Delta-matroids and Vassiliev invariants

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Abstract

Vassiliev (finite type) invariants of knots can be described in terms of weight systems. These are functions on chord diagrams satisfying so-called 4-term relations. The goal of the present paper is to show that one can define both the first and the second Vassiliev moves for binary delta-matroids and introduce a 4-term relation for them in such a way that the mapping taking a chord diagram to its delta-matroid respects the corresponding 4-term relations.

Understanding how the 4-term relation can be written out for arbitrary binary delta-matroids motivates introduction of the graded Hopf algebra of binary delta-matroids modulo the 4-term relations so that the mapping taking a chord diagram to its delta-matroid extends to a morphism of Hopf algebras. One can hope that studying this Hopf algebra will allow one to clarify the structure of the Hopf algebra of weight systems, in particular, to find reasonable new estimates for the dimensions of the spaces of weight systems of given degree.

Vassiliev (finite type) invariants of knots can be described in terms of weight systems. These are functions on chord diagrams satisfying so-called 4-term relations. In the study of the $\mathfrak{sl}_2$ weight system in [8], it was shown that its value on a chord diagram depends on the intersection graph of the diagram rather than on the diagram itself. Moreover, it was shown that the value of this weight system on an intersection graph depends on the cycle matroid of the graph rather than on the graph itself. This result arose the question whether there is a natural way to introduce a 4-term relation on the space spanned by matroids, similar to the one for graphs [13]. It happened however that the answer is negative: there are graphs having isomorphic cycle matroids such that applying the "second Vassiliev move" to a

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pair of corresponding vertices a, b of the graphs we obtain two graphs with nonisomorphic matroids.

The goal of the present paper is to show that the situation is different for binary delta-matroids: one can define both the first and the second Vassiliev moves for binary delta-matroids and introduce a 4-term relation for them in such a way that the mapping taking a chord diagram to its delta-matroid respects the corresponding 4-term relations. Moreover, this mapping admits a natural extension to chord diagrams on several circles, which correspond to singular links. Delta-matroids were introduced by A. Bouchét [4]. Bouchét used them, in particular, to study embedded graphs, whence their relationship with (multiloop) chord diagrams is by no means unexpected. Some evidence for the existence of such a relationship can be found, for example, in [2], where an analogue of the Tutte polynomial for embedded graphs has been introduced. The authors show that this polynomial satisfies the Vassilev 4-term relation when restricted to chord diagrams, and it is shown in [9] that the polynomial is, in fact, delta-matroidal.

Understanding how the 4-term relation can be written out for arbitrary binary delta-matroids motivates introduction of the graded Hopf algebra of binary delta-matroids modulo the 4-term relations so that the mapping taking a chord diagram to its delta-matroid extends to a morphism of Hopf algebras. One can hope that studying this Hopf algebra will allow one to clarify the structure of the Hopf algebra of weight systems, in particular, to find reasonable new estimates for the dimensions of the spaces of weight systems of given degree. Note that the classical approach to study links through link diagrams also leads to a connection with delta=matroids, see, for example, [18]. Also it would be interesting to find a relationship between the Hopf algebras arising in this paper with a very close to them in spirit bialgebras of Lagrangian subspaces in [11].

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1 Algebra of set systems

A *set system* $(E;\Phi)$ is a finite set E together with a subset Φ of the set 2^E of subsets in E. The set E is called the *ground set* of the set system, and elements of Φ are its *feasible sets*. Two set systems $(E_1;\Phi_1)$, $(E_2;\Phi_2)$ are said to be *isomorphic* if there is a one-to-one map $E_1 \to E_2$ identifying the subset $\Phi_1 \subset 2^{E_1}$ with the subset $\Phi_2 \subset 2^{E_2}$. Below, we make no difference between isomorphic set systems.

A set system $(E;\Phi)$ is *proper* if Φ is nonempty. Below, we consider only proper set systems, without indicating this explicitly.

1.1 The graded vector space of set systems

Let $\mathcal{S}_n$ denote the vector space (over the field of complex numbers $\mathbb{C}$, for definiteness) freely spanned by set systems whose ground set consists of n elements, $\mathcal{S}_0$ being the field $\mathbb{C}$ itself. The direct sum

$$\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \dots$$

is an infinite dimensional graded vector space.

Example 1.1 The vector space $\mathcal{S}_0$ is 1-dimensional. It is spanned by the only set system on zero elements, namely, the set system $\{\emptyset;\{\emptyset\}\}$.

The vector space $\mathcal{S}_1$ is 3-dimensional. It is spanned by the three set systems

$$s_{11} = \{\{1\};\{\emptyset\}\}, \quad s_{12} = \{\{1\};\{\emptyset,\{1\}\}\}, \quad s_{13} = \{\{1\};\{\{1\}\}\}.$$

Here and below, in our notation s_{ij} for set systems, the first index i denotes the number of elements in the ground set, while the second one is chosen ambiguously.

Remark 1.2 Note that the set systems $\{\emptyset;\{\emptyset\}\}$ and s_{11} *are* proper. Indeed, in both cases the corresponding set of subsets is not empty: it contains one element, namely, the empty set.

1.2 Multiplication of set systems

The *direct sum* of two set systems $D_1 = (E_1;\Phi_1)$, $D_2 = (E_2;\Phi_2)$ with disjoint ground sets E_1, E_2 is defined to be

$$D_1D_2 = (E_1 \sqcup E_2; \{\phi_1 \sqcup \phi_2 | \phi_1 \in \Phi_1, \phi_2 \in \Phi_2\}). \tag{1}$$

Since we consider set systems up to isomorphism, we will always assume that, when considering direct sums, the ground sets E_1 and E_2 of the summands are disjoint. Below, we will also refer to the direct sum as to the *product* of set systems. This operation extends by linearity to a bilinear multiplication

$$m : \mathcal{S} \otimes \mathcal{S} \to \mathcal{S}, \qquad m(D_1 \otimes D_2) = D_1 D_2,$$

which is graded (meaning that $m : \mathcal{S}_k \otimes \mathcal{S}_\ell \to \mathcal{S}_{k+\ell}$ for all $k, \ell \geq 0$), and commutative. The unit of this multiplication is the set system $(\emptyset; \{\emptyset\})$, which is the generator of S_0.

Example 1.3 The vector space $\mathcal{S}_2$ is 11-dimensional. It is spanned by the six set systems that are products of set systems on one element sets, namely,

$$\begin{aligned}
s_{11}^2 &= \{\{1,2\}; \{\{\emptyset\}\}\}, \\
s_{12}^2 &= \{\{1,2\}; \{\emptyset, \{1\}, \{2\}, \{1,2\}\}\}, \\
s_{13}^2 &= \{\{1,2\}; \{\{1,2\}\}\}, \\
s_{11}s_{12} &= \{\{1,2\}; \{\emptyset, \{1\}\}\} = \{\{1,2\}; \{\emptyset, \{2\}\}\}, \\
s_{11}s_{13} &= \{\{1,2\}; \{\{1\}\}\} = \{\{1,2\}; \{\{2\}\}\}, \\
s_{12}s_{13} &= \{\{1,2\}; \{\{1\}, \{1,2\}\}\} = \{\{1,2\}; \{\{2\}, \{1,2\}\}\},
\end{aligned}$$

and the five other set systems

$$\begin{aligned}
s_{21} &= \{\{1,2\}; \{\emptyset, \{1,2\}\}\}, \\
s_{22} &= \{\{1,2\}; \{\emptyset, \{1\}, \{1,2\}\}\} = \{\{1,2\}, \{\emptyset, \{2\}, \{1,2\}\}\}, \\
s_{23} &= \{\{1,2\}; \{\emptyset, \{1\}, \{2\}\}\}, \\
s_{24} &= \{\{1,2\}; \{\{1\}, \{2\}\}\}, \\
s_{25} &= \{\{1,2\}; \{\{1\}, \{2\}, \{1,2\}\}\}.
\end{aligned}$$

2 Generalities on delta-matroids

In this section we briefly reproduce the general facts about delta-matroids that we will require further. We follow the approach and terminology of [9], but use slightly different notation.

2.1 Delta-matroids

Let Δ denote the symmetric difference of sets, $A \Delta B = (A \setminus B) \sqcup (B \setminus A)$. A *delta-matroid* is a set system $D = (E; \Phi)$ satisfying the following Symmetric Exchange Axiom (SEA):

For any $\phi_1, \phi_2 \in \Phi$ and for any $e \in (\phi_1 \Delta \phi_2)$ there is an element $e' \in (\phi_2 \Delta \phi_1)$ such that $\phi_1 \Delta \{e, e'\} \in \Phi$.

It is easy to check that all the set systems on 1 or 2 elements, which are enlisted in Sec. 1, are delta-matroids. However, there are set systems that are not delta-matroids already among set systems on three elements. For example, if, for the set system $(\{1,2,3\}; \{\emptyset, \{1,2,3\}\})$ we take $\phi_1 = \emptyset$, $\phi_2 = \{1,2,3\}$, then the SEA will not be satisfied.

2.2 Delta-matroids of embedded graphs

An *embedded graph* is, essentially, a graph drawn on a compact surface in such a way that its complement is a disjoint union of disks. We will always assume that the graph is connected. Edges in an embedded graph are also called *ribbons*, or *handles*, and we make no distinction between embedded and ribbon graphs. Generalities on embedded graphs can be found, for example, in [15].

If otherwise is not stated explicitly, we allow both orientable and nonorientable surfaces. A loop in an embedded graph, that is, an edge connecting a vertex with itself, can be orientable or disorienting (half-twisted). If there is a disorienting loop in an embedded graph, then the graph itself is nonorientable. However, a nonorientable ribbon graph does not necessarily contain a disorienting loop: it suffices that there exists a disorienting cycle, not necessarily of length 1, in it.

To each embedded graph Γ, its delta-matroid $D(\Gamma) = (E(\Gamma); \Phi(\Gamma))$ is associated. The ground set of the delta-matroid is the set $E(\Gamma)$ of the edges of Γ. A subset $\phi \subset E(\Gamma)$ is feasible, $\phi \in \Phi(\Gamma)$, if the boundary of the embedded spanning subgraph of Γ formed by the set ϕ is connected, that is, consists of a single connected component. This means, in particular, that the spanning subgraph of Γ formed by the set ϕ is connected (otherwise, each connected component of the spanning subgraph would add at least one connected component to the boundary). Since, for a plane graph, this requirement coincides with the requirement that ϕ is a spanning tree, feasible sets for graphs embedded into a surface of arbitrary genus are called *quasi-trees*. For graphs embedded in surfaces of positive genus, not all of quasi-trees necessarily are trees, although each subset of edges forming a spanning tree is feasible.

Delta-matroids of orientable embedded graphs are *even*, meaning that all the feasible sets in them have cardinality of the same parity.

Example 2.1 All the set systems in Sec. 1 are delta-matroids of embedded graphs. Thus, s_{11} is the delta-matroid of the embedded graph with one vertex and an orientable loop, s_{12} is the delta-matroid of the embedded graph with one vertex and a half-twisted loop, while s_{13} is the delta-matroid of the embedded graph with two vertices and an edge connecting them. The delta-matroids s_{11}, s_{13} correspond to orientable embedded graphs, and are even, while s_{12} is not even.

The following statement is straightforward.

Proposition 2.2 ([9]) *If Γ_1, Γ_2 are two embedded graphs with the delta-matroids $D(\Gamma_1), D(\Gamma_2)$, respectively, then the delta-matroid of the embedded graph $\Gamma_1\#\Gamma_2$ obtained by gluing Γ_1, Γ_2 along a vertex is the product of the delta-matroids of the summands, $D(\Gamma_1\#\Gamma_2) = D(\Gamma_1)D(\Gamma_2)$.*

Here the gluing $\Gamma_1\#\Gamma_2$ of embedded graphs Γ_1, Γ_2 along a vertex is defined in the following way: we choose an arbitrary vertex in Γ_1 and an arbitrary vertex in Γ_2, and glue the two vertices together so that the half-edges of Γ_1 leave the joint vertex in the same cyclic order, followed by the those of Γ_2. The above proposition means, in particular, that the delta-matroid of the resulting graph depends neither on the choice of the two vertices to be glued, nor on the choice of the breaking point inside each vertex. Note that the number of vertices in the result of gluing of two graphs is one less than the total number of vertices in the graphs.

Example 2.3 The delta-matroid s_{13}^2 is represented by the only embedded graph with three vertices and two edges.

3 Δ-matroids of abstract graphs and binary delta-matroids

Certain abstract graphs can be represented as intersection graphs of *chord diagrams*, which are embedded graphs with a single vertex. In spite of the fact that one graph can be the intersection graph of different chord diagrams, all these diagrams have one and the same delta-matroid, which is, therefore, associated to the graph itself. Bouchét extended this construction to arbitrary abstract graphs.

3.1 Binary delta-matroids

Let G be an (abstract) undirected graph. We say that G is nondegenerate if its adjacency matrix $A(G)$, considered as a matrix over the field of two elements, is nondegenerate. Define the set system $(V(G);\Phi(G))$, $\Phi(G)\subset 2^{V(G)}$, by

$$\begin{array}{rcl} V(G) & \text{is} & \text{the set of vertices of } G, \\ \Phi(G) & = & \{U\subset V(G) | G_U \text{ is nondegenerate}\}, \end{array}$$

where G_U is the subgraph in G induced by the subset U of vertices.

Theorem 3.1 ([4]) *The set system $(V(G);\Phi(G))$ is a delta-matroid.*

We call this delta-matroid the *nondegeneracy delta-matroid* of the graph G.

For an orientable embedded graph Γ with a single vertex, denote by $\gamma(\Gamma)$ its intersection graph, that is, the graph whose vertices correspond one-to one to the ribbons of Γ, and two vertices are connected by an edge iff the ends of the corresponding ribbons alternate along the vertex.

Theorem 3.2 ([4]) *Let Γ be an orientable ribbon graph with a single vertex. Then its Δ-matroid $(E(\Gamma);\Phi(\Gamma))$ coincides with the nondegeneracy delta-matroid of the intersection graph $\gamma(\Gamma)$ of G.*

According to the theorem from [17], the number of connected components of the boundary of a ribbon graph Γ with a single vertex is equal to $\operatorname{corank}(A(\gamma(\Gamma)))+1$, where the adjacency matrix is considered over the field with two elements. In particular, the boundary has a single component iff the matrix $A(\gamma(\Gamma))$ is nondegenerate.

Theorem 3.1 is naturally generalized to framed graphs and nonorientable embedded graphs. Recall the definition of a framed graph from [14].

Definition 3.1 A *framed graph* is an (abstract) graph G together with a *framing*, that is, a mapping $f:V(G)\to\{0,1\}$. In the adjacency matrix $A(G)$ of a framed graph, the diagonal element corresponding to a vertex $v\in V(G)$ is $f(v)$, while nondiagonal elements are defined as usual.

For a framed graph G, the set system $(V(G);\Phi(G))$, is defined in the same way as for an unframed one.

Now let Γ be a ribbon graph with a single vertex, not necessarily orientable. The intersection graph $\gamma(\Gamma)$ of the ribbon graph Γ is the framed graph such that each nonoriented loop is taken to a vertex with framing 1. The theorem from [17] has the following framed analogue.

Theorem 3.3 *For an embedded graph Γ with a single vertex, not necessarily orientable, let $A(\gamma(\Gamma))$ be the adjacency matrix of its framed intersection* *graph. Then the number of connected components of the boundary of* Γ *is equal to* $\operatorname{corank}(A(\gamma(\Gamma)))+1$.

As a consequence, we obtain a generalization of Theorem 3.2 for not necessarily orientable ribbon graph with a single vertex.

Corollary 3.2 *Let* Γ *be a ribbon graph with a single vertex. Then its delta-matroid* $(E(\Gamma);\Phi(\Gamma))$ *coincides with the nondegeneracy* Δ*-matroid of the intersection graph* $\gamma(\Gamma)$ *of* Γ*.*

Nondegeneracy delta-matroids of abstract framed graphs are examples of binary delta-matroids. In order to define the notion of binary delta-matroid, we will need the twist operation. For a set system $D=(E;\Phi)$ and a subset $E'\subset E$ define the *twist* $D*E'$ of D with respect to E' by

$$D*E'=(E;\Phi\Delta E')=(E;\{\phi\Delta E'|\phi\in\Phi\}).$$

Theorem 3.4 ([6]) *Any twist of a nondegeneracy delta-matroid of a framed graph is a delta-matroid.*

Bouchét calls the delta-matroids obtained as twists of nondegenracy delta-matroids of framed graphs *binary delta-matroids*. In particular, he shows that

Theorem 3.5 ([6]) *Delta-matroids of embedded graphs are binary.*

Below, we will consider the algebra of binary delta-matroids. It is well-defined due to the following statement.

Theorem 3.6 ([9]) *The product of two binary delta-matroids is a binary delta-matroid.*

This theorem means that we can consider the graded commutative algebra of binary delta-matroids, which is a graded subalgebra in the algebra $\mathcal{S}$ of set systems. We will denote this algebra by $\mathcal{B}$:

$$\mathcal{B}=\mathcal{B}_0\oplus\mathcal{B}_1\oplus\mathcal{B}_2\oplus\dots.$$

The graded subalgebra $\mathcal{B}^e$ in $\mathcal{B}$ is spanned by even binary delta-matroids. Recall that a delta-matroid $(E;\Phi)$ is *even* if the parity of the cardinality is the same for all sets in Φ.

3.2 Comultiplication of binary delta-matroids

In addition to multiplication, we are going to introduce a comultiplication μ on the space $\mathcal{B}$ of binary delta-matroids, $\mu : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$. By definition, the *coproduct* $\mu(D)$ of a delta-matroid $D = (E; \Phi)$ is

$$\mu(D) = \sum_{E' \subset E} D_{E'} \otimes D_{E \setminus E'}. \qquad (2)$$

Here, for a subset $E' \subset E$ of the ground set E of a delta-matroid D, we denote by $D_{E'}$ the restriction of D to E'.

Let us recall the definition of restriction from [9]. It requires some other notions, which we collect together in a single paragraph.

Definition 3.3 Let $D = (E; \Phi)$ be a delta-matroid. An element $e \in E$ is a *coloop* if it enters all feasible sets in D. If e is not a coloop, then the delta-matroid D *delete* e, $D \setminus \{e\}$ is

$$D \setminus \{e\} = (E \setminus \{e\}; \{\phi \in \Phi | \phi \subset E \setminus \{e\}\}).$$

An element $e \in E$ is a *loop* if it does not enter any feasible set in D. If e is not a loop, then the delta-matroid D *contract* e, $D/\{e\}$ is

$$D/\{e\} = (E \setminus \{e\}; \{\phi \setminus \{e\} | \phi \in \Phi \text{ and } \phi \ni e\}).$$

If e is a coloop, then, by definition, $D \setminus \{e\} = D/\{e\}$. If e is a loop, then, by definition, $D/\{e\} = D \setminus \{e\}$. A *minor* of D is a delta-matroid obtained from D by a sequence of deletions and contractions. The *restriction* $D_{E'}$ of D to a subset $E' \subset E$ is the result of deleting all elements in $(E \setminus E') \subset E$ in D.

All these notions are well-defined. This means, in particular, that the deletion and contraction of a delta-matroid are delta-matroids as well, and any sequence of deletions and contractions leads to the same delta-matroid independently of the order of the elements in the sequence (which are assumed to be pairwise distinct). In the notation below, we will often omit braces around one-element sets, writing $E \setminus e$ instead of $E \setminus \{e\}$, and so on.

Proposition 3.4 ([9]) *If $D(\Gamma) = (E(\Gamma); \Phi(\Gamma))$ is the delta-matroid of an embedded graph Γ and $E' \subset E(\Gamma)$ is a subset of its edges such that the corresponding spanning subgraph is connected, then $D_{E'}$ is the delta-matroid of the spanning subgraph $(V(\Gamma); E')$. Moreover, if $E' \subset E(\Gamma)$ is an arbitrary subset, and $\Gamma'_1, \dots, \Gamma'_k$ are the connected components of the corresponding spanning subgraph of Γ, then the delta-matroid $D(\Gamma)_{E'}$ coincides with the product $D(\Gamma'_1) \dots D(\Gamma'_k)$ of the delta-matroids $D(\Gamma'_1), \dots, D(\Gamma'_k)$.*

Theorem 3.7 ([9]) *For a binary delta-matroid $D = (E;\Phi)$, its restriction $D_{E'}$ to an arbitrary subset $E' \subset E$ is a binary delta-matroid.*

The following statement shows that the coproduct defined above is compatible with the product.

Proposition 3.5 *Let $D_1 = (E_1;\Phi_1)$, $D_2 = (E_2;\Phi_2)$ be two delta-matroids. Then*

$$\mu(D_1D_2) = \mu(D_1)\mu(D_2).$$

Proof. Consider a subset $E' \subset E_1 \sqcup E_2$. Such a subset is represented as $E' = E_1' \sqcup E_2'$ with $E_1' \subset E_1$, $E_2' \subset E_2$. Therefore,

$$\mu(D_1D_2) = \sum_{E_1'\subset E_1, E_2'\subset E_2} D_{1E_1'}D_{2E_2'} \otimes D_{1E_1\setminus E_1'}D_{2E_2\setminus E_2'},$$

since $(D_1D_2)_{E_1'\sqcup E_2'} = D_{1E_1'}D_{2E_2'}$. Therefore,

$$\mu(D_1D_2) = \sum_{E_1'\subset E_1} D_{1E_1'} \otimes D_{1E_1\setminus E_1'} \sum_{E_2'\subset E_2} D_{2E_2'} \otimes D_{2E_2\setminus E_2'}.$$

The converse statement also is clear, which proves the Proposition.

The coproduct μ extends by linearity to a comultiplication of the graded vector space spanned freely by the delta-matroids. Below, we will use it only for binary delta-matroids, and we consider the comultiplication

$$\mu : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}.$$

The counit for the comultiplication is the algebra homomorphism $\mathcal{B} \to \mathbb{C}$, which is isomorphism when restricted to $\mathcal{B}_0$, and zero when restricted to $\mathcal{B}_i$ for $i = 1, 2, \dots$.

The proof of the following theorem is a routine checking of axioms, which we omit.

Theorem 3.8 *The vector space $\mathcal{B}$ endowed with the comultiplication* (2) *and the multiplication* (1) *is a graded commutative cocommutative Hopf algebra. The subalgebra $\mathcal{B}^e \subset \mathcal{B}$ spanned by even binary delta-matroids forms a Hopf subalgebra in this Hopf algbera.*

According to the Milnor–Moore theorem, each commutative cocommutative graded Hopf algebra is nothing but the Hopf algebra of polynomials in its primitive elements. Recall that an element p of a Hopf algebra is *primitive* if

$$\mu(p) = 1 \otimes p + p \otimes 1,$$

and that primitive elements form a vector subspace in the algebra. For a graded Hopf algebra, its vector subspace of primitive elements also is graded.

Example 3.6 The elements s_{11}, s_{12}, s_{13} in $\mathcal{B}_1$ are primitive, and $\mathcal{B}_1$ coincides with its primitive subspace. The elements $s_{21}, s_{22}, s_{23}, s_{24}, s_{25}$ are not primitive. Nevertheless, the dimension of the primitive subspace in $\mathcal{B}_2$ is 5: any space $\mathcal{B}_n$ can be represented as the direct sum of its primitive subspace and subspace of decomposable elements, which is spanned by polynomials in elements of smaller degrees. In $\mathcal{B}_2$, the subspace spanned by decomposable elements is 6-dimensional and spanned by $s_{11}^2, s_{12}^2, s_{13}^2, s_{11}s_{12}, s_{11}s_{13}, s_{12}s_{13}$.

Similarly, $\mathcal{B}_1^e$ coincides with its subspace of primitive elements and is 2-dimensional, while $\mathcal{B}_2^e$ is the direct sum of the 3-dimensional subspace spanned by decomposable elements and the 2-dimensional primitive subspace.

Due to the proposition below, the Hopf algebra structure above can be restricted to binary delta-matroids such that the empty set is feasible.

Proposition 3.7 *Let $D = (E; \Phi)$ be a binary delta-matroid such that the empty set is feasible, $\emptyset \in \Phi$. Then for the restriction of D to any subset in E, the empty set also is feasible.*

Indeed, D cannot contain coloops: otherwise $\emptyset$ would not be feasible. And if $e \in E$ is not a coloop, then $\emptyset$ is a feasible set for $D \setminus e$ as well.

Therefore, both multiplication and comultiplication in $\mathcal{B}$ and $\mathcal{B}^e$ preserve the subspaces spanned by binary delta-matroids with feasible emptysets. We denote the corresponding Hopf algebras by $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \dots$ and $\mathcal{K}^e = \mathcal{K}_0^e \oplus \mathcal{K}_1^e \oplus \mathcal{K}_2^e \oplus \dots$, respectively. (The notation reflects the fact that these Hopf algebras are related to chord diagrams and embedded graphs with a single vertex, that is, to knots, rather than to links). The corresponding dimensions of the spaces of primitive elements are 2 for $\mathcal{K}_1$, 3 for $\mathcal{K}_2$, 1 for $\mathcal{K}_1^e$, and 1 for $\mathcal{K}_2^e$.

4 Four-term relations

Vassiliev's theory of finite order knot invariants [19] associates to a knot invariant of order at most n a *weight system* of order n, that is, a function on chord diagrams (= embedded graphs with a single vertex) with n chords satisfying 4-term relations. This construction has a straightforward generalization to chord diagrams of links, which are essentially embedded graphs with the number of vertices equal to the number of connected components of the link.

The definition of the 4-term relations requires the definition of two operations, namely, exchanging of handle ends (the first Vassiliev move) and handle sliding (the second Vassiliev move). The handle sliding for binary delta-matroids was defined in [16]. Below, we give the description of this operation, and define the operation of exchanging handle ends. As a result, we can introduce 4-term relations for binary delta-matroids and the corresponding quotient Hopf algebra.

It was shown in [16] that for the delta-matroids of embedded graphs, the operation of handle sliding, when applied to two ribbons with neighboring ends, coincides with the handle sliding for embedded graphs. We prove a similar statement for the operation of exchanging handle ends. Although handle sliding and exchanging handle ends do not preserve the class of delta-matroids of embedded graphs, they preserve a wider class of binary delta matroids. As a result, we are able to construct a Hopf algebra of binary delta-matroids modulo 4-term relations.

Any function on binary delta-matroids satisfying the 4-term relations defines a weight system, whence a link invariant. Therefore, studying these functions can help to construct knot invariants and clarify their nature.

Note that the connected sum of chord diagrams is well defined only if 4-term relations are imposed. This property allows one to define the Hopf algebra of chord diagrams modulo 4-term relations. It was asked in [14] whether imposing the 4-term relations allows one to define multiplication on framed chord diagrams as well. Recently, D. P. Ilyutko and V. O. Manturov [10] answered this question in negative. The results of the present section show, however, that on the level of (binary) delta-matroids we obtain Hopf algebra structures not only for framed chord diagrams, but for arbitrary embedded graphs as well. Multiplication in these Hopf algebras is well defined independently of whether the 4-term relations are imposed.

4.1 The second Vassiliev move: handle sliding

Let $D = (E; \Phi)$ be a set system, $a, b \in E$ be two different elements.

Definition 4.1 ([16]) The result of *sliding of the element a over the element b* is the set system $\widetilde{D}_{ab} = (E; \widetilde{\Phi}_{ab})$, where $\widetilde{\Phi}_{ab} = \Phi\Delta\{\phi \sqcup \{a\} | \phi \sqcup \{b\} \in \Phi \text{ and } \phi \subset E \setminus \{a, b\}\}$.

It is proved in [16] that if $D = (E(\Gamma); \Phi(\Gamma))$ is the delta-matroid of an embedded graph Γ and a, b are two ribbons in Γ with neighboring ends, then the delta-matroid of the ribbon graph $\widetilde{\Gamma}_{ab}$ obtained from Γ by sliding the handle a over the handle b coincides with the delta-matroid $\widetilde{D}_{ab}$. However, if the ends of the ribbons a, b in Γ are not neighboring, then the handle sliding of the above definition can lead to a set system that is not isomorphic to the delta-matroid of any embedded graph. Moreover, the following example from [16] shows that a handle sliding applied to a delta-matroid can produce a set system that is not a delta-matroid.

Example 4.2 For the delta-matroid

$$D = (\{1, 2, 3\}; \{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\})$$

the set system $\widetilde{D}_{12} = (\{1, 2, 3\}; \{\emptyset, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\})$ is a delta-matroid no longer.

Nevertheless, the following theorem is valid.

Theorem 4.1 ([16]) *If $D = (E; \Phi)$ is a binary delta-matroid and a, b are two distinct elements in E, then $\widetilde{D}_{ab}$ is a binary delta-matroid.*

In the next section we prove a similar theorem for the other Vassiliev move, the first one.

In [13], the second Vassiliev move was defined for abstract graphs. We are going to show that this definition is, in fact, consistent with the definition above. Let us recall the definition from [13] (together with its extension to framed graphs in [14]). For a framed abstract graph G and a pair of vertices $a, b \in V(G)$ in it, the graph $\widetilde{G}_{ab}$ is defined as a graph on the same set $V(G)$ of vertices such that the adjacency of any vertex c to a, $c \neq a, b$, toggles iff c is adjacent to b in G. In addition, the adjacency of a and b toggles if the framing of b is 1.

If G is the intersection graph of a chord diagram, and a, b are two chords with neighboring ends in the diagram, then this move indeed corresponds to sliding of the handle a along the handle b [14].

Theorem 4.2 *For an abstract framed graph G, we have*

$$D(\widetilde{G}_{ab}) = \widetilde{D(G)}_{ab}.$$

Proof. Indeed, the adjacency matrix $A(G)$ of an abstract framed graph G can be considered as the matrix of a symmetric binary form over the field of two elements $\mathbb{F}_2$ in the vector space $\mathbb{F}_2^{V(G)}$ spanned by the vertices of the graph. The second Vassiliev move $G \mapsto \widetilde{G}_{ab}$ does not modify the form, but changes the basis:

$$(a, b, c, \dots) \mapsto (a + b, b, c, \dots).$$

(Note that this property justifies the name of the move: on the homology level the second Kirby move in topology of 3-manifolds does exactly the same thing, but over $\mathbb{Z}$ rather than over $\mathbb{F}_2$).

Of course, this change of basis does not affect the (non)degeneracy property of any subset of vertices in G not containing a or containing both $\{a,b\}$. Now, if a subset $U \sqcup \{b\} \subset V(G)$ does not contain a and is nondegenerate, then the nondegeneracy of $U \sqcup \{a\}$ toggles between G and $\widetilde{G}_{ab}$.

4.2 The first Vassiliev move: exchanging handle ends

For an embedded graph Γ and two distinct ribbons $a, b \in E(\Gamma)$ such that one of the ends of a is a neighbor of one of the ends of b along some vertex, the first Vassiliev move consists in exchanging these neighboring ends. The following definition mimics what happens with the underlying delta-matroids under this operation.

Let $D = (E; \Phi)$ be a set system, $a, b \in E$ be two different elements.

Definition 4.3 The result of *exchanging of the ends of the ribbon a and the ribbon b* is the set system $D'_{ab} = (E; \Phi'_{ab})$, where $\Phi'_{ab} = \widetilde{(\Phi * b)}_{ab} * b$.

Note that, in contrast to the second Vassiliev move, the first Vassiliev move is symmetric with respect to the ribbons a, b whose neighboring ends we exchange, $D'_{ab} = D'_{ba}$.

Since the operation $*$ preserves the class of binary delta-matroids, Theorem 4.1 immediately implies

Proposition 4.4 *If $D = (E; \Phi)$ is a binary delta-matroid and a, b are two distinct elements in E, then D'_{ab} is a binary delta-matroid.*

Theorem 4.3 *If $D = (E(\Gamma); \Phi(\Gamma))$ is the delta-matroid of an embedded graph Γ and a, b are two ribbons in Γ with neighboring ends, then the delta-matroid of the ribbon graph Γ'_{ab} obtained from Γ by exchanging the ends of the handles a and b coincides with the delta-matroid D'_{ab}.*

Proof. The set system $D * b$ is the delta-matroid of the partial dual embedded graph Γ^b, see [7] or [9]. After taking the partial dual along b, sliding the neighboring end of the handle a along the new b and returning b to its original place, we obtain exactly the neighboring ends exchange move.

Vassiliev moves for binary delta-matroids possess properties similar to those for embedded graphs:

Proposition 4.5 *The following statements about the Vassiliev moves are valid:*

- *the first Vassiliev move is an involution, $(D'_{ab})'_{ab} = D$;*

- *the second Vassiliev move is an involution,* $\widetilde{(\widetilde{D}_{ab})}_{ab} = D$*;*

- *the first and the second Vassiliev moves commute,* $(\widetilde{D}_{ab})'_{ab} = \widetilde{(D'_{ab})}_{ab}$.

4.3 The four-term relation for binary delta-matroids

As usual, we say that an invariant f of embedded graphs *satisfies the four-term relation* if for any embedded graph Γ and any pair a, b of its distinct edges having neighboring ends we have

$$f(\Gamma) - f(\Gamma'_{ab}) = f(\widetilde{\Gamma}_{ab}) - f(\widetilde{\Gamma}'_{ab}). \tag{3}$$

Similarly, we say that an invariant f of binary delta-matroids *satisfies the four-term relation* if for any binary delta-matroid D and a pair of distinct elements a, b in its ground set we have

$$f(D) - f(D'_{ab}) = f(\widetilde{D}_{ab}) - f(\widetilde{D}'_{ab}). \tag{4}$$

Theorem in [16] and Theorem 4.3 above mean that

Theorem 4.4 *Any invariant of binary delta-matroids satisfying the* 4*-term relation* (4) *defines a weight system, whence a link invariant.*

4.4 Hopf algebras of binary delta-matroids modulo 4-term relations

The Hopf algebra $\mathcal{B}$ of binary delta-matroids, as well as its Hopf subalgebra $\mathcal{B}^e$ of even binary delta-matroids can be factorized modulo the 4-term relations. Denote by $\mathcal{FB}$ (respectively, $\mathcal{FB}^e$) the graded quotient space of the space of binary matroids (respectively, even binary matroids) modulo the 4-term relations:

$$\begin{aligned} \mathcal{FB}_i &= \mathcal{B}_i / \langle D - D'_{ab} - \widetilde{D}_{ab} + \widetilde{D}'_{ab} \rangle, \quad i = 0, 1, 2, \dots \\ \mathcal{FB}^e_i &= \mathcal{B}^e_i / \langle D - D'_{ab} - \widetilde{D}_{ab} + \widetilde{D}'_{ab} \rangle, \quad i = 0, 1, 2, \dots . \end{aligned}$$

Theorem 4.5 *The multiplication m and the comultiplication μ induce on the spaces $\mathcal{FB}$ and $\mathcal{FB}^e$ the structure of graded commutative cocommutative Hopf algebras.*

Example 4.6 The vector spaces $\mathcal{FB}^e_i$ for $i = 0, 1,$ and 2 coincide with the vector space $\mathcal{B}^e_i$, since the even 4-term relations are trivial for these values of i. In contrast, there is a (single) nontrivial 4-term relation for $i = 2$ in the noneven case:

$$s_{11}s_{12} - s_{22} = s_{23} - s_{12}^2.$$

Therefore, $\mathcal{FB}_2 = \mathcal{B}_2/\langle s_{11}s_{12} - s_{22} - s_{23} + s_{12}^2\rangle$, $\dim \mathcal{FB}_2 = 10$, and the primitive subspace in it is 4-dimensional. Indeed, none of the elements s_{22}, s_{23} is decomposable, but their sum is.

Since both the first and the second Vassiliev move preserve the class of binary delta-matroids with feasible empty set, the quotients $\mathcal{FK}$ and $\mathcal{FK}^e$ of the Hopf algebras $\mathcal{K}$ and $\mathcal{K}^e$, respectively, modulo the 4-term relations also are Hopf algebras. For $n = 1, 2$ the corresponding 4-term relations are trivial.

Let us collect the computed dimensions of the spaces of primitive elements into a table.

n	$\mathcal{B}_n$	$\mathcal{B}_n^e$	$\mathcal{FB}_n$	$\mathcal{FB}_n^e$	$\mathcal{K}_n$	$\mathcal{K}_n^e$	$\mathcal{FK}_n$	$\mathcal{FK}_n^e$
1	3	2	3	2	2	1	2	1
2	5	2	4	2	3	1	3	1

Table 1: Dimensions of the primitive subspaces

Example 4.7 The weight system w_C on framed chord diagrams corresponding to the Conway invariant of knots can be defined as the function taking on a chord diagram value 1 if the corresponding one-vertex ribbon graph has a connected boundary and 0 otherwise. This weight system admits a natural extension to binary delta-matroids: for a binary delta-matroid $D = (E; \Phi)$, define $w_C(D) = 1$ if $E \in \Phi$ and 0 otherwise. This function satisfies the 2-term relation: $w_C(D) = w_C(\widetilde{D}_{ab})$ for any pair of distinct elements $a, b \in E$, whence the 4-term relation. We extend it to $\mathcal{FB}$ by linearity.

The function w_C obviously is multiplicative, $w_C(D_1, D_2) = w_C(D_1)w_C(D_2)$ for any pair of binary delta-matroids D_1, D_2. Therefore, its logarithm is well defined. The value of this logarithm on chord diagrams is known to be related to the weight system $\mathfrak{sl}_2$, see details in [1, 12]. Hence, the value of $\log w_C$ on binary delta-matroids can be considered as a manifestation of the existence of a yet unknown construction of an $\mathfrak{sl}_2$-weight system on binary delta-matroids extending that for chord diagrams. This construction is unknown yet even for (framed) graphs, see [12].

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