# ON MONODROMY GROUPS OF DEL PEZZO SURFACES

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Abstract. We show that the monodromy group acting on $H^1(\cdot, \mathbb{Z})$ of a smooth hyperplane section of a del Pezzo surface over $\mathbb{C}$ is the entire group $\mathrm{SL}_2(\mathbb{Z})$. For smooth surfaces with $b_1 = 0$ and hyperplane section of genus $g > 2$, there exist examples in which a similar assertion is false. Actually, if hyperplane sections of a smooth surface are hyperelliptic curves of genus $g \geq 3$, then the monodromy group acting on the integer $H^1$ on hyperplane sections is a proper subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

## Introduction

Suppose that $X \subset \mathbb{P}^n$ is a smooth projective surface over $\mathbb{C}$ and that the curve $Y \subset X$ is its smooth hyperplane section. Variation of $Y$ gives rise to an action of $\pi_1((\mathbb{P}^n)^* \setminus X^*)$ on $H^1(Y, \mathbb{Z})$, where $(\mathbb{P}^n)^*$ is the dual projective space and $X^* \subset (\mathbb{P}^n)^*$ is the dual variety. The subgroup of $\mathrm{Aut}(H^1(Y, \mathbb{Z}))$ that corresponds to this action will be called *hyperplane monodromy group* of the surface $X$ (see Definition 3.1).

Surfaces for which hyperplane monodromy group is trivial were classified in [Zak73]. The aim of this paper is to compute these groups in the simplest case in which they are non-trivial.

Suppose that the hyperplane section $Y$ is a curve of genus $g$. Since monodromy preserves the cup-product, the hyperplane monodromy group is a subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ (symplectic group with respect to a symplectic form on $\mathbb{Z}^{2g}$ with Pfaffian 1). In particular, if $g = 1$ then the hyperplane monodromy group is a subgroup of $\mathrm{SL}(2, \mathbb{Z})$.

If genus of hyperplane sections of a smooth surface $X$ equals 1, then either $b_1(X) = 0$, and in this case $X$ is a del Pezzo surface embedded by the anticanonical class (see for example [Ion84, Proposition 2.4] or [Fuj80, Theorem 1.9]), or $b_1(X) = 1$, in which case the main result of [Zak73] shows that $X$ is ruled by lines. In the second case the hyperplane monodromy group is trivial. Our main result concerns the first case.

**Theorem 0.1.** *If $X \subset \mathbb{P}^n$ is a del Pezzo surface embedded by (a subsystem of) the anticanonical linear system $|-K_X|$, then its hyperplane monodromy group is the entire $\mathrm{SL}_2(\mathbb{Z})$.*

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This theorem may give the impression that it is a particular case of a general result to the effect that the hyperplane monodromy group of any smooth surface $X$ with sectional genus $g$ and $H^1(X,\mathbb{Z}) = 0$ is always the entire $\mathrm{Sp}_{2g}(\mathbb{Z})$ (if $H^1(X,\mathbb{Z}) \neq 0$, then the monodromy group *must* be a proper subgroup of the entire symplectic group since it fixes the elements of $H^1(Y,\mathbb{Z})$ coming from $H^1(X,\mathbb{Z})$). This is not the case. To wit, in Section 4 we show that the monodromy group acting on integral $H^1$ of fibers of any family of hyperelliptic curves of genus $g > 2$ is a proper subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ (Corollary 4.2). In particular, this is the case for families of hyperelliptic hyperplane sections; surfaces $X$ with $H^1(X,\mathbb{Z}) = 0$ and hyperelliptic hyperplane sections of genus $g$ exist for any $g \geq 2$.

The main results of this paper are based on A'Campo's computation of monodromy in some families of elliptic and hyperelliptic curves [A'C79].

The paper is organized as follows. In Section 1 we list several simple results on monodromy groups in smooth families; at the end of this section we state some of A'Campo's results in the form that is convenient for our purposes.

In Section 2 we prove an auxiliary result on monodromy in families of elliptic curves (Proposition 2.3), which may be of some independent interest.

In Section 3 we prove Theorem 0.1, and in Section 4 we prove the above mentioned result on monodromy for surfaces with hyperelliptic hyperplane sections.

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**Notation and conventions.** All our algebraic varieties are defined over $\mathbb{C}$ and reduced, so they are essentially identified with their sets of closed points; the only exception is the proof of Lemma 2.6. If $X$ is an algebraic variety, then $X_{\mathrm{sm}}$ is its smooth locus and $X_{\mathrm{sing}}$ is it singular locus.

If $X$ is an algebraic variety and $\mathcal{R}$ is a coherent sheaf of reduced $\mathcal{O}_X$-algebras, we denote its relative spectrum (which is a scheme over $X$) by $\mathbf{Spec}\,\mathcal{R}$. Needless to say, under our assumptions $\mathbf{Spec}\,\mathcal{R}$ is an algebraic variety and the canonical morphism $\mathbf{Spec}\,\mathcal{R} \to X$ is finite.

When we write "a general $X$ has property $Y$", this always means "property $Y$ holds for a Zariski open and dense set of $X$'s". The word "generic" is used in the scheme-theoretic sense (only once).

By *del Pezzo surface* we mean a smooth projective surface $X$ for which the anticanonical class $-K_X$ is ample.

If $B$ is an algebraic variety and $p\colon \mathcal{X} \to B$ is a proper and smooth morphism, we will say that $\mathcal{X} \xrightarrow{p} B$ (or just $\mathcal{X}$ if there is no danger of confusion) is a *family of smooth varieties* over $B$. If $p\colon \mathcal{X} \to B$ is a family of smooth varieties over $B$ and $f\colon B' \to B$ is a morphism, then by $p'\colon f^*\mathcal{X} \to B'$ we mean the the pull-back of $\mathcal{X}$ along $f$.

If $p \in \mathbb{P}^n$ is a point and $L \subset \mathbb{P}^n$ is a linear subspace, then $\overline{p, L}$ denotes the linear span of $\{p\} \cup L$.

If $A_1, \dots, A_4$ are points on the affine line with coordinates $a_1, \dots, a_4$, then by their cross-ratio we mean

$$[A_1, A_2, A_3, A_4] = \frac{a_3 - a_1}{a_3 - a_2} \Big/ \frac{a_4 - a_1}{a_4 - a_2}.$$

By $\pi_1$ of an algebraic variety over $\mathbb{C}$ we always mean fundamental group in the classical (complex) topology.

## 1. Generalities on monodromy groups

Suppose that $B$ is an irreducible variety and $p\colon \mathcal{X} \to B$ is a family of smooth varieties.

If $b \in B_{\mathrm{sm}}$, $k \in \mathbb{N}$, and $G$ is an abelian group, then the fundamental group $\pi_1(B_{\mathrm{sm}}, b)$ acts on $H^k(p^{-1}(b), G)$.

**Definition 1.1.** The image (corresponding to this action) of $\pi_1(B_{\mathrm{sm}}, b)$ in $\mathrm{Aut}(H^k(p^{-1}(b), G))$ will be called *monodromy group* of the family $\mathcal{X}$ at $b$ and denoted $\mathrm{Mon}(\mathcal{X}, b)$ (we suppress the mention of $k$ and $G$; there will be no danger of confusion).

Since $B$ is irreducible, monodromy groups at $b$ are isomorphic for all smooth $b \in B$. These isomorphisms need not be canonical: if $b, b_1 \in B_{\mathrm{sm}}$, the isomorphisms $\mathrm{Aut}(H^k(p^{-1}(b), \mathbb{Z})) \to \mathrm{Aut}(H^k(p^{-1}(b_1), \mathbb{Z}))$ induced by two paths from $b$ to $b_1$ lying in $B_{\mathrm{sm}}$, differ by an inner automorphism of $\mathrm{Aut}(H^k(p^{-1}(b_1), G))$. Thus, if we fix once and for all the group $A = \mathrm{Aut}(H^k(p^{-1}(b_0), G))$ for some $b_0 \in B_{\mathrm{sm}}$, then all the groups $\mathrm{Mon}(\mathcal{X}, b)$ define the same conjugacy class of subgroups of $A$; this class (or, abusing the language, any subgroup belonging to this class) will be denoted by $\mathrm{Mon}(\mathcal{X})$.

*Remark* 1.2. In the sequel we will be working with families of smooth curves of genus $g$ as fibers and monodromy action on $H^1$ of the fiber. Since monodromy preserves the intersection form, the subgroups $\mathrm{Mon}(\mathcal{X})$, where $\mathcal{X}$ is such a family, will be defined up to an inner automorphism of the symplectic group $\mathrm{Sp}_{2g}$.

Below we list some simple properties of monodromy groups.

**Proposition 1.3.** *Suppose that $B$ is an irreducible variety, $U \subset B$ is a non-empty Zariski open subset, and $\mathcal{X}$ is a smooth family over $B$. Then* $\mathrm{Mon}(\mathcal{X}|_U) = \mathrm{Mon}(\mathcal{X})$.

*Proof.* The result follows from the fact that, for any $b \in U \cap B_{\mathrm{sm}}$, the natural homomorphism $\pi_1(U \cap B_{\mathrm{sm}}, b) \to \pi_1(B_{\mathrm{sm}}, b)$ is epimorphic (see for example [FL81, 0.7(B) ff.]). □

**Proposition 1.4.** *If $B'$ and $B$ are irreducible varieties, $\mathcal{X}$ is a smooth family over $B$, and $f\colon B' \to B$ is a morphism such that $f(B') \cap B_{\mathrm{sm}} \neq \varnothing$, then* $\mathrm{Mon}(f^*\mathcal{X}) \subset \mathrm{Mon}(X)$.

(Here, if the subgroups $A_1, A_2 \subset A$ are defined up to a conjugation, by $A_1 \subset A_2$ we mean that $h^{-1}A_1h \subset A_2$ for some $h \in A$.)

*Proof.* Obvious from Proposition 1.3. □

**Proposition 1.5.** *Suppose that $B'$ and $B$ are irreducible varieties and $\mathcal{X}$ is a smooth family over $B$. If $f\colon B' \to B$ is a dominant morphism such that a general fiber of $f$ is irreducible, then* $\operatorname{Mon}(f^*\mathcal{X}) = \operatorname{Mon}(\mathcal{X})$.

*Proof.* It follows from [Ver76, Corollary 5.1] and the algebraic version of Sard's theorem that there exists a Zariski open non-empty $U \subset B_{\mathrm{sm}}$ such that all the fibers of $f$ over points of $U$ are irreducible, the induced mapping $f'\colon f^{-1}U \cap B'_{\mathrm{sm}} \to U$ is a locally trivial bundle in the complex topology, and the fibers of $f$ over points of $U$ are smooth. Since fibers of $f$ over points of $U$ are irreducible, fibers of the mapping $f'$, being irreducible and smooth, are connected. Since $f'$ is a locally trivial bundle in the complex topology, the mapping $f'_*\colon \pi_1(f^{-1}U \cap B'_{\mathrm{sm}}, b') \to \pi_1(U, f(b'))$, where $b'$ is a smooth point of $f^{-1}(U)$, is surjective. This, together with Proposition 1.3, implies the assertion. □

To conclude this section, we state some results of A'Campo which will be used in the sequel.

**Notation 1.6.** For any integer $n \geq 3$, denote by $U_n \subset \mathbb{A}^n$ the open subset consisting of the $n$-tuples $a_0, \dots, a_{n-1}$ for which the polynomial $t^n + a_{n-1}t^{n-1} + \dots + a_0$ has no multiple roots, and denote by $\mathcal{A}_n \to U_n$ the family of hyperelliptic curves of genus $g = \lfloor (n-1)/2 \rfloor$ in which the fiber over $a_0, \dots, a_{n-1} \in U_n$ is the smooth projective model of the curve defined by the equation

$$y^2 = x^n + \sum_{j=0}^{n-1} a_j t^j$$

(sometimes we will regard elements of $U_n$ as polynomials rather than just points of $\mathbb{A}^n$).

We denote by $V_n \subset U_n$ the subset consisting of polynomials with zero coefficient at $t^{n-1}$; let $\mathcal{B}_n$ be the restriction of the family $\mathcal{A}_n$ to $V_n$. Part of A'Campo's results may be stated as follows.

**Theorem 1.7** (A'Campo [A'C79, Corollary to Theorem 1])**.** *If $g = 1$, then*

$$\operatorname{Mon}(\mathcal{A}_{2g+1}) = \operatorname{Mon}(\mathcal{B}_{2g+1}) = \operatorname{Mon}(\mathcal{A}_{2g+2}) = \operatorname{Mon}(\mathcal{B}_{2g+2}) = \operatorname{Sp}_2(\mathbb{Z}) = \operatorname{SL}_2(\mathbb{Z}).$$

*If $g \geq 2$, then*

$$
\begin{aligned}
(\mathrm{Sp}_{2g}(\mathbb{Z}) : \mathrm{Mon}(\mathcal{A}_{2g+2})) &= (\mathrm{Sp}_{2g}(\mathbb{Z}) : \mathrm{Mon}(\mathcal{B}_{2g+2})) \\
&= \frac{2^{g^2}(2^{2g}-1)(2^{2(g-1)}-1)\cdot\dots\cdot(2^2-1)}{(2g+2)!}, \\
(\mathrm{Sp}_{2g}(\mathbb{Z}) : \mathrm{Mon}(\mathcal{A}_{2g+1})) &= (\mathrm{Sp}_{2g}(\mathbb{Z}) : \mathrm{Mon}(\mathcal{B}_{2g+1})) \\
&= \frac{2^{g^2}(2^{2g}-1)(2^{2(g-1)}-1)\cdot\dots\cdot(2^2-1)}{(2g+1)!}.
\end{aligned}
$$

*In particular,* $\mathrm{Mon}(\mathcal{A}_6) = \mathrm{Sp}_4(\mathbb{Z})$*, and* $(\mathrm{Sp}_{2g}(\mathbb{Z}) : \mathrm{Mon}(\mathcal{A}_{2g+2})) > 1$ *if* $g \geq 3$.

(See Remark 1.2 for the explanation why we regard $\mathrm{Mon}(\mathcal{B}_n)$ and $\mathrm{Mon}(\mathcal{A}_n)$ as conjugacy classes of subgroups in the symplectic group.)

*Remark* 1.8. Instead of the family $\mathcal{B}_n$, A'Campo studied the family of affine curves with the equation $y^2 = x^n + a_{n-2}x^{n-2} + \dots + a_0$. If $n = 2g+1$ is odd, such a curve is the corresponding compact Riemann surface $Y$ of genus $g$ with a puncture at the point $p$ lying over the point $\infty$ of $\mathbb{P}^1$ with the affine coordinate $x$; A'Campo computed the monodromy action on $H_1(Y \setminus \{p\}, \mathbb{Z})$, with intersection pairing. Since $H_1(Y \setminus \{p\}, \mathbb{Z}) \cong H_1(Y, \mathbb{Z})$, index of this monodromy group in $\mathrm{Sp}_{2g}(\mathbb{Z})$ is equal to index of the monodromy group acting on $H^1(Y, \mathbb{Z})$ (observe that the monodromy group acting on $H^1(Y, \mathbb{Z})$ differs from that acting on $H_1(Y, \mathbb{Z})$ by the involutive automorphism $M \mapsto (M^t)^{-1}$ of $\mathrm{Sp}_{2g}(\mathbb{Z})$). If $n = 2g+2$ is even, a fiber is of the form $Y \setminus \{p, q\}$, where $Y$ is a compact Riemann surface of genus $g$; in this case, A'Campo computed monodromy action on $H_1(Y \setminus \{p, q\})/\langle (p), (q) \rangle$, where $(p), (q) \in H_1(Y \setminus \{p, q\})$ are classes of small positive loops around $p$ and $q$ (and $(p) + (q) = 0$). Since $H_1(Y \setminus \{p, q\}, \mathbb{Z})/\langle (p), (q) \rangle \cong H_1(Y, \mathbb{Z})$, index of this monodromy group in $\mathrm{Sp}_{2g}(\mathbb{Z})$ is the same as that of the monodromy group acting on $H^1(Y, \mathbb{Z})$.

The result for the family $\mathcal{A}_n$ is the same as that for $\mathcal{B}_n$. It follows from Proposition 1.5 since $\mathcal{A}_n = \pi^*\mathcal{B}_n$, where $\pi \colon U_n \to V_n$ is the mapping sending a polynomial with roots $\alpha_0, \dots, \alpha_{n-1}$ to the polynomial with roots

$$
\alpha_0 - \frac{\alpha_0 + \dots + \alpha_{n-1}}{n}, \dots, \alpha_{n-1} - \frac{\alpha_0 + \dots + \alpha_{n-1}}{n},
$$

and the fibers of $\pi$, being isomorphic to $\mathbb{A}^1$, are irreducible.

## 2. A result on families of elliptic curves

Firstly, we fix some terminology and notation concerning elliptic curves.

Following Miranda [Mir89], we distinguish between curves of genus 1 and elliptic curves: by elliptic curve over a field $K$ we mean a smooth projective curve over $K$ of genus 1 with a distinguished $K$-rational point.

Similarly, by a *family of curves of genus* 1 we will mean a smooth and proper morphism $p \colon \mathcal{X} \to B$ such that its fibers are curves of genus 1, and by a *family of elliptic curves* we mean a pair $(\mathcal{X}, s)$, where $\mathcal{X} \to B$ is a family of curves of genus 1 and $s\colon B \to \mathcal{X}$ is a section.

To each curve $C$ of genus 1 over a filed $K$ one can assign its *$j$-invariant* $j(C) \in K$; recall that if $C$ is (the smooth projective model of) the curve defined by the Weierstrass equation $y^2 = x^3 + px + q$, then

$$j(C) = 1728 \cdot \frac{4p^3}{4p^3 + 27q^2}.$$

Two curves of genus 1 over $\mathbb{C}$ are isomorphic if and only if their $j$-invariants are equal.

We begin with a folklore result for which I do not know an adequate reference.

**Proposition 2.1.** *Suppose that $p\colon \mathcal{X} \to B$ is a (smooth and proper) family of curves of genus 1, where $B$ is a variety (i.e., a reduced scheme of finite type over $\mathbb{C}$). Then the mapping from $B$ to $\mathbb{C}$ that assigns $j$-invariant $j(f^{-1})(b)$ to a point $b \in B$, is induced by a morphism from $B$ to $\mathbb{A}^1$.*

*Proof.* If $\mathcal{X}' = \operatorname{Pic}^0(\mathcal{X}/B)$ (relative Picard variety, see [Kle, Section 5]), then the family $p'\colon \mathcal{X}' \to \mathbb{C}$ has a section (to wit, the zero section) and induces the same mapping from $B$ to $\mathbb{C}$ since $\operatorname{Pic}^0(C) \cong C$ if $C$ is a smooth curve of genus 1 over $\mathbb{C}$. Thus, without loss of generality one may assume that the family in question has a section; in this case see [Del75, § 5]. □

**Notation 2.2.** If $\mathcal{X} \to B$ is a family of curves of genus 1, then the morphism $B \to \mathbb{A}^1$ assigning the $j$-invariant $j(p^{-1}(b))$ to a point $b \in B$, will be denoted by $J_{\mathcal{X}}$.

In a family of smooth projective curves of genus 1, the monodromy group acting on $H^1$ of the fiber is contained in $\mathrm{SL}_2(\mathbb{Z})$.

**Proposition 2.3.** *Suppose that $p\colon \mathcal{X} \to B$ is a family of curves of genus 1, where $B$ is irreducible. If the morphism $J_{\mathcal{X}}\colon B \to \mathbb{A}^1$ is dominant and its general fiber is irreducible, then the monodromy group acting on $H^1$ of fibers of $p$ is either the entire $\mathrm{SL}_2(\mathbb{Z})$ or a subgroup of index 2 in $\mathrm{SL}_2(\mathbb{Z})$.*

*Remark* 2.4. It is well known that a subgroup of index 2 in $\mathrm{SL}_2(\mathbb{Z})$ is unique. This is the subgroup of automorphisms of $\mathbb{Z}^2$ with determinant 1 that, after reduction modulo 2, induce even permutations of non-zero elements of $(\mathbb{Z}/2\mathbb{Z})^2$. The uniqueness of such a subgroup follows from the fact that an epimorphism from $\mathrm{SL}_2(\mathbb{Z})$ onto $\mathbb{Z}/2\mathbb{Z}$ is unique since the abelianization of $\mathrm{SL}_2(\mathbb{Z})$ is the cyclic group of order 12.

We begin with two lemmas.

**Lemma 2.5.** *Suppose that $p\colon \mathcal{X} \to B$ is a family of curves of genus 1. Then there exists a family of elliptic curves $p'\colon \mathcal{X}' \to B$ such that the induced mappings $J_{\mathcal{X}}, J_{\mathcal{X}'}\colon B \to \mathbb{A}^1$ are the same and $\operatorname{Mon}(\mathcal{X}') \subset \mathrm{SL}_2(\mathbb{Z})$ is conjugate to $\tau(\operatorname{Mon}(\mathcal{X}))$, where $\tau\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$ is the automorphism defined by the formula $M \mapsto (M^t)^{-1}$.*

*Proof.* Put $\mathcal{X}' = \mathrm{Pic}^0(\mathcal{X}/B)$. As we have seen in the proof of Proposition 2.1, $J_{\mathcal{X}} = J_{\mathcal{X}'}$ and the family $p'\colon \mathcal{X}' \to B$ has a section. Finally, $R^1p'_*\mathbb{Z} \cong \underline{\mathrm{Hom}}(R^1p_*\mathbb{Z}, \mathbb{Z})$, where by $\mathbb{Z}$ we mean the constant sheaf with the stalk $\mathbb{Z}$ (see for example [Mum70, § 9]), and this implies the assertion about monodromy. □

**Lemma 2.6.** *Suppose that $p_1\colon \mathcal{X}_1 \to B$ and $p_2\colon \mathcal{X}_2 \to B$ are families of elliptic curves over the same irreducible base $B$ such that $J_{\mathcal{X}_1} = J_{\mathcal{X}_2}$ and the morphism $J_{\mathcal{X}_1} = J_{\mathcal{X}_2}$ is not constant; put $G_i = \mathrm{Mon}(\mathcal{X}_i)$, $i = 1, 2$. Then there exist subgroups $H_1 \subset G_1$ and $H_2 \subset G_2$ such that $(G_i : H_i) \le 2$ and $H_1$ is conjugate to $H_2$.*

*Proof.* Put $K = \mathbb{C}(B)$ (the field of rational functions). For $m = 1, 2$ denote by $\tilde{X}_m$ the generic fiber of $p_m$ (over $\mathrm{Spec}\, K$). The curves $\tilde{X}_1$ and $\tilde{X}_2$ are elliptic curves over $K$ with the same $j$-invariant, and this $j$-invariant is not equal to 0 or 1728 since the morphism $J_{\mathcal{X}_1} = J_{\mathcal{X}_2}$ is not constant. Hence, either $\tilde{X}_1 \cong \tilde{X}_2$ or there exists a quadratic extension $L \supset K$ such that $\tilde{X}_1 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L \cong \tilde{X}_2 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$ (see [Sil09, Chapter X, Proposition 5.4]). This implies that there exists a Zariski open $U \subset B$ and a finite étale morphism $\alpha\colon V \to U$ of degree 2 that $(i \circ \alpha)^*\mathcal{X}_1 \cong (i \circ \alpha)^*\mathcal{X}_2$, where $i\colon U \to B$ is the natural inclusion. If $b \in V$, then, for $m = 1, 2$, the group $\mathrm{Mon}((i \circ \alpha)^*\mathcal{X}_m, b)$ is a subgroup of index at most 2 in $\mathrm{Mon}(\mathcal{X}_m, \alpha(b))$. In view of Proposition 1.3 this implies the lemma. □

*Proof of Proposition 2.3.* Since a subgroup of index 2 in $\mathrm{SL}_2(\mathbb{Z})$ is unique (see Remark 2.4), the automorphism $M \mapsto (M^t)^{-1}$ maps this subgroup onto itself, so Lemma 2.5 implies that we may assume that the family $p\colon \mathcal{X} \to B$ in question has a section. Assuming that, put

$$V_3 = \{(p, q) \in \mathbb{A}^2 \colon 4p^3 + 27q^2 \neq 0\}$$

(see Notation 1.6) and consider the family of elliptic curves $\mathcal{B}_3 \to V_3$ in which the fiber over $(p, q)$ is the smooth projective model $C_{p,q}$ of the curve with equation $y^2 = x^3 + px + q$ and the section assigns to $(p, q)$ the "point at infinity" of this model. According to A'Campo's Theorem 1.7, one has $\mathrm{Mon}(\mathcal{B}_3) = \mathrm{SL}_2(\mathbb{Z})$.

Now put $\mathbb{A}^1_0 = \mathbb{A}^1 \setminus \{0, 1728\}$ and

$$V_{3,0} = J_{\mathcal{B}_3}^{-1}(\mathbb{A}^1_0) = \{(p, q \in \mathbb{A}^2 \colon p \neq 0, q \neq 0)\}.$$

Let $\mathcal{B}_{3,0}$ be the restriction of the family $\mathcal{B}_3$ to $V_{3,0}$; put $B_0 = J_{\mathcal{X}}^{-1}(\mathbb{A}^1_0)$, and let $\mathcal{X}_0$ be the restriction of $\mathcal{X}$ to $B_0$. Proposition 1.3 implies that $\mathrm{Mon}(\mathcal{X}_0) = \mathrm{Mon}(\mathcal{X})$ and $\mathrm{Mon}(\mathcal{B}_{3,0}) = \mathrm{Mon}(\mathcal{B}_3) = \mathrm{SL}_2(\mathbb{Z})$.

Observe that there exists an isomorphism $g\colon V_{3,0} \to (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1_0$ such that the diagram

$$\begin{tikzcd} V_{3,0} \arrow[rr, "g"] \arrow[dr, "J_{\mathcal{X}_0}"'] & & (\mathbb{A}_1 \setminus \{0\}) \times \mathbb{A}^1_0 \arrow[dl, "\mathrm{pr}_2"] \\ & \mathbb{A}^1_0 & \end{tikzcd}$$

is commutative. Indeed, one can define $g$ by the formula $(p,q) \mapsto (q/p, j(C_{p,q}))$, and the inverse morphism will be

$$(\lambda, j) \mapsto \left( \frac{\lambda^2}{\frac{4}{27}\left(\frac{1728}{j} - 1\right)}, \frac{\lambda^3}{\frac{4}{27}\left(\frac{1728}{j} - 1\right)} \right).$$

Hence, in the fibered product

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "u"'] & V_{3,0} \arrow[d, "J_{\mathcal{B}_{3,0}}"] \\ B_0 \arrow[r, "J_{\mathcal{X}_0}"'] & \mathbb{A}^1_0 \end{tikzcd}$$

(we mean fibered product in the category of reduced algebraic variates, so $W$ is the scheme theoretic fibered product modulo nilpotents) the variety $W$ is isomorphic to $(\mathbb{A}^1 \setminus \{0\}) \times B_0$; in particular, $W$ is irreducible.

A general fiber of the morphism $J_{\mathcal{X}_0}$ is irreducible according to the hypothesis. Thus, a general fiber of the morphism $f$ is also irreducible, and a general fiber of the morphism $u$ is irreducible since it is isomorphic to $\mathbb{A}^1 \setminus \{0\}$. Now Proposition 2.3 implies that for the pull-back families $f^*\mathcal{B}_{3,0}$ and $u^*\mathcal{X}_0$ on $W$ one has $\mathrm{Mon}(f^*\mathcal{B}_{3,0}) = \mathrm{Mon}(\mathcal{B}_{3,0}) = \mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{Mon}(u^*\mathcal{X}_0) = \mathrm{Mon}(\mathcal{X}_0)$ (as usual, the second equation holds up to a conjugation).

It is clear that $J_{f^*\mathcal{B}_{3,0}} = J_{\mathcal{B}_{3,0}} \circ f = J_{\mathcal{X}_0} \circ u = J_{u^*\mathcal{X}_0}$. Applying Lemma 2.6 to the families $f^*\mathcal{B}_{3,0}$ and $u^*\mathcal{X}_0$, one obtains the result. □

*Remark* 2.7. I do not know whether there exists a family satisfying the hypotheses of Proposition 2.3 such that the monodromy group is not the entire $\mathrm{SL}_2(\mathbb{Z})$.

## 3. Main result

In this section we prove Theorem 0.1.

If $X \subset \mathbb{P}^n$ is a smooth projective variety and $X^* \subset (\mathbb{P}^n)^*$ is its projective dual, one can define the "universal smooth hyperplane section of $X$", that is, the family

$$\mathcal{U}_X = \{(x, \alpha) \in X \times ((\mathbb{P}^n)^* \setminus X^*) \colon x \in H_\alpha\}, \tag{1}$$

where $H_\alpha \subset \mathbb{P}^n$ is the hyperplane corresponding to the point $\alpha \in (\mathbb{P}^n)^*$. The morphism $p\colon (x, \alpha) \mapsto \alpha$ makes $\mathcal{X}$ a family of smooth $n$-dimensional projective varieties over $(\mathbb{P}^n)^* \setminus X^*$; for any natural $d$, this family induces a monodromy action of $\pi_1((\mathbb{P}^n)^* \setminus X^*)$ on $H^d(Y,\mathbb{Z})$, where $Y$ is a smooth hyperplane section of $X$.

**Definition 3.1.** In the above setting, the image of $\pi_1((\mathbb{P}^n)^* \setminus X^*)$ in the group $\operatorname{Aut}(H^n(Y,\mathbb{Z}))$ will be called *hyperplane monodromy group* of $X$.

One can give a similar definition for hyperplane monodromy group acting on cohomology with arbitrary (constant) coefficients; besides $\mathbb{Z}$, we will need $\mathbb{Z}/2\mathbb{Z}$.

We begin with a general fact.

**Lemma 3.2.** *Suppose that $X \subset \mathbb{P}^n$ is a smooth projective variety and that $p \in \mathbb{P}^n \setminus X$ is a point such that the projection with center $p$ induces an isomorphism $\pi_p \colon X \to X' \subset \mathbb{P}^{N-1}$. If $H \ni p$ is a hyperplane that is transversal to $X$, then, after identifying $Y = X \cap H$ with $Y' = \pi_p(Y) = X' \cap \pi_p(H)$, the hyperplane monodromy groups acting on $H^n(Y,\mathbb{Z})$ and $H^n(Y',\mathbb{Z})$, are the same.*

The proof that is sketched below was suggested to me by Jason Starr.

*Sketch of proof.* Denote by $H_p \subset (\mathbb{P}^n)^*$ the hyperplane corresponding to the point $p \in \mathbb{P}^n$. It is clear that $H_p$ is naturally isomorphic to $(\mathbb{P}^{N-1})^*$ and that $(X')^* = X^* \cap H_p$. Moreover, the hyperplane $H_p$ is transversal to $X^*$ at any smooth point of $X^*$ (indeed, if $H_p$ is tangent to $X^*$ at a smooth point, then $p \in (X^*)^* = X$, which contradicts the hypothesis).

To prove the lemma it suffices to show that $\pi_1(H_p \setminus (X')^*)$ surjects onto $\pi_1((\mathbb{P}^n)^* \setminus X^*)$. To that end observe that there exists a line $\ell \subset H_p$ that is transversal to the smooth part of $X^* \cap H_p = (X')^*$ (in particular, $\ell$ does not pass through singular points of $X^* \cap H_p$). It follows from the transversality of $H_p$ to the smooth part of $X^*$ that $\ell$ is transversal to the smooth part of $X^*$, too. Thus, $\pi_1(\ell \setminus X^*)$ surjects both onto $\pi_1(H_p \setminus (X')^*)$ and onto $\pi_1((\mathbb{P}^n)^* \setminus X^*)$, whence the desired surjectivity. □

Lemma 3.2 implies that when studying hyperplane monodromy groups one may always assume that the variety in question is embedded by a complete linear system. Recall that if a del Pezzo surface $X \subset \mathbb{P}^n$ is embedded by the complete linear system $|-K_X|$ then $\deg X = n \le 9$; besides, if $n > 3$, $p \in X$ is a general point, and $\bar{X}$ is the blow-up of $X$ at $p$, then the projection $\pi_p \colon X \dashrightarrow \mathbb{P}^{n-1}$ induces an isomorphism $\bar{\pi}_p \colon \bar{X} \to X' = \overline{\pi_p(X)} \subset \mathbb{P}^{n-1}$ and $X' \subset \mathbb{P}^{n-1}$ is a del Pezzo surface embedded by $|-K_{X'}|$.

**Lemma 3.3.** *In the above setting, suppose that the hyperplane monodromy group of $X'$ is the entire $\mathrm{SL}_2(\mathbb{Z})$. Then the hyperplane monodromy group of $X$ is the entire $\mathrm{SL}_2(\mathbb{Z})$ as well.*

*Proof.* Informally the proof may be summed up in one phrase: if variation of hyperplanes passing through $p$ and transversal to $X$ is enough to obtain the entire group $\mathrm{SL}_2(\mathbb{Z})$, then *a fortiori* this is the case for all hyperplanes transversal to $X$. A formal argument follows.

Assume that the $\mathbb{P}^{n-1}$ into which the surface $X$ is projected is a hyperplane in $\mathbb{P}^n$, $\mathbb{P}^{n-1} \not\ni x$. If a hyperplane $H \subset \mathbb{P}^n$ contains the point $p \in X$, then $H = \overline{p, h}$, where $h = H \cap \mathbb{P}^{n-1}$ is a hyperplane in $\mathbb{P}^{n-1}$. If $H$ is transversal to $X$, then $\pi_p$ induces an isomorphism between the curve $H \cap X$, which is a smooth hyperplane section of $X$, and the curve $h \cap X'$, which is a smooth hyperplane section of $X'$. Put

$$V = \{\alpha \in (\mathbb{P}^{n-1})^* \colon \overline{p, h_\alpha} \text{ is transversal to } X\},$$

where $h_\alpha \subset \mathbb{P}^{n-1}$ is the hyperplane corresponding to the point $\alpha \in (\mathbb{P}^{n-1})^*$, and set

$$\mathcal{U} = \{(\alpha, x) \in V \times X' \colon x \in h_\alpha\}.$$

In the diagram

$$\begin{array}{ccccc} \mathcal{U}_{X'} & \longrightarrow & \mathcal{U} & \longrightarrow & \mathcal{U}_X \\ \downarrow{\scriptstyle q'} & & \downarrow & & \downarrow{\scriptstyle q} \\ (\mathbb{P}^{n-1})^* \setminus (X')^* & \xleftarrow{\;j\;} & V & \xrightarrow{\;r\;} & (\mathbb{P}^n)^* \setminus X^* \end{array} \tag{2}$$

where $\mathcal{U}_{X'}$ and $\mathcal{U}_X$ are universal smooth hyperplane sections of $X'$ and $X$, $j$ is an open embedding, and $r$ maps a hyperplane $h_\alpha \subset \mathbb{P}^{n-1}$ to the hyperplane $\overline{p, h_\alpha}$ in $\mathbb{P}^n$, both squares are Cartesian. Pick a point $\alpha \in V$; the hyperplane $\overline{p, h_\alpha} \subset \mathbb{P}^n$ is $H_{r(\alpha)}$, where $r(\alpha) \in (\mathbb{P}^n)^*$. If $Y' = {q'}^{-1}(\alpha)$ and $Y = X \cap H_{r(\alpha)} = q^{-1}(r(\alpha))$, then in the commutative diagram

$$\begin{array}{ccccc} \pi_1((\mathbb{P}^{n-1})^* \setminus (X')^*, \alpha) & \xleftarrow{\;w\;} & \pi_1(V, \alpha) & \longrightarrow & \pi_1((\mathbb{P}^n)^* \setminus X^*, r(\alpha)) \\ & \searrow{\scriptstyle u} & \downarrow & & \downarrow{\scriptstyle v} \\ & & \mathrm{Aut}(H^1(Y', \mathbb{Z})) & = & \mathrm{Aut}(H^1(Y, \mathbb{Z})) \end{array}$$

the mapping $w$ is an epimorphism since $V$ is Zariski open in $(\mathbb{P}^{n-1})^* \setminus (X')^*$, whence $\mathrm{Im}\, u \subset \mathrm{Im}\, v$. This proves the lemma. □

Projecting del Pezzo surfaces in $\mathbb{P}^n$, $n > 3$, consecutively from general points on them, one arrives at a cubic in $\mathbb{P}^3$; Lemma 3.3 implies that it suffices to prove Theorem 0.1 for this surface.

The next lemma reduces the problem to the case of "del Pezzo surfaces of degree 2", for which the anticanonical linear system defines a ramified covering of degree 2 over $\mathbb{P}^2$.

Suppose that $X \subset \mathbb{P}^3$ is a smooth cubic and $p \in X$ is a general point. Let $\bar{X}$ be the blow-up of $X$ at $p$. Then the projection $\pi_p \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ induces a finite morphism $\bar{\pi}_p \colon \bar{X} \to \mathbb{P}^2$ of degree 2; the branch locus of this morphism is a smooth curve $C \subset \mathbb{P}^2$ of degree 4. For $\alpha \in (\mathbb{P}^2)^*$, denote the corresponding line by $\ell_\alpha \subset \mathbb{P}^2$. If $\ell_\alpha$ is transversal to $C$ (i.e., $\alpha \notin C^*$), then $\bar{\pi}_p^{-1}(\ell_\alpha)$ is smooth, irreducible, and isomorphic to $X \cap \overline{p, \ell_\alpha}$.

**Lemma 3.4.** *Put*

$$\mathcal{X} = \{(\alpha, x) \in ((\mathbb{P}^2)^* \setminus C^*) \times \bar{X} \colon \bar{\pi}_p(x) \in \ell_\alpha\} \tag{3}$$

*and denote the morphism $(\alpha, x) \mapsto \alpha$ by $q\colon \mathcal{X} \to (\mathbb{P}^2)^* \setminus C^*$. If $R$ is a commutative ring and $\operatorname{Mon}(\mathcal{X}, R) = \mathrm{SL}_2(R)$, then the hyperplane monodromy group of $X$ with coefficients in $R$ is also equal to $\mathrm{SL}_2(R)$.*

*Proof.* The proof is similar to that of Lemma 3.3. It suffices to make the following modifications in the diagram (2): put $n = 3$, replace $\mathcal{U}_{X'}$ by $\mathcal{X}$, replace $(X')^*$ by $C^*$, and put $V = (\mathbb{P}^2)^* \setminus (C^* \cup \{\beta\})$, where $\ell_\beta = T_pX \cap \mathbb{P}^2$. □

**Lemma 3.5.** *The monodromy group acting on $H^1$ of fibers of the family* (3) *with coefficients $\mathbb{Z}/2\mathbb{Z}$, is $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$.*

*Proof.* Suppose that $Y = \bar{\pi}_p^{-1}(\ell)$, where $\ell \subset \mathbb{P}^2$ is transversal to $C$, is a fiber of the family (3). We are to show that the monodromy group in question performs all the permutations of the non-zero elements of $H^1(Y, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^2$.

Recall that $H^1(Y, \mathbb{Z}/2\mathbb{Z}) \cong (\operatorname{Pic}(Y))_2$ (the subgroup of elements of order at most 2). If $\ell \cap C = \{P_1, P_2, P_3, P_4\}$ and $Q_j = \bar{\pi}_p^{-1}(P_j)$, then, since $Y$ is a two-sheeted covering of $\ell$ with ramification at $Q_1, \dots, Q_4$, any point of order 2 in $\operatorname{Pic}(Y)$ is represented by a divisor of the form $Q_i - Q_j$. Thus, to prove the lemma it suffices to show that, as $\ell$ varies in the family of lines transversal to $C$, the monodromy on the set $\ell \cap C$ performs all the permutations of the set $\ell \cap C$. The latter assertion is a particular case of [BH86, Theorem on p. 906]. □

The following lemma, which will be used in the proof of Proposition 3.7, is valid over algebraically closed fields of arbitrary characteristic.

**Lemma 3.6.** *Suppose that $W$ is a smooth irreducible variety of dimension $n$, $L$ is a smooth irreducible curve* (*we do not assume that $W$ or $L$ is projective*), *and $\varphi\colon W \to L$ is a proper and surjective morphism with $(n-1)$-dimensional fibers. Put $Z = \mathbf{Spec}\, \varphi_*\mathcal{O}_W$ and let $v\colon Z \to L$ be the natural morphism.*

*If there exists a point $p \in L$ such that $\varphi^{-1}(p)$ is irreducible and the morphism $\varphi$ has maximal rank at a general point of $\varphi^{-1}(p)$, then the natural morphism $v\colon Z \to L$ is an isomorphism.*

*Proof.* It is clear that $Z$ is an irreducible and reduced curve. Since $\varphi$ is proper and $\varphi^{-1}(p)$ is connected, the stalk $(\varphi_*\mathcal{O}_W)_p$ is a local ring, so $v^{-1}(p)$ consists of one point; denote this point by $z$. I claim that that $z$ is a smooth point of $Z$ and the morphism $v$ is unramified at $z$. Indeed, let $\tau \in \mathcal{O}_{L,p}$ be a generator of the maximal ideal. Its image $v^*\tau \in \mathcal{O}_{Z,z}$ can be represented by a regular function $f \in \mathcal{O}_W(\varphi^{-1}(U))$, where $U \subset L$ is a Zariski neighborhood of $p$. Since the morphism $\varphi$ has maximal rank at a general point of $\varphi^{-1}(p)$, the function $v^*\tau$ vanishes on the irreducible divisor $\varphi^{-1}(p)$ with multiplicity 1. Since regular functions on $\varphi^{-1}(U)$ must be constant on the fibers of the proper morphism $\varphi$, any element of the maximal ideal of the local ring $\mathcal{O}_{Z,z}$ is representable by a regular function $g \in \mathcal{O}_W(\varphi^{-1}(V))$, where $V$ is a Zariski neighborhood of $p$, such that the zero locus of $g$ in $\varphi^{-1}(V)$ coincides with $u^{-1}(z)$. Hence, $v^*\tau$ generates the maximal ideal of $\mathcal{O}_{Z,z}$, which proves our claim.

Since $v^{-1}(p) = \{z\}$, $Z$ is smooth at $z$, and $v$ is unramified at $z$, we conclude that the finite morphism $v$ has degree 1. Since $L$ is smooth, Zariski main theorem implies that $v$ is an isomorphism. □

**Proposition 3.7.** *Suppose that $\pi\colon X \to \mathbb{P}^2$ is a finite morphism of degree 2 branched over a smooth quartic $C \subset \mathbb{P}^2$, where $X$ is smooth. If $J\colon (\mathbb{P}^2)^* \setminus C^* \to \mathbb{A}^1$ is the morphism $\alpha \mapsto j(\pi^{-1}(\ell_\alpha))$, where $\ell_\alpha$ is the line in $\mathbb{P}^2$ corresponding to $\alpha \in (\mathbb{P}^2)^*$, then a general fiber of $J$ is irreducible.*

*Proof.* Let us show that the morphism $J$ extends to a morphism

$$J_1\colon (\mathbb{P}^2)^* \setminus (C^*)_{\text{sing}} \to \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}.$$

Indeed, if $\ell \subset \mathbb{P}^2$ is a line and $\ell \cap C = \{P_1, P_2, P_3, P_4\}$, then the curve $\pi^{-1}(\ell)$ is a curve of genus 1 and

$$j(\pi^{-1}(\ell)) = 256\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1-\lambda)^2}, \tag{4}$$

where $\lambda$ is the cross-ratio $[P_1, P_2, P_3, P_4]$, in no matter what order (see for example [Sil09, Chapter III, Proposition 1.7b]). If $\alpha$ is a smooth point of $C^* \subset (\mathbb{P}^2)^*$, then the line $\ell_\alpha$ is tangent to $C$ at exactly one point that is not an inflection point. Thus, as the line $\ell$ tends to $\ell_\alpha$, exactly two intersection points from $\ell \cap C$ merge, so the cross-ratio of these four points tends to 0 (or 1, or $\infty$, depending on the ordering), and formula (4) shows that $j(\pi^{-1}(\ell))$ tends to $\infty$. This proves the existence of the desired extension.

Our argument shows that $J_1^{-1}(\infty) = C^* \setminus (C^*)_{\text{sing}}$; if we regard $J_1$ as a rational mapping from $(\mathbb{P}^2)^*$ to $\mathbb{P}^1$ and if

$$\begin{array}{ccc} W & & \\ \downarrow{\scriptstyle\sigma} & \searrow^{J_2} & \\ (\mathbb{P}^2)^* & \overset{J_1}{\dashrightarrow} & \mathbb{P}^1 \end{array} \tag{5}$$

is a minimal resolution of indeterminacy for $J_1$, then $J_2^{-1}(\infty)$ equals the strict transform of $C^*$ with respect to $\sigma$.

Now I claim that, at a general point of $J_2^{-1}(\infty)$, derivative of $J_2$ has rank 1. It suffices to prove this assertion for $J_1$ and a general smooth point of $C^*$. To that end it suffices to construct an analytic mapping $\gamma\colon D \to (\mathbb{P}^2)^*$, where $D$ is a disk in the complex plane with center at 0, such that $\gamma(D \setminus \{0\}) \subset (\mathbb{P}^2)^* \setminus C^*$, $\gamma(0)$ is a smooth point of $C^*$, and $|j(\pi^{-1}(\ell_{\gamma(t)}))| \sim \text{const}/|t|$.

Suppose that a point $c \in C$ is not an inflection point nor a tangency point of a bitangent; if $\ell_\alpha \subset \mathbb{P}^2$ is the tangent line to $C$ at $c$, then $\alpha$ s a smooth point of $C^*$. Now choose affine $(x, y)$-coordinates in $\mathbb{P}^2$ so that $c = (0, 0)$, the tangent $\ell_\alpha$ has equation $y = 0$, and $\ell_\alpha \cap C = \{c, (C, 0), (D, 0)\}$, where $C, D \neq 0$ (so the remaining two points of $\ell_\alpha \cap C$ are in the finite part of $\mathbb{P}^2$ with respect to the chosen coordinate system). If $\ell_{\gamma(t)}$ is the line with affine equation $y = t$, then, for all small enough $t$, one has $\ell_{\gamma(t)} \cap C = \{A(t), B(t), C(t), D(t)\}$, where the $x$-coordinates of $A(t)$ and $B(t)$ are $\sqrt{t} + o(\sqrt{|t|})$ (for both values of $\sqrt{t}$), while the $x$ coordinates of $C(t)$ and $D(t)$ tend to finite and non-zero numbers $C$ and $D$. Hence,

$$\big|[C(t), A(t), B(t), D(t)]\big| \sim \frac{\text{const}}{\sqrt{|t|}} \quad \text{as } t \to 0;$$

formula (4) implies that $|j(\pi^{-1}(\ell_t))| \sim \text{const}/|t|$, as desired.

Let

$$\begin{array}{ccc} W & \xrightarrow{\;J_2\;} & \mathbb{P}^1 \\ & {\scriptstyle u}\searrow \quad \nearrow{\scriptstyle v} & \\ & Z & \end{array}$$

be the Stein factorization in which $W$ is a blow-up of $(\mathbb{P}^2)^*$ (see (5)), $Z = \mathbf{Spec}(J_2)_*\mathcal{O}_W$, and $v$ is a finite morphism. Applying Lemma 3.6 with $L = \mathbb{P}^1$, $\varphi = J_2$, and $p = \infty$, we conclude that $v$ is an isomorphism. Thus, fibers of $J_2$ coincide with fibers of $u$; since the latter are connected, fibers of $J_2$ are connected as well. Bertini theorem implies that a general fiber of $J_2$ is smooth; since it is connected, it must be irreducible. This implies that a general fiber of $J$ is irreducible. □

*Proof of Theorem 0.1.* In view of Proposition 3.2 and Lemmas 3.3 and 3.4, it suffices to prove that $\mathrm{Mon}(\mathcal{X}) = \mathrm{SL}_2(\mathbb{Z})$, where $\mathcal{X}$ is the family defined by (3).

Applying Proposition 3.7 to the surface $\bar{X}$ (blow-up of a cubic at a general point $p$) and the mapping $\bar{\pi}_p \colon \bar{X} \to \mathbb{P}^2$ (induced by the projection with center $p$), we see that the family $\mathcal{X}$ defined by formula (3) satisfies the hypothesis of Proposition 2.3. Hence, $\mathrm{Mon}(\mathcal{X})$ is either the entire $\mathrm{SL}_2(\mathbb{Z})$ or its subgroup of index 2. In the first case we are done, and the second case is impossible: Lemma 3.5 tells us that $\mathrm{Mon}(\mathcal{X})$ induces, on the cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$, the entire group $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \cong S_3$, while the only subgroup of index 2 in $\mathrm{SL}_2(\mathbb{Z})$ induces only even permutations of non-zero elements of $(\mathbb{Z}/2\mathbb{Z})^2$ (see Remark 3.5). This completes the proof. □

## 4. Families of hyperelliptic curves

**Proposition 4.1.** *If $\pi \colon \mathcal{X} \to B$ is a family of hyperelliptic curves of genus $g > 2$, then*

$$(\mathrm{Sp}_{2g}(\mathbb{Z}) : \mathrm{Mon}(\mathcal{X})) \geq \frac{2^{g^2}(2^{2g}-1)(2^{2(g-1)}-1)\cdot \dots \cdot (2^2-1)}{(2g+2)!}.$$

**Corollary 4.2.** *If $\pi \colon \mathcal{X} \to B$ is a family of hyperelliptic curves of genus $g > 2$, then $\mathrm{Mon}(\mathcal{X})$ is a proper subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$.*

*Proof of Proposition 4.1.* In this proof, $\mathrm{Mon}(\mathcal{X},\mathbb{Z})$ will denote the monodromy group acting on the integer $H^1$ of a fiber of $\mathcal{X}$, and $\mathrm{Mon}(\mathcal{X},\mathbb{Z}/2\mathbb{Z})$ will stand for the monodromy group acting on cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Since the reduction modulo 2 mapping $\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/2\mathbb{Z})$ is surjective, one has

$$(\mathrm{Sp}_{2g}(\mathbb{Z}) : \mathrm{Mon}(\mathcal{X},\mathbb{Z})) \geq (\mathrm{Sp}_{2g}(\mathbb{Z}/2\mathbb{Z}) : \mathrm{Mon}(\mathcal{X},\mathbb{Z}/2\mathbb{Z})),$$

so it suffices to show that

(6)
$$(\mathrm{Sp}_{2g}(\mathbb{Z}/2\mathbb{Z}) : \mathrm{Mon}(\mathcal{X},\mathbb{Z}/2\mathbb{Z})) \geq \frac{2^{g^2}(2^{2g}-1)(2^{2(g-1)}-1)\cdot\dots\cdot(2^2-1)}{(2g+2)!}.$$

To that end, let $X$ be a hyperelliptic curve of genus $g \geq 2$ that is a fiber of $\mathcal{X}$; denote its Weierstrass points by $P_1,\dots,P_{2g+2}$. It is well known (see for example [Cor01, Lemma 2.1]) that the 2-torsion subgroup $(\mathrm{Pic}(X))_2 \subset \mathrm{Pic}(X)$ is generated by classes of divisors $P_i - P_j$. Since $\mathrm{Pic}(X)_2 \cong H^1(X,\mathbb{Z}/2\mathbb{Z})$, the action of $\pi_1(B_{\mathrm{sm}})$ on $H^1(X,\mathbb{Z}/2\mathbb{Z})$ is completely determined by the permutations of the Weierstrass points $P_1,\dots,P_{2g+2}$ it induces. Thus, order of $\mathrm{Mon}(\mathcal{X},\mathbb{Z}/2\mathbb{Z})$ is at most $(2g+2)!$. Since

$$(\mathrm{Sp}_{2g}(\mathbb{Z}/2\mathbb{Z}) : 1) = 2^{g^2}(2^{2g}-1)(2^{2(g-1)}-1)\cdot\dots\cdot(2^2-1),$$

the proposition follows. □

*Remark* 4.3. The bound in Proposition 4.1 is sharp since, according to A'Campo [A'C79], it is attained for the family $\mathcal{B}_{2g+2}$.

**Corollary 4.4.** *If $X \subset \mathbb{P}^n$ is a smooth surface such that $H^1(X,\mathbb{Z}) = 0$, its smooth hyperplane sections are hyperelliptic curves of genus $g \geq 3$, and $X$ is not ruled by lines, then the hyperplane monodromy group of $X$ is a non-trivial proper subgroup of* $\mathrm{Sp}_{2g}(\mathbb{Z})$.

*Proof.* The main result of [Zak73] implies that the hyperplane monodromy group of $X$ is non-trivial, and the rest follows from Corollary 4.2. □

Surfaces satisfying the hypothesis of Corollary 4.4 exist for any $g \geq 3$. For example, $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^{3g+5}$ by the complete linear system of bidegree $(2, g+1)$ has the required properties.

*Remark* 4.5. Corollary 4.4 does not tell anything about surfaces with hyperplane sections of genus 2. Actually, a complete list of such surfaces is known (see for example [Liv84] and [Som79]), so it may be possible to compute their hyperplane monodromy groups "by hand".

In Corollary 4.4 hyperplane sections of the surface $X$ were hyperelliptic. I do not know whether there exists a surface $X$ with non-hyperelliptic general hyperplane section of genus $g > 2$ such that $H^1(X,\mathbb{Z}) = 0$ and the hyperplane monodromy group is different from $\mathrm{Sp}_{2g}(\mathbb{Z})$ and non-trivial. It is easy to show that for the Veronese surface $X = v_4(\mathbb{P}^2) \subset \mathbb{P}^{14}$, as well as for its projection from one or two points lying on $X$, the hyperplane monodromy group is the entire $\mathrm{Sp}_{2g}(\mathbb{Z})$ (where $g = 3$).

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