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ENDOMORPHISMS OF PROJECTIVE BUNDLES OVER A CERTAIN CLASS OF VARIETIES

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Abstract. Let B be a simply-connected projective variety such that the first cohomology groups of all line bundles on B are zero. Let E be a vector bundle over B and $X = \mathbb{P}(E)$. It is easily seen that a power of any endomorphism of X takes fibers to fibers. We prove that if X admits an endomorphism which is of degree greater than one on the fibers, then E splits into a direct sum of line bundles.

Introduction

During the last 20 years, the question which smooth projective varieties have endomorphisms of degree greater than one (which we shall sometimes simply call "endomorphisms", as opposed to automorphisms) has attracted some attention for both geometric and dynamical reasons (see e.g. [3], [4], [9], [10] - this is only a beginning of the list). Though in this generality it is still far from being solved, there is a number of partial results suggesting that varieties with such endomorphisms generally come from two obvious cases (abelian and toric varieties) by means of simple geometric constructions such as taking a product with another smooth projective variety or taking a quotient by a finite freely acting group. For instance, Nakayama proved in the beginning of 2000's that a rational smooth projective surface with endomorphisms must be toric. Around the same time, one of the authors of the present note has considered the case of a projective bundle X over a projective base B, $p : X \to B$, and proved the following result.

Theorem 1 ([1, p. 17]). *X has an endomorphism commuting with the projection onto the base if and only if X is a quotient of a product $B' \times \mathbb{P}^r$ by a finite freely acting group.*

A simple remark on endomorphisms of projective bundles $X = \mathbb{P}(E)$, where E is a vector bundle ([1, p. 18]) is that a power of any $f : X \to X$ sends fibers to fibers and thus must be over an endomorphism of the base; so if by any

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chance we know that all endomorphisms of B are of finite order - for instance when B is of general type (see for example [8]) - then this result describes the situation completely.

The argument in Theorem 1 (the "only if" part, the "if" part being rather standard) proceeds as follows. One considers the space of all morphisms $R^m(\mathbb{P}(V))$ from $\mathbb{P}^n = \mathbb{P}(V)$ to itself given by degree m polynomials (well-known to be an affine variety) and its quotient M by $PGL(V)$ (that is, the spectrum of the ring of the invariants). It turns out that for m big enough, $PGL(V)$ acts with finite stabilizers, so M is the geometric quotient (i.e., actually parameterizes the orbits of the action). Now let $X = \mathbb{P}(E)$ be a projective bundle over B. An endomorphism f of X over B naturally induces a morphism from B to M. Its image must be a point since B is projective and M is affine. Let t be a lift of this point to $R^m(\mathbb{P}(V))$. Trivializing the bundle over a suitably fine open covering (U_α) of B we have, denoting by f_α the restriction of f to $p^{-1}(U_\alpha)$ written in the trivialization: $f_\alpha = h_\alpha \cdot t$, where h_α is in $PGL_{n+1}(\mathcal{O}_{U_\alpha})$. Denote by $g_{\alpha\beta}$ the transition functions of our projective bundle, it follows that $h_\alpha^{-1} g_{\alpha\beta} h_\beta \in Stab(t)$, in other words, by changing the trivialization we make the transition functions of X constant with values in a finite group.

In general, for an endomorphism f of $\mathbb{P}(E)$ we may suppose that f is over an endomorphism Φ of the base; there are then two cases to be treated: the case where f induces isomorphisms of fibers (considered as exceptional; when $X = \mathbb{P}(E)$ it means that $\Phi^* E$ is a twist of E by a line bundle) and the case where the degree of f on the fibers is greater than one. In [1], only the rank-two case (that of projective line bundles) was considered. It was established that either X is a finite quotient of a product or E has a subbundle. This last statement has been pursued further to yield that E must split into a direct sum of line bundles after a finite, not necessarily étale, base change ([1], theorem 2); from a different point of view, one can restrict to a specific class of bases to obtain a stronger statement. For instance, if B satisfies the condition $H^1(B, L) = 0$ for any line bundle L, then having a subbundle is equivalent to splitting for rank-two bundles. It therefore follows from the results of [1] that if B is simply connected and $H^1(B, L) = 0$ for any line bundle L on B, then an X with endomorphisms of degree greater than one on fibers must be the projectivization of a split rank-two bundle.

The purpose of the present note is to prove this result in the case of arbitrary rank projective bundles over such specific bases.

Theorem 2. *Let B be a simply-connected projective variety such that for any line bundle $\mathcal{L}$ its first cohomology $H^1(B, \mathcal{L}) = 0$. Let E be a vector bundle of rank $n+1$ on B. If there exists a fiberwise endomorphism*

$$\begin{tikzcd} \mathbb{P}(E) \arrow[r, "\phi"] \arrow[d, "\pi"] & \mathbb{P}(E) \arrow[d, "\pi"] \\ B \arrow[r, "\Phi"] & B \end{tikzcd} \tag{1}$$

of degree greater than one on the fibers, then E splits into a direct sum of line bundles:

$$E = \overset{n}{\underset{i=0}{\oplus}} \mathcal{L}_i \tag{2}$$

What we show is in fact slightly more general, as in [1].

Theorem 3. *Let B be as in the previous theorem and E and F vector bundles of the same rank $n+1$ on B. If there exists a morphism*

$$\phi : \mathbb{P}(E) \to \mathbb{P}(F) \tag{3}$$

over B which is of degree greater than $n+1$, then E and F both split into a direct sum of line bundles.

Obviously, Theorem 2 follows from this statement: consider the endomorphism ϕ as a morphism $\mathbb{P}(E) \to \mathbb{P}(\Phi^*(E))$, and apply Theorem 3 after replacing ϕ by a suitably high power.

To sum up, one has the following.

Corollary 1. *Let B be as in the previous theorems and E a vector bundle over B. Assume that $\mathbb{P}(E)$ has an endomorphism f of degree greater than one. Then either $g^*E \cong E \otimes L$ for some endomorphism g of B and some line bundle L over B* (*and then a power of f is induced by that pull-back*), *or E splits into a direct sum of line bundles.*

Indeed, a power of f is over an endomorphism of B; according to Theorem 2, we are in the second case whenever the degree of this power on the fibers is greater than one. If this degree is equal to one, this means that we are in the first case.

In the ideal situation, one would like to prove the statement of Theorem 2 for an arbitrary toric base B. The reason is that the projectivization of a vector bundle over a toric base is itself toric if and only if the bundle is split ([7]). This would therefore strongly support the principle that varieties with endomorphisms are closely related to toric or abelian varieties. However few toric bases (e.g. $\mathbb{P}^n, n \geq 2$, or products of such) actually satisfy the cohomology vanishing condition as above; so more work is needed to obtain such a result. It is probably related to the fact that we never make use of a condition like $F = g^*E$ in Theorem 3.

The paper is organized as follows. We recall the general set-up in Section 1 and make a fiberwise invariant-theoretic calculation in Section 2. This calculation implies the fiberwise existence of invariant subspaces with certain properties. In Section 3, we glue them together in a subbundle and conclude by induction.

1. Reduction to invariant theory

Let V and W be vector spaces of dimension $n+1$. Denote by $R^m(\mathbb{P}(V),\mathbb{P}(W))$ the set of all morphisms between $\mathbb{P}(V)$ and $\mathbb{P}(W)$ given by homogeneous polynomials of degree m without a common zero except at $(0,0,\dots,0)$:

$$\begin{aligned} y_0 &= f_0(x_0,x_1,\dots,x_n),\\ y_1 &= f_1(x_0,x_1,\dots,x_n),\\ &\cdots\\ y_n &= f_n(x_0,x_1,\dots,x_n). \end{aligned} \tag{4}$$

This is an affine variety, indeed the complement to the hypersurface defined by the resultant of the f_i in the projective space $\mathbb{P}(\mathrm{Hom}(W^*, S^mV^*))$, with the action of $PGL(V)\times PGL(W)$ given by

$$((g,h)\cdot f)(x) = h^{-1}(f(g(x))). \tag{5}$$

The quotient M of $R^m(\mathbb{P}(V),\mathbb{P}(W))$ by this action (i.e., the spectrum of the ring of invariants), in contrast with the case of the action of $PGL(V)$ when $V = W$ (see [1], Proposition 1.1) is not a geometric quotient: indeed some points have infinite stabilizers, and all the adherent orbits give the same point on the quotient. Let us denote by M_0 the "bad subset" of M (by definition it consists of points corresponding to orbits not separated by the invariants).

When some fiber of a vector bundle E over B is identified with V and that of F with W, a morphism of projective bundles $\mathbb{P}(E)\to\mathbb{P}(F)$ over a base B gives, in the same way as in the paper [1], a map from B to M, which must be constant as soon as B is projective.

The following claim is proved exactly as in [1, p. 22] (this argument is also recalled in the introduction to the present paper).

Claim 1. *If the image point is not in M_0, then $\mathbb{P}(E)$ and $\mathbb{P}(F)$ trivialize after a finite unramified base change.*

If B is simply-connected, this yields that these are already trivial on B, and in particular they split into a direct sum of line bundles.

So the interesting case is when the image point lands in M_0. In this situation, we strive to deduce some information about the geometry of our morphism. We aim to show that E and F have subbundles E' and F' such that the inverse image of $\mathbb{P}(F')$ is $\mathbb{P}(E')$ and that the map f induces a morphism on the quotients. This shall enable us to conclude by induction in the case when the cohomological condition on B is satisfied.

Let us also remark that replacing our original endomorphism ϕ of $\mathbb{P}(E)$ by a power, we may assume that m is greater than the rank $n+1$ of the vector bundles E and F, as we shall for the computations in the next section.

2. Unstable morphisms

In this section we consider two vector spaces V and W of dimension $n+1$ and a morphism f between their projectivisations of degree $d = m^n$. First of all assume f is stabilized by an infinite subgroup $Stab(f)$ in $PGL(V) \times PGL(W)$. Recall from [1]:

Lemma 1 ([1], Lemma 1.2)**.** *If $m > n+1$, then a unipotent element u of $PGL(V) \times PGL(W)$ does not stabilize any element of $R^m(\mathbb{P}(V), \mathbb{P}(W))$.*

By this lemma the subgroup $Stab(f) \subset PGL(V) \times PGL(W)$ consists of semisimple elements. Take any of these elements and consider the minimal subgroup in the stabilizer that contains this element. The connected component of the unity of this subgroup is an algebraic torus or trivial. If it is trivial for any element in $Stab(f)$, then the stabilizer is discrete and therefore is finite. If $Stab(f)$ is infinite, it contains a subgroup isomorphic to $\mathbb{G}_m$. Lifting its action on $\mathbb{P}(V)$ and $\mathbb{P}(W)$ to an action on V and W we assume that it is given by

$$\begin{aligned} &g_{b,c} : \mathbb{G}_m \to GL(V) \times GL(W) \\ &g_{b,c}(\lambda) = (diag(\lambda^{c_0}, \lambda^{c_1}, \dots, \lambda^{c_n}); diag(\lambda^{b_0}, \lambda^{b_1}, \dots, \lambda^{b_n})) \end{aligned} \tag{6}$$

in appropriate coordinates on V and W.

In these coordinates, let the morphism f be given by $(f_0, f_1, \dots, f_n)$ with

$$\begin{aligned} y_0 &= f_0(x_0, \dots, x_n) = \sum_{|I|=m} a_{0,I} x^I, \\ y_1 &= f_1(x_0, \dots, x_n) = \sum_{|I|=m} a_{1,I} x^I, \\ &\dots \\ y_n &= f_n(x_0, \dots, x_n) = \sum_{|I|=m} a_{n,I} x^I. \end{aligned} \tag{7}$$

Here $I = (i_0, i_1, \dots, i_n)$ is a multiindex and $|I| = i_0 + i_1 + \dots + i_n$.

Applying an element of the diagonal group in $g_{b,c}(\lambda) \in Stab(f)$, we get the following formulae for $g_{b,c} \cdot f$:

$$y_j = \sum \lambda^{\langle c, I\rangle - b_j} a_{j,I} x^I. \tag{8}$$

Here $\langle -, - \rangle$ denotes the scalar product between multiindexes:

$$F(I) := \langle c, I \rangle = \sum_{j=0}^{n} c_j i_j. \tag{9}$$

Since $g_{b,c}$ stabilizes f there exists a constant C, such that for any j, I with $a_{j,I} \neq 0$

$$\langle c, I \rangle - b_j = C. \tag{10}$$

Consider the $(n+1)$-dimensional lattice $\Lambda \cong \mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1} = \Lambda \otimes \mathbb{R}$. Denote by $p_i \in \Lambda$ the vertex corresponding to the i-th base vector $(0, \dots, 0, m, 0, \dots, 0)$. For any subset $\{p_{i_1}, \dots, p_{i_k}\}$ denote by $\Delta(p_{i_1}, \dots, p_{i_k}) \subset \Lambda$ the simplex of dimension $k-1$ with vertexes $p_{i_1}, \dots, p_{i_k}$. Set

$$\Delta = \Delta(p_0, p_1, \dots, p_n) \subset \Lambda. \tag{11}$$

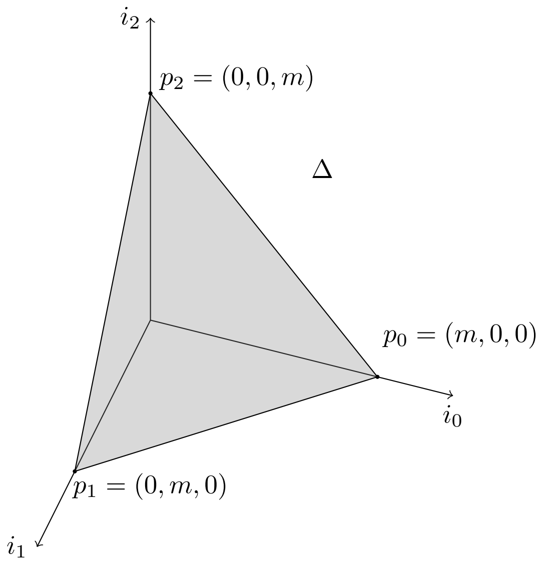

FIGURE 1. The simplex Δ in the case $n = 2$

Equations (10) define $n+1$ hyperplanes in $\mathbb{R}^n$ (not necessarily distinct). Let us denote them by Π_j.

Now let us consider the Newton polyhedron of f_j:

$$NP(f_j) := Conv\{I \in \Lambda \mid a_{j,I} \neq 0\} \tag{12}$$

and prove some easy facts about Newton polyhedra of the morphism f.

Proposition 1. *If f has infinite stabilizer, then $NP(f_j) \subset \Pi_j \cap \Delta$.*

Proof. As the degree of f_j equals m, the polyhedron $NP(f_j)$ lies in the simplex Δ. By the previous calculation we see that if $g_{b,c}$ stabilizes f, then (10) holds and consequently the multi-indices of the monomials of f_j lies in the hyperplane Π_j. $\square$

Lemma 2. *If f is a morphism of projective spaces, then every vertex of Δ is contained in one of the hyperplanes Π_j.*

Proof. Assume the vertex $p_0 = (m, 0, \dots, 0)$ does not lie in any Π_j. Consequently no polynomial f_j contains the monomial x_0^m. Then all f_j vanish at the point $(1 : 0 : \cdots : 0) \in \mathbb{P}(V)$, so f is not a morphism. $\square$

Lemma 3. *Each hyperplane Π_j contains some vertex of Δ. Moreover a hyperplane repeated exactly $k+1$ times (i.e., corresponding to the polynomials $f_0, \dots, f_k$, up to renumbering) contains exactly $k+1$ vertices of Δ.*

Proof. Since all the hyperplanes Π_j are parallel, if they contain a common vertex they coincide. There is a natural partition of the set $H(f)$ of equations

$$H_1 \sqcup H_2 \sqcup \cdots \sqcup H_l = H(f), \tag{13}$$

where a subset H_i consists of equations corresponding to the same hyperplane Π_i, as well as of the set of vertices

$$V(\Delta) = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_l, \tag{14}$$

where V_i consists of vertices lying in Π_j.

Since $|V(\Delta)| = n+1 = |H(f)|$ it follows that either the statement of the lemma is true or $k+1 = |V_i| > |H_i| = s+1$ for some i.

Assume $|V_i| > |H_i|$. The polynomials f_i indexed by H_i contain monomials depending only on the variables indexed by V_i, but the others do not: up to renumbering, $f_{s+1}, \dots, f_n$ are zero as soon as $x_{k+1} = \cdots = x_n = 0$. Then $f_0, \dots, f_s$ define a regular map of the subspace of $\mathbb{P}(V)$ given by the vanishing of $x_{k+1}, \dots, x_n$ to the subspace of $\mathbb{P}(W)$ given by the vanishing of $y_{s+1}, \dots, y_n$, but this is impossible since the dimension of the source would then be greater than that of the target. $\square$

From these assertions we deduce the following statement.

Proposition 2. *Let f be a morphism between $\mathbb{P}(V)$ and $\mathbb{P}(W)$ with infinite stabilizer in $PGL(V) \times PGL(W)$. There exist $V' \subset V(\Delta)$ and $H' \subset H(f)$ such that $|V'| = |H'| < n+1$ and*

$$NP(f_j) \subset \Delta(V') \tag{15}$$

for any $f_j \in H'$.

Proof. Let us recall the function F from (9). Denote $M' = \max\{F(p_i)\}$, where p_i runs through the set of vertices of Δ. Set

$$H' = \{f_j \mid F|_{\Pi_j} = M'\}. \tag{16}$$

As F is not constant on Δ, $\emptyset \subsetneq H' \subsetneq H(f)$. Denote by V' the set of vertices on the hyperplane corresponding to the equations in H'. By the previous lemma $|V'| = |H'|$. Obviously, $\Pi_j \cap \Delta = \Delta(V')$ and so the polynomials $f_j \in H'$ depend only on the variables corresponding to the vertices in V'. $\square$

So far, we have discussed the morphisms of projective spaces with infinite stabilizer in $PGL(V) \times PGL(W)$. But our goal is to study the morphisms f with non-closed orbits under the group action. By a generalization of the

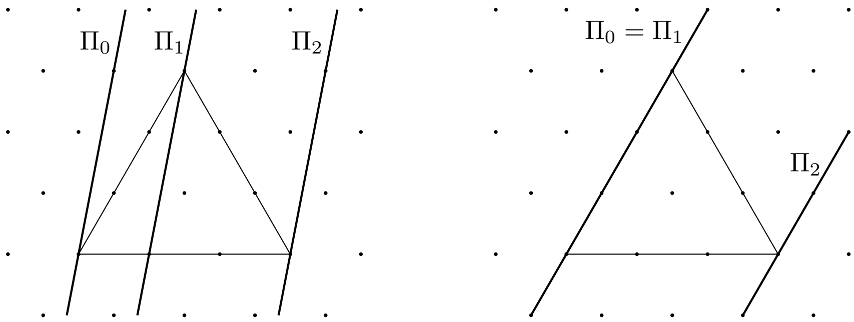

FIGURE 2. Two types of Newton polyhedra of f_0, f_1 and f_2 in the case $n = 2$.

Hilbert–Mumford criterion ([5] Theorem 4.2), we reach the boundary of the orbit $(PGL(V) \times PGL(W)) \cdot f$ while acting on f by one-parameter subgroups $g_{b,c}(\mathbb{G}_m)$ as in (6). As earlier, the map $g_{b,c}(\lambda) \cdot f$ is given by the equations (8). Let us introduce a new notation

$$K_j = \min_{\{I \mid a_{j,I} \neq 0\}} \{\langle c, I\rangle - b_j\}. \tag{17}$$

Set $K = \min_j \{K_j\}$. Then we can describe the limit of $g_{b,c}(\lambda) \cdot f$ when λ goes to zero.

Lemma 4. *Denote* $\bar{f} = \lim_{\lambda \to 0}(g_{b,c}(\lambda) \cdot f)$*, then*

$$\bar{f}_j(x_0, \ldots, x_n) = \sum_{\langle c,I\rangle - b_j = K} a_{j,I} x^I \tag{18}$$

and the original map was of type:

$$f_j = \sum_{\langle c,I\rangle - b_0 = K} a_{j,I} x^I + \sum_{\langle c,I\rangle - b_0 > K} a_{j,I} x^I. \tag{19}$$

The proof is a straightforward calculation.

Obviously, the group $g_{b,c}(\mathbb{G}_m)$ stabilizes the morphism $\bar{f}$, so $\bar{f}$ has infinite stabilizer and in Proposition 2 we have a description of its Newton polyhedron. Now consider the set of half-spaces

$$\Pi_j^+ = \{I \in \Lambda \otimes \mathbb{R} \mid \langle c, I\rangle - b_j \geq K \subset \mathbb{R}^n\}. \tag{20}$$

Lemma 4 implies that $NP(f_j) = \Pi_j^+ \cap \Delta$. From the proof of Proposition 2 we see that there is always a hyperplane Π_j intersecting our simplex Δ by a face and such that the rest of the simplex is below Π_j. Thus the following holds.

Proposition 3. *If f is an unstable morphism between $\mathbb{P}(V)$ and $\mathbb{P}(W)$, then there are nonempty sets $V' \subset V(\Delta)$ and $H' \subset H(f)$ such that $|V'| = |H'| < n + 1$ and*

$$NP(f_j) \subset \Delta(V') \tag{21}$$

for any $f_j \in H'$.

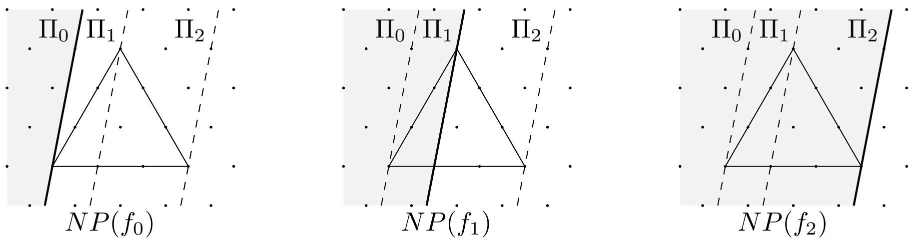

FIGURE 3. Here are Newton polyhedra of f_i in the case $n = 2$. On each picture $NP(f_i)$ is shadowed.

Proof. Actually, consider the set H' from the previous lemma. As for any $\Pi_j \in H'$, the restriction of function F to Π_j equals $M' = \max\{F(p_i)\}$, then

$$\Delta \subset \{I|\ F(I) \leq M'\}. \tag{22}$$

Therefore for any $\Pi_j \in H'$ the half-space Π_j^+ also intersects Δ by $\Delta(V')$. □

In the language of equations this means that the first $s+1$ equations depend only on the first $s+1$ variables.

3. Proof of the theorem

By our analysis from the previous section, the fiberwise morphisms have the following property: after a suitable identification $\mathbb{P}(V) \cong \mathbb{P}(W) \cong \mathbb{P}^n$, there is a projective subspace $Z = \mathbb{P}^k \subset \mathbb{P}^n$ such that $f^{-1}(Z) = Z$. We would like to show that these fiberwise subspaces glue into global subbundles $\mathbb{P}(E_0) \subset \mathbb{P}(E)$, $\mathbb{P}(F_0) \subset \mathbb{P}(F)$, and proceed by induction. The problem here is that the subspace Z is in general not unique, and so the above claim is not obvious. Our strategy is to show that these subspaces are finitely many in each fiber, and that the variety $\mathcal{Z}$ parameterizing them in the relative Grassmannian $Gr_k(\mathbb{P}(E))$ is unramified over B. Since we consider a simply-connected B, this means that a component of $\mathcal{Z}$ is a section, producing a subbundle of $\mathbb{P}(E)$, and similarly for $\mathbb{P}(F)$.

If $h : \mathbb{P}^n \to \mathbb{P}^n$ is an endomorphism given by degree m polynomials, then the subspaces $\mathbb{P}^k = Z \subset \mathbb{P}^n$ with the property $h^{-1}Z = Z$, sometimes called *completely invariant*, have been considered before, in particular, because of their importance for holomorphic dynamics (see for example [2], [6]). If Z is such a subspace, then at a general point $x \in Z$ one has $\deg_x(h) = m^{n-k}$, where $\deg_x$ is the *local degree* at x, which means the number of branches of h coming together at x. More precisely, one defines $\deg_x(h)$ as the number of preimages of a general point y close to $h(x)$ in a small neighbourhood of x. The set of points x where the local degree $\deg_x(h) \geq N$ is a closed algebraic subset, so for any irreducible algebraic subset A it makes sense to define $\deg_A(h)$ as the local degree at a general point of A. In our case, $\deg_Z(h) = \frac{\deg(h)}{\deg(h|_Z)} = m^{n-k}$. The following lemma on local degrees is due to Briend and Duval ([6], Erratum).

Lemma 5. *Possibly after replacing h by a power, one has $\deg_A(h) \leq m^p$ for all codimension-p subvarieties A.*

As an immediate consequence, one gets:

Corollary 2. *The number of completely invariant $\mathbb{P}^k$'s is finite for any k.*

Proof. Indeed, as any subset completely invariant by h is also completely invariant by h^m for any m, one may suppose that h itself satisfies the conclusion of the lemma. Let A be a component of the Zariski closure of the union of the completely invariant subspaces of dimension k. If A is of dimension $> k$, we have $\deg_A h < m^{n-k}$. On the other hand A has a Zariski-dense subset along which the local degree is at least m^{n-k}, but this is a contradiction since the subsets $\{x \mid \deg_x(h) \geq N\}$ are closed. □

Remark 1. Note also that given a family of endomorphisms $f_t : \mathbb{P}^n \to \mathbb{P}^n$ parameterized by a base T and a point $0 \in T$, two distinct codimension-p totally invariant subspaces for f_t cannot converge to the same subspace in the limit f_0. Indeed otherwise the local degree of any iterate f_0^l at this limiting subspace would be strictly greater than m^{lp} (the local degree of f_t^l along each of the two subspaces), contradicting Lemma 5.

Now we are ready to deduce from the previous section a key proposition about morphisms between projective bundles:

Proposition 4. *Assume $\phi : \mathbb{P}(E) \to \mathbb{P}(F)$ is a morphism over the base B of degree $d > 1$, such that its restriction to a fiber corresponds to an unstable orbit in $R^m(\mathbb{P}(V), \mathbb{P}(W))$. Then there are subbundles $E_0 \subsetneq E$ and $F_0 \subsetneq F$, such that*

$$\phi^{-1}(\mathbb{P}(F_0)) = \mathbb{P}(E_0) \tag{23}$$

and $0 < rk(E_0) = rk(F_0) < rk(E) = rk(F)$.

Proof. By the results in the previous section, in any fiber of $\mathbb{P}(F)$ there are coordinates in which for any $0 \leq j \leq s$

$$y_j = f_j(x_0, \ldots, x_s). \tag{24}$$

We claim that the preimage of the subspace $H = \{y_0 = \cdots = y_s = 0\}$ is the subspace $\{x_0 = \cdots = x_s = 0\}$. Indeed the last subspace is certainly contained in the preimage of the first one. If there is another point $P = (p_0 : \cdots : p_s : p_{s+1} : \cdots : p_n)$ in that preimage, consider the projective subspace generated by P and the last $n - s$ base vectors: its dimension is $n - s$, so it must have nonempty intersection with the subvariety given by the equations

$$f_{s+1}(x_0, \ldots, x_n) = \cdots = f_n(x_0, \ldots, x_n) = 0 \tag{25}$$

which has dimension at least s. Any point in this intersection must be an indeterminacy point of f, a contradiction.

It remains to show that these subspaces H fit together in a subbundle $F_0 \subsetneq F$ and the same happens to their preimages, giving a subbundle $\phi^{-1}(F_0) = E_0 \subsetneq E$. Our source of inspiration is Corollary 2 together with Remark 1. These have to be adjusted since our source and target are not canonically identified and the subspace H becomes completely invariant only after a choice of such an identification; if another choice is taken, H ceases to be completely invariant and a new completely invariant subspace might appear. Nevertheless all such subspaces H have a geometric property independent on choices: the local degree along H is $m^{codim(H)}$ and its image is again a projective subspace. This property also subsists under iteration. The argument of Corollary 2 thus shows the finiteness of possible such H in each fiber of our projective bundle.

Finally, consider the parameter space for such subspaces in the fibers of, say, $\mathbb{P}(E)$, finite over the base B. Take any of its irreducible components $\mathcal{Z}$ dominating B. If $\mathcal{Z}$ ramifies over $b \in B$, this means, after trivializing and identifying both projective bundles over a small analytic neighbourhood of b, that two projective subspaces of a neighbouring fiber, with the largest possible (for a subvariety of given codimension) local degree along them, tend to the same limit projective subspace in the central fiber. This gives a contradiction as the local degree along this limit subspace must be even greater (as in Remark 1). In conclusion, since B is simply-connected, $\mathcal{Z}$ maps one-to-one to B and the universal family over $\mathcal{Z}$ gives a gluing of the subspaces in the fibers into a subbundle of $\mathbb{P}(E)$. $\square$

To complete the proof of the theorem let us consider a linear mapping induced by the morphism ϕ:

$$\phi^* : F^* \to S^m E^*. \tag{26}$$

As we have shown we have subbundles E_0 and F_0, such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & (F/F_0)^* \arrow[r] \arrow[d, "(\phi/\phi_0)^*"'] & F^* \arrow[r] \arrow[d, "\phi^*"'] & F_0^* \arrow[r] \arrow[d, "\phi_0^*"'] & 0 \\ 0 \arrow[r] & (S^m E/S^m E_0)^* \arrow[r] & S^m E^* \arrow[r] & S^m E_0^* \arrow[r] & 0 \end{tikzcd} \tag{27}$$

Consider the bundle $(S^m E/S^m E_0)^*$ and write

$$(S^m E/S^m E_0)^* \cong \oplus_{i=0}^{m-1} S^i E_0^* \otimes S^{m-i}(E/E_0)^*. \tag{28}$$

In particular there is a projection

$$(S^m E/S^m E_0)^* \xrightarrow{pr_0} S^m(E/E_0)^* \tag{29}$$

and $pr_0 \circ (\phi/\phi_0)^* = \psi^*$ induces a map between projective bundles $\mathbb{P}(E/E_0)$ and $\mathbb{P}(F/F_0)$ given by degree m polynomials. In fact this map is regular, that is, a morphism. To check this one observes that one may view $(x_0 : \dots : x_s)$ and $(y_0 : \dots : y_s)$ from Corollary 4 as coordinates on the projectivization of the quotients, and the map of these projectivizations is then given by $f_0, \dots f_s$. To say that this map has no indeterminacy point $(p_0 : \dots : p_s)$ is the same as to say that the preimage of $\mathbb{P}(F_0)$ from Corollary 4 contains nothing but $\mathbb{P}(E_0)$.

Proof of Theorem 3. We argue by induction on $n+1 = rkE$. If $rkE = 1$, then E is already linear, so the base of induction is trivial.

Suppose now, that for all ranks less then $n+1$ the statement is true. The restriction of the morphism ϕ to a fiber gives us an element in $R^m(\mathbb{P}(V), \mathbb{P}(W))/(PGL(V) \times PGL(W))$.

If this element corresponds to a stable orbit in $R^m(\mathbb{P}(V), \mathbb{P}(W))$, then the argument in the proof of Theorem 1 in [1, p. 22] (see also Claim 1 and introduction of loc.cit.), proves that after a finite étale base change both $\mathbb{P}(E)$ and $\mathbb{P}(F)$ trivialize. As the variety B is simply-connected, there are no nontrivial étale base changes, so both $\mathbb{P}(E)$ and $\mathbb{P}(F)$ are trivial and hence split.

If we get an unstable orbit, then by Corollary 4 the bundles E and F sit in short exact sequences:

$$\begin{aligned} &0 \to E_0 \to E \to E/E_0 \to 0, \\ &0 \to F_0 \to F \to F/F_0 \to 0 \end{aligned} \tag{30}$$

and there are morphisms given by polynomials of the same degree $m > 1$ between the projectivisations of bundles E_0, F_0, E/E_0 and F/F_0, namely

$$\begin{aligned} &\phi_0 : \mathbb{P}(E_0) \to \mathbb{P}(F_0), \\ &\psi : \mathbb{P}(E/E_0) \to \mathbb{P}(F/F_0). \end{aligned} \tag{31}$$

By the inductive assumption all these bundles must split into direct sums of line bundles. Since for any line bundle $\mathcal{L}$ on B, its first cohomology $H^1(B, \mathcal{L}) = 0$, we see that

$$Ext^1(E/E_0, E_0) = Ext^1(F/F_0, F_0) = 0. \tag{32}$$

So the extensions are trivial too. Consequently E and F split into a direct sum of line bundles. □

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