# Geodesic description of Heavy-Light Virasoro blocks

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ABSTRACT: We continue to investigate the dual description of the Virasoro conformal blocks arising in the framework of the classical limit of the $\mathrm{AdS_3/CFT_2}$ correspondence. To give such an interpretation in previous studies, certain restrictions were necessary. Our goal here is to consider a more general situation available through the worldline approximation to the dual AdS gravity. Namely, we are interested in computing the spherical conformal blocks without the previously imposed restrictions on the conformal dimensions of the internal channels. The duality is realised as an equality of the so-called heavy-light limit of the $n$-point conformal block and the action of $n-2$ particles propagating in some AdS-like background with either a conical singularity or a BTZ black hole. We describe a procedure that allows relaxing the constraint on the intermediate channels. To obtain an explicit expression for the conformal block on the CFT side, we use a recently proposed recursion procedure and find full agreement between the results of the boundary and bulk computations.

# 1  Introduction

Computing correlators in conformal field theory (CFT) requires knowing the conformal blocks [1]. They define the contributions of particular sets of primary fields and their descendants to the intermediate channels when the operator product expansion (insertion of a complete set of states) is applied. Although symmetry algebras in principle fix the coefficients of the series expansions for these functions, conformal blocks are known explicitly only in particular cases. The AdS/CFT duality reveals an interesting interpretation of conformal blocks [2]. Basically, this interpretation is available in a semiclassical treatment (but see, e.g. [3, 4]). Nevertheless, despite a certain progress in understanding the dual description of conformal blocks, there are still some open questions, one of which we address here.

We focus on $AdS_3/CFT_2$ in the case of a pure Virasoro symmetry. The main issue is to define bulk degrees of freedom appropriate to the observables of the boundary CFT. According to the Brown–Henneaux relation $c \sim 1/G_N$, we are dealing with the large central charge CFT limit and the classical gravity in the bulk [5]. The large central charge behaviour of the conformal blocks was first considered in the context of their analytic properties [6]. It was noticed that the classical limit of the conformal block has an interesting exponentiation property: it becomes an exponent of a function depending on appropriately rescaled conformal dimensions (called classical). This function is known as the *classical conformal block*. To be interpreted as a classical action, a certain linearisation procedure [2, 7] must be applied to the classical block. The obtained object is called a heavy-light block. An important statement from the previous studies [8–17] implies that the heavy-light block equals to the classical action up to some easily tractable terms. To verify this statement, some methods of calculating the CFT counterpart of the duality were developed [18–20]. In this paper, we discuss the CFT on a sphere, but the duality can also be extended to conformal blocks on higher-genus surfaces [21–24].

We consider the AdS/CFT correspondence in the example of the 5-point heavy-light block. The dual description [25–27] for this object is as following: two *heavy* fields form the BTZ or conical singularity [28], and three remaining *light* fields correspond to three particles propagating in this background (see Fig. 2). Such an interpretation prescribes that two dimensions of the heavy fields are equal. Calculating the minimal action of the dual configuration involves solving a variation problem. The solution can be found exactly when one of light external particles is absent and the configuration hence corresponds to a 4-point heavy-light block. The problem can then be solved using perturbation theory with the third particle mass regarded as a small parameter. But one additional constraint on the internal dimensions was imposed by this method. This approach led to the configuration with equal dimensions of the internal fields. This point was not obvious, because there are no physical reasons for this constraint. Hence, there is a question about principal possibility to solve this problem with unequal internal dimensions.

In this note, we compute the action corresponding to the 5-point heavy-light block. We use an approach that allows analysing the bulk configuration perturbatively without the

previously imposed restriction on the internal channels. We compare the obtained action with the boundary CFT calculation and find agreement. The CFT side of the problem requires an effective method for computing conformal block coefficients. The most convenient method for our purpose is $c$-recursion. It was first developed for the 4-point conformal block by Zamolodchikov [29] in the form of the pole decomposition over the $c$-plane with the residues proportional to conformal blocks with shifted parameters. This property leads to the recursion procedure of calculating the coefficients of conformal blocks level by level. This idea was recently extended to $n$-point spherical and toric conformal blocks [30]. We apply these results here to compute the 5-point block.

## 2   Conformal side of conformal blocks

We are interested in the conformal blocks of the correlation function with two heavy asymptotic states. Using projective invariance to fix $z_5 = \infty$, $z_4 = 1$, and $z_1 = 0$ we obtain the block depending on two modules $z_2$ and $z_3$. The corresponding diagram is depicted on the Fig. 1.
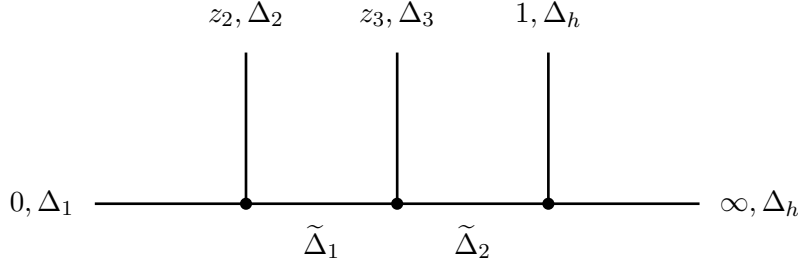


**Figure 1**. 5-point conformal block.

Introducing new variables $q_2 = z_3$ and $q_1 = z_2/z_3$ and fixing intermediate channels, the conformal block

$$F(\Delta_{1,2,3,4,5}, \widetilde{\Delta}_{1,2}|q_1, q_2) = \sum_{r,s=0}^{\infty} F_{r,s}\, q_1^r\, q_2^s \,, \tag{2.1}$$

with the expansion coefficients:

$$F_{r,s} = \sum_{r=|k|=|l|=0}^{\infty} \sum_{s=|m|=|n|=0}^{\infty} \mathcal{M}_{k,l}^{-1}(\tilde{\Delta}_1)\mathcal{M}_{m,n}^{-1}(\tilde{\Delta}_2)\langle\Delta_1|\mathcal{O}_2\mathcal{L}_{-k}|\widetilde{\Delta}_1\rangle \times,$$

$$\langle\widetilde{\Delta}_1|\mathcal{L}_l\mathcal{O}_3\mathcal{L}_{-m}|\widetilde{\Delta}_2\rangle \, \langle\widetilde{\Delta}_2|\mathcal{L}_n\mathcal{O}_h|\Delta_h\rangle \,. \tag{2.2}$$

Here $\mathcal{M}_{m,n}(\tilde{\Delta}_{1,2})$ are Gram matrices for Virasoro module states, $\mathcal{L}_{-n} = L_{-k}^{i_k}...L_{-1}^{i_1}$ are elements of the universal enveloping of Virasoro algebra with $|n| = i_1 + ... + k i_k$ and $\mathcal{L}_m = \mathcal{L}_{-m}^\dagger$.

## 2.1 Recursive representation

We provide here the recursive representation of the 5-point conformal block. The most efficient way of computing the coefficients of the block is so-called q-recursion (see [31]), but for our purposes it is more convenient to use c-recursion because it is more suitable for comparison with the bulk computation.

Let us shortly review the recursive representation of the 4-point block. Calculations based on the definition quickly becomes tedious with the increasing of the level. In [29] the method based on the analysis of the analytical properties of 4-point blocks has been proposed. Following [32] it was noticed that conformal blocks, considered as function of $c$, have only simple poles in special points:

$$c_{r,s}(\tilde{\Delta}) = 13 + 6(b_{r,s}^2(\tilde{\Delta}) + b_{r,s}^{-2}(\tilde{\Delta})),$$
$$b_{r,s}^2(\tilde{\Delta}) = \frac{1}{1-r^2}\left(2\tilde{\Delta} + rs - 1 + \sqrt{(r-s)^2 + 4(rs-1)\tilde{\Delta} + 4\tilde{\Delta}^2}\right), \tag{2.3}$$

where $\Delta$ is the intermediate dimension. These poles are related to degenerate values of the intermediate conformal dimension. If the fusion rules for the external fields are satisfied the conformal block is regular. Consequently, the residues in points $c = c_{r,s}$ must contain so-called fusion polynomials which have zeros cancelling zeros of the denominator and can be derived from this requirement.

This idea can be generalized on an arbitrary number of intermediate channels [30]. Here we provide necessary formulas for computing the 5-point conformal block. The conformal block has poles in $c = c_{r,s}(\tilde{\Delta}_1)$ and $c = c_{r,s}(\tilde{\Delta}_2)$ (2.3) related to degenerate values in the first and in the second intermediate channels

$$F_{k,m} = G_{k,m} + \sum_{r\geq 2, s\geq 1} \frac{V_{r,s}^{(1)}}{c - c_{r,s}(\tilde{\Delta}_1)} + \sum_{r\geq 2, s\geq 1} \frac{V_{r,s}^{(2)}}{c - c_{r,s}(\tilde{\Delta}_2)}, \tag{2.4}$$

where $V_{r,s}^{(1)}$ and $V_{r,s}^{(2)}$ are the corresponding residues and $G_{k,m}$ gives the asymptotics of $F_{k,m}$ as $c \to \infty$. The residues are given explicitly by

$$V_{r,s}^{(1)} = -\frac{\partial c_{r,s}(\tilde{\Delta}_1)}{\partial \tilde{\Delta}_1} A_{rs}^{c_{rs}(\tilde{\Delta}_1)} P_{c_{rs}(\tilde{\Delta}_1)}^{rs} \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} P_{c_{rs}(\tilde{\Delta}_1)}^{rs} \begin{bmatrix} \tilde{\Delta}_2 \\ \Delta_3 \end{bmatrix} F_{k-rs,m}(\tilde{\Delta}_1 \to \tilde{\Delta}_1 + rs, c \to c_{rs}(\tilde{\Delta}_1)), \tag{2.5}$$

$$V_{r,s}^{(2)} = -\frac{\partial c_{r,s}(\tilde{\Delta}_2)}{\partial \tilde{\Delta}_2} A_{rs}^{c_{rs}(\tilde{\Delta}_2)} P_{c_{rs}(\tilde{\Delta}_2)}^{rs} \begin{bmatrix} \Delta_4 \\ \Delta_5 \end{bmatrix} P_{c_{rs}(\tilde{\Delta}_2)}^{rs} \begin{bmatrix} \Delta_3 \\ \tilde{\Delta}_1 \end{bmatrix} F_{k,m-rs}(\tilde{\Delta}_2 \to \tilde{\Delta}_2 + rs, c \to c_{rs}(\tilde{\Delta}_2)), \tag{2.6}$$

where $A_{rs}^{c_{rs}(\Delta)}$ are the norms of the degenerated vectors:

$$A_{rs}^{c_{rs}(\Delta)} = \frac{1}{2} \prod_{m=1-r}^{r} \prod_{n=1-s}^{s} (mb_{rs}(\Delta) + nb_{rs}(\Delta)^{-1})^{-1}, \quad (m,n) \neq (0,0), (r,s), \tag{2.7}$$

and $P^{rs}_{c_{rs}(\Delta)}\begin{bmatrix}\Delta_i\\\Delta_j\end{bmatrix}$ are the fusion polynomials:

$$P^{rs}_{c_{rs}(\Delta)}\begin{bmatrix}\Delta_i\\\Delta_j\end{bmatrix} = \prod_{p=1-r \text{ step } 2}^{r-1}\prod_{q=1-s \text{ step } 2}^{s-1} \frac{\lambda^i_{rs}(\Delta) + \lambda^j_{rs}(\Delta) + pb_{rs}(\Delta) + qb_{rs}(\Delta)^{-1}}{2} \times$$

$$\frac{\lambda^i_{rs}(\Delta) - \lambda^j_{rs}(\Delta) + pb_{rs}(\Delta) + qb_{rs}(\Delta)^{-1}}{2} \ , \quad (2.8)$$

where

$$\lambda^i_{rs}(\Delta) = \frac{\sqrt{(b_{rs}(\Delta) + b_{rs}(\Delta)-1)^2 - 4\Delta}}{4} \ . \quad (2.9)$$

The last ingredient is $G_{k,m}$ [26]:

$$G_{k,m} = \frac{(\widetilde{\Delta}_1 + \Delta_2 - \Delta_1)_k\,(\widetilde{\Delta}_2 + \Delta_4 - \Delta_5)_m}{(2\widetilde{\Delta}_1)_k\,(2\widetilde{\Delta}_2)_m}\,\tau_{k,m} \ , \quad (2.10)$$

where the function $\tau_{k,m}$ is given by

$$\tau_{k,m} = \sum_{p=0}^{\min[k,m]}(-1)^p\frac{(-2\widetilde{\Delta}_2 - m + 1)_p(\widetilde{\Delta}_2 + \Delta_3 - \widetilde{\Delta}_1)_{m-p}(\widetilde{\Delta}_1 + \Delta_3 - \widetilde{\Delta}_2 + p - m)_{k-p}}{p!(k-p)!(m-p)!} \ .$$

$$(2.11)$$

These formulas provide an efficient way to compute the conformal block coefficients.

## 2.2  Classical block

Exponentiation of the conformal block in the semiclassical limit when all dimensions are heavy was firstly observed in [1, 6] and has natural physical interpretation in the framework of Liouville field theory consideration. The deep relation between AdS$_3$ physics and Liouville theory explains our interest to the classical blocks (see, e.g. [33–35]). Our study of conformal blocks in this context is motivated by the goal is to better understand this connection.

From the quantum block we obtain the classical block taking the $c \to \infty$ limit:

$$f_{cl}(\epsilon_{1,2,3}, \epsilon_h, \tilde{\epsilon}_{1,2}|q_1, q_2) =$$
$$- \lim_{c\to\infty}\frac{6}{c}\log F(\Delta_{1,2,3} \to \frac{c}{6}\epsilon_{1,2,3}, \Delta_h \to \frac{c}{6}\epsilon_h, \tilde{\Delta}_{1,2} \to \frac{c}{6}\tilde{\epsilon}_{1,2}, c|q_1, q_2) \ . \quad (2.12)$$

Let us notice that the existence of this limit imposes non-trivial constrains on the coefficients of the conformal blocks.

## 2.3  Heavy-light block

Here we define and calculate the heavy-light block. Firstly, we notice that the introduced notation can be confusing because the dimensions of the fields are usually classified according their central charge scaling properties. Now we introduce a smallness parameter $\delta$, which

corresponds to the weakness of the gravity interactions between two particles. We call the first coefficient of the expansion of the classical block in $\delta$ the heavy-light block. Namely, we change

$$\epsilon_{1,2,3} \to \delta\epsilon_{1,2,3}, \quad \tilde{\epsilon}_{1,2} \to \delta\tilde{\epsilon}_{1,2}, \quad \epsilon_h \to \epsilon_h , \tag{2.13}$$

and perform an expansion of the classical block in this parameter:

$$f_{cl} = \delta f_{hl} + O(\delta^2) . \tag{2.14}$$

This procedure allows to take into account the fact that two fields are "heavy" in the sense of the gravitational interaction. In other words, this takes into account the fact that two fields form the background and all others are "light" so that back-reaction is neglected.

In order to compare with the bulk computation we perform the following substitutions

$$\epsilon_h = \frac{1-\alpha^2}{4}, \qquad \tilde{\epsilon}_2 = \tilde{\epsilon}_1 + \mu\epsilon_3, \qquad \epsilon_2 = \epsilon_1 . \tag{2.15}$$

This yields the following expansion

$$f_{hl}(\epsilon_1, \epsilon_1, \nu, \alpha, \tilde{\epsilon}_1, t | q_1, q_2) = -\sum_{m=0}^{m} \sum_{n=0} g_{mn}(\epsilon_1, \tilde{\epsilon}_1)\epsilon_3^m \mu^n . \tag{2.16}$$

Below the explicit results for the first three coefficients are presented:

$$g_{00} = \epsilon_1 \frac{q_1^2 q_2^2}{12}\left(1-\alpha^2\right) + \tilde{\epsilon}_1\left(\tfrac{1}{2}q_1 q_2 + \tfrac{5}{24}q_1^2 q_2^2 - \tfrac{1}{48}\alpha^2 q_1^2 q_2^2\right) , \tag{2.17}$$

$$g_{10} = \frac{q_1}{2} + \frac{q_2}{2} + \frac{q_1^2}{8} - \frac{q_1 q_2}{4} + \frac{7q_2^2}{24} + \frac{q_1^3}{24} + \frac{q_1^2 q_2}{8} - \frac{q_1 q_2^2}{24} + \frac{5q_2^3}{24} + \frac{q_1^4}{64} + \frac{q_1^3 q_2}{16} - \frac{11q_1^2 q_2^2}{96} - \frac{q_1 q_2^3}{48} + \frac{469q_2^4}{2880}$$
$$+\alpha^2\left(-\frac{q_2^2}{6} + \frac{q_1 q_2^2}{6} - \frac{q_1^2 q_2^2}{24} - \frac{q_2^3}{6} + \frac{q_1 q_2^3}{12} - \frac{11}{72}q_2^4\right) + \frac{1}{180}\alpha^4 q_2^4 ,$$
$$\tag{2.18}$$

$$g_{11} = -\frac{q_1}{2} + \frac{q_2}{2} - \frac{q_1^2}{8} + \frac{q_1 q_2}{4} + \frac{5q_2^2}{24} - \frac{q_1^3}{24} - \frac{q_1^2 q_2}{8} + \frac{q_1 q_2^2}{24} + \frac{q_2^3}{8} - \frac{q_1^4}{64} - \frac{q_1^3 q_2}{16} + \frac{11q_1^2 q_2^2}{96} + \frac{q_1 q_2^3}{48} + \frac{251q_2^4}{2880}$$
$$+\alpha^2\left(-\frac{q_2^2}{12} + \frac{q_1 q_2^2}{12} - \frac{q_1^2 q_2^2}{48} - \frac{q_2^3}{12} + \frac{q_1 q_2^3}{24} - \frac{11q_2^4}{144}\right) + \frac{7\alpha^4 q_2^4}{1440} .$$
$$\tag{2.19}$$

Let us notice that the heavy-light block depends linearly on the light dimensions. This observation allows to think of it as describing a solution of some classical problem. In the next section we will get an appropriate action.

## 3 Bulk treatment

Here we consider a classical gravity of probe particles in the AdS background with an angle defect. We provide necessary formulas in a very brief form (refer to [7, 25] for more details). We have a triple of coordinates $(t, \rho, \phi)$ and the interval is

$$ds^2 = \frac{\alpha^2}{\cos^2 \rho}\left(-dt^2 + \sin^2 \rho \, d\phi^2 + \frac{1}{\alpha^2}d\rho^2\right) , \tag{3.1}$$

where $\alpha$ corresponds to the conical defect. One can find the geodesic length for one particle in this background:

$$S = \ln \frac{\sqrt{\eta}}{\sqrt{1+\eta} + \sqrt{1-s^2\eta}} \bigg|_{\eta'}^{\eta''} , \qquad (3.2)$$

with $\eta'$ and $\eta''$ being initial and final radial coordinates of the geodesic. Angular coordinate $\phi$ is cyclic and, as a consequence, we have conserved angular momenta. Here we have introduced the following notation

$$s = \frac{|p_\phi|}{\alpha} . \qquad (3.3)$$

The main hypothesis is the equivalence between the heavy-light block and the classical action of 5-line configuration

$$f_{hl} \sim S_{cl} , \qquad (3.4)$$

where $S_{cl}$ is the sum of the geodesic lengths with the particle masses being identified with the block scaled dimension parameters.[1]

### 3.1 Five-line configuration

The goal of this section is to find the 5-particle geodesic length for the configuration depicted on the Fig. **2**
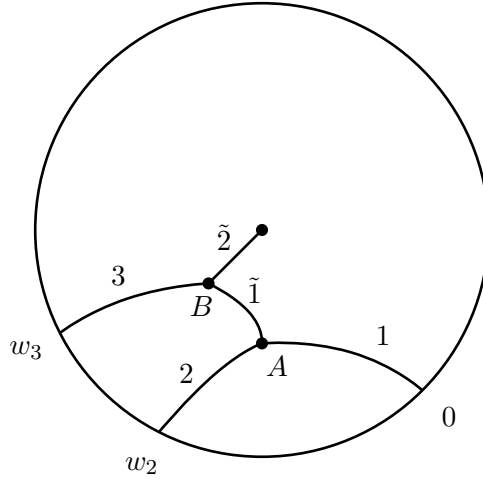


**Figure 2**. Five-lines geodesic configuration at fixed time disk related to the 5-pt conformal block.

We denote by $S_i$ and $S_{\tilde{i}}$ the contribution of each particle to the total action

$$S = \epsilon_1 S_1 + \epsilon_2 S_2 + \epsilon_3 S_3 + \tilde{\epsilon}_1 S_{\tilde{1}} + \tilde{\epsilon}_2 S_{\tilde{2}} . \qquad (3.5)$$

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[1]The equality is achieved with an appropriate identification between $q_{1,2}$ and the geodesic configuration boundary points.

We minimize (3.5) and obtain as consequence equilibrium conditions in the vertexes:

$$\left(\tilde{\epsilon}_1 \tilde{p}_\mu^1 + \epsilon_1 p_\mu^1 + \epsilon_2 p_\mu^2\right)\Big|_A = 0 \,, \quad \left(\tilde{\epsilon}_1 \tilde{p}_\mu^1 + \tilde{\epsilon}_2 \tilde{p}_\mu^2 + \epsilon_3 p_\mu^3\right)\Big|_B = 0 \,, \tag{3.6}$$

arising as a result of the variation with respect to the vertexes positions. The angular momenta conservation laws imply

$$\tilde{s}_2 = 0 \,, \qquad \epsilon_3 s_3 - \tilde{\epsilon}_1 \tilde{s}_1 = 0 \,, \qquad \epsilon_1 s_1 - \epsilon_2 s_2 - \tilde{\epsilon}_1 \tilde{s}_1 = 0 \,, \tag{3.7}$$

where we suppose that proper time is growing with moving from the vertexes and it allows us to fix signs. We can resolve the radial equilibrium conditions with respect to the radial positions of the vertexes:

$$\eta_A = \frac{1 - \sigma^2}{s_1^2 + s_2^2 - 2s_1 s_2 \sigma} \,, \qquad \sigma = \frac{\epsilon_1^2 + \epsilon_2^2 - \tilde{\epsilon}_1^2}{2\epsilon_1 \epsilon_2} \,, \tag{3.8}$$

$$\eta_B = -\frac{(\epsilon_3 - \tilde{\epsilon}_1 - \tilde{\epsilon}_2)(\epsilon_3 + \tilde{\epsilon}_1 - \tilde{\epsilon}_1)(\epsilon_3 - \tilde{\epsilon}_1 + \tilde{\epsilon}_2)(\epsilon_3 + \tilde{\epsilon}_1 + \tilde{\epsilon}_2)}{4\,\epsilon_3^2\,\tilde{\epsilon}_2^2\,s_3^2} \,. \tag{3.9}$$

The equations (3.7),(3.8),(3.9) can be resolved for $s_1$ and $s_3$. We want to relate these two momenta with angular positions of two particles $w_2$ and $w_3$ (the first coordinate is fixed as reference point due to the axial symmetry). We can integrate angular coordinate along the path and express the final angular position through the integration constants and the coordinates of vertexes:

$$e^{i\alpha w_2} = \frac{\left(\sqrt{1 - s_1^2\,\eta_A} - is_1\,\sqrt{1 + \eta_A}\right)\left(\sqrt{1 - s_2^2\,\eta_A} - is_2\,\sqrt{1 + \eta_A}\right)}{(1 - is_1)(1 - is_2)} \,, \tag{3.10}$$

$$e^{i\alpha w_3} = \frac{\left(\sqrt{1 - s_3^2\eta_B} - is_3\sqrt{1 + \eta_B}\right)\left(\sqrt{1 - \tilde{s}_1^2\eta_B} - i\tilde{s}_1\sqrt{1 + \eta_B}\right)\left(\sqrt{1 - s_1^2\eta_A} - is_1\sqrt{1 + \eta_A}\right)}{(1 - is_3)\left(\sqrt{1 - \tilde{s}_1^2\eta_A} - i\tilde{s}_1\sqrt{1 + \eta_A}\right)(1 - is_1)} \,. \tag{3.11}$$

The system of equations (3.7)–(3.11) completely defines the momenta and the radial coordinates of the vertexes as functions of the rescaled dimensions and two boundary coordinates. Partial lengths are following:

$$S_1 = -\ln \frac{\sqrt{\eta_1}}{\sqrt{1 + \eta_1} + \sqrt{1 - s_1^2\eta_1}} - \ln 2\Lambda \,, \tag{3.12}$$

$$S_2 = -\ln \frac{\sqrt{\eta_1}}{\sqrt{1 + \eta_1} + \sqrt{1 - s_2^2\eta_1}} - \ln 2\Lambda \,, \tag{3.13}$$

$$S_3 = -\ln \frac{\sqrt{\eta_2}}{\sqrt{1 + \eta_2} + \sqrt{1 - s_3^2\eta_2}} - \ln 2\Lambda \,, \tag{3.14}$$

$$S_{\tilde{1}} = \ln \frac{\sqrt{\eta_1}}{\sqrt{1 + \eta_1} + \sqrt{1 - (\tilde{s}_1)^2\eta_1}} - \ln \frac{\sqrt{\eta_2}}{\sqrt{1 + \eta_2} + \sqrt{1 - (\tilde{s}_1)^2\eta_2}} \,, \tag{3.15}$$

$$S_{\tilde{2}} = \ln \frac{\sqrt{\eta_2}}{1 + \sqrt{1 + \eta_2}} \, , \tag{3.16}$$

where we have introduced cutoff $\Lambda$.

## 3.2 Perturbative solution

In order to make final formulas less bulky, we set $\epsilon_1 = \epsilon_2$. Similar consideration can be easily performed for unequal external dimensions. We start from one in some sense trivial solution of angular equations. Setting

$$\begin{cases} \epsilon_3 = 0 \, , \\ \tilde{\epsilon}_1 = \tilde{\epsilon}_2 \, , \end{cases} \tag{3.17}$$

one obtains

$$\tilde{s}_1 = \tilde{s}_2 = 0 \, , \tag{3.18}$$

so that the four-point heavy-light block corresponds to the following action [10]:

$$S(\epsilon_3 = 0, \tilde{\epsilon}_1 = \tilde{\epsilon}_2) = -2\epsilon_1 \ln \sin \theta_2 + \tilde{\epsilon}_1 \ln \tan \frac{\theta_2}{2} \, , \tag{3.19}$$

where $\theta_{2,3} = \frac{\alpha w_{2,3}}{2}$. Due to existence of this solution one can wish to perturb with aim to obtain the 5-pt case with one external extra-light field. In order to obtain the 5-pt block, we have to take into account an additional external field. Following [25], the perturbed solution can be found with the assumptions:

$$\begin{cases} \nu = \frac{\epsilon_3}{\tilde{\epsilon}_1} \ll 1 \, , \\ \tilde{\epsilon}_1 = \tilde{\epsilon}_2 \, . \end{cases} \tag{3.20}$$

The calculation in this setting leads to the action, which corresponds to the 5-point heavy-light block with two equal intermediate dimensions:

$$S(\epsilon_3 \neq 0, \tilde{\epsilon}_1 = \tilde{\epsilon}_2) = -2\epsilon_1 \ln \sin \theta_2 + \tilde{\epsilon}_1 \ln \tan \frac{\theta_2}{2} - \epsilon_3 \ln \sin(2\theta_3 - \theta_2) + \mathcal{O}(\epsilon_3^2) \, . \tag{3.21}$$

We consider now the case, where $\tilde{\epsilon}_1 \neq \tilde{\epsilon}_2$. Naively, such an assumption leads to the divergence of the radial coordinate of B-vertex (see eq.(3.9)). However, this problem can be solved in the following way. The character of the divergence implies that the difference between the two internal dimensions must be proportional to $\nu$. We introduce a new parameter $\mu$ in such a way that the previous case (3.20) is reproduced when this parameter is equal to zero, and vice versa, if $\nu = 0$ we must obtain (3.17) because the internal dimensions are related, in fact, to one field. As a result, we can express the conditions in the following form:

$$\begin{cases} \nu = \frac{\epsilon_3}{\tilde{\epsilon}_1} \ll 1 \, , \\ \tilde{\epsilon}_2 = \tilde{\epsilon}_1(1 + \nu\mu) \, . \end{cases} \tag{3.22}$$

With these assumptions we obtain the following angular momenta

$$s_1 = -\cot\theta_2 - \nu\frac{\tilde{\epsilon}_1}{2\epsilon_1}\cot(2\theta_3 - \theta_2) + \frac{\tilde{\epsilon}_1}{2\epsilon_1}\csc\theta_2 + \mu\nu\frac{\tilde{\epsilon}_1}{2\epsilon_1}\csc(2\theta_3 - \theta_2)\,, \tag{3.23}$$

$$s_2 = -\cot\theta_2 + \nu\frac{\tilde{\epsilon}_1}{2\epsilon_1}\cot(2\theta_3 - \theta_2) + \frac{\tilde{\epsilon}_1}{2\epsilon_1}\csc\theta_2 - \mu\nu\frac{\tilde{\epsilon}_1}{2\epsilon_1}\csc(2\theta_3 - \theta_2)\,, \tag{3.24}$$

$$s_3 = -\cot(2\theta_3 - \theta_2) + \frac{\mu}{\sin(2\theta_3 - \theta_2)} - \frac{\nu}{4}\frac{(3 + \cos(4\theta_3 - 2\theta_2) - 2\cos(2\theta_3 - 2\theta_2) - 2\cos 2\theta_3)}{\sin(2\theta_3 - \theta_2)^3}$$
$$- \frac{\mu\nu}{4}\frac{(6\cos\theta_2 + \cos(4\theta_3 - \theta_2) + \cos(4\theta_3 - 3\theta_2) - 8\cos(2\theta_3 - \theta_2))}{\sin(2\theta_3 - \theta_2)^3}\,, \tag{3.25}$$

$$\tilde{s}_1 = -\cot(2\theta_3 - \theta_2) + \mu\nu\csc(2\theta_3 - \theta_2)\,, \tag{3.26}$$

and the following positions of vertexes:

$$\eta_A = \frac{4\epsilon_1 - \tilde{\epsilon}_1^2}{(\tilde{\epsilon}_1\csc\theta_2 - 2\epsilon_1\cot\theta_2)^2}\,, \tag{3.27}$$

$$\eta_B = \tan(2\theta_3 - \theta_2)^2 + 2\mu\frac{\tan(2\theta_3 - \theta_2)^2}{\cos(2\theta_3 - \theta_2)} - \nu\frac{1 - 2\cos\theta_2\sec(2\theta_3 - \theta_2) + \sec(2\theta_3 - \theta_2)^3}{\cos(2\theta_3 - \theta_2)}$$
$$+ \mu\nu\left(1 - 3\sec(2\theta_3 - \theta_2)^4 + 2\cos\theta_2\sec(2\theta_3 - \theta_2)(2\sec(2\theta_3 - \theta_2)^2 - 1)\right)\,. \tag{3.28}$$

Now we substitute the obtained momenta and the vertexes positions in the expression for geodesic lengths:

$$S_1 = -\log(\sin\theta_2) - \nu\cot(2\theta_3 - \theta_2)\sin\theta_2 + \mu\nu\frac{\sin\theta_2}{\sin(2\theta_3 - \theta_2)}\,, \tag{3.29}$$

$$S_2 = -\log(\sin\theta_2) + \nu\cot(2\theta_3 - \theta_2)\sin\theta_2 - \mu\nu\frac{\sin\theta_2}{\sin(2\theta_3 - \theta_2)}\,, \tag{3.30}$$

$$S_3 = -\log(-\sin(2\theta_3 - \theta_2)) + O(\nu)\,, \tag{3.31}$$

$$S_{\tilde{1}} = \log(\tan\frac{\theta_2}{2}) - \log\frac{(2\epsilon_1\cos\theta_2 - \tilde{\epsilon}_1)\sin(2\theta_3 - \theta_2)}{(\cos\theta_2 + 1)(\mu - \cos(2\theta_3 - \theta_2))} + \nu\frac{1}{4\sin(2\theta_3 - \theta_2)^2}\times$$
$$\times\left(3 + \cos(4\theta_3 - 2\theta_2) - 4\cos\theta_2\cos(2\theta_3 - \theta_2) + 4\mu(\cos\theta_2 - \cos(2\theta_3 - \theta_2))\right)\,, \tag{3.32}$$

$$S_{\tilde{2}} = \log(\tan(\theta_3 - \frac{\theta_2}{2})) - \nu\frac{1}{4\sin(2\theta_3 - \theta_2)^2}\times$$
$$\times\left(3 + \cos(4\theta_3 - 2\theta_2) - 4\cos\theta_2\cos(2\theta_3 - \theta_2) + 4\mu(\cos\theta_2 - \cos(2\theta_3 - \theta_2))\right)\,, \tag{3.33}$$

We now have assembled all the ingredients that we need in order to write down the total action. Using

$$\theta_2 = \frac{i\alpha}{2}\log(1 - q_1 q_2)\,, \quad \theta_3 = \frac{i\alpha}{2}\log(1 - q_2) \tag{3.34}$$

and expanding the total geodesic length up to the first order in $\epsilon_3$, we get

$$S = -2\epsilon_1 \log \sinh\left(\frac{\alpha}{2}\log(1 - q_1 q_2)\right) + \tilde{\epsilon}_1 \log \tanh\left(\frac{\alpha}{4}\log(1 - q_1 q_2)\right)$$
$$-\epsilon_3 \log \sinh\left(\alpha \frac{2\log(1 - q_2) - \log(1 - q_1 q_2)}{2}\right) + \epsilon_3 \mu \log \tanh\left(\alpha \frac{2\log(1 - q_2) - \log(1 - q_1 q_2)}{4}\right).$$
$$(3.35)$$

The corresponding result for the heavy-light block in the same approximation is

$$f_{hl} = -S - \epsilon_3 \log\left(-\frac{\alpha q_2}{1 - q_2}\right) + \epsilon_3 \mu \log\left(-\frac{\alpha q_2}{2}\right) + \tilde{\epsilon}_1 \log\left(-\frac{\alpha q_1 q_2}{4}\right) - 2\epsilon_1 \log\left(-\frac{\alpha q_1 q_2}{2}\right)$$
$$+ \epsilon_1 \log(1 - q_1 q_2). \quad (3.36)$$

The terms in the right hand side are connected with the standard conformal block prefactor and the prefactor, related to the conformal transformation of the fields (3.34).

## 4  Conclusion

We have considered the AdS/CFT correspondence between 5-point heavy-light classical conformal blocks in the boundary CFT with a pure Virasoro algebra and classical worldline actions in the bulk. We calculated the action for the 5-line geodesic configuration, and after the conformal transformation (3.34), we obtained the equivalence between action (3.35) and the 5-point heavy-light block (2.16)–(2.19) computed using $c$-recursion formulas (2.4). We thus demonstrated that there exists a dual interpretation of the 5-point heavy-light block with unequal internal dimensions.

It is interesting to study possible generalizations of the holographic interpretation of conformal blocks on a sphere and on higher-genus surfaces to the case of the boundary CFT with an extended symmetry algebra and, in particular, to the case of W- and super-symmetry. Another interesting question is to investigate the $1/c$ corrections for the conformal blocks in the context of the correspondence.

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