Universal cohomological expressions for singularities in families of genus 0 stable maps

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Abstract

We consider families of curve-to-curve maps that have no singularities except those of genus 0 stable maps and that satisfy a versality condition at each singularity. We provide a universal expression for the cohomology class Poincaré dual to the locus of any given singularity. Our expressions hold for any family of curve-to-curve maps satisfying the above properties.

1 Introduction

1.1 A family of curve-to-curve maps

A family of curve-to-curve maps is a triple of smooth complex manifolds X, Y and B and a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {}_{p}\searrow \quad \swarrow{}_{q} & \\ & B & \end{array} \qquad (1)$$

satisfying the following properties.

1. The map p is a flat family of reduced not necessarily compact nodal curves.

2. The map q is a flat family of reduced not necessarily compact smooth curves.

3. The map f is proper.

In Section 1.4 we will introduce a list of allowed singularities for the map f. These will be precisely the singularities of genus 0 stable maps. For each singularity we will also require a versal deformation property: informally, the map f should locally present all possible deformations of each singularity. Assuming that f has allowed singularities only and satisfies the versal deformation property, we will compute the cohomology class of every singularity locus.

The forthcoming paper [4] extends the results of the present one to the study of multisingularities.

1.2 Thom polynomials: a motivation

Let $f : X \to Y$ be a sufficiently generic map between two complex compact manifolds. Denote by

$$c(f) = c(f^*(T_*Y))/c(T_*X) = 1 + c_1(f) + c_2(f) + \ldots,$$

the full Chern class of the map f, $c_i(f) \in H^{2i}(X)$.

Given a singularity type α, one can consider the locus of points $x \in X$ such that f has a singularity of type α at x. This locus is a submanifold, and we will denote the Poincaré dual cohomology class of its closure by $[\alpha] \in H^*(X, \mathbb{Z})$.

In [8] Thom proved that *for every singularity type α and for every generic map f we have $[\alpha] = P_\alpha(c_1(f), c_2(f), \ldots)$, where $P_\alpha(c_1, c_2, \ldots)$ is a polynomial independent of f.*

For instance, we have $[A_1] = c_1(f)$, thus $P_{A_1} = c_1$.

The polynomials P_α are called *Thom polynomials*. It is, in general, quite hard to compute them (Thom's proof is not constructive); however computing even a single Thom polynomial may provide a solution for many seemingly unrelated problems of enumerative geometry.

This paper grew out of an attempt to "apply" Thom's theorem to the map f of Eq. (2), a case to which it is not applicable since the genericity condition for f brakes down due to existence of nonisolated singularities.

In Section 1.4 we describe all singularity types that can appear in a family of genus 0 stable maps. In Section 1.7 we introduce a family of cohomology classes (larger than just the classes $c_i(f)$) that we will call *basic classes*. Then we prove that the class determined by every singularity type has a

universal polynomial expression in terms of the basic classes, independent of the family f. Moreover, we give a simple way to compute these expressions.

1.3 Local models

Consider two families of curve-to-curve maps.

$$
\begin{array}{ccc}
X \xrightarrow{\ f\ } Y & \qquad & X' \xrightarrow{\ f'\ } Y' \\
p \searrow \ \swarrow q & & p' \searrow \ \swarrow q' \\
B & & B'
\end{array}
\tag{2}
$$

Let $y \in Y$ and $b = q(y) \in B$; $y' \in Y'$ and $b' = q'(y') \in B'$. Let X_0 be a closed subset of $f^{-1}(y)$ and X_0' a closed subset of $(f')^{-1}(y')$. In practice, X_0 will be either a point or a connected component of $f^{-1}(y)$ contracted by f.
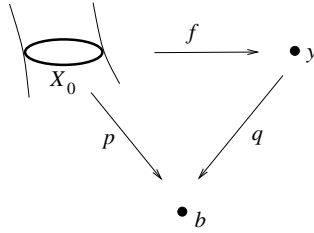
Definition 1.1 We say that the family (X', Y', B') is a *local model* for the family (X, Y, B) if there exist three holomorphic submersions $\varphi_X : X \to X'$, $\varphi_Y : Y \to Y'$, $\varphi_B : B \to B'$ defined in a neighborhood of X_0, y, and b respectively, identifying X_0 with X_0' and commuting with the maps p, q, f.

1.4 Singularity types

Let $x \in X$ be any point, $b = p(x) \in B$ and $y = f(x) \in Y$ its images in B and Y, respectively, and X_b the fiber of p through x. We will describe several types of singularities of f at x, namely, ramification points, nodes, and contracted components. These are the only singularities that can arise in families of genus 0 stable maps. A map $C \to \mathbb{CP}^1$ of a genus 0 nodal curve C to $\mathbb{CP}^1$ is said to be *stable* if its automorphism group is finite. In other words, a map is stable if each irreducible component of the domain C that is contracted to a point has at least three points of intersection with other irreducible components of C.

For each singularity type we will define a versality condition using local models. Both points and nodes are isolated singularities, while contracted

components are nonisolated ones. Universal polynomials for isolated singularities in the absence of non-isolated ones were found in [2, 3]. They determine the singularity classes modulo cohomology classes supported on the subvariety in X consisting of contracted components. Including contracted components completes the computation of universal polynomials for genus 0 curves.

1.4.1   Ramification points

Definition 1.2 We say that f presents a *singularity of type A_k at x* if x is a smooth point of X_b and $f|_{X_b}$ has a ramification of order $k+1$ at x.

The local model for the A_k singularity is given by the family

$$v(u, \gamma_1, \ldots, \gamma_{k-1}) = u^{k+1} + \gamma_1 u^{k-1} + \cdots + \gamma_{k-1} u.$$

Here:
$(\gamma_1, \ldots, \gamma_{k-1})$ is a system of coordinates in B',
$(\gamma_1, \ldots, \gamma_{k-1}, u)$ a system of coordinates in X',
$(\gamma_1, \ldots, \gamma_{k-1}, v)$ a system of coordinates in Y'.

Definition 1.3 Let $x \in X$ be a point at which f presents an A_k singularity. We say that the family (X, Y, B) is *versal* at x if it is locally modeled at the neighborhood of x by the family (X', Y', B') above.

1.4.2   Nodes

Definition 1.4 We say that f presents a *singularity of type I_{k_1,k_2} at x* if x is a node of X_b and f has ramifications of orders k_1 and k_2 on the branches of X_b at x.

A local model for the singularity I_{k_1,k_2} is given by the family

$$\beta = u_1 u_2,$$

$$v = u_1^{k_1} + \gamma_1^{(1)} u_1^{k_1-1} + \cdots + \gamma_{k_1-1}^{(1)} u_1 + u_2^{k_2} + \gamma_1^{(2)} u_2^{k_2-1} + \cdots + \gamma_{k_2-1}^{(2)} u_2.$$

Here:
$(\gamma_1^{(1)}, \ldots, \gamma_{k_1-1}^{(1)}, \gamma_1^{(2)}, \ldots, \gamma_{k_2-1}^{(2)}, \beta)$ is a system of coordinates in B',
$(\gamma_1^{(1)}, \ldots, \gamma_{k_1-1}^{(1)}, \gamma_1^{(2)}, \ldots, \gamma_{k_2-1}^{(2)}, u_1, u_2)$ a system of coordinates in X',
$(\gamma_1^{(1)}, \ldots, \gamma_{k_1-1}^{(1)}, \gamma_1^{(2)}, \ldots, \gamma_{k_2-1}^{(2)}, \beta, v)$ a system of coordinates in Y'.

Definition 1.5 Let $x \in X$ be a point at which f presents an I_{k_1,k_2} singularity. We say that the family (X, Y, B) is *versal* at x if it is locally modeled at the neighborhood of x by the family (X', Y', B') above.
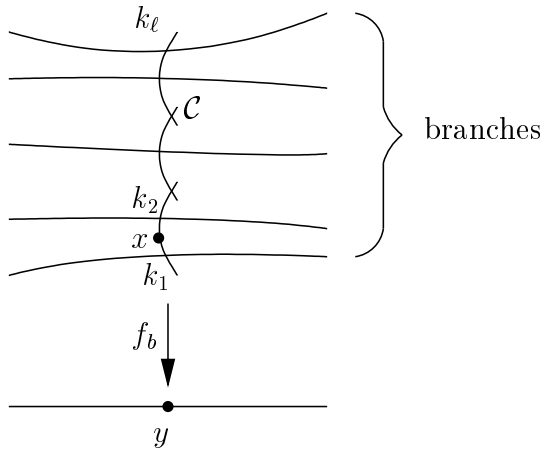
Figure 1: The contracted part $\mathcal{C}$ of a point x. Branches are indicated, together with the orders $k_1, \ldots, k_\ell$ of the restriction of the function f_b to the corresponding branch

1.4.3 Contracted components

Stable maps of genus 0 can also present non-isolated singularities. Suppose that x happens to lie on an irreducible component of X_b contracted by f. Then we denote by $\mathcal{C}$ the *connected* component of $f_b^{-1}(y) \subset X_b$ containing x. Note that it necessarily includes the irreducible component of X_b containing x, but may include other irreducible components as well. We call $\mathcal{C}$ the *contracted part* of x. Finally we call *branches* the connected components of the neighborhood of $\mathcal{C}$ in $X_b \setminus \mathcal{C}$. If, on $\mathcal{C}$, we mark its intersection points $x_1, \ldots, x_\ell$ with the branches, it becomes a stable curve.

Definition 1.6 For $\ell \geq 3$, we say that f presents a *singularity of type* $I_{k_1, k_2, \ldots, k_\ell}$ *at* x if

- f is constant on the irreducible component of X_b containing x;

- the contracted part $\mathcal{C}$ has genus 0;

- there are ℓ branches and they can be numbered from 1 to ℓ in such a way that f has ramification orders $k_1, \ldots, k_\ell$ at their intersection points with $\mathcal{C}$.

To define a local model for this singularity we need to introduce the space of relative stable maps; however, the local model may actually be only a subspace of this space, see Definition 1.7.

Let $\overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}(\mathbb{CP}^1;\infty)$ be the space of stable maps of genus 0 to $\mathbb{CP}^1$, defined up to an additive constant, and relative to ∞ with ramification profile $(k_1,\ldots,k_\ell)$. This space has a distinguished sub-orbifold $\overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}^Z(\mathbb{CP}^1;\infty)$ that we will call the *zero locus*. It parametrizes maps from curves with $\ell+1$ components: one contracted rational component and ℓ rational components more that intersect the contracted one and on which the map equals z^{k_i}, $1 \leq i \leq \ell$. As an orbifold, the zero locus is a gerb with base $\overline{\mathcal{M}}_{0;\ell}$ and group $\prod_{i=1}^{\ell} \mathbb{Z}/k_i\mathbb{Z}$. Let

$p : \overline{\mathcal{C}}_{0;k_1,\ldots,k_\ell}(\mathbb{CP}^1;\infty) \to \overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}(\mathbb{CP}^1;\infty)$ be the universal curve,

$q : \mathbb{CP}^1 \times \overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}(\mathbb{CP}^1;\infty) \to \overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}(\mathbb{CP}^1;\infty)$ the universal target curve, and

$f : \overline{\mathcal{C}}_{0;k_1,\ldots,k_\ell}(\mathbb{CP}^1;\infty) \to \mathbb{CP}^1 \times \overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}(\mathbb{CP}^1;\infty)$ the universal map. The union of contracted parts of fibers over the zero locus $\overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}^Z(\mathbb{CP}^1;\infty)$ forms the zero locus $\overline{\mathcal{C}}_{0;k_1,\ldots,k_\ell}^Z(\mathbb{CP}^1;\infty)$ in the universal curve.

Let $x \in X$ be a point at which f presents an $I_{k_1,\ldots,k_\ell}$ singularity and $\mathcal{C}$ the corresponding contracted component. Let $b \in \overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}^Z(\mathbb{CP}^1;\infty)$ be the point of the zero locus such that the contracted component is isomorphic to $\mathcal{C}$. Consider a local chart of $\overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}(\mathbb{CP}^1;\infty)$ at b. Now, let B' be any sub-manifold of this local chart that is transversal to the zero locus and of dimension at least

$$\dim \overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}(\mathbb{CP}^1;\infty) - \dim \overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}^Z(\mathbb{CP}^1;\infty) = \sum k_i.$$

Let X' be a neighborhood of $\mathcal{C}$ in the preimage $(p')^{-1}(B')$ and Y' a neighborhood of $y = f(x)$ in the preimage $(q')^{-1}(B')$.

Definition 1.7 Let $x \in X$ be a point at which f presents an $I_{k_1,\ldots,k_\ell}$ singularity and $\mathcal{C}$ the corresponding contracted component. We say that f is *versal* at x if there exists a family (X,Y,B) as above such that (X,Y,B) is locally modeled on (X',Y',B') at a neighborhood of $\mathcal{C}$.

1.5 Versal families

Definition 1.8 A *genus 0 versal family of maps* is a family of maps that only has singularities of types A_k and $I_{k_1,\ldots,k_\ell}$ for $\ell \geq 2$ and satisfies the versality condition for each singularity.

Genus 0 versal families are precisely the object of our study. Before proceeding we give several examples of genus 0 versal families.

Example 1.9 Consider the space $\overline{\mathcal{M}}_{0;d}(\mathbb{CP}^1)$ of stable genus 0 degree d maps to $\mathbb{CP}^1$ without marked points. The universal map over this space is a genus 0 versal family.

Example 1.10 Given a line bundle $L \to B$ over a smooth base B, consider the space
$$(L \setminus \{\text{zero section}\}) \times_{\mathbb{C}^*} \overline{\mathcal{M}}_{0;d}(\mathbb{CP}^1) \to B.$$
Here $\mathbb{C}^*$ acts on $\overline{\mathcal{M}}_{0;d}(\mathbb{CP}^1)$ via the natural action on the target $\mathbb{CP}^1$. The universal map over this space is a genus 0 versal family.

Example 1.11 Given a principal $\mathrm{PSL}(2,\mathbb{C})$ bundle $G \to B$ over a smooth base B, consider the space

$$G \times_{\mathrm{PSL}(2,\mathbb{C})} \overline{\mathcal{M}}_{0;d}(\mathbb{CP}^1) \to B.$$

Here $\mathrm{PSL}(2,\mathbb{C})$ acts on $\overline{\mathcal{M}}_{0;d}(\mathbb{CP}^1)$ via the natural action on the target $\mathbb{CP}^1$. The universal map over this space is a genus 0 versal family.

Example 1.12 Consider the space $\overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}(\mathbb{CP}^1; \infty)$ of stable genus 0 maps to $\mathbb{CP}^1$ relative to ∞ with ramification profile $(k_1, \ldots, k_n)$. Consider the universal map over this space. Remove the ∞ section in the target curve and its preimages in the source curve. The family thus obtained is a genus 0 versal family.

Example 1.13 Given a line bundle $L \to B$ over a smooth base B, consider the space

$$(L \setminus \{\text{zero section}\}) \times_{\mathbb{C}^*} \overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}(\mathbb{CP}^1; \infty) \to B.$$

Here $\mathbb{C}^*$ acts on $\overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}(\mathbb{CP}^1; \infty)$ via the natural action on the target $\mathbb{CP}^1$. Consider the universal map over this space. Remove the ∞ section in the target curve and its preimages in the source curve. The family thus obtained is a genus 0 versal family.

Example 1.14 Consider the space $\overline{\mathcal{M}}_{g;d}(C)$ of degree d genus g stable maps to a given smooth curve C. Consider the universal map over this space. Remove from the universal map all the source curves that contain contracted components of genus greater than 0. Also remove the images of these curves in the target. The family thus obtained is a genus 0 versal family.

Example 1.15 Let $z = z_1, \ldots, z = z_\ell$ be a given set of points on $\mathbb{C}\mathrm{P}^1$ with affine coordinate z. Consider the family of maps

$$f(z) = \sum_{i=1}^{\ell} \left[\left(\frac{u_i}{z - z_i} \right)^{k_i} + \gamma_{i,1} \left(\frac{u_i}{z - z_i} \right)^{k_i - 1} + \cdots + \gamma_{i,k_i-1} \left(\frac{u_i}{z - z_i} \right) \right].$$

This family is extended to $u_i = 0$ in the following way. The source $\mathbb{C}\mathrm{P}^1$ acquires a bubble with global coordinate w_i. The point $w_i = 0$ is attached to $z = z_i$. The function on the bubble is given by

$$w_i^{k_i} + \gamma_{i,1} w_i^{k_i - 1} + \cdots + \gamma_{i,k_i-1} w_i.$$

This family is a genus 0 versal family.

Example 1.16 This is an example of a family that is NOT versal. Let $z' = z'_1, \ldots, z' = z'_{\ell_1+1}$ be a given set of points on the projective line with global coordinate z' and $z'' = z''_1, \ldots, z'' = z''_{\ell_2+1}$ be a given set of points on another projective line with global coordinate z''. The point z'_{ℓ_1+1} is identified with the point z''_{ℓ_2+1}. Consider the following family of maps on the nodal curve thus obtained:

$$f(z') = \sum_{i=1}^{\ell_1} \left[\left(\frac{u'_i}{z' - z'_i} \right)^{k'_i} + \gamma_{i,1} \left(\frac{u'_i}{z' - z'_i} \right)^{k'_i - 1} + \cdots + \gamma_{i,k'_i-1} \left(\frac{u'_i}{z' - z'_i} \right) \right] + C';$$

$$f(z'') = \sum_{i=1}^{\ell_2} \left[\left(\frac{u''_i}{z'' - z''_i} \right)^{k''_i} + \gamma_{i,1} \left(\frac{u''_i}{z'' - z''_i} \right)^{k''_i - 1} + \cdots + \gamma_{i,k''_i-1} \left(\frac{u''_i}{z'' - z''_i} \right) \right] + C'';$$

where the constants C' and C'' are such that $f(z'_{\ell_1+1}) = f(z''_{\ell_2+1}) = 0$. This family is extended to $u'_i = 0$ and $u''_i = 0$ as in the previous example.

This family is not versal. Indeed, even though it satisfies the versal property for the singularity $I_{k'_1,\ldots,k'_{\ell_1},k''_1,\ldots,k''_{\ell_2}}$, a generic map of this family also has an $I_{1,1}$ singularity that does not satisfy the versal property because the node is never smoothened.

1.6   Singularity loci

Every singularity from the list in Sec. 1.4 determines a cycle in X, namely the closure of the set of points $x \in X$ where f presents a given singularity. By abuse of notation we will usually denote in the same way the singularity type and the corresponding cycle in X.

For instance, the cycle $I_{1,1}$ is the set of nodes of the fibers X_b.

The versal deformation property insures that the codimension of the A_k locus equals k, that of the I_{k_1,k_2} locus equals $k_1 + k_2$, and that of the $I_{k_1,...,k_\ell}$ locus equals $\sum k_i$.

In the notation $I_{k_1,...,k_\ell}$ the indices form a multiset, that is, the order of the indices is immaterial. Since the cycle $I_{k_1,...,k_\ell}$ is the *closure* of the set where f presents the corresponding singularity, the ramification orders of f at the attaching points of the branches can actually be greater than (but not smaller than) k_i. Thus, for instance, the cycle $I_{1,2}$ presents a self-intersection along the cycle $I_{2,2}$. Therefore the normalization of $I_{1,2}$ contains two copies of $I_{2,2}$.

Let $|\mathrm{Aut}\{k_1, \ldots, k_\ell\}|$ be the number of permutations $\sigma \in S_\ell$ such that $k_{\sigma(i)} = k_i$ for every i.

Fix an order of the indices $k_1, \ldots, k_\ell$. Let $\widehat{I}_{k_1,...,k_\ell}$ be the space of couples $(x \in I_{k_1,...,k_\ell}, \text{numbering of the branches})$, such that the ramification order at the branch number i is equal to k_i (greater than or equal to k_i in the closure). Then $\widehat{I}_{k_1,...,k_\ell}$ is a finite nonramified covering of the normalization of $I_{k_1,...,k_\ell}$. Its degree equals $|\mathrm{Aut}\{k_1, \ldots, k_\ell\}|$.

Notation 1.17 We denote by $a_k \in H^{2k}(X, \mathbb{Q})$ the cohomology class Poincaré dual to the cycle A_k. We denote by $i_{k_1,...,k_\ell} \in H^{2\sum k_i}(X, \mathbb{Q})$ the cohomology class Poincaré dual to the cycle $I_{k_1,...,k_\ell}$ multiplied by $|\mathrm{Aut}\{k_1, \ldots, k_\ell\}|$.

Remark 1.18 Including the factor $|\mathrm{Aut}\{k_1, \ldots, k_\ell\}|$ in the definition of cohomology classes will greatly simplify the formulas in the sequel.

Notation 1.19 We denote by $U \colon \widehat{I}_{k_1,...,k_\ell} \to I_{k_1,...,k_\ell}$ the covering map and by $V \colon \widehat{I}_{k_1,...,k_\ell} \to \overline{\mathcal{M}}_{0,\ell+1}$ the natural map from the covering $\widehat{I}$ to the moduli space of curves:
$$I_{k_1,...,k_\ell} \xleftarrow{U} \widehat{I}_{k_1,...,k_\ell} \xrightarrow{V} \overline{\mathcal{M}}_{0,\ell+1}.$$
Given a class $\alpha \in H^*(\overline{\mathcal{M}}_{0,\ell+1})$, we denote by $\alpha i_{k_1,...,k_\ell}$ the class $U_* V^*(\alpha)$.

Note that the notation is coherent: if $\alpha = 1$ we have $U_* V^*(1) = i_{k_1,...,k_\ell}$ as defined previously, because the map u has degree $|\mathrm{Aut}\{k_1, \ldots, k_\ell\}|$.

Definition 1.20 The cohomology classes a_k and $\alpha i_{k_1,...,k_\ell}$ in $H^*(X, \mathbb{Q})$ are called *singularity classes*.

1.7 Basic classes

Now we are going to introduce a different set of cohomology classes in X. We will call them *basic classes*.

Definition 1.21

Let $\mathcal{L}_X \to X$ be the line bundle whose fiber over $x \in X$ is the *cotangent line to X_b at x*. This line bundle is well-defined outside $I_{1,1}$ and can be uniquely extended to $I_{1,1}$ (in the standard way). *We denote by $\psi = c_1(\mathcal{L}_X) \in H^2(X, \mathbb{Q})$ its first Chern class.*

Let $\mathcal{L}_Y \to X$ be the line bundle whose fiber over $x \in X$ is the *cotangent line to $f(X_b)$ at $y = f(x)$*. This line bundle is well-defined everywhere and *we denote by $\xi = c_1(\mathcal{L}_Y) \in H^2(X, \mathbb{Q})$ its first Chern class.*

Consider the cycle $I_{1,\dots,1}$ with ℓ subscripts. Over the $\ell!$-sheeted covering $\widehat{I}_{1,\dots,1}$ consider the ℓ line bundles whose fibers are the *cotangent lines to the branches at their intersection points with the contracted part*. Denote their first Chern classes by $\nu_1, \dots, \nu_\ell$. In a more traditional notation, we have $\nu_i = V^* \psi_i$, $i = 1 \dots, \ell$.

Let $\alpha \in H^*(\overline{\mathcal{M}}_{0,\ell+1})$ be a cohomology class. We denote by $\alpha\delta_{m_1,\dots,m_\ell}$ the cohomology class $U_*(\nu_1^{m_1} \cdots \nu_\ell^{m_\ell} V^* \alpha)$.

Definition 1.22 The classes ψ^k, and $\alpha\delta_{k_1,\dots,k_\ell}$ are called *basic cohomology classes*.

Our aim is now to express singularity classes in terms of the basic classes and vice versa.

1.8   Main results

We start with an explicit equality.

Theorem 1 *For every $m \geq 1$ we have $\prod_{r=1}^{m}(r\psi - \xi) = a_m + P_m$, where P_m is a linear combination of terms of the form $\xi^q \psi^p i_{k_1,\dots,k_\ell}$. Moreover $P_1 = 0$ and*

$$P_m = \left(\sum_{\ell \geq 2} \frac{1}{\ell!} \sum_{k_1 + \dots + k_\ell = m} k_1 \cdots k_\ell \, i_{k_1,\dots,k_\ell} \right) + (m\psi - \xi) P_{m-1}$$

for $m \geq 2$.

Sample calculations for $1 \leq m \leq 6$ are given in the appendix.

The equality of Theorem 1 does not express a singularity class via basic classes or a basic class via singularity classes. However it is the key result in the proof of the following theorem.

Theorem 2 *Every singularity class can be expressed as a linear combination of basic classes multiplied by powers of ξ. Every basic class can be expressed as a linear combination of singularity classes multiplied by powers of ξ.*

The proof of this theorem is constructive, that is, the expressions can be effectively computed. In the appendix we write out the expressions for all classes up to codimension 5.

Finally, in the following theorem, we give an explicit formula for certain coefficients in the expressions of basic classes in terms of singularity classes.

Introduce the following polynomials in variables x_k:

$$X_m = \sum_\ell \frac{1}{\ell!} \sum_{\substack{k_1,\ldots,k_\ell \\ \sum k_i = m-\ell+2}} \frac{m!}{(m-\ell+2)!} \prod_{i=1}^\ell k_i x_i.$$

Denote by $\alpha_\ell \in H^{\mathrm{top}}(\overline{\mathcal{M}}_{0;\ell+1})$ the cohomology class Poincaré dual to a point.

Theorem 3 *Choose m and $k_1,\ldots,k_\ell$ so that $m + 2 = \ell + \sum_{i=1}^\ell k_i$. The coefficient of $\alpha_\ell i_{k_1,\ldots,k_\ell}$ in the expression for ψ^m is equal to the coefficient of $x_{k_1} \cdots x_{k_\ell}$ in X_m, that is, to*

$$\frac{1}{|\mathrm{Aut}\{k_1,\ldots,k_\ell\}|} \frac{m!}{(m-\ell+2)!} \prod k_i.$$

Choose $m_1,\ldots,m_s$ and $k_1,\ldots,k_\ell$ so that $2s + \sum_{j=1}^s m_j = \ell + \sum_{i=1}^\ell k_i$. The coefficient of $\alpha_\ell i_{k_1,\ldots,k_\ell}$ in the expression for $\alpha_s \delta_{m_1,\ldots,m_s}$ is equal to the coefficient of $x_{k_1} \cdots x_{k_\ell}$ in $X_{m_1} \cdots X_{m_s}$.

In Section 3 we relate these coefficients with Okounkov and Pandharipande's *completed cycles*.

2 Computations with singularity and basic classes

2.1 Proof of Theorem 1

The proof goes by induction. The equality $\psi - \xi = A_1$ follows from the fact that df is a section of the line bundle $\mathcal{L}_X \otimes \mathcal{L}_Y^*$ (see Definition 1.21) that has a simple zero precisely on the A_1 locus.

Assume that the equality is true for $m-1$ and prove it for m. By the induction assumption we have

$$\prod_{r=1}^m (r\psi - \xi) = (m\psi - \xi)(a_{m-1} + P_{m-1}).$$

The term $(m\psi - \xi)P_{m-1}$ is a part of the expression for P_m. Thus our main task is to compute the product $(m\psi - \xi)a_{m-1}$.

To do that, note that $d^m f$ is a section of the line bundle $\mathcal{L}_X^{\otimes m} \otimes \mathcal{L}_Y^*$ restricted to the locus A_{m-1}. Thus we must describe the components of the zero locus of $d^m f$ and their multiplicities.

The obvious component of the zero locus is $A_m \subset A_{m-1}$. On this component the section $d^m f$ has a simple zero as follows from the versality condition for A_m. Indeed, in the local model of Definition 1.3 the stratum A_{m-1} is parametrized by one variable a via

$$u^{m+1} + \gamma_1 u^{m-1} + \ldots \gamma_{m-1} u = (u - a)^m (u + ma) + ma^m;$$

the equation $d^m f$ in this parametrization reads $a = 0$.

The other components are the cycles $I_{k_1, \ldots, k_\ell}$ that lie in the closure of A_{m-1} and are of codimension 1 in A_{m-1}, that is, $\sum k_i = m$. We are going to show that all cycles $I_{k_1, \ldots, k_\ell}$ with $\sum k_i = m$ lie in the closure of A_{m-1} and that the vanishing multiplicity of $d^m f$ on the cycle $I_{k_1, \ldots, k_\ell}$ equals $k_1 \cdots k_\ell$. To do that, we study the local models for each of these singularities.

We will assume that $\ell \geq 3$ leaving the simpler and very similar case $\ell = 2$ to the reader (see also [3]).

Let $\overline{\mathcal{M}}_{0;k_1, \ldots, k_\ell}(\mathbb{C}P^1; \infty)$ be the space of stable maps of genus 0 to $\mathbb{C}P^1$, defined up to an additive constant, and relative to ∞ with ramification profile $(k_1, \ldots, k_\ell)$. Denote by $\overline{\mathcal{C}}_{0;k_1, \ldots, k_\ell}(\mathbb{C}P^1; \infty)$ the universal curve and let $\overline{\mathcal{C}}_{0;k_1, \ldots, k_\ell}^Z(\mathbb{C}P^1; \infty) \subset \overline{\mathcal{C}}_{0;k_1, \ldots, k_\ell}(\mathbb{C}P^1; \infty)$ be the zero locus in the universal curve (see the discussion before Definition 1.7).

Lemma 2.1 *The natural section $d^m f : A_{m-1} \to \mathcal{L}_X^{\otimes m} \otimes \mathcal{L}_Y^*$ on the stratum $A_{m-1} \subset \overline{\mathcal{C}}_{0;k_1, \ldots, k_\ell}(\mathbb{C}P^1; \infty)$ has a vanishing of order $k_1 \cdots k_\ell$ along $\overline{\mathcal{C}}_{0;k_1, \ldots, k_\ell}^Z(\mathbb{C}P^1; \infty)$, whenever $\sum k_i = m$.*

From the lemma we conclude that

$$(m\psi - \xi)a_{m-1} = a_m + \sum_\ell \sum_{\substack{k_1 \leq \cdots \leq k_\ell \\ \sum k_i = m}} k_1 \cdots k_\ell \, [I_{k_1, \ldots, k_\ell}]$$

$$= a_m + \sum_\ell \frac{1}{\ell!} \sum_{\substack{k_1, \ldots, k_\ell \\ \sum k_i = m}} k_1 \cdots k_\ell \, i_{k_1, \ldots, k_\ell}.$$

This holds for any family that satisfies the versality condition of Definition 1.7. Theorem 1 is proved. ◇

Proof of the lemma. We say that a point of $\overline{\mathcal{C}}_{0;k_1,\ldots,k_\ell}(\mathbb{C}\mathrm{P}^1;\infty)$ is *generic* if the contracted part is smooth. We will study the vanishing order at a generic point.

Introduce a coordinate z on the contracted part and denote by $x, z_1, \ldots, z_\ell$ the marked point and the intersection points with the branches. It is proved in [1] that $\overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}(\mathbb{C}\mathrm{P}^1;\infty)$ is a cone over its zero locus and that the fiber of this cone is parametrized by coordinates u_i, $1 \le i \le \ell$ and a_{ij}, $1 \le j \le k_i - 1$ if we write the stable map in the following form:

$$f(z) = \sum_{i=1}^{\ell} \left[\left(\frac{u_i}{z - z_i} \right)^{k_i} + a_{i,1} \left(\frac{u_i}{z - z_i} \right)^{k_i - 1} + \cdots + a_{i,k_i-1} \left(\frac{u_i}{z - z_i} \right) \right] . \quad (3)$$

Let

$$f(z) = \frac{(z - x)^m}{\prod_{i=1}^{\ell}(z - z_i)^{k_i}}.$$

This is the unique, up to transformations $f \mapsto cf + b$, rational function with poles of orders k_i at the points z_i and such that $f'(x) = f''(x) = \cdots = f^{(m-1)}(x) = 0$. Expand f in the form (3) and denote by $\bar{u}_i$ and $\bar{a}_{ij}$ the coefficients thus obtained. Further, let

$$K = \mathrm{LCM}(k_1, \ldots, k_\ell)$$

and

$$r_i = \frac{K}{k_i} \quad \text{for} \quad 1 \le i \le \ell.$$

Then the stratum A_{m-1} in $\overline{\mathcal{M}}_{0;k_1,\ldots,k_\ell}(\mathbb{C}\mathrm{P}^1;\infty)$ consists of $k_1 \cdots k_\ell/L$ irreducible components parametrized by a complex parameter c in the following way:

$$u_i = \alpha_i c^{r_i} \bar{u}_i, \quad a_{ij} = \alpha_i^j c^{jr_i} \bar{a}_{ij}.$$

Here $(\alpha_1, \ldots, \alpha_\ell)$ is a collection of roots of unity, $\alpha_i^{k_i} = 1$. We should choose one such collection in each of the d orbits of $\mathbb{Z}/K\mathbb{Z}$ in $\mathbb{Z}/k_1\mathbb{Z} \times \cdots \times \mathbb{Z}/k_\ell\mathbb{Z}$ and this choice determines the irreducible component of the stratum A_{m-1}.

In each component of the stratum, the mth derivative at x of the function corresponding to parameter c is equal to $c^L \cdot f^{(m)}(x)$. Thus the vanishing order of the mth derivative at $c = 0$ equals K for each component. Since there are $k_1 \cdots k_\ell/K$ components, we get the total vanishing order equal to $k_1 \cdots k_\ell$. $\diamond$

2.2 Proof of Theorem 2

2.2.1 Classes in $\overline{\mathcal{M}}_{0;\ell+1}$

The definition of both basic and singularity classes involves a class $\alpha \in H^*(\overline{\mathcal{M}}_{0;\ell+1})$. Before proving the theorem we must introduce notation for these classes. They will be described using marked trees.

Definition 2.2 An ℓ-tree or just a *tree* is a rooted tree with ℓ leaves; the valencies of all vertices except the leaves and the root are at least 3; the valency of the root is 1 (but we don't call it a leaf). A *marked ℓ-tree* or just a *marked tree* is an ℓ-tree whose all vertices except the root are marked with nonnegative integers.

The picture below shows a marked tree.
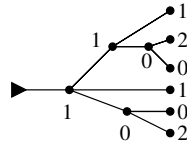
If T is a marked tree we will denote by t the underlying tree obtained by forgetting the markings.

Take a marked tree T and number its leaves in an arbitrary way. This tree with numbered leaves determines a cohomology class in $\overline{\mathcal{M}}_{0;\ell+1}$ as follows. Consider the set of curves in $\overline{\mathcal{M}}_{0;\ell+1}$ whose dual graph[1] is isomorphic to t, the marked point x being the root of the tree. Every interior vertex of t corresponds to an irreducible component of the curve. On this irreducible component there is exactly one marked point that leads to x (in other words, from every vertex v of t exactly one edge leads to the root). The first Chern class of the cotangent line bundle to this point is denoted by ψ_v. The cohomology class that we assign to T with numbered leaves is given by $[T] = \prod_{v \in V(T)} \psi_v^{m_v}$ supported on the boundary stratum corresponding to t, where the product runs over all internal vertices v of t and m_v is the corresponding marking. Note that this class does not depend on the markings of the leaves.

Using the class $[T]$ we can now assign to a marked tree T both a singularity class and a basic class. These classes will not depend on the numbering of the leaves of T, but will depend on their markings. Denote by $m_1, \ldots, m_\ell$ the integer markings on the leaves.

---

[1] The dual graph of a curve is obtained by replacing every irreducible component by a vertex, every node by an edge, every marked point by a leaf, and the point x by the root.

Basic class. The basic class that we assign to T is given by

$$[T]_{\text{basic}} = U_*\left(V^*[T]\prod_l \nu_l^{m_l}\right),$$

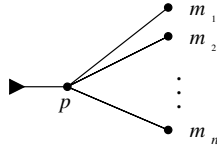where $U\colon \widehat{I}_{1,\dots,1} \to I_{1,\dots,1}$ and $V\colon \widehat{I}_{1,\dots,1} \to \overline{\mathcal{M}}_{0;\ell+1}$ are as in Definition 1.22.

The singularity class. The singularity class assigned to T is

$$[T]_{\text{sing}} = U_*(V^*[T]),$$

where $U\colon \widehat{I}_{m_1+1,\dots,m_\ell+1} \to I_{m_1+1,\dots,m_\ell+1}$ and $V\colon \widehat{I}_{m_1+1,\dots,m_\ell+1} \to \overline{\mathcal{M}}_{0;\ell+1}$ are as in Notation 1.19 and Definition 1.20.

Example 2.3 If T is the tree

then $[T]_{\text{sing}} = \psi^p i_{m_1+1,\dots,m_\ell+1}$, $[T]_{\text{basic}} = \psi^p \delta_{m_1,\dots,m_\ell}$.

Notation 2.4 If T is the tree

we let, by convention, $[\blacktriangleright\!\!-\!\!\bullet\, m]_{\text{basic}} = \psi^m$, $[\blacktriangleright\!\!-\!\!\bullet\, m]_{\text{sing}} = a_m$, $[\blacktriangleright\!\!-\!\!\bullet 0]_{\text{basic}} = [\blacktriangleright\!\!-\!\!\bullet 0]_{\text{sing}} = 1$.

Thus every singularity class and every basic class can be represented as a linear combination of trees.

Remark 2.5 Keel [5] proved that the cohomology classes Poincaré dual to the boundary strata of $\overline{\mathcal{M}}_{0;\ell+1}$ span the whole cohomology ring of $\overline{\mathcal{M}}_{0;\ell+1}$. Therefore our tree notation is sufficient to express any class, but is strongly redundant. For instance, whenever the integer d_v assigned to an interior vertex of T is greater than the valency of v minus 3, the corresponding class vanishes. As another example of redundancy, we have $[T_1]_{\text{sing}} = 3[T_2]_{\text{sing}}$, where
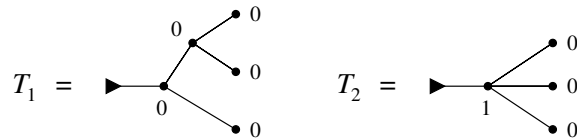
However the classes in our expressions naturally appear in the form of marked ℓ-trees, and we chose not to simplify them further, since we do not know any simple nonredundant expression for the result of such simplification.

We finish this section with a definition that we will need soon.

Definition 2.6 Suppose ℓ marked trees $T_1, \ldots, T_\ell$ are assigned to the leaves of a marked ℓ-tree T. The *substitution* of $T_1, \ldots, T_\ell$ into T is the tree obtained by erasing the leaves of T and gluing the roots of $T_1, \ldots, T_\ell$ into the vertices on which the leaves grew. Note that every vertex of the substitution (except the root) inherits an integer from either T or one of the T_i's, but not both.
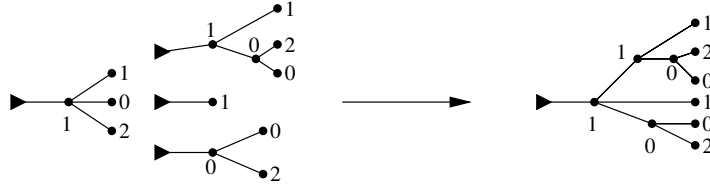
If instead of the trees $T_1, \ldots, T_\ell$ we have ℓ linear combinations of trees we can, of course, extend the operation of substitution by multi-linearity.

2.2.2 Basic classes via singularity classes

Recall the list of basic classes: ψ^k and $[T]_{\text{basic}}$, where T is an ℓ-tree, $\ell \geq 2$. We must express every class like that as a linear combination of singularity classes multiplied by powers of ξ.

Expressions for ψ^m. Theorem 1 gives an expression for $\prod_{r=1}^{m} (r\psi - \xi)$ as a linear combination of singularity classes multiplied by powers of ξ. Writing

$$
\begin{aligned}
\psi &= (\psi - \xi) + \xi, \\
\psi^2 &= \frac{1}{2} \cdot (2\psi - \xi)(\psi - \xi) + \frac{3}{2}\xi \cdot (\psi - \xi) + \xi^2, \\
\psi^3 &= \frac{1}{6} \cdot (3\psi - \xi)(2\psi - \xi)(\psi - \xi) + \frac{11}{12}\xi \cdot (2\psi - \xi)(\psi - \xi) + \frac{7}{4}\xi^2 \cdot (\psi - \xi) + \xi^3,
\end{aligned}
$$

and so on, we obtain similar expressions for powers of ψ.

Example 2.7 We have

$$
\psi = a_1 + \xi, \qquad \psi^2 = \frac{1}{2}a_2 + \frac{1}{4}i_{1,1} + \frac{3}{2}\xi a_1 + \xi^2.
$$

More computations are given in the appendix.

To sum up, for every m there exists a linear combination L_m of trees with coefficients in $\mathbb{Q}[\xi]$ such that $\psi^m = [L_m]_{\text{sing}}$.

The general basic class. Consider class $[T]_{\text{basic}}$ and denote by $m_1, \ldots, m_\ell$ the integers on the leaves of T.

Proposition 2.8 *Let $\widehat{T}$ be the substitution of $L_{m_1}, \ldots, L_{m_\ell}$ into the tree T. Then the class $[T]_{\text{basic}}$ is equal to $[\widehat{T}]_{\text{sing}}$.*

Proof. This proposition is almost obvious. It suffices to note that every class ν_i plays the role of the class ψ on the corresponding branch, and therefore we can use the expressions for ψ^m to express $\nu_i^{m_i}$ for each i. $\diamondsuit$

Example 2.9 Let us compute the expression of $\delta_{0,1,2}$ in singularity classes. We have

$$\delta_{0,1,2} = \left[\begin{array}{c} \blacktriangleright \!\!-\!\!\overset{0}{\underset{1}{\underset{2}{\diagdown}}} \end{array} \right]_{\text{basic}},$$

$$1 = \left[\blacktriangleright\!\!-\!\!\bullet 0 \right]_{\text{sing}}, \qquad \psi = \left[\blacktriangleright\!\!-\!\!\bullet 1 \right]_{\text{sing}} + \xi \left[\blacktriangleright\!\!-\!\!\bullet 0 \right]_{\text{sing}},$$

$$\psi^2 = \frac{1}{2} \left[\blacktriangleright\!\!-\!\!\bullet 2 \right]_{\text{sing}} + \frac{1}{4} \left[\begin{array}{c} \blacktriangleright\!\!-\!\!\overset{0}{\underset{0}{\diagdown}} \end{array} \right]_{\text{sing}} + \frac{3}{2} \xi \left[\blacktriangleright\!\!-\!\!\bullet 1 \right]_{\text{sing}} + \xi^2 \left[\blacktriangleright\!\!-\!\!\bullet 0 \right]_{\text{sing}}.$$

Substituting the last three expressions into the first tree, we obtain a linear combination of 8 trees that simplifies to

$$\delta_{0,1,2} = \frac{1}{2} i_{1,2,3} + \frac{3}{2} \xi i_{1,2,2} + \frac{1}{2} \xi i_{1,1,3} + \frac{5}{2} \xi^2 i_{1,1,2} + \xi^3 i_{1,1,1} +$$

$$+ \frac{1}{4} \left[\begin{array}{c} \blacktriangleright\!\!\!\diagup\!\!\!\overset{0}{\underset{0}{\diagdown}} \\ {}_0 \; {}_0 \end{array} \right]_{\text{sing}} + \frac{1}{4} \xi \left[\begin{array}{c} \blacktriangleright\!\!\!\diagup\!\!\!\overset{0}{\underset{0}{\diagdown}} \\ {}_0 \; {}_0 \end{array} \right]_{\text{sing}}.$$

2.2.3 Singularity classes via basic classes

Let T be a marked ℓ-tree with the markings of the leaves equal to $m_1, \ldots, m_\ell$. We call $m_1 + \cdots + m_\ell$ the *weight* of T. It follows from the previous section that the expression of $[T]_{\text{basic}}$ via singularity classes has the form

$$[T]_{\text{basic}} = \frac{1}{m_1! \cdots m_\ell!} [T]_{\text{sing}} + \text{ lower weight terms.}$$

Therefore these expressions form a triangular change of basis and can be inverted.

The expressions for the simplest singularity classes are computed in the appendix.

2.3 Proof of Theorem 3

The expression for ψ^m. The first claim of Theorem 3 gives an explicit formula for the coefficient of the class $\alpha_\ell i_{k_1,\ldots,k_\ell}$ in the expression of ψ^m, where α_ℓ is the class of a point in $\overline{\mathcal{M}}_{0;\ell+1}$. We will prove this formula using Theorem 1; however, since the terms we are interested in do not contain the class ξ, we can reduce the formula of Theorem 1 modulo ξ. We obtain:

$$m!\psi^m = a_m + \widetilde{P}_m \quad (\mathrm{mod} \ \xi),$$

where $\widetilde{P}_1 = 0$, and

$$\widetilde{P}_m = \left(\sum_{n \geq 2} \frac{1}{\ell!} \sum_{k_1 + \cdots + k_\ell = m} k_1 \cdots k_\ell \ i_{k_1,\ldots,k_\ell} \right) + m\psi \, \widetilde{P}_{m-1}$$

for $m \geq 2$.

Thus we see that the term $i_{k_1,\ldots,k_\ell}$ first "appears" in $\widetilde{P}_{m-\ell+2}$ with coefficient

$$\frac{k_1 \cdots k_\ell}{|\mathrm{Aut}\{k_1,\ldots,k_\ell\}|}$$

and then gets multiplied by $(m-\ell+3)\psi$, $\ldots$, $m\psi$, until it becomes

$$\frac{k_1 \cdots k_\ell}{|\mathrm{Aut}\{k_1,\ldots,k_\ell\}|} \frac{m!}{(m-\ell+2)!} \psi^{n-2} i_{k_1,\ldots,k_\ell}.$$

Since the class ψ^{n-2} is precisely the class of a point in $\overline{\mathcal{M}}_{0;\ell+1}$ (see, for instance, [1]), we obtain precisely the same coefficient as in the formulation of the theorem.

The expression for $\alpha_s \delta_{m_1,\ldots,m_s}$. Recall that $\alpha_\ell \in H^{n-2}(\overline{\mathcal{M}}_{0;\ell+1})$ is the class of a point.

According to Proposition 2.8, to obtain the expression of $\alpha_s \delta_{m_1,\ldots,m_s}$ we must substitute the expressions for ψ^{m_i} into the marked ℓ-tree T:
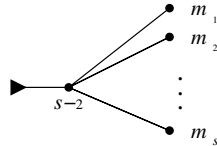
It is easy to determine when a substitution of ℓ marked trees $T_1, \ldots, T_\ell$ into T contributes to the class $\alpha_\ell i_{k_1,\ldots,k_\ell}$. First of all, the marked trees T_j must have ℓ leaves altogether with markings $k_1 - 1, \ldots, k_\ell - 1$. Second, if a marked tree

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T_j has ℓ_j leaves with markings $k_1^{(j)} - 1, \ldots, k_{n_j}^{(j)} - 1$, then it must describe a singularity class proportional to $\alpha_{n_j} i_{k_1^{(j)}, \ldots, k_{n_j}^{(j)}}$. The coefficient of the singularity class $\alpha_{n_j} i_{k_1^{(j)}, \ldots, k_{n_j}^{(j)}}$ in the expression for $\nu_j^{m_j}$ is equal to the coefficient of the monomial $x_{k_1^{(j)}} \cdots x_{k_{n_j}^{(j)}}$ in the polynomial X_{m_j}. Therefore, as claimed in the theorem, the coefficient of the singularity class $\alpha_\ell i_{k_1, \ldots, k_\ell}$ in the expression for $\alpha_s \delta_{m_1, \ldots, m_s}$ is equal to the coefficient of the monomial $x_{k_1} \cdots x_{k_\ell}$ in the polynomial $X_{m_1} \cdots X_{m_s}$.

3 Completed cycles

3.1 Completed cycles and the classes ψ^m

Let $\mathbb{C}S_N$ be the group algebra of the symmetric groupt S_N. If $f \in \mathbb{C}S_N$ is a central element of the group algebra, and λ a partition of N, we denote by $f(\lambda)$ the scalar by which f acts in the irreducible representation λ. (By Schur's lemma every central element of $\mathbb{C}S_N$ acts by a scalar in every irreducible representation of S_N.) This identifies the center of $\mathbb{C}S_N$ with the algebra of functions on the set of partitions of N. Let $C_{k_1, \ldots, k_\ell}$ be the sum of all permutations in S_N with ℓ numbered cycles of lengths $k_1, \ldots, k_\ell$ respectively, and $N - \sum k_i$ non-numbered fixed points. For instance, C_2 is the sum of all transpositions, $C_{1,1}$ is $N(N-1)$ times the identity permutation, etc. Note that $C_{k_1, \ldots, k_\ell}$ defines a central element in $\mathbb{C}S_N$ *simultaneously for all N*. For instance, the sum of transpositions C_2 is a well-defined element of $\mathbb{C}S_N$ for any N. If $N < \sum k_i$, then the corresponding central element vanishes. Thus $C_{k_1, \ldots, k_\ell}$ is actually a family of central elements in the group algebras $\mathbb{C}S_N$ for all N.

Definition 3.1 Given the positive integers $k_1, \ldots, k_\ell$, the family of central elements $C_{k_1, \ldots, k_\ell} \in \mathbb{C}S_N$ for $N \geq 1$ is called a *stable[2] central element*.

Theorem 4 (Kerov, Olshanski [6]) *For every $m \geq 0$ there exists a unique linear combination $\overline{C}_{m+1}$ of stable central elements $C_{k_1, \ldots, k_\ell}$ such that*

$$\overline{C}_{m+1}(\lambda) = \frac{1}{(m+1)!} \left[(\lambda_i - i + 1/2)^{m+1} - (-i + 1/2)^{m+1} \right]$$

for every partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ of every integer N.

---

[2]This has nothing to do with stable maps.

Example 3.2 We have

$$
\begin{aligned}
\overline{C}_1 &= C_1, \\
\overline{C}_2 &= C_2, \\
\overline{C}_3 &= \frac{1}{2}C_3 + \frac{1}{4}C_{1,1} + \frac{1}{24}C_1, \\
\overline{C}_4 &= \frac{1}{6}C_4 + \frac{1}{3}C_{1,2} + \frac{5}{24}C_2, \\
\overline{C}_5 &= \frac{1}{24}C_5 + \frac{1}{8}C_{1,3} + \frac{1}{12}C_{2,2} + \frac{1}{36}C_{1,1,1} + \frac{11}{48}C_3 + \frac{1}{32}C_{1,1} + \frac{1}{1920}C_1.
\end{aligned}
$$

Definition 3.3 The linear combination $\overline{C}_{m+1}$ is called the *completed $(m+1)$-cycle.*

Remark 3.4 The set $S_\lambda = \{(\lambda_i - i + 1/2)_{i \geq 1}\} \subset \mathbb{Z} + \frac{1}{2}$ is a uniform way to encode a partition of any size. The empty partition corresponds to the set $S_\emptyset$ of negative half-integers. The set S_λ differs from $S_\emptyset$ by a finite number of elements. The function $\overline{C}_{m+1}$ is basically the renormalized sum of $(m+1)$-st powers of the elements of S_λ divided by $(m+1)!$.

Denote by $S(z)$ the power series

$$
S(z) = \frac{\sinh(z/2)}{z/2}.
$$

Proposition 3.5 (Okounkov, Pandharipande, [7], Proposition 3.2)
We have
$$
\overline{C}_{m+1} = \sum_{\substack{g \geq 0, \\ n \geq 1}} \frac{1}{\ell!} \sum_{\substack{k_1, \ldots, k_\ell \\ \sum k_i + \ell + 2g - 2 = m}} \rho_{g;k_1,\ldots,k_\ell} C_{k_1,\ldots,k_\ell},
$$
where the rational constant $\rho_{g;k_1,\ldots,k_\ell}$ is the coefficient of z^{2g} in the power series
$$
\frac{\prod k_i}{K!} S(z)^{K-1} \prod_{i=1}^{\ell} S(k_i z), \qquad K = \sum_{i=1}^{\ell} k_i.
$$

Remark 3.6 Our normalization of completed cycles differs from that of [7] in three ways. First, by a factor of $m!$; second, by the absence of a constant term corresponding to $\ell = 0$; third, by the fact that, contrary to [7], for us $C_{k_1,\ldots,k_\ell}$ is the sum of permutations with ℓ *numbered cycles*, which changes the corresponding coefficient in the completed cycle by a factor $|\mathrm{Aut}\{k_1, \ldots, k_\ell\}|$.

Every constant $\rho_{g;k_1,\ldots,k_\ell}$ is associated to a singularity of a higher genus stable map. Namely, by analogy with Definition 1.6, we can say that f presents *a singularity of type* $I_{g;k_1,\ldots,k_\ell}$ at $x \in C$ if x lies on a contracted part of genus g meeting ℓ branches of C at ramification points of orders $k_1, \ldots, k_\ell$. It is well-known that if a stable map f presents a singularity of type $I_{g;k_1,\ldots,k_\ell}$ at x, then the image $f(x)$ must be considered a branch point of multiplicity $m = \sum k_i + \ell + 2g - 2$. In particular, if the stable map can be deformed into a generic smooth map, the branch point will split into m simple branch points.

Since in this paper we only study stable maps of genus 0, we will be only interested in the genus 0 part of the completed cycles, that is, the part that corresponds to the $g = 0$ terms in the sum of Proposition 3.5, or in other words, to the terms satisfying $\sum k_i + \ell = m + 2$.

Proposition 3.7 *If $\sum k_i + n = m + 2$, then the coefficient of $C_{k_1,\ldots,k_\ell}$ in $\overline{C}_{m+1}$ is the same as the coefficient of $\alpha_\ell i_{k_1,\ldots,k_\ell}$ in the expression of ψ^m via the singularity classes, where α_ℓ is the class of a point in $\overline{\mathcal{M}}_{0;\ell+1}$.*

Proof. According to Theorem 3 and Proposition 3.5 both are equal to

$$\frac{1}{|\mathrm{Aut}\{k_1,\ldots,k_\ell\}|} \cdot \frac{\prod k_i}{(\sum k_i)!}.$$

$\diamondsuit$

Remark 3.8 In [7] Okounkov and Pandharipande established a relation between Gromov-Witten invariants of $\mathbb{C}\mathrm{P}^1$ and the completed cycles. Therefore the result of Proposition 3.7 was to be expected. Note, however, that Gromov-Witten invariants are only intersection *numbers*, while here we are working with *cohomology classes*. Therefore we can expect to get more information than what is contained in the completed cycles. And indeed, our expressions for ψ^m involve terms that do not appear in the completed cycles and do not contribute to the computation of Gromov-Witten invariants.

3.2 Products of completed cycles and the classes $\alpha_s \delta_{m_1,\ldots,m_s}$

Proposition 3.9 ([6]) *The product of two stable central elements is a finite linear combination of stable central elements.*

Example 3.10 We have $C_2^2 = C_{2,2} + 3C_3 + \frac{1}{2}C_{1,1}$. Indeed, C_2 is the sum of all transpositions, so C_2^2 is the sum of products of all possible pairs of

transpositions. A product of two transpositions can give either a permutation with two disjoint 2-cycles (in a unique way if we number the 2-cycles in the same order as the transpositions), or a 3-cycle (and every 3-cycle can be decomposed into a product of two transpositions in three possible ways), or an identity permutation with two marked elements (but we can number these marked element in two different ways).

The proof in the general case is a simple generalization of the above example and is left to the reader.

Proposition 3.11 *Let $\alpha_\ell \in H^{\mathrm{top}}(\overline{\mathcal{M}}_{0;\ell+1})$ be the class Poincaré dual to a point. Choose $m_1, \ldots, m_s$ and $k_1, \ldots, k_\ell$ such that $2s + \sum m_j = \ell + \sum k_i$. Then the coefficient of $\alpha_\ell i_{k_1,\ldots,k_\ell}$ in the expression of $\alpha_s \delta_{m_1,\ldots,m_s}$ in terms of singularity classes equals the coefficient of the stable central element $C_{k_1,\ldots,k_\ell}$ in the product $\overline{C}_{m_1+1} \cdots \overline{C}_{m_s+1}$.*

Proof. Accroding to Theorem 3, the coefficient of $\alpha_\ell i_{k_1,\ldots,k_\ell}$ in the expression for $\alpha_s \delta_{m_1,\ldots,m_s}$ is equal to the coefficient of the monomial $x_{k_1} \cdots x_{k_\ell}$ in the product of polynomials $X_{m_1} \cdots X_{m_s}$. Recall that the polynomials X_m are defined as

$$X_m = \sum_\ell \frac{1}{\ell!} \sum_{\substack{k_1,\ldots,k_\ell \\ \sum k_i = m-\ell+2}} \frac{m!}{(m-\ell+2)!} \prod_{i=1}^{\ell} k_i x_i.$$

They are transformed into the genus 0 part of the completed cycles $\overline{C}_m$ if we replace every monomial $x_{k_1} \cdots x_{k_\ell}$ by the stable central element $C_{k_1,\ldots,k_\ell}$.

Let us call $\ell + \sum k_i$ the *order* of the stable central element $C_{k_1,\ldots,k_\ell}$. Then the genus g terms of a completed cycle $\overline{C}_m$ have order $m + 2 - 2g$. The genus 0 elements have the biggest possible order $m + 2$.

Consider two stable central elements $C_{k_1,\ldots,k_\ell}$ and $C_{l_1,\ldots,l_s}$. Denote their orders by d_1 and d_2. Then we have

$$C_{k_1,\ldots,k_\ell} \cdot C_{l_1,\ldots,l_s} = C_{k_1,\ldots,k_\ell,l_1,\ldots,l_s} + \text{ lower order terms,}$$

where the order of the first term equals $d_1 + d_2$. We conclude that *if we only keep the highest order terms, then stable central elements multiply like monomials*:

$$\left(x_{k_1} \cdots x_{k_\ell} \right) \cdot \left(x_{l_1} \cdots x_{l_s} \right) = x_{k_1} \cdots x_{k_\ell} x_{l_1} \cdots x_{l_s}.$$

This is enough to prove the second assertion of Theorem 3. Indeed, we have already identified the highest order terms of a completed cycle $\overline{C}_{m+1}$ with the coefficients of the polynomial X_m. Therefore the highest order terms of the product $\overline{C}_{m_1+1} \cdots \overline{C}_{m_s+1}$ are identified with the coefficients of the product of polynomials $X_{m_1} \cdots X_{m_s}$.

4 Appendix: sample computations

4.1 Expressions for $\prod(r\psi - \xi)$

Theorem 1 leads to the following expressions for $\prod(r\psi - \xi)$:

- $\psi - \xi =$

  a_1

- $(\psi - \xi)(2\psi - \xi) =$

  $a_2 + \frac{1}{2}i_{1,1}$

- $(\psi - \xi)(2\psi - \xi)(3\psi - \xi) =$

  $a_3 + 2i_{1,2} + \frac{1}{6}i_{1,1,1} - \frac{1}{2}\xi i_{1,1}.$

- $(\psi - \xi)(2\psi - \xi)(3\psi - \xi)(4\psi - \xi) =$

  $a_4 + 3i_{1,3} + 2i_{2,2} + i_{1,1,2} + \frac{1}{24}i_{1,1,1,1} + \frac{2}{3}\psi i_{1,1,1} - \xi\left(2i_{1,2} + \frac{1}{6}i_{1,1,1}\right) + \frac{1}{2}\xi^2 i_{1,1}$

- $(\psi - \xi)(2\psi - \xi)(3\psi - \xi)(4\psi - \xi)(5\psi - \xi) =$

  $a_5 + 4i_{1,4} + 6i_{2,3} + \frac{3}{2}i_{1,1,3} + 2i_{1,2,2} + \frac{1}{3}i_{1,1,1,2} + \frac{1}{120}i_{1,1,1,1,1} + 5\psi i_{1,1,2}$

  $+ \frac{5}{24}\psi i_{1,1,1,1} - \xi\left(3i_{1,3} + 2i_{2,2} + i_{1,1,2} + \frac{1}{24}i_{1,1,1,1} + \frac{3}{2}\psi i_{1,1,1}\right) + \xi^2\left(2i_{1,2} + \frac{1}{6}i_{1,1,1}\right)$

  $- \frac{1}{2}\xi^3 i_{1,1}$

4.2 Expressions for powers of ψ

By taking linear combinations of the equalities of Section 4.1, we obtain the following expressions for the powers of ψ (the underlined terms appear in Okounkov and Pandharipande's completed cycles):

- $\psi = \underline{a_1} + \xi$

- $\psi^2 = \underline{\frac{1}{2}a_2} + \underline{\frac{1}{4}i_{1,1}} + \frac{3}{2}\xi a_1 + \xi^2$

- $\psi^3 = \underline{\frac{1}{6}a_3} + \underline{\frac{1}{3}i_{1,2}} + \underline{\frac{1}{36}i_{1,1,1}} + \frac{11}{12}\xi a_2 + \frac{3}{8}\xi i_{1,1} + \frac{7}{4}\xi^2 a_1 + \xi^3.$

- $\psi^4 = \underline{\frac{1}{24}a_4} + \underline{\frac{1}{8}i_{1,3}} + \underline{\frac{1}{12}i_{2,2}} + \underline{\frac{1}{24}i_{1,1,2}} + \underline{\frac{1}{576}i_{1,1,1,1}} + \underline{\frac{1}{36}\psi i_{1,1,1}} + \frac{25}{72}\xi a_3 + \frac{11}{18}\xi i_{1,2}$

  $+ \frac{11}{216}\xi i_{1,1,1} + \frac{85}{72}\xi^2 a_2 + \frac{7}{16}\xi^2 i_{1,1} + \frac{15}{8}\xi^3 a_1 + \xi^4$

- $\psi^5 = \underline{\frac{1}{120}a_5} + \underline{\frac{1}{30}i_{1,4}} + \underline{\frac{1}{20}i_{2,3}} + \underline{\frac{1}{80}i_{1,1,3}} + \underline{\frac{1}{60}i_{1,2,2}} + \underline{\frac{1}{360}i_{1,1,1,2}} + \underline{\frac{1}{14400}i_{1,1,1,1,1}}$

  $+ \underline{\frac{1}{24}\psi i_{1,1,2}} + \underline{\frac{1}{576}\psi i_{1,1,1,1}} + \frac{137}{1440}\xi a_4 + \frac{25}{96}\xi i_{1,3} + \frac{25}{144}\xi i_{2,2} + \frac{25}{288}\xi i_{1,1,2}$

  $+ \frac{25}{6912}\xi i_{1,1,1,1} + \frac{11}{216}\xi\psi i_{1,1,1} + \frac{415}{864}\xi^2 a_3 + \frac{85}{108}\xi^2 i_{1,2} + \frac{85}{1296}\xi^2 i_{1,1,1} + \frac{575}{432}\xi^3 a_2$

  $+ \frac{15}{32}\xi^3 i_{1,1} + \frac{31}{16}\xi^4 a_1 + \xi^5$

4.3 Basic via singularity classes

Using the expressions for the powers of ψ we obtain the expressions for the other basic classes:

Codimension 2:

- $\delta_{0,0} = i_{1,1}$

Codimension 3:

- $\delta_{0,0,0} = i_{1,1,1}$
- $\delta_{0,1} = i_{1,2} + \xi i_{1,1}$

Codimension 4:

- $\delta_{0,0,0,0} = i_{1,1,1,1}$
- $\delta_{0,0,1} = i_{1,1,2} + \xi i_{1,1,1}$
- $\psi \delta_{0,0,0} = \psi i_{1,1,1}$
- $\delta_{1,1} = i_{2,2} + 2\xi i_{1,2} + \xi^2 i_{1,1}$
- $\delta_{0,2} = \frac{1}{2} i_{1,3} + \frac{1}{4} \psi i_{1,1,1} + \frac{3}{2} \xi i_{1,2} + \xi^2 i_{1,1}$

Codimension 5:

- $\delta_{0,0,0,0,0} = i_{1,1,1,1,1}$
- $\delta_{0,0,0,1} = i_{1,1,1,2} + \xi i_{1,1,1,1}$
- $\alpha \delta_{0,0,0,0} = \alpha i_{1,1,1,1}$ for any class $\alpha \in H^2(\overline{\mathcal{M}}_{0;5})$
- $\delta_{0,1,1} = i_{1,2,2} + 2\xi i_{1,1,2} + \xi^2 i_{1,1,1}$
- $\delta_{0,0,2} = \frac{1}{2} i_{1,1,3} + \frac{1}{4} \left[\begin{array}{c} \vcenter{\hbox{}} \end{array} \right]_{\text{sing}} + \frac{3}{2} \xi i_{1,1,2} + \xi^2 i_{1,1,1}$
- $\psi \delta_{0,0,1} = \psi i_{1,1,2} + \xi \psi i_{1,1,1}$
- $\delta_{1,2} = \frac{1}{2} i_{2,3} + \frac{1}{4} \psi i_{1,1,2} + \frac{1}{2} \xi i_{1,3} + \frac{3}{2} \xi i_{2,2} + \frac{1}{4} \xi \psi i_{1,1,1} + \frac{5}{2} \xi^2 i_{1,2} + \xi^3 i_{1,1}$

- $\delta_{0,3} = \frac{1}{6}i_{1,4} + \frac{1}{3}\psi i_{1,1,2} + \frac{1}{36}\left[\begin{array}{c}\text{(tree diagram)}\end{array}\right]_{\text{sing}} + \frac{11}{12}\xi i_{1,3} + \frac{3}{8}\xi\psi i_{1,1,1} + \frac{7}{4}\xi^2 i_{1,2} + \xi^3 i_{1,1}$

4.4 Singularity via basic classes

Inverting the relations of Sections 4.2 and 4.3 we obtain the following expressions for the singularity classes in terms of basic classes:

Codimension 1:

- $a_1 = \psi - \xi$

Codimension 2:

- $i_{1,1} = \delta_{0,0}$

- $a_2 = 2\psi^2 - \frac{1}{2}\delta_{0,0} - 3\xi\psi + \xi^2$

Codimension 3:

- $i_{1,1,1} = \delta_{0,0,0}$

- $i_{1,2} = \delta_{0,1} - \xi\delta_{0,0}$

- $a_3 = 6\psi^3 - 2\delta_{0,1} - \frac{1}{6}\delta_{0,0,0} - 11\xi\psi^2 + \frac{5}{2}\xi\delta_{0,0} + 6\xi^2\psi - \xi^3$

Codimension 4:

- $i_{1,1,1,1} = \delta_{0,0,0,0}$

- $i_{1,1,2} = \delta_{0,0,1} - \xi\delta_{0,0,0}$

- $\psi i_{1,1,1} = \psi\delta_{0,0,0}$

- $i_{2,2} = \delta_{1,1} - 2\xi\delta_{0,1} + \xi^2\delta_{0,0}$

- $i_{1,3} = 2\delta_{0,2} - \frac{1}{2}\psi\delta_{0,0,0} - 3\xi\delta_{0,1} + \xi^2\delta_{0,0}$

- $a_4 = 24\psi^4 - 6\delta_{0,2} - \delta_{1,1} - \delta_{0,0,1} + \frac{13}{6}\psi\delta_{0,0,0} - \frac{1}{24}\delta_{0,0,0,0} - 50\xi\psi^3 + \frac{1}{5}\xi\delta_{0,1} + \frac{7}{6}\xi\delta_{0,0,0} + 35\xi^2\psi^2 - \frac{15}{2}\xi^2\delta_{0,0} - 10\xi^3\psi + \xi^4$

References

[1] **T. Ekedahl, S. Lando, M. Shapiro, A. Vainshtein.** *Hurwitz numbers and intersections on moduli spaces of curves.* – Invent. Math. **146** (2001), no. 2, 297–327.

[2] **M. Kazaryan, S. Lando.** *On intersection theory on Hurwitz spaces.* (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 68 (2004), no. 5, 91–122; translation in Izv. Math. 68 (2004), no. 5, 935964.

[3] **M. Kazarian, S. Lando.** *Thom polynomials for maps of curves with isolated singularities.* – Tr. Mat. Inst. Steklova, **258** (2007), Anal. i Osob. Ch. 1, 93–106 (Russian) ; translation in Proc. Steklov Inst. Math., **258** (2007), no. 1, `arXiv:  0706:1523`.

[4] **M. Kazarian, S. Lando., D. Zvonkine** in preparation.

[5] **S. Keel.** *Intersection theory of moduli space of stable n-pointed curves of genus zero.* – Trans. Amer. Math. Soc., **330** (1992), no. 2, 545–574.

[6] **Kerov, Olshanski.** *Polynomial functions on the set of Young diagrams.* – C. R. Acad. Sci. Paris, Sér. I, Math. **319** (1994), no. 2, 121–126.

[7] **A. Okounkov, R. Pandharipande.** *Gromov-Witten theory, Hurwitz theory, and completed cycles.* Ann. of Math.**163** (2006), no. 2, 517–560.

[8] **R. Thom.** *Quelques propriétés globales des variétés différentiables.* – Comment. Math. Helv., **28** (1954), 17–86.