations following SDE (18) is contained in the probability density function, $p[l(t), t]$ which is a solution to the forward Kolmogorov (or Fokker-Planck) equation:

$$\frac{\partial}{\partial t}p[l(t), t] = -\frac{\partial}{\partial l}\{[l(t)(a - bl(t)]p[l(t), t]\} + \frac{1}{2}\frac{\partial^2}{\partial l^2}\{\sigma^2 l^2(t)p[l(t), t]\} \quad (22)$$

with boundaries and initial conditions specified from the actual process considerations.

Equations (21) and (22) are closely related to each other; namely, the process satisfying SDE (18) has the probability density function which is the solution to the KFP eq. (22). Generally, if the drift and the noise coefficients of a random process $x(t)$ are given by the Ito processes $A[x(t), t]$ and $B[x(t), t]$, respectively, then its SDE:

$$dx(t) = A[x(t), t]\,dt + \sqrt{B[x(t), t]}\,dW(t),$$

and the KFP equation with regard to the probability density function, $p[x(t), t]$:

$$\frac{\partial}{\partial t}p[x(t), t] = -\frac{\partial}{\partial l}\{A[x(t), y]\,p[l(t), t]\} + \frac{1}{2}\frac{\partial^2}{\partial l^2}\{B[x(t), t]\,p[x(t), t]\},$$

are equivalent in the representation of the process (Gardiner 1997, 118).

Accordingly, the quadratic noise coefficient, $\sigma^2 l^2(t)$, in the KFP equation of the logistic leverage model implies that its SDE (18) has the linear noise coefficient, $\sigma\, l(t)$. It follows that the SDE for the relative leverage rate of change, $dl(t)/l(t)$, has to have the drift, linearly dependent upon leverage while its noise is independent of leverage. Any other plausible noise and drift structuring, like analysed thoroughly in (Pasquali 2001), would have led to different models and, as such, are subject to appropriate economic validation.[10]

Solving the KFP eq. (22) poses a formidable problem, and it is easier to infer information about asymptotic leverage out of its stationary distribution which is independent of time and initial state, $l(t_0)$. This approach is widely used in finance to study asymptotical characteristics of instruments with different maturities, and, quite evidently, the stationary distribution of leverage might reveal important long term characteristics of the global financial system. Existence of a stationary distribution for logistic stochastic processes with structural parameters and volatility independent of time is a firmly established fact (Dennis and Patil 1984; Liu and Shen 2015) though ergodic qualities of these processes require further economic investigation.

Stationary probability density function, $p(l)$, of a random process that follows SDE (18) is a non-trivial solution to the ordinary differential KFP equation:

$$-\frac{\partial}{\partial l}[l(a - bl)p(l)] + \frac{1}{2}\frac{\partial^2}{\partial l^2}[\sigma^2 l^2 p(l)] = 0 \quad (23)$$

The straightforward (though complicated) solving ODE (20) defines a two-parametric probability density function of asymptotical gamma distribution of leverage:

$$p(l;\alpha,\beta) = [\beta^{\alpha}/\Gamma(\alpha)]\, l^{\alpha-1}e^{-\beta l} \qquad (24)$$

The shape of the distribution depends upon parameter, $\alpha = (2a/\sigma^2) - 1$, and its rate, or scale $1/\beta$, is defined by parameter, $\beta = 2b/\sigma^2$. Stationary distribution exists for the positive parameter α, or $0 < \sigma^2 < 2a$ (Pasquali 2001). Three forms of gamma distribution that could be met in the leverage analysis are shown in Figure 8.

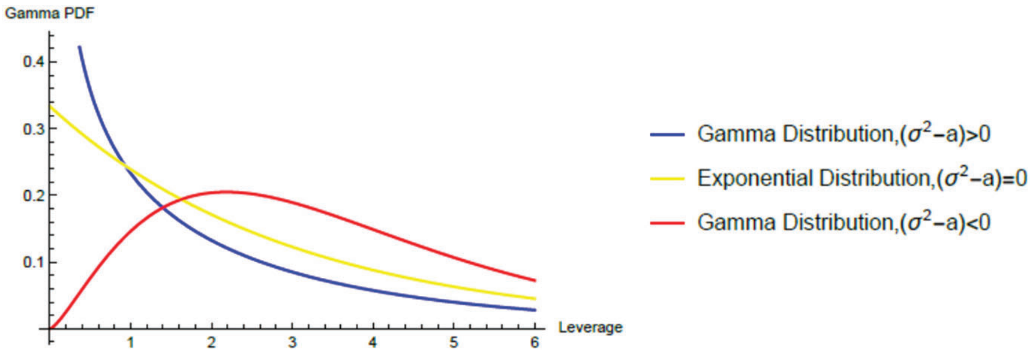

Figure 8: Three forms of gamma distribution.

Probability density function (21) is associated with unimodal gamma distribution (red curve) if $\sigma^2 < a$; it is transformed into an exponential distribution (yellow curve) for $\sigma^2 - a = 0$, and becomes a decreasing "J-shaped" distribution (blue curve) for $a < \sigma^2 < 2a$ (Walck 1996). The two-parametric gamma distribution is no more complicated than the widely used Gaussian distribution and, due to its asymmetry and long tails is well-suited to financial applications.

The most probable value of a peaked (unimodal) gamma and exponentially distributed leverage is its mode:

$$Mode[L] = (\alpha - 1)/\beta = K - \sigma^2/b;\ \alpha \geq 1 \qquad (25)$$

which is not defined for a decreasing "J-shaped" distribution. The mode, if it exists, is smaller than expectation:

$$\langle L\rangle = \alpha/\beta = K - (\sigma^2/2b)$$

due to skewness of the gamma distribution to the right.

Stability of leverage dynamics is measured by the stochastic Lyapunov exponent (SLE), λ, which is a generalization of the deterministic Jacobian:

$$\lambda = [\beta^{\alpha}/\Gamma(\alpha)] \int_0^{\infty} (a - 2bl)l^{\alpha-1}\exp[-\beta l]dl \qquad (26)$$

Since the averaging in (23) goes over the ensemble of asymptotic leverages with stationary gamma distribution, SLE takes a very simple form:

$$\lambda = \langle a - 2bL\rangle = a - 2b\langle L\rangle = \sigma^2 - a \qquad (27)$$

where $\langle . \rangle$ means the ensemble averaging (Dennis et al. 2003). Roughly speaking, distributions of a random leverage process, associated with negative SLE (23), converge in the long run to a unimodal stationary distribution with a well-defined mode. Otherwise, divergent realizations of leverage would have positive SLE: their stationary distribution is a decreasing "J-shaped" distribution for $a < \sigma^2 < 2a$. The zero Lyapunov exponent forms a boundary case of a one-parametric exponential distribution with its mode at the origin.

SLE (23) in the logistic leverage model could be viewed as a simple measure of the market confidence in the debt sustainability. In the case of inequality, $\sigma^2 < a$, the aggregate of creditors are expecting to receive positive spreads on their investments because they are confident, implicitly, in the ultimate debt redemption. Otherwise, the variance, larger than the structural parameter a, would undermine their beliefs in the debt sustainability thus convincing creditors either to tighten the terms of credit or to refuse making loans outright.

Thus, the focus of our interests, from the economic point of view, is illustrated by Figure 8. If the stationary leverage distribution is given by the decreasing "J-shaped" gamma distribution (blue curve), the uncertainty is high, and the debt accumulation is unsustainable; leverage cannot be stabilized at any value, $l/gt1$, admissible in the modern financial system. Hence economic policies decreasing the uncertainty of financial intermediation have to be implemented: their impact would transform the leverage distribution into a peaked gamma distribution (red curve) thus stabilizing the long term leverage around its most probable value. The boundary case between these two scenarios is formed by the exponential distribution (yellow curve).

In the subsequent sections two scenarios, or patterns, of stochastic long run leverage dynamics are investigated assuming constant positive spreads and different variances.

8 Scenario of "Empirical" Leverage Dynamics

One of the most intricate issues of stochastic dynamics is a relationship between leverage, as a measure of debt accumulation, and the system riskiness that depends upon perspectives of debt redemption. For example, it is a well-recognized fact that, *ceteris paribus*, larger debt and leverage would make the financial system riskier. But is the reverse always true, as well? For example, was the global financial system riskier in 2011 when its leverage was 5.43 than in 2007, when the leverage was 3.53?

A conceivable answer to these questions could be found via the computation of standard hazard, failure and survival functions associated with asymptotical leverage distribution. If a debt system characteristic "lifetime" is measured by its leverage, then its probability density function, $p(l) = \Pr[l < L \leq l + dl]$, by implication, would carry information about instantaneous unconditional failures that might happen in a process of debt redemption. It follows that the conditional instantaneous rate of the global debt default is given by its hazard rate, $h(l)$:

$$h(l) = p(l)/(1 - CDF(l)) \equiv p(l)/S(l) \tag{28}$$

where unconditional probability of a failure to redeem debt outstanding is recovered from the leverage cumulative distribution, $CDF(l) = \Pr[L \leq l]$. Accordingly, the system would survive with probability $S(l) = 1 - CDF(l) = \Pr[L/gtl]$ if the global debt is redeemed at any particular leverage. Note, that the survival function, $S(l) = \exp[-\int_0^l h(z)dz]$, measuring debt sustainability, should be evaluated under boundary condition $l = 1$, if the modern financial system is analyzed.

Given parameters $a_H = 0.058$, $b_H = 0.011$, $\sigma^2 = 0.0735$, $\alpha = 0.578$, $\beta = 0.272$, $1/\beta = 3.676$, estimated on the *GFSR* data, the random global leverage process following SDE (18) had asymptotical probability density function $p(l) = 0.306\, l^{-0.478} \exp[-0.272\, l]$. For the modern financial system containing debts, a boundary condition, $S(l_0) = 1,\ l_0 = 1.062$, was introduced that shifted to the right asymptotic pdf and other risk-related functions in Figure 9.

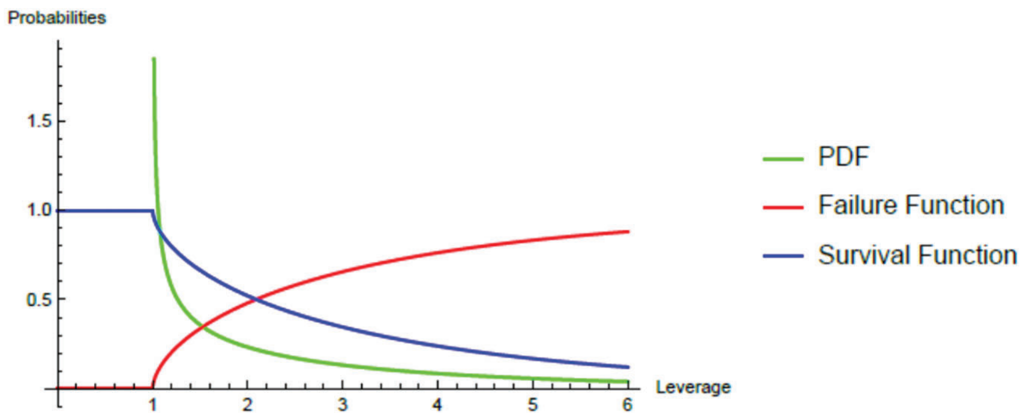

Figure 9: Scenario of "empirical" leverage dynamics.

As seen, positive Lyapunov exponent, $\lambda = 0.016$, yielded to a decreasing, or "J-shaped", stationary gamma distribution (green curve) that has no mode: with time the bulk of the leverage realizations would concentrate around $l = 1$ meaning the collapse of the modern financial system since all debts would be extinct; hence the debt system cannot be stabilized in any pragmatic sense of this term.[11] The form of stationary distribution suggested interpretation of unconditional debt redemption: failure rates associated with smaller, typically new, debts were higher than instantaneous failures of larger (and more mature) ones. In other words, the high riskiness of loaning to the new small enterprises, the dominant feature of the current debt system, was extrapolated into the distant future.[12]

Quick decrease of the survival function indicated poor debt performance and foreboded its extreme fragility in the future. Were the future volatility as high as in the period of 2003–2013, leverage realizations would concentrate with high probability around $l = 1$ (or zero without the system translation to the right) in the long run, thus simulating stochastic analogue of a Fisherian debt-deflation process. The failure and survival functions warns about debt unsustainability: as seen in Figure 9, the global debt had (approximately) equal chances to survive, or to fail, at a low leverage, $l_N = 2.18$, which is the root of equation $S(l) = CDF(l) = 0.5$. This equation has another economic interpretation: deleveraging from its current macrofinancial level, $l = 4$, is desirable only if accomplished without alarmingly high volatility in the debt markets. Five important points of empirical asymptotic distribution are given in Table 4; particular leverage $l_N = 2.18$ has an important economic meaning to be explained in Section 9.

Table 4: Empirical scenario of leverage dynamics.

Leverage	$l_1 = 1.062$	$l_N = 2.18$ *	$l_2 = 2.45$	$l_3 = 4.0$	$K = 5.27$
Empirical pdf	5.637	0.215	0.182	0.087	0.053
Failure function	0.0	0.507	0.561	0.756	0.843
Survival function	1.0	0.493	0.439	0.244	0.157

Parameter l_N is estimated as the mode of stationary gamma distribution.

Does it mean that the global leverage would evolve according to the "pessimistic" scenario? The answer is: it depends. The gloomy perspectives, produced by high empirical volatility, $\sigma^2 = 0.0735$ per annum, could be negated by smaller uncertainty, driving distributions of stochastic leverage to its peaked stationary gamma distribution. Smaller noise, facilitating leverage convergence, in its turn, could result from the more efficient balancing of supply and demand for debts by market forces in the long run.

9 The Long Run and Anchor Leverage

Annually calculated empirical macrofinancial rates of return, ROE and ROA, are not equal to each other implying the existence of unbalanced debt supply and demand. But, if the true wealth (*pretium verum*) of any society consists of its real resources (including knowledge, culture and social organization), then the "fair" value of financial assets (*pretium justum*) exists only subject to a balance between debt supply and demand; otherwise, the entire mankind could have been enriched by a mere issuance of money and other "paper" instruments – in a way similar to Baron Munchausen pulling himself out of a mire by his hair. Since, fortunately or unfortunately, it is impossible, revenues of creditors and borrowers cannot deviate significantly from each other in the longer term.

In the long run aggregate borrowers are assumed to be taking additional debts contingent on a current weighted average cost of capital, μ, and on the expected rate of investment return, r, thus on spread, $a = \mu - r$. In other words, for any viable leverage the expected rate of return on equity, $\rho(l)$, is an indicator of the long run supply of debt:

$$\rho(l) = r + (\mu - r)l \tag{29}$$

Simultaneously, but independently of borrowers, the collective of debt holders (creditors) are expecting to receive a positive return on assets, given the current rate of return on equity, ρ, and another spread, $c = \rho - r$. Hence the long term demand for debt, measured in the leverage scale, is indicated as:

$$\mu(l^{-1}) = r + (\rho - r)l^{-1} \tag{30}$$

where the demand for debt follows changes in its major factor, capital, implicitly,[13] hence should be measured along the leverage scale as l^{-1}.

Generally, at any stationary leverage, $l^*/gt1$, current rates of return on assets and on equities are different, as for example, $r < \mu < \rho$. Contrary to that, in the long term competitive adjustments between creditors and borrowers would balance the demand for debt with its supply. Thus the expected future rates of return of creditors and borrowers, ROA and ROE, as functions of leverage, should be equal:

$$\mu(l_N^{-1}) = \rho(l_N) \tag{31}$$

The positive root of equality (27), l_N, by its economic meaning, defines the *anchor leverage*:

$$l_N \equiv K^{0.5} = (a/b)^{0.5} \tag{32}$$

at which indicators of the long run supply of and demand for debts are equal.

In fact, eq. (32), taking into account financial balances of states and flows (2–3), generalizes the microfinancial collateralized loan balance, $l = l/(l-1)$, that follows from a single eq. (2). By definition, a robust long term relation between aggregate debt and aggregate capital exists at the anchor leverage, l_N, and it forms an equilibrium between marginal effects of additional debts for collective borrowers and creditors, $d\rho(l) + d\mu(l^{-1}) = 0$. To the left of anchor leverage, relatively abundant capital backs up a relatively scarce debt. This process is reflected in *ROE* larger than *ROA*, and could be considered as a definition of an overcapitalized financial system.[14] If the system moves to the right of anchor leverage, then the increase in *ROE* is accompanied by decreasing *ROA*, and their changes would reflect growing debts and decreasing capital. Hence the region to the right of anchor leverage defines, generally, an overindebted financial system.

The above said is easily adapted to the standard macroeconomic analysis of aggregate saving and investment. Firstly, for the constant leverage, indicators of the demand for debt, $\mu(l^{-1})$, and the supply of debt, $\rho(l)$, are easily transformed into linear functions of the long interest rate (investment return), r; and secondly, appropriate equations are to be interpreted asymmetrically as indicators of the supply of funds, $\mu(r)$, (saving) and of the demand for funds (investment), $\rho(r)$, accordingly.

10 Scenario of Leverage Convergence

In a stochastic context the anchor leverage is be interpreted as the most probable value of leverage at which market forces tend to equalize the demand and supply of debts in the very long run. Hence, if leverage distributions converged to the unimodal stationary gamma distribution, it would tantamount to the long run leverage stabilization around its anchor value associated, by economic meaning, with the mode of stationary distribution. Such an "optimistic" scenario of financial evolution could have been realized, for example, via coordinated, comprehensive and consistent reforms of financial markets decreasing dramatically the global uncertainty. Smaller uncertainty, in its turn, would transform stationary gamma distribution into a peaked (unimodal) one, with well-defined mode. In short, the mode of stationary distribution, if it exists, is to be identified with the anchor leverage:

$$Mode[L] = l_N \tag{33}$$

From the definition of the mode (22) equality (29), in its turn, implies, with necessity, that the expected variance has to be no larger than its critical value:

$$\sigma_c^2 = a - \sqrt{ab} \tag{34}$$

Thus the structure of leverage dynamics defines the critical variance necessary to secure its convergence in the long run. If satisfied, condition (30) facilitates simulation of the debt sustainability with the same, as in Section 8, structural parameters but smaller variance, $\sigma_c^2 = 0.034$. The peaked gamma distribution of leverage, simulated on the empirical data, is given by the following pdf: $p(l) = 0.279\, l^{1.412} \exp[-0.647\, l]$, and the boundary condition, $S(l_0) = 1$, that is satisfied for $l_0 \cong 0$. Scenario of leverage convergence is presented in Figure 10 and Table 5.

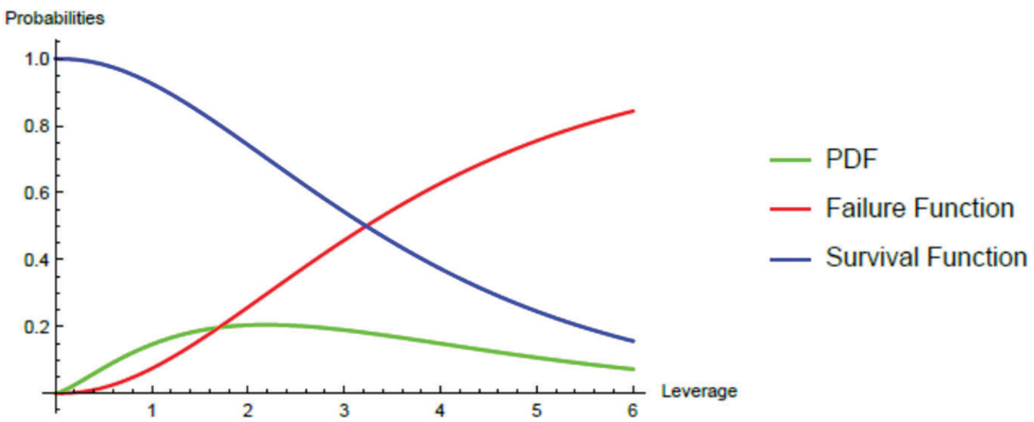

Figure 10: Scenario of leverage convergence.

Table 5: The "optimistic" scenario of leverage dynamics.

Leverage	$l_1 = 1.0$	$l_N = 2.18$ *	$l_4 = 3.23$	$\langle L \rangle = 3.73$	$K = 5.27$
Estimated pdf	0.146	0.205	0.181	0.155	0.097
Failure function	0.074	0.294	0.5	0.586	0.782
Survival function	0.926	0.706	0.5	0.414	0.218

Parameter l_N is estimated as the mode of stationary gamma distribution.

As seen in Figure 10, the failure and survival functions are very different from their analogues in the "pessimistic" scenario. Contrary to the "new debt high fragility" system, this scenario simulated riskiness of the "mature" debt. As a consequence, the survival function of the mature debt system (0.706) at the anchor leverage is much higher than its unconditional failure function (0.294); even at the expected leverage (3.73) the debt system survival is a bit smaller than its failure. It suggested that the sustainable debt market could have existed (in the medium term, perhaps) around much higher leverage, $l_4 = 3.23$, at which chances of the global debt survival and failure are equal.

Asymptotic leverage probability density functions of "pessimistic" and "optimistic" scenarios, together with the long term debt demand and supply indicators, are presented in Figure 11. If stationary probability density function is an "empirical" one, given by the decreasing green curve, then the debt cannot be stabilized, and the system would have gone towards its imminent collapse. The global leverage, after fluctuating without discernible tendencies around 4.6 in a period of 2003–2013, could have, nevertheless, decreased asymptotically to some positive value, depending upon the sign of its SLE. As it was seen, a variance no larger than 0.034, would drive the global leverage from its initial position (4.52) in the year 2013 towards anchor value, $l_N = 2.18$.

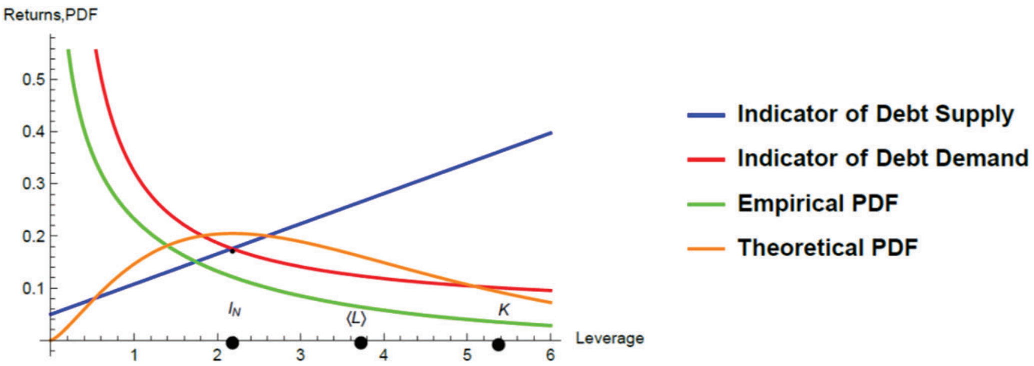

Figure 11: Gamma distributed leverage and its anchor.

Many runs of simulation validated the model structure, including magnitudes and location of anchor leverage, l_N, expected value, $\langle L \rangle$, and the deterministic attractor, K, as in Figure 11. Their comparison with empirical leverage, showing overindebtedness of the current financial system, stressed the necessity of comprehensive financial reforms fostering smaller uncertainty and leverage convergence in distribution to the unimodal gamma distribution with anchor leverage as its most probable value. It follows, in paricular, that a small, no-zero variance, by influencing market participants to be wary and cautious, would lead to the leverage lesser than in a completely deterministic environment.

11 Collateral ratio for the Global Finance

Development of financial markets is intertwined with the evolution of real markets. As a small step towards a unified macrofinancial-cum-macroeconomic theory we propose to solve this general problem as an estimation of the expected aggregate collateral ratio. Collateral ratio, l_Y, is a measure of correspondence between total financial assets and real resources (approximated by the world GDP) that can be decomposed as a product:

$$l_Y = l_t * q_t \tag{35}$$

where ratio of equities-to-the world GDP,[15] $q_t = e_t/Y_t$, reflects the market valuation of global equities. Information about decomposition (31) is given in Table 6.

Table 6: The global collateral ratio components.

	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013
l_t	3.97	3.89	4.08	3.74	3.53	6.4	4.92	4.54	5.43	5.12	4.52
q_t	0.87	0.91	0.84	1.05	1.19	0.55	0.82	0.88	0.67	0.73	0.84
l_Y	3.45	3.54	3.42	3.93	4.2	3.52	4.03	3.99	3.64	3.74	3.79

As it was shown above, financial leverage, l_t, in the long run converges to the anchor leverage, l_N, hence the problem of global financial assets valuation might be reduced to valuation of the global equity market. The latter, in its turn, could be viewed as a value of market fundamentals, Y, associated with the global GDP, that is multiplied by the coefficient of valuation, q, as shown by the breakdown of the long run value of global equities, $e = Y * q$.

Coefficients of valuation, q, were originally introduced by J. Tobin (Tobin and Brainard 1977) to explain fluctuations in equity values by intensity of the firms financing activities. Extended to macrofinance, Tobin's hypothesis would imply that parameter q, if growing, would reflect increases in the value of equities due to "optimistic" business expectations regarding future profits and dividends fostered by subsequent growth of aggregate investment and GDP. Accordingly, the expectations of a stagnated economy would drive the value of equities, hence the Tobin coefficients q, down.

As seen in Table 6, macrofinancial coefficient q_t fluctuated around 0.85 in the period of 2003–2013 due to huge losses in the year of the "credit crunch". Assuming that the long run valuation of the equities market is "fair", random fluctuations of the Tobin's coefficients are conceivably cancelled out with the high probability of making the fully collateralized equity market equal (approximately) to the world GDP. Formally, the above said is equivalent to the assumption of global equity-to-GDP coefficients, $Q(t)$, following the Ornstein-Ulenbeck stochastic process:

$$Q(t) = 1 + (Q_0 - 1)\exp[-\kappa t] + \sigma \int_0^t \exp[\kappa(u - t)]dW(u) \quad (36)$$

where 1.0 is the long run average, $\kappa$is the mean reverting parameter, and $\sigma$is the diffusion parameter. Some realizations of this process with $Q_0 = 0.87; \kappa = 0.5; \sigma = 0.055$ are represented in Figure 12 where the black curve shows the expected values of Tobin's coefficients up to ten years.[16]

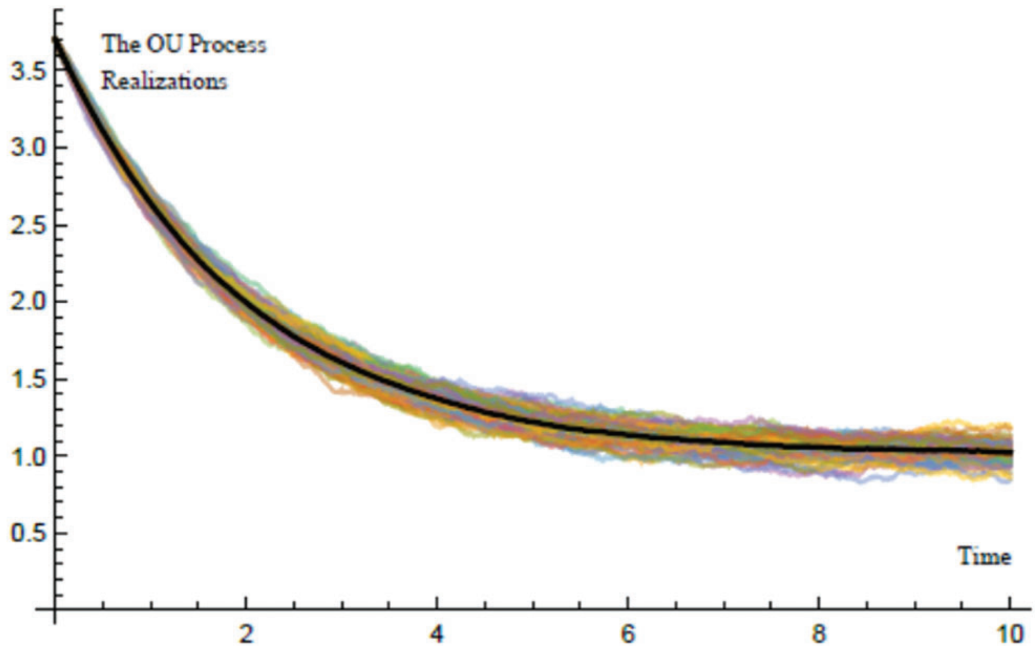

Figure 12: Global equity-to-GDP realizations.

The immediate corollary from the above deliberations is as follows: a fully collateralized equity market would make the expected long term ratio, l_Y, to be asymptotically the same, or close to the anchor leverage:

$$\lim_{t\to\infty} l_Y(t) = l_N \tag{37}$$

Relation (33) could be used at the preliminary stages of numerical evaluations of different configurations between financial and real markets. For example, if the global equity market is evaluated properly, $\langle Q\rangle = 1.0$, then the global anchor leverage (equaled to the mode of its stationary distribution, $l_N = 2.18$) would define the "fair" long run value of financial assets. Accordingly, the "true" debt-to-capital ratio 1.18, indicating the upper limit of sustainable global debt accumulation in the long run, made operational the idea of a reasonable squeezing of the bloated financial system (Kay 2015) to the size supported by real resources.

12 Discussion and Some Comments

Regarding complexity of the global financial system its logistic model was very simple, yet nevertheless, it addressed properly important issues of debt dynamics and stability. In particular, rapid debt accumulation in the period of 2003–2013 was explained as a consequence of excessive leveraging that had caused the global "credit crunch" (the system bifurcation) in 2007–2008. Patterns of sustainable and unsustainable debt accumulation, resembling economic realities, were recognized by the model simulations. Thus, positive SLE indicated highly uncertain perspectives of debt amortization: high volatility, similar to a heavy tax, would force market participants either to restrict lending or refuse it altogether. Such outcome, foreboding ultimate collapse of the system, could be avoided only by the implementation of comprehensive market reforms, decreasing leverage and improving debt servicing and redemption (BIS 2014). Accordingly, negative SLE would imply realization of effective macroeconomic and macroprudential policies fostering the leverage convergence in distribution to the stationary gamma distribution. The anchor leverage is much smaller than its actual values hence stochastic convergence is tantamount to the system restructuring towards a stable configuration of financial and real markets with slower debt accumulation.

An interesting aspect of the model application might be outlined briefly. Many financial experts expressed recently their concerns about hectic market speculation being fostered by the policy of "quantitative easing", QE. In particular, J. Hussman, the well-known trader and analyst, persuasively stated that by decreasing ROI almost to zero monetary policy convinced investors to equate current yields to YTM and distorted their expectations. Meanwhile expected future investment returns might remain positive only due to overvalued financial assets and bloated leverage (Weekly Market Comments, May 16, Hussman). The same story is said by the logistic model if speculative activities were propagated by the increased liquidity. Under the circumstances, as seen in Figure 5, the stable leverage trajectory (yellow curve) would be translated upward to coincide with a convex trajectory representing the blowing Minsky bubble (red curve).

Methodologically, the model simulations were based on two basic hypotheses. Firstly, different patterns in macrofinancial behaviour were formed by particular combinations of leverage and rates of return. The model parameters were subject to sampling errors and their reliability was expected to increase as far as samples would enlarge, and their expert evaluations would improve in further explorations of financial systems. Secondly, the noise was incorporated into logistic model in a particular way: it was linearly dependent upon leverage in the SDE (18). This presumption made it possible to recognize different patterns of economic growth and debt accumulation. Looking from this angle, the anchor leverage implied a healthy and robust debt accumulation that was directly opposed to economic stagnation of the extinct indebtness (which is also stable). It follows that the stable growth is not contingent on the debt extermination; instead, indebtness has to be kept within reasonable limits, justified by appropriate economic conditions.

All in all, the model recommendations are not excessively detailed at the moment, but everything is relative. The logistic methodology has produced many mathematically sophisticated models and their generalizations like the Lotka-Volterra system. Exploring along these lines could provide new theoretically substantiated arguments helping to address more rigorously the important issues of debt stabilization.

Acknowledgements

The author expresses his gratitude to participants of several scientific seminars in Moscow and London where different aspects of the model were discussed.

Notes

[1]Logistic maps (difference equations), once popular in economic literature about chaos, are beyond the scope of the paper. Besides, a continuous one-dimensional logistic equation does not produce chaotic behavior.

[2]The *t-test* gave the same result though it required an *a priori* assumption of each observation to be sampled from a bivariate normal population.

[3]In our opinion, the adherence to the straightforward, and very general, interpretations of equations similar to (1) was a cause of economists' indifference to logistic models, especially, in their continuous form.

[4]Numerical values of the model parameters, as represented in Figure 2, were calculated to satisfy equation: $\rho_t = r_t + (\mu_t - r_t) l_{t-1}$ for every year in the period of 2003–2013. It follows the economic logic of the market participants: their current leverage decisions predetermine, by and large, financial parameters in a subsequent year.

[5]The long term dynamics will be studied later for a particular regime with the steady leverage defined by the balance between long term debt demand and supply.

[6]A vertical asymptote in Figure 5 exists only for negative parameters and $l_0 > l^*$.

[7]Wicksellian "differentials", analogous to spreads in our model, were thoroughly investigated by Aubrey (2013).

[8]The latter, probably, is another reason of the economists' indifference towards logistic models.

[9]Since Brownian motion has no derivative, eq. (21), in fact, is a symbolic representation of the integral equation: $l(t) - l(0) = \int_0^t l(u)[a - bl(u)]du + \int_0^t \sigma l(u) dW(u)$.

[10]For example, a general logistic SDE with quadratic noise component was analysed in (Mao, Marion, and Renshaw 2000). This process has a stationary pdf which is very different from the gamma distribution (Pasquali 2001; Gora 2005). Economic substantiation of this particular noise structuring and the subsequent behaviour of the model are to be investigated specially.

[11]Since noise suppresses deterministic drift of the system this pattern is in agreement with the general logistic model proposed in (Mao, Marion, and Renshaw 2000). But the latter has noise structuring different from our model (18–19) hence its stationary pdf is not of the gamma distribution as well.

[12]This scenario seems to be similar to the "high infant mortality" version routinely considered in the theory of system reliability.

[13]Note that a "benign" coordinate transformation: from leverage to capital intensity, like in Section 6, changed the system, its dynamics and its steady states.

[14]Many modern equity markets operate with LtV (loan-to-value ratio) up to 50 percent for collateralized loans with margin calls.

[15]J. Hussman (Weekly Market Comment, September 12, **2016b**) used the ratio of nonfinancial market capitalization to nominal GDP as a measure of long run valuation on the US equity market. His index is, evidently, qualitatively the same as the global equity-to-GDP coefficient, q. In the log scale Hussman's index fluctuated around 1.0 for the last 20 years.

[16]The detailed analysis might come with any constant, close to 1.0, as a "fair" long run coefficient of the global equity market valuation.

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