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Grothendieck-Teichmüller group, operads and graph complexes: a survey

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Abstract. The paper offers a more or less self-contained introduction into the theory of the Grothendieck-Teichmüller group and Drinfeld associators using the theory of operads and graph complexes.

1. Introduction

The Grothendieck-Teichmüller group GT (and its graded version GRT) is one of the most interesting and mysterious objects in modern mathematics. This group together with its principal homogeneous space — the set of V. Drinfeld's associators — plays a central role in many seemingly unrelated areas of mathematics: V. Drinfeld pioneered its applications of in the number theory and the theory of quasi-Hopf algebras [**Dr2**]; P. Etingof and D. Kazhdan [**EK, ES**] used it to solve the Drinfeld quantization conjecture for Lie bialgebras; the formality theories of M. Kontsevich [**K1, K2, K3**] and D. Tamarkin [**T1, T2**] unravels the role of the group GRT in deformation quantizations of Poisson structures; A. Alekseev and C. Torossian applied it to the solution of the Kashiwara-Vergne problem in the Lie theory [**AT2**]; the authors of [**MW5**] found its interpretation as a symmetry group of the involutive Lie bialgebra properad playing a key role in the string topology; A. Alekseev, N. Kawazumi, Y. Kuno and F. Naef [**AKKN1, AKKN2**] and G. Massuyeau [**Mas**] proved its importance in the Goldman-Turaev theory of spaces of free homotopy loops in Rieman surfaces of genus g with n punctures; this group — more precisely its Lie algebra $\mathfrak{grt}$ — plays a central role in the recent spectacular advances of M. Chan, S. Galatius and S. Payne [**CGP1**] in the theory of the cohomology groups of moduli spaces $\mathcal{M}_g$ of genus g algebraic curves. T. Willwacher [**W1, W3**] established a very important link between $\mathfrak{grt}$ and cohomologies of some graph complexes which found many applications.

Our main purpose is to give a more or less short and self-contained introduction into the theory of the Grothendieck-Teichmüller group and its applications through its operadic, properadic and graph complexes incarnations, and explain without proofs some of the results mentioned in the previous paragraph.

The Grothendieck-Teichmüller theory first appeared in A. Grothendieck's famous *Esquisse d'un programme* (1981) with the purpose to give a new geometric description of the universal Galois group $Gal(\mathbb{Q})$ of the field of rational numbers; A. Grothendieck's main tool to approach this problem was the Deligne-Mumford moduli stack $\mathcal{M}$ of algebraic curves of arbitrary genus with marked points and a very insightful observation that the (outer) automorphism group $Out(\pi_1^{geom}(\mathcal{M}))$ of the geometric fundamental group of this stack can be described more or less explicitly as the set of elements of a free profinite group on two generators satisfying a small number of equations (see [**L**] and [**LS**] for more details and references). A few years later (1989) V. Drinfeld introduced [**Dr2**] $\mathbb{K}$-prounipotent Grothendieck-Teichmüller groups GT and GRT for an arbitrary field $\mathbb{K}$ of characteristic zero while studying braided quasi-Hopf algebras and their universal deformations. The profinite version $\widehat{GT}$ of the first group contains $Gal(\mathbb{Q})$ and is essentially $Out(\pi_1^{geom}(\mathcal{M}_0))$, where $\mathcal{M}_0$ is the of moduli spaces of genus zero curves with marked points.

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V. Drinfeld has written down explicitly [**Dr2**] systems of algebraic equations defining elements of both prounipotent groups GT and GRT. Thanks to the works of D. Bar-Natan [**B-N**] the algebro-geometric meanings of these two systems of equations are well-understood by now with the help of the theory of operads — GT is essentially the automorphism group of the topological operad $\widehat{\mathcal{P}a\mathcal{B}}$ of parenthesized braids while GRT is the automorphism group of a much simpler graded analogue $\widehat{\mathcal{P}a\mathcal{C}\mathcal{D}}$ of the operad $\widehat{\mathcal{P}a\mathcal{B}}$. The set of Drinfeld associators can be identified with the set of isomorphisms $\widehat{\mathcal{P}a\mathcal{B}} \to \widehat{\mathcal{P}a\mathcal{C}\mathcal{D}}$ of operads. This part of the story explaining GT and GRT as automorphism groups of operads is covered in §§1-6 of this survey; it is based on the notes of a lecture course read by the author at the University of Stockholm in the fall of 2011. B. Fresse has written a book [**Fr**] explaining this approach to GT and GRT in much more detail. In the second part, §§7-8, we try to explain a very different T. Willwacher's approach to the group GRT via the graph complexes (and compactified configuration spaces), and its applications to the deformation quantization theory of Poisson structures and Lie bialgebras, Lie theory, Goldman-Turaev theory, and the theory of moduli spaces $\mathcal{M}_g$. We explain various graph complexes incarnations of the Lie algebra $\mathfrak{grt}$ of GRT; some of them appear quite mysterious at present.

2. Completions of groups, Lie algebras and algebras

2.1. Completed filtered vector spaces, Lie algebras and algebras. A topological ring (in particular, a field) is a ring R which is also a topological space such that both the addition and the multiplication operations define *continuous* maps $R \times R \to R$.

A topological vector space V is a vector space over a topological field $\mathbb{K}$ (say, $\mathbb{R}$ or $\mathbb{C}$ equipped with their standard topologies or with the discrete topology) which is endowed with a topology making the vector addition $V \times V \to V$ and scalar multiplication $\mathbb{K} \times V \to V$ operations into continuous maps. One can define similarly *topological $\mathbb{K}$-(co)algebras*, *topological Lie algebras over $\mathbb{K}$*, *topological Hopf algebras over $\mathbb{K}$*, etc.

Topological vector spaces, rings and $\mathbb{K}$-algebras are particular examples of uniform spaces in which it make sense to talk about Cauchy sequences and hence about their completeness. Any such a space V can be completed in an essentially unique way with the help of equivalence classes of Cauchy sequences; V sits in its completion $\widehat{V}$ as a dense subset and the induced topology coincides with the original one. The completion of the tensor product $\widehat{V} \otimes \widehat{W}$ is denoted by $\widehat{V} \widehat{\otimes} \widehat{W}$ (sometimes we abbreviate the latter notation to $\widehat{V} \otimes \widehat{W}$ when no confusion may arise).

We shall be interested in this survey in a class of topological spaces, rings and (co)algebras whose topologies are determined by filtrations. Their completions — often called *complete filtered* spaces, rings and (co)algebras — can be constructed in an explicit way as inverse limits. We shall explain the phenomenon for algebras; the adoption of all the constructions below to Lie algebras, coalgebras, etc. is straightforward (see, e.g., [**Ei**]).

Let A be a $\mathbb{K}$-algebra equipped with a descending filtration by ideals,

$$A = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \mathfrak{m}_2 \supset \dots \tag{2.1}$$

Hence we get a directed system of quotient rings and their morphisms,

$$\dots \longrightarrow A/\mathfrak{m}_{i+1} \longrightarrow A/\mathfrak{m}_i \longrightarrow \dots \longrightarrow A/\mathfrak{m}_2 \longrightarrow A/\mathfrak{m}_1 \longrightarrow 0.$$

The associated completed filtered algebra is defined as the inverse limit,

$$\widehat{A} := \lim_{\longleftarrow} A/\mathfrak{m}_i := \lim_{\longleftarrow} A/\mathfrak{m}_i := \left\{ a = (a_1, a_2, \dots) \in \prod_{i=1}^{\infty} A/\mathfrak{m}_i \mid a_j = a_i \bmod \mathfrak{m}_i \ \text{ for all } j > i \right\}$$

The algebra $\widehat{A}$ has an induced filtration by ideals,

$$\widehat{\mathfrak{m}}_i := \left\{ a = (a_1, a_2, \dots) \in \widehat{A} \mid a_j = 0 \ \text{ for all } j \leq i \right\}.$$

It is clear that the quotient algebras $A/\mathfrak{m}_i$ and $\widehat{A}/\widehat{\mathfrak{m}}_i$ can be identified. The completed tensor product of such completed algebras is defined by

$$\widehat{A} \widehat{\otimes} \widehat{A'} := \lim_{\underleftarrow{i,j}} A/\mathfrak{m}_i \otimes A'/\mathfrak{m}'_i.$$

The filtered algebra A and and its completion $\widehat{A}$ can be made into topological spaces by defining a basis of open neighborhoods of a point a in A or, respectively, in $\widehat{A}$ to be

$$\left\{a+\mathfrak{m}^i\right\}_{i\in\mathbb{N}} \quad \text{or, respectively,} \quad \left\{a+\widehat{\mathfrak{m}}^i\right\}_{i\in\mathbb{N}}.$$

Such a topology on a filtered algebra is called the *Krull* topology. It is not hard to see that $\widehat{A}$ is the completion of A as a topological algebra. Indeed, let $\{a_i\}_{i\geq 1}$ be a Cauchy sequence in A equipped with the Krull topology, that is, a sequence which satisfies the condition: for any open neighborhood U of zero in A there is a number N_U such that for any $i,j > N_U$ one has $a_i - a_j \in U$. Equivalently, $\{a_i\}_{i\geq 1}$ is a Cauchy sequence in A if and only if for any integer n there exists an integer N_n such that

$$a_i - a_j \in \mathfrak{m}_i \quad \text{for all} \quad i,j > N_n.$$

Such a sequence always converges in $\widehat{A}$ to the point a whose n-th coordinate in $\prod_n A/\mathfrak{m}_n$ is equal, by definition, to $a_{N_n} \mod \mathfrak{m}_n$. Reversely, any point in $\widehat{A}$ gives rise to a Cauchy sequence in A.

We shall work below with filtrations generated by powers $\mathfrak{m}_i := I^i$ of a fixed ideal I in A. The associated completion $\widehat{A}$ is often called the *I-adic completion* of A, and the associated topology on A is called the *I-adic topology*.

2.1.1. **Examples.**

(i) Let $A = \mathbb{K}[x_1,\dots,x_n]$ be a polynomial algebra over $\mathbb{K}$, and let I be its maximal ideal. The I-adic completion of A,

$$\widehat{A} = \mathbb{K}[[x_1,\dots,x_n]],$$

is the algebra of formal power series over $\mathbb{K}$.

(ii) Let $\mathfrak{g}$ be a positively graded Lie algebra, $\mathfrak{g} = \bigoplus_{i=1}^{\infty}\mathfrak{g}_i$, with $\mathfrak{g}_i$ being finite dimensional. It can be equipped with a descending filtration as in (2.1) given by the Lie ideals $\mathfrak{m}_i := \bigoplus_{j\geq i}\mathfrak{g}_j$. The completion of $\mathfrak{g}$ with respect to this filtration is then equal to

$$\widehat{\mathfrak{g}} = \prod_{i\geq 1}\mathfrak{g}_i$$

and is called *the degree completion* of $\mathfrak{g}$ (similarly one can define the degree completion of positively graded vector spaces, rings, algebras, etc).

Consider next the universal enveloping algebra $U(\widehat{\mathfrak{g}})$ of the completed graded Lie algebra $\widehat{\mathfrak{g}}$. It is, by definition, the quotient of the tensor algebra $\otimes^\bullet\widehat{\mathfrak{g}}$ by the ideal J generated by all elements of the form $x\otimes y - y\otimes x - [x,y]$, $x,y\in\widehat{\mathfrak{g}}$. The tensor algebra inherits a positive gradation from $\widehat{\mathfrak{g}}$ in the standard way,

$$\otimes^\bullet\widehat{\mathfrak{g}} = \bigoplus_{i\in 1}^{\infty}(\otimes^\bullet\widehat{\mathfrak{g}})^i \quad \text{with} \quad (\otimes^\bullet\widehat{\mathfrak{g}})^i := \bigoplus_{\substack{i=i_1+\dots+i_n\\ n\geq 0, i_1,\dots,i_n\geq 1}} \widehat{\mathfrak{g}}_{i_1}\otimes\dots\otimes\widehat{\mathfrak{g}}_{i_n}.$$

As $[\widehat{\mathfrak{g}}_i,\widehat{\mathfrak{g}}_j]\subset\widehat{\mathfrak{g}}_{i+j}$, the ideal J is homogeneous with respect to this gradation, i.e. it is generated by the homogeneous elements. Therefore $U(\widehat{\mathfrak{g}})$ comes equipped with an induced gradation and we can define its graded completion $\widehat{U}(\widehat{\mathfrak{g}})$.

2.1.2. **Remark.** To define the completed universal enveloping algebra $\widehat{U}(\widehat{\mathfrak{g}})$ we need only a descending filtration of $\mathfrak{g}$ as in (2.1), a weaker structure than the positive gradation. However, in applications below such a filtration always comes from a positive gradation on $\mathfrak{g}$ as in the above Example 2.1.1.

2.1.3. **Exercise.** Use the Krull topology on $\widehat{U}(\widehat{\mathfrak{g}})$ to show that $\widehat{U}(\widehat{\mathfrak{g}})$ is a completed filtered Hopf algebra (with the coproduct, $\Delta : \widehat{U}(\widehat{\mathfrak{g}}) \to \widehat{U}(\widehat{\mathfrak{g}})\widehat{\otimes}\widehat{U}(\widehat{\mathfrak{g}})$, taking values in the completed tensor product).

2.2. Baker-Campbell-Hausdorff formula. Recall that one can associate a group $G := \exp(\mathfrak{g})$ to a completed positively graded Lie algebra over a field $\mathbb{K}$ of characteristic zero,

$$\mathfrak{g} = \prod_{k=1}^{\infty}\mathfrak{g}_k,$$

with all graded components being of finite dimension. The group G coincides with $\mathfrak{g}$ as a set (and the identification map $\mathfrak{g} \to G$ is denoted by exp while its inverse $G \to \mathfrak{g}$ by log), and the group multiplication is defined by the Campbell-Hausdorff formula:

$$\exp(x) \cdot \exp(y) := \exp(\mathfrak{bch}(x, y)),$$

where

$$\mathfrak{bch}(x, y) = \log(e^x e^y) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] - \frac{1}{12}[y, [y, x]] \ldots, \tag{2.2}$$

and $e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}$ and $\log(1 + x) := \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k}$. This formal power series satisfies obviously the associativity relation,

$$\mathfrak{bch}(x, \mathfrak{bch}(y, z)) = \mathfrak{bch}(\mathfrak{bch}(x, y), z) = \log(e^x e^y e^z),$$

in the ring $\mathbb{K}\langle\langle x, y, z\rangle\rangle$.

2.2.1. **Example.** Let $\mathfrak{lie}_n$ stand for the free Lie algebra over $\mathbb{K}$ generated by n letters $x_1, \ldots, x_n$ and $\widehat{\mathfrak{lie}}_n$ for its degree completion (with degrees of the generators $x_1, \ldots, x_n$ set to be 1). Let $\mathfrak{ass}_n := \mathbb{K}\langle x_1, \ldots, x_n\rangle$ be the free associative algebra generated by $x_1, \ldots, x_n$ and $\widehat{\mathfrak{ass}}_n = \mathbb{K}\langle\langle x_1, \ldots, x_n\rangle\rangle$ its degree completion. One can make $\mathfrak{ass}_n$ and $\widehat{\mathfrak{ass}}_n$ into Lie algebras by setting $[X, Y] = XY - YX$. Then $\mathfrak{lie}_n \subset \mathfrak{ass}_n$ and $\widehat{\mathfrak{lie}}_n \subset \widehat{\mathfrak{ass}}_n$ are Lie subalgebras (in fact they are the smallest Lie subalgebras containing all the generators).

One can make $\widehat{\mathfrak{ass}}_n$ into a bialgebra (in fact, a Hopf algebra) by defining a coproduct,

$$\Delta : \widehat{\mathfrak{ass}}_n \longrightarrow \widehat{\mathfrak{ass}}_n \widehat{\otimes} \widehat{\mathfrak{ass}}_n,$$

by first setting

$$\Delta(x_i) := 1 \otimes x_i + x_i \otimes 1$$

and then extending to arbitrary monomials in $\widehat{\mathfrak{ass}}_n$ by the rule

$$\Delta(x_{i_1} x_{i_2} \cdots x_{i_k}) = \Delta(x_{i_1})\Delta(x_{i_2}) \cdots \Delta(x_{i_k}).$$

An element $g \in \widehat{\mathfrak{ass}}_n$ is called *grouplike* if $\Delta(g) = g \otimes g$. An element $p \in \widehat{\mathfrak{ass}}_n$ is called *primitive* if $\Delta(p) = 1 \otimes p + p \otimes 1$. The following is true:

- Let f be a formal power series from $\widehat{\mathfrak{ass}}_n$ whose constant term is zero. Then $e^f \in \widehat{\mathfrak{ass}}_n$ is grouplike if and only if f is primitive.
- The set of primitive elements in $\widehat{\mathfrak{ass}}_n$ is a Lie subalgebra of $(\widehat{\mathfrak{ass}}_n, [\ ,\ ])$ which can be identified with $\widehat{\mathfrak{lie}}_n$.
- The set of grouplike elements in $\widehat{\mathfrak{ass}}_n$ forms a group with respect to the multiplication in $\widehat{\mathfrak{ass}}_n$ which can be identified with $\exp(\widehat{\mathfrak{lie}}_n)$.

As a Hopf algebra, $\widehat{\mathfrak{ass}}_n$ is precisely the completed universal enveloping algebra of $\widehat{\mathfrak{lie}}_n$.

2.3. Prounipotent completions. A *linear algebraic group* over a field $\mathbb{K}$ is, by definition, an algebraic group that is isomorphic to an algebraic subgroup of the group of invertible $n \times n$ matrices $GL(n, \mathbb{K})$, that is, to a subgroup defined by polynomial equations. Such a group G is called *unipotent* if its image under the embedding $G \to GL(n, \mathbb{K})$ lies in the set $\{g \in GL(n, \mathbb{K}) | (g - 1)^n = 0\}$. An example of a unipotent group is the group of all upper-triangular matrices in $GL(n, \mathbb{K})$ with 1's on the main diagonal.

A group $\widehat{G}$ is called *pro-unipotent* if it is equal to the inverse limit,

$$\widehat{G} = \varprojlim G_i := \left\{ \vec{g} \in \prod_{i \in I} G_i \mid f_{ij}(g_j) = g_i \right\},$$

of some series of unipotent groups $\{G_i\}_{i \in I}$ parameterized by a directed set $(I, \leq)$ and equipped with a system of homomorphisms

$$\{f_{ij} : G_j \to G_i\}_{i,j \in I \text{ such that } i \leq j}$$

satisfying the conditions[1]: $f_{ii} = \mathrm{Id}$ and $f_{ik} = f_{ij} \circ f_{jk}$ for any $i \leq j \leq k$. The Lie algebra $\mathfrak{g}$ of the prounipotent group G is defined as the inverse limit,

$$\widehat{\mathfrak{g}} = \lim_{\longleftarrow} \mathfrak{g}_i$$

of the Lie algebras $\mathfrak{g}_i$ of the groups G_i. There exist mutually inverse maps $\exp : \widehat{\mathfrak{g}} \to \widehat{G}$ and $\log : \widehat{G} \to \widehat{\mathfrak{g}}$.

DEFINITION 2.1. The *prounipotent completion over a field* $\mathbb{K}$ of an abstract group G is a prounipotent group $\widehat{G}(\mathbb{K})$ together with a homomorphism $i : G \to \widehat{G}(\mathbb{K})$ satisfying the following universality property: if $h : G \to \widehat{H}$ is a homomorphism of G into a prounipotent group $\widehat{H}$ over $\mathbb{K}$, then there is a unique homomorphism $f : \widehat{G}(\mathbb{K}) \to \widehat{H}$ such that h factors as the composition $h : G \xrightarrow{i} \widehat{G}(\mathbb{K}) \xrightarrow{f} \widehat{H}$.

Prounipotent completions are often called *Malcev completions* in the literature. One denotes sometimes the prounipotent completion over $\mathbb{K}$ of a group G simply by $\widehat{G}$ omitting thereby the reference to $\mathbb{K}$.

2.4. Quillen's construction of the prounipotent completion. Let G be an abstract group, and

$$\mathbb{K}[G] := \left\{ \sum_{g \in G} \lambda_g g \mid \lambda_g \in \mathbb{K} \text{ such that only finitely many } \lambda_g \text{ can be non-zero} \right\}$$

its group algebra over a field $\mathbb{K}$. There is an algebra homomorphism,

$$\begin{array}{rccc} \varepsilon : & \mathbb{K}[G] & \longrightarrow & \mathbb{K} \\ & : \sum_{g \in G} \lambda_g g & \longrightarrow & \sum_{g \in G} \lambda_g, \end{array}$$

called the *augmentation*. Let $I_{\mathbb{K}} := \mathsf{Ker}\, \varepsilon$ be the augmentation ideal, and let $\widehat{\mathbb{K}[G]}$ be the $I_{\mathbb{K}}$-adic completion[2] of $\mathbb{K}[G]$ (see §2.1). The ideal $I_{\mathbb{K}}$ is generated by elements of the form $g - \mathbb{1}$, where $\mathbb{1}$ is the unit in G (often denoted by 1 by abuse of notation).

The algebra $\mathbb{K}[G]$ can be maid into a Hopf algebra with the coproduct defined as follows

$$\begin{array}{rccc} \Delta : & \mathbb{K}[G] & \longrightarrow & \mathbb{K}[G] \otimes \mathbb{K}[G] \\ & \sum_{g \in G} \lambda_g g & \longrightarrow & \sum_{g \in G} \lambda_g g \otimes g, \end{array}$$

This coproduct is continuous in the $I_{\mathbb{K}}$-adic topology on $\mathbb{K}[G]$ and hence induces by continuity a coproduct

$$\Delta : \widehat{\mathbb{K}[G]} \longrightarrow \widehat{\mathbb{K}[G]} \mathbin{\widehat{\otimes}} \widehat{\mathbb{K}[G]}$$

where $\widehat{\otimes}$ is the completed tensor product. This gives $\widehat{\mathbb{K}[G]}$ the structure of a complete filtered Hopf algebra.

Consider the set of group-like elements,

$$\widehat{G} := \left\{ x \in \widehat{\mathbb{K}[G]} : \Delta(x) = x \otimes x \text{ and } \varepsilon(x) = 1 \right\}. \tag{2.3}$$

This is a subgroup of the group of units in $\widehat{\mathbb{K}[G]}$.

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[1] Such a system $\{G_i, f_{ij}\}$ is called an *inverse system* of groups. Analogously one defines an inverse system of algebras, modules etc.

[2] The $I_{\mathbb{K}}$-adic topology on $\mathbb{K}[G]$ is Hausdorff as the intersection of all powers of I is the zero ideal. In this case one can define the I-adic topology as the unique *metric topology* associated with the following metric on $\mathbb{K}[G]$:

$$d(x, x') := ||x - x'||,$$

where, for any $x \in \mathbb{K}[G]$, its norm $||x||$ is defined to be 2^{-n} with n being the largest natural number such that $x \in I^n$. The algebra $\widehat{\mathbb{K}[G]}$ can be identified with the metric completion of $\mathbb{K}[G]$.

2.4.1. **Remark.** If $char(\mathbb{K}) = 0$, then the $\mathbb{K}$-powers of elements $g \in G$ are well-defined in the ring $\widehat{\mathbb{K}[G]}$,

$$g^\lambda = (\mathbb{1} - (\mathbb{1} - g))^\lambda := \sum_{i=0}^{\infty} \frac{\lambda(\lambda-1)\cdots(\lambda-i+1)}{i!}(\mathbb{1}-g)^i.$$

In some cases the group $\widehat{G}$ is generated by such powers; the corresponding Lie algebra $\widehat{\mathfrak{g}}$ is generated by elements $\log g = \log(\mathbb{1} - (\mathbb{1} - g)) := \sum_{i=1}^{\infty}(\mathbb{1}-g)^i$. Note that the above formula for g^λ implies,

$$hg^\lambda h^{-1} = (hgh^{-1})^\lambda, \tag{2.4}$$

and

$$g^\lambda g^\mu = g^{\lambda+\mu}$$

for any $g, h \in G$ and $\lambda, \mu \in \mathbb{K}$.

2.4.2. **Theorem** [**Ha**]**.** *If $I_{\mathbb{K}}/I_{\mathbb{K}}^2$ is a finite-dimensional vector space over $\mathbb{K}$, then $\widehat{G}$ is a prounipotent completion of G.*

If G is finitely generated, then the condition of this Theorem is satisfied and the above explicit construction gives us the prounipotent completion of G.

It is often useful for practical computations to consider an increasing filtration of the group (2.3),

$$\ldots \subseteq \widehat{G}_{l+1} \subseteq \widehat{G}_l \subseteq \ldots \subseteq \widehat{G}_1 \subseteq \widehat{G},$$

defined by

$$\widehat{G}_l = \widehat{G} \cap (\mathbb{1} + \widehat{I_{\mathbb{K}}}^l).$$

If the condition of Theorem 2.4.2 is satisfied, then each group,

$$\widehat{G}^l := \widehat{G}/\widehat{G}_{l+1}$$

is a unipotent group, and one can compute the prounipotent completion of G as the inverse limit

$$\widehat{G} = \varprojlim \widehat{G}^l.$$

2.4.3. **The case** $char(\mathbb{K}) = 0$**.** Consider the set of *primitive elements*, $\widehat{\mathfrak{g}}$, of the completed group algebra $\widehat{\mathbb{K}[G]}$ defined by

$$\widehat{\mathfrak{g}} := \left\{ x \in \widehat{I_{\mathbb{K}}} \mid \Delta(x) = \mathbb{1} \otimes x + x \otimes \mathbb{1} \right\}.$$

The bracket $[x, y] = xy - yx$ makes $\widehat{\mathfrak{g}}$ into a Lie algebra. The $\widehat{I_{\mathbb{K}}}$-adic topology of $\widehat{\mathbb{K}[G]}$ induces a topology on $\widehat{\mathfrak{g}}$ making the latter into a complete topological Lie algebra. The logarithm and exponential functions give us well-defined and mutually inverse homeomorphisms,

$$\log : \mathbb{1} + \widehat{I_{\mathbb{K}}} \longrightarrow \widehat{I_{\mathbb{K}}} \quad \text{and} \quad \exp : \widehat{I_{\mathbb{K}}} \longrightarrow \mathbb{1} + \widehat{I_{\mathbb{K}}}$$

which restrict to the mutually inverse homeomorphisms,

$$\log : \widehat{G} \longrightarrow \widehat{\mathfrak{g}} \quad \text{and} \quad \exp : \widehat{\mathfrak{g}} \longrightarrow \widehat{G}.$$

Thus one can describe $\widehat{G}$ in terms of its Lie algebra $\widehat{\mathfrak{g}}$ which is sometimes a simpler object to define. The latter is the inverse limit of the Lie algebras of the prounipotent groups $\widehat{G}^l$ defined above.

2.5. Examples of prounipotent completions.

(i) Let G be a group of finite type, that is, a quotient of the free group generated by a finite set $\{g_1, \ldots, g_n\}$ by the normal subgroup generated by a finite number of relations

$$\{R_i(g_1, \ldots, g_n) = 1\}_{i \in [m]}, \quad m \in \mathbb{N}.$$

Let $\mathbb{K}$ be a field of characteristic zero, and let $\widehat{\mathfrak{g}}$ be the quotient of the completed free Lie algebra $\widehat{\mathfrak{lie}}_n$ generated by symbols $\{\gamma_1, \ldots, \gamma_n\}$ by the ideal generated by the relations,

$$\{\log R_i(e^{\gamma_1}, \ldots, e^{\gamma_n}) = 0\}_{i \in [m]}.$$

Then the group $\widehat{G}(\mathbb{K}) := \exp \widehat{\mathfrak{g}}$ gives us the prounipotent completion of G over $\mathbb{K}$ [**Br**].

(ii) If F_n is the free group in n letters, $x_1, \ldots, x_n$, then its prounipotent completion, $\widehat{F}_n(\mathbb{K})$, over a field $\mathbb{K}$ of characteristic zero is equal to $\exp(\widehat{\mathfrak{lie}_n})$, where $\widehat{\mathfrak{lie}_n}$ is the completed filtered Lie algebra over $\mathbb{K}$ generated by the following set,

$$\left\{ \log x_i = \log(1-(1-x_i)) = \sum_{k=1}^{\infty} \frac{1}{k}(1-x_i)^k \right\}_{i\in[n]} .$$

Put another way, $F_n(\mathbb{K})$ is generated by elements of the form x_i^{λ}, $\lambda \in \mathbb{K}$.

The next examples are taken from [**K**].

(iii) Let $G = \mathbb{Z}$ and let $\mathbb{K}$ be a field of characteristic zero. As $\mathbb{Z}$ is isomorphic to the free group generated by a symbol t, its prounipotent completion $\widehat{\mathbb{Z}}(\mathbb{K})$ can be computed using the construction in Example (i) above to give

$$\widehat{\mathbb{Z}}(\mathbb{K}) = \exp(\widehat{\mathfrak{lie}_1}) \simeq \mathbb{K} \ \text{ (as Abelian groups).}$$

One can get the same result using Quillen's construction as follows [**K**]. The group algebra $\mathbb{K}[\mathbb{Z}]$ is the ring of Laurent polynomials $\mathbb{K}[t,t^{-1}]$ with the augmentation ideal $I_{\mathbb{K}}$ being a principal one generated by $t-1$. The completion $\widehat{\mathbb{K}[\mathbb{Z}]}$ is the formal power series $\mathbb{K}[[T]]$ with the inclusion $\mathbb{K}[\mathbb{Z}] \to \widehat{\mathbb{K}[\mathbb{Z}]}$ given by

$$\begin{array}{ccc} \mathbb{K}[t,t^{-1}] & \longrightarrow & \mathbb{K}[[T]] \\ (t,t^{-1}) & \longrightarrow & (1+T, \sum_{k=1}^{\infty}(-1)^k T^k). \end{array}$$

The ideal $\widehat{I_{\mathbb{K}}}$ is therefore the principal one generated by the symbol T (that is, the maximal ideal of $\mathbb{K}[[T]]$) so that the augmentation map $\mathbb{K}[[T]] \to \mathbb{K}$ has the form $T \to 0$. The coproduct is given by $\Delta(T) = 1\otimes T + T \otimes 1 + T \otimes T$.

If $\mathbb{K}$ has characteristic zero, then the set of group-like elements in $\widehat{\mathbb{K}[\mathbb{Z}]}$, i.e. the prounipotent completion of $\mathbb{Z}$ over $\mathbb{K}$, is given by

$$\widehat{\mathbb{Z}}(\mathbb{K}) = \{(1+T)^{\alpha} = \exp(\alpha \log(1+T)) \in \mathbb{K}[[T]] \mid \alpha \in \mathbb{K}\} \simeq \mathbb{K}.$$

Note that in this case $\widehat{\mathbb{Z}}(\mathbb{K})^l = \widehat{\mathbb{Z}}(\mathbb{K}) \cap (1+\widehat{I_{\mathbb{K}}}^l) = \{1\}$ for $l \geq 2$ so that $\widehat{\mathbb{Z}}(\mathbb{K})_l = \widehat{\mathbb{Z}}(\mathbb{K})$ for all l.

If $\mathbb{K}$ has characteristic p, then the element $1+T$ has finite order in each group

$$\widehat{\mathbb{Z}}(\mathbb{K})_l = \widehat{\mathbb{Z}}(\mathbb{K}) / \widehat{\mathbb{Z}}(\mathbb{K}) \cap (1+\widehat{I_{\mathbb{K}}}^{l+1})$$

as

$$(1+T)^p = 1+T^p \in 1+\widehat{I_{\mathbb{K}}}^p.$$

It follows [**K**] that the prounipotent completion of $\mathbb{Z}$ over $\mathbb{K}$ is equal to $\mathbf{Z}_p$, the Abelian group (in fact, the ring) of p-adic integers.

(iv) Let $G = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, $n \geq 2$. The associated group algebra over a field $\mathbb{K}$ is $\mathbb{K}[\mathbb{Z}_n] = \mathbb{K}[t]/(t^n-1)$, the quotient of the polynomial algebra $\mathbb{K}[t]$ by the principal ideal generated by (t^n-1). The augmentation ideal $I_{\mathbb{K}} \subset \mathbb{K}[\mathbb{Z}_n]$ is the principal one generated by $t-1$. Note that t is a unit with $t^{-1} = t^{n-1}$. There is a factorization in $\mathbb{K}[\mathbb{Z}_n]$,

$$\begin{aligned} (t-1)^2 &= t^2 - 2t + 1 \\ &= t^n + t^2 - 2t \\ &= t(t^{n-2} + \ldots + t + 2)(t-1). \end{aligned}$$

If characteristic of $\mathbb{K}$ is prime to n or equal to zero, then $t^{n-2} + \ldots + t + 2$ is also a unit in $\mathbb{K}[\mathbb{Z}_n]$ so that $I_{\mathbb{K}}^2 = I_{\mathbb{K}}$ and hence

$$\widehat{\mathbb{K}[\mathbb{Z}_n]} = \varprojlim \mathbb{K}[\mathbb{Z}_n]/I_{\mathbb{K}}^l = \mathbb{K}$$

Hence the prounipotent completion of $\mathbb{Z}_n$ in this case is the trivial group, $\widehat{\mathbb{Z}_n}(\mathbb{K}) = \{1\}$ (cf. Example (iii) above).

If $char(\mathbb{K}) = p$ divides n, then one can assume[3] that $n = p^d$ for some $d \geq 1$. As $(1-t)^{p^d} = 1 - t^{p^d} = 1 - t^n = 0$, one has $I_{\mathbb{K}}^{p^d} = 0$ so that $\widehat{\mathbb{Z}_n}(\mathbb{K}) = \mathbb{Z}_n(\mathbb{K})$ and hence the prounipotent completion is given by the following set of group-like elements,

$$\widehat{\mathbb{Z}_n}(\mathbb{K}) = \{t^i \mid 0 \leq i \leq p^d\} \simeq \mathbb{Z}_{p^d}$$

(v) Examples (ii) and (iii) are special cases of a more general statement [**K**]: if G is a finitely generated Abelian group, then

$$\widehat{G}(\mathbb{K}) := \begin{cases} G \otimes_{\mathbb{Z}} \mathbb{K} & \text{if } char(\mathbb{K}) = 0 \\ G \otimes_{\mathbb{Z}} \mathbf{Z}_p & \text{if } char(\mathbb{K}) = p. \end{cases}$$

2.6. Profinite completions. Let G be an abstract group and $\{I\}$ a family of all normal subgroups $I \subset G$ of finite index. This family can be made into a partially ordered set with respect to the inclusion. As $I_1 I_2 \in \{I\}$ for any $I_1, I_2 \in \{I\}$, this partially ordered set is directed so that $\{I\}$ gives us an inverse system of ideals and one can consider the inverse limit,

$$\widetilde{G} := \varprojlim G/I,$$

which is called the *profinite completion of* G. For example, $\widetilde{\mathbb{Z}} = \prod_p \mathbf{Z}_p$.

3. Monoidal categories and monoidal functors

For a category C its class of objects is denoted by $Ob(\mathsf{C})$, and the set of morphisms from an object A to an object B by $\mathrm{Mor}_{\mathsf{C}}(A, B)$. If $Ob(\mathsf{C})$ is a set, then the category is called *small*. The composition of morphisms $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$ is denoted by $\circ$.

3.1. Monoidal categories (see e.g. [ES]). A category C is called a *semigroup category* if it is equipped with a bifunctor

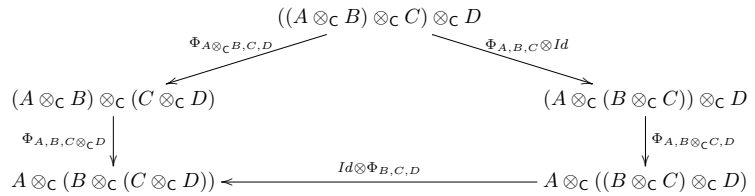
$$\begin{array}{rccc} \cdot \otimes_{\mathsf{C}} \cdot & \mathsf{C} \times \mathsf{C} & \longrightarrow & \mathsf{C} \\ & A, B \in Ob(\mathsf{C}) & \longrightarrow & A \otimes_{\mathsf{C}} B \in Ob(\mathsf{C}), \end{array}$$

and an isomorphism of trifunctors,

$$\begin{array}{rccc} \Phi: & (\,\cdot \otimes_{\mathsf{C}} \cdot\,) \otimes_{\mathsf{C}} \cdot & \longrightarrow & \cdot \otimes_{\mathsf{C}} (\,\cdot \otimes_{\mathsf{C}} \cdot\,) \\ \Phi_{A,B,C}: & (A \otimes_{\mathsf{C}} B) \otimes_{\mathsf{C}} C & \longrightarrow & A \otimes_{\mathsf{C}} (B \otimes_{\mathsf{C}} C) \end{array}$$

such that the diagram

(3.1)



commutes for any $A, B, C, D \in Ob(\mathsf{C})$.

Commutativity of the diagram (3.1) is called the *pentagon axiom*. A remarkable fact is that the pentagon axiom implies commutativity of all similar diagrams in the category $\mathcal{S}$. More precisely, one has

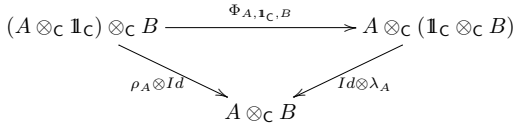
[3]The prounipotent completion of the cartesian product of two groups is the cartesian product of their prounipotent completions.

3.1.1. **Mac-Lane cohenerence theorem.** *Let* C *be a semigroup category and consider an arbitrary collection* $A_1, \ldots, A_n$ *of objects* C*. Given any two complete bracketings of the formal expression* $A_1 \otimes_{\mathsf{C}} \otimes A_2 \otimes_{\mathsf{C}} \ldots \otimes_{\mathsf{C}} A_n$*, then all isomorphisms from one bracketing to another composed of the associativity isomorphisms* Φ *and their inverses, are equal to each other.*

A semigroup category C is called *monodial* if it has an object $\mathbb{1}_{\mathsf{C}}$ (called a *unit*) together with isomorphisms

$$\lambda_A : \mathbb{1}_{\mathsf{C}} \otimes_{\mathsf{C}} A \longrightarrow A, \qquad \rho_A : A \otimes_{\mathsf{C}} \mathbb{1}_{\mathsf{C}} \longrightarrow A$$

such that the diagram



commutes for any $A, B \in Ob(\mathsf{C})$.

A monoidal category is called *strict* if all the isomorphisms Φ, λ and ρ are identities.

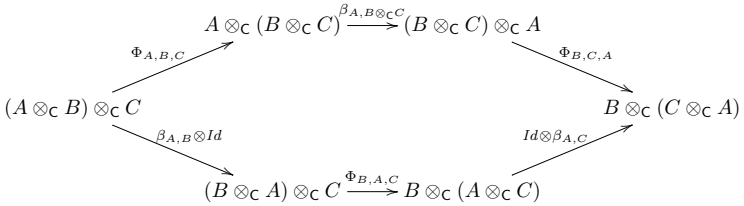
3.1.2. **Symmetric monoidal categories.** A monoidal category is called *symmetric* if it is equipped with an isomorphism of bifunctors,

$$\begin{array}{rccc} \beta: & \cdot \otimes_{\mathsf{C}} \cdot & \longrightarrow & \cdot \otimes_{\mathsf{C}}^{op} \cdot \\ \beta_{A,B} & A \otimes_{\mathsf{C}} B & \longrightarrow & B \otimes_{\mathsf{C}} A \end{array} \tag{3.2}$$

such that

$$\beta_{B,A} \circ \beta_{A,B} = Id \tag{3.3}$$

and the diagram



(3.4)

commutes for any $A, B, C \in Ob(\mathsf{C})$.

Commutativity of diagram (3.4) is called the *hexagon axiom*.

There is a symmetric monoidal analogue of the above Mac Lane coherence theorem which says, that given any complete bracketings of the formal expressions $A_1 \otimes_{\mathsf{C}} \otimes A_2 \otimes_{\mathsf{C}} \ldots \otimes_{\mathsf{C}} A_n$ and $A_{\sigma(1)} \otimes_{\mathsf{C}} \otimes A_{\sigma(2)} \otimes_{\mathsf{C}} \ldots \otimes_{\mathsf{C}} A_{\sigma(n)}$, $\sigma \in \mathbb{S}_n$, then all isomorphisms from one bracketing to another composed of the associativity isomorphisms Φ, symmetry isomorphisms β and their inverses, are equal to each other.

3.1.3. **Braided monoidal categories.** A monoidal category is called *braided* if it is equipped with an isomorphism of functors (3.2) which makes the diagram (3.4) commutative while its inverse β^{-1} makes a version of the diagram (3.4) in which the symbols $\beta_{A,B\otimes_{\mathsf{C}}C}$, $\beta_{A,B}$ and $\beta_{A,C}$ are replaced with $\beta^{-1}_{B\otimes_{\mathsf{C}}C,A}$, $\beta^{-1}_{B,A}$ and $\beta^{-1}_{C,A}$ respectively commutative as well.

Mac Lane type coherence property of a braided monoidal category will be explained below in terms of the braid groups.

3.2. Examples of monodial categories.

(i) Let $\mathsf{Vect}_{\mathbb{K}}$ be the category whose objects are vector spaces[4] over a field $\mathbb{K}$ and whose morphisms, $f : A \longrightarrow B$, are continuous linear maps. This is a strict symmetric monoidal category with

$$\begin{aligned} A \otimes_{\mathsf{Vect}_{\mathbb{K}}} B &:= A \otimes_{\mathbb{K}} B, \quad \text{the ordinary (completed) tensor product of vector spaces over } \mathbb{K}, \\ \mathbb{1}_{\mathsf{Vect}_{\mathbb{K}}} &:= \mathbb{K} \end{aligned}$$

and the symmetry morphism is given by $\beta_{A,B}(A \otimes_{\mathbb{K}} B) = B \otimes_{\mathbb{K}} A$.

(ii) Let $\mathsf{Ass}_{\mathbb{K}}$ be the category of associative algebras (with unit) over a field $\mathbb{K}$. This is a subcategory of $\mathsf{Vect}_{\mathbb{K}}$. Moreover, $\mathsf{Ass}_{\mathbb{K}}$ is a strict symmetric monoidal category with respect to the structures induced from $\mathsf{Vect}_{\mathbb{K}}$.

(iii) Let $\mathsf{Hopf}_{\mathbb{K}}$ be the category of Hopf algebras over a field $\mathbb{K}$. This a subcategory of $\mathsf{Ass}_{\mathbb{K}}$ which is is strict symmetric monoidal with respect to the structures induced from $\mathsf{Ass}_{\mathbb{K}}$.

(iv) Let $\mathsf{coAss}_{\mathbb{K}}$ be the category of associative coalgebras over a field $\mathbb{K}$. This a subcategory of $\mathsf{Vect}_{\mathbb{K}}$ which is is strict symmetric monoidal with respect to the structures induced from $\mathsf{Vect}_{\mathbb{K}}$.

(v) Let $\mathsf{Lie}_{\mathbb{K}}$ be the category whose objects are (complete Lie algebras over a field $\mathbb{K}$ and whose morphisms, $f : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$, are linear maps preserving Lie brackets, i.e. $f \circ [\cdot,\cdot] = [f(\cdot), f(\cdot)]$. This is a strict symmetric monoidal category with

$$\begin{aligned} \mathfrak{g} \otimes_{\mathsf{Lie}_{\mathbb{K}}} \mathfrak{h} &:= \mathfrak{g} \oplus \mathfrak{h}, \quad \text{the direct sum of vector spaces over } \mathbb{K}, \\ \mathbb{1}_{\mathsf{Lie}_{\mathbb{K}}} &:= 0, \end{aligned}$$

and the symmetry morphism given by $\beta_{\mathfrak{g},\mathfrak{h}}(\mathfrak{g} \oplus \mathfrak{h}) = \mathfrak{h} \oplus \mathfrak{g}$.

(i-v)′ There are obvious *dg*, that is, *differential graded*, versions — $\mathsf{dgVect}_{\mathbb{K}}$, $\mathsf{dgAss}_{\mathbb{K}}$, $\mathsf{dgHopf}_{\mathbb{K}}$, $\mathsf{dgCoAss}_{\mathbb{K}}$, $\mathsf{dgLie}_{\mathbb{K}}$ — of all the above symmetric monoidal categories whose objects are *dg* vector spaces, *dg* associative algebras etc.

We often use the following notation. Let $V = \oplus_{i\in\mathbb{Z}} V^i$ be a $\mathbb{Z}$-graded vector space, then for any integer $k \in \mathbb{Z}$ the symbol $V[k]$ stands for the $\mathbb{Z}$-graded vector space with $V[k]^i := V^{i+k}$ and and s^k for the associated isomorphism $V \to V[k]$; for $v \in V^i$ one denotes $|v| := i$. For a pair of $\mathbb{Z}$-graded vector spaces V_1 and V_2, the symbol $\mathrm{Hom}_i(V_1, V_2)$ stands for the space of homogeneous linear maps of degree i, and $\mathrm{Hom}(V_1, V_2) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_i(V_1, V_2)$; for example, $s^k \in \mathrm{Hom}_{-k}(V, V[k])$. Furthermore, we use the notation $\odot^n V$ for the n-fold symmetric product of the vector space V.

(vi) Let Cat be the category of small categories (with functors as morphisms). This is a strict symmetric monoidal category with

$$Ob(C' \otimes_{\mathsf{Cat}} C'') := Ob(C') \times Ob(C''),$$

and

$$\mathrm{Mor}_{C'\otimes_{\mathsf{Cat}} C''}(X' \times X'', Y' \times Y'') = \mathrm{Mor}_{C'}(X', Y') \times \mathrm{Mor}_{C''}(X'', Y'').$$

Its full subcategory consisting of small groupoids (that is, small categories with every morphism being an isomorphism) is denoted by CatG.

(vii) Let $\mathsf{Cat}(\mathsf{Vect}_{\mathbb{K}})$ be a category of small $\mathbb{K}$-linear categories, that is, a category of small categories which satisfy two conditions: (1) morphisms between any pair of objects form a vector space, and (2) the composition of morphisms is a bilinear map. This is a subcategory of Cat but we make it into a symmetric monoidal category in a slightly different way:

$$Ob(C' \otimes_{\mathsf{Cat}(\mathsf{Vect}_{\mathbb{K}})} C'') := Ob(C') \times Ob(C''),$$

and

$$\mathrm{Mor}_{C'\otimes_{\mathsf{Cat}(\mathsf{Vect}_{\mathbb{K}})} C''}(X' \times X'', Y' \times Y'') = \mathrm{Mor}_{C'}(X', Y') \otimes_{\mathbb{K}} \mathrm{Mor}_{C''}(X'', Y'').$$

[4]The completed filtered versions of the categories listed below will be denoted by the same kernel symbol, but equipped with a wide hat, e.g., $\widehat{\mathsf{Vect}}_{\mathbb{K}}$, $\widehat{\mathsf{Ass}}_{\mathbb{K}}$ etc.

For future reference we denote by $\mathsf{Cat}(\mathsf{coAss}_{\mathbb{K}})$ the (strict symmetric monoidal) subcategory of $\mathsf{Cat}(\mathsf{Vect}_{\mathbb{K}})$ such that the space of morphisms between any two objects is a $\mathbb{K}$-coalgebra, and the composition is compatible with the coalgebra structures.

(viii) Let Set be the category of sets. This is a strict symmetric monoidal category with

$$\begin{aligned} I \otimes_{\mathsf{Set}} J &:= I \times J, \text{ the Cartesian product of sets } I \text{ and } J, \\ \mathbb{1}_{\mathsf{Set}} &:= \text{one element set,} \end{aligned}$$

and the symmetry morphism given by $\beta_{A,B}(I \times J) = J \times I$.

(ix) Let Group be the subcategory of Set whose objects are groups and homomorphisms as morphisms. This is a strict symmetric monoidal category with respect to the the Catersian product; the unit is given by the trivial group 1.

(x) Let Top be the subcategory of Set whose objects are Hausdorff spaces with compactly generated topology and continuous maps as morphisms. This is a strict symmetric monoidal category with respect to the structures inherited from Set.

3.3. Monoidal functors. Let $(\mathsf{C}, \otimes_{\mathsf{C}}, \mathbb{1}_{\mathsf{C}})$ and $(\mathsf{D}, \otimes_{\mathsf{D}}, \mathbb{1}_{\mathsf{D}})$ be monoidal categories. A functor $F : \mathsf{C} \to \mathsf{D}$ is called *monoidal* if it comes with a morphism $\mathbb{1} : \mathbb{1}_{\mathsf{D}} \to F(\mathbb{1}_{\mathsf{C}})$ and a natural transformation

$$\begin{aligned} \phi : &\quad F(\,\cdot\,) \otimes_{\mathsf{D}} F(\,\cdot\,) &\longrightarrow &\quad F(\,\cdot \otimes_{\mathsf{C}} \,\cdot\,) \\ \phi_{A,B} : &\quad F(A) \otimes_{\mathsf{D}} F(B) &\longrightarrow &\quad F(A \otimes_{\mathsf{C}} B) \end{aligned}$$

such that the following diagrams,

$$\begin{array}{ccc} (F(A) \otimes_{\mathsf{D}} F(B)) \otimes_{\mathsf{D}} F(C) & \xrightarrow{\Phi_{F(A),F(B),F(C)}} & F(A) \otimes_{\mathsf{D}} (F(B)) \otimes_{\mathsf{D}} F(C)) \\ \downarrow{\scriptstyle \phi_{A,B} \otimes Id} & & \downarrow{\scriptstyle Id \otimes \phi_{B,C}} \\ F(A \otimes_{\mathsf{C}} B) \otimes_{\mathsf{D}} F(C) & & F(A) \otimes_{\mathsf{D}} F(B \otimes_{\mathsf{C}} C) \\ \downarrow{\scriptstyle \phi_{A \otimes_{\mathsf{C}} B, C}} & & \downarrow{\scriptstyle \phi_{A, B \otimes_{\mathsf{C}} C}} \\ F((A \otimes_{\mathsf{C}} B) \otimes_{\mathsf{C}} C) & \xrightarrow[\Phi_{A,B,C}]{} & F(A \otimes_{\mathsf{C}} (B \otimes_{\mathsf{C}} C)) \end{array}$$

$$\begin{array}{ccc} \mathbb{1}_{\mathsf{D}} \otimes_{\mathsf{D}} F(A) & \xrightarrow{\mathbb{1} \otimes Id} & F(\mathbb{1}_{\mathsf{C}}) \otimes_{\mathsf{D}} F(A) \\ \downarrow{\scriptstyle \lambda_{F(A)}} & & \downarrow{\scriptstyle \phi_{\mathbb{1}_{\mathsf{C}}, A}} \\ F(A) & \xleftarrow{F(\lambda_A)} & F(\mathbb{1}_{\mathsf{C}} \otimes_{\mathsf{C}} A)) \end{array} \qquad \begin{array}{ccc} F(A) \otimes_{\mathsf{D}} \mathbb{1}_{\mathsf{D}} & \xrightarrow{Id \otimes \mathbb{1}} & F(A) \otimes_{\mathsf{D}} F(\mathbb{1}_{\mathsf{C}}) \\ \downarrow{\scriptstyle \rho_{F(A)}} & & \downarrow{\scriptstyle \phi_{A, \mathbb{1}_{\mathsf{C}}}} \\ F(A) & \xleftarrow{F(\rho_A)} & F(A \otimes_{\mathsf{C}} \mathbb{1}_{\mathsf{C}})) \end{array}$$

commute for any $A, B, C \in Ob(\mathsf{C})$.

A functor F between braided monoidal categories is called *braided monoidal* if it is monoidal and makes the diagram

$$\begin{array}{ccc} F(A) \otimes_{\mathsf{D}} F(B) & \xrightarrow{\beta} & F(B) \otimes_{\mathsf{D}} F(A) \\ \downarrow{\scriptstyle \phi_{A,B}} & & \downarrow{\scriptstyle \phi_{B,A}} \\ F(A \otimes_{\mathsf{C}} B) & \xrightarrow[F(\beta)]{} & F(B \otimes_{\mathsf{C}} A) \end{array}$$

commutative for any $A, B \in Ob(\mathsf{C})$.

A braided monoidal functor between symmetric monoidal categories is called *symmetric monoidal*.

3.4. Examples of monoidal functors.

(i) The contravariant functor

$$\begin{aligned} Lin_{\mathbb{K}} : &\quad \mathsf{Set} &\longrightarrow &\quad \mathsf{Vect}_{\mathbb{K}} \\ &\quad I &\longrightarrow &\quad \mathbb{K}[I] := \text{the space of functions } I \to \mathbb{K} \end{aligned}$$

is symmetric monoidal.

(ii) The (completed) universal enveloping functor,

$$\begin{array}{rccc} U: & \mathsf{Lie}_{\mathbb{K}} & \longrightarrow & \mathsf{Hopf}_{\mathbb{K}} \\ & \mathfrak{g} & \longrightarrow & U(\mathfrak{g}) \end{array}$$

is symmetric monoidal because $U(\mathfrak{g} \oplus \mathfrak{h}) = U(\mathfrak{g}) \otimes_{\mathbb{K}} U(\mathfrak{h})$.

(iii) Consider a functor,

$$\Delta_{\mathbb{K}} : \mathsf{CatG} \longrightarrow \mathsf{Cat}(\mathsf{coAss}_{\mathbb{K}})$$

which is identity on objects $I \in Ob(\mathsf{CatG})$,

$$\Delta_{\mathbb{K}}(I) = I,$$

and on morphisms it coincides with $Lin_{\mathbb{K}}$ defined above in (i),

$$\Delta_{\mathbb{K}}(\mathrm{Mor}_{\mathsf{CatG}}(I,J)) = \mathbb{K}[\mathrm{Mor}_{\mathsf{CatG}}(I,J)]$$

where $\mathbb{K}[\mathrm{Mor}_{\mathsf{Cat}}(I,J)]$ is equipped with the unique coalgebra structure in which all elements f of $\mathrm{Mor}_{\mathsf{Cat}}(I,J)$ are group-like, $\Delta(f) = f \otimes_{\mathbb{K}} f$. This functor is symmetric monoidal. Note that $\Delta_{\mathbb{K}}(\mathbb{S}_n) = \mathbb{K}[\mathbb{S}_n]$, the group algebra of $\mathbb{S}_n = \mathrm{Mor}_{\mathsf{CatG}}([n],[n])$ equipped with its standard coalgebra structure.

(iv) Let $(\mathfrak{g},\ [\ ,\ ])$ be a Lie algebra, and $CE_\bullet(\mathfrak{g})$ its *Chevalley-Eilenberg complex* (also known as the *bar constriction*) which is, by definition, a dg coalgebra

$$CE^\bullet(\mathfrak{g}) := \odot^\bullet(\mathfrak{g}[1]) = \bigoplus_{n=0}^{\infty} (\otimes^n \mathfrak{g}[1])_{\mathbb{S}_n},$$

with the coproduct $\Delta : CE^\bullet(\mathfrak{g}) \to CE^\bullet(\mathfrak{g}) \otimes_{\mathbb{K}} CE^\bullet(\mathfrak{g})$ given by

$$\Delta : s(x_1) \odot \ldots \odot s(x_n) := \sum_{\substack{[n]=I_1 \sqcup I_2 \\ \#I_1, I_2 \geq 0}} (-1)^{\sigma(I_1,I_2)} x_{I_1} \otimes x_{I_2},$$

where, for a naturally ordered subset $I = \{i_1, \ldots, i_k\}$ of $[n]$, we set

$$x_I := s(x_{i_1}) \odot \ldots \odot s(x_{i_k}),$$

and $(-1)^{\sigma(I_1,I_2)}$ is the sign of the permutation $[n] \to [I_1 \sqcup I_2]$. The differential, $d : CE^\bullet(\mathfrak{g}) \to CE^\bullet(\mathfrak{g})$, is given by

$$ds(x_1) \odot \ldots \odot s(x_n) := \sum_{1 \leq i < j \leq n} (-1)^{i+j} s([x_i, x_j]) \odot s(x_1) \odot \ldots \odot \widehat{s(x_i)} \odot \ldots \odot \widehat{s(x_j)} \odot \ldots \odot s(x_n).$$

As $CE^\bullet(\mathfrak{g} \oplus \mathfrak{h}) = CE^\bullet(\mathfrak{g}) \otimes_{\mathbb{K}} CE^\bullet(\mathfrak{h})$, the functor

$$\begin{array}{rccc} CE^\bullet: & \mathsf{Lie}_{\mathbb{K}} & \longrightarrow & \mathsf{dgCoAss}_{\mathbb{K}} \\ & \mathfrak{g} & \longrightarrow & CE^\bullet(\mathfrak{g}) \end{array}$$

is symmetric monoidal.

(v) An associative $\mathbb{K}$-algebra A with unit can be interpreted as a category with one object, $\bullet$, and with $\mathrm{Mor}(\bullet,\bullet) := A$. This defines a functor

$$\mathsf{Ass}_{\mathbb{K}} \longrightarrow \mathsf{Cat}(\mathsf{Vect}_{\mathbb{K}})$$

which is symmetric monoidal.

(vi) Let $I^n := [0,1]^n \subset \mathbb{R}^n$ be the n-cube (we set I^0 to be a point), and X a topological space. A continuous map $f : I^n \to X$ is called a *singular n-cube*. A singular cube f is called *degenerate* if the function f does not depend on (at least one) one of the n canonical coordinates, $(t^1, t^2, \ldots, t^n)$ on I^n. Define a $\mathbb{Z}$-graded vector space of (cubical) chains,

$$Chains_\bullet(X) = \bigoplus_{n=0}^{\infty} Chains_{-n}(X),$$

by setting $Chains_n(X)$ to be the quotient of the free Abelian group consisting of formal linear combinations,

$$\sum_i n_i f_i, \quad n_i \in \mathbb{Z},$$

of non-degenerate singular cubes $f_i : I^n \to X$ modulo the relation,

$$f = (-1)^\sigma f \circ \sigma,$$

for any permutation $\sigma \in \mathbb{S}_n$ acting on the cube as follows,

$$\begin{array}{rccc} \sigma: & I^n & \longrightarrow & I^n \\ & (t_1, t_2, \ldots, t_n) & \longrightarrow & (t_{\sigma(1)}, t_{\sigma(2)}, \ldots, t_{\sigma(n)}) \end{array}$$

The differential, ∂, on $Chains_\bullet(X)$ is defined in the usual way,

$$\begin{array}{rccl} \partial: & Chains_n(X) & \longrightarrow & Chains_{n-1}(X) \\ & f(t_1, \ldots, t_n) & \longrightarrow & (\partial f)(t_1, \ldots, t_{n-1}) := \sum_{i=1}^n (-1)^i f(t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_n) - \\ & & & \qquad - \sum_{i=1}^n (-1)^i f(t_1, \ldots, t_{i-1}, 1, t_i, \ldots, t_n) \end{array}$$

The homology of the complex $(Chains_\bullet(X), \partial)$ is denoted by $H_\bullet(X, \mathbb{Z})$ and is called the *singular (cubical) homology* of the topological space X. This homology is equal to the singular simplicial homology of X (so that at the homology level the adjective *cubical* can be omitted). For a field $\mathbb{K}$ we denote $Chains_\bullet(X, \mathbb{K}) := Chains_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{K}$ and its homology by $H_\bullet(X, \mathbb{K})$.

For any finite collection $\{X_i\}_{i \in I}$ of topological spaces, there is a product map

$$\bigotimes_{i \in I} Chains_\bullet(X_i) \longrightarrow Chains_\bullet\left(\prod_{i \in I} X_i\right),$$

which is a *natural* homomorphism of complexes.

The cubical chains functor

$$\begin{array}{ccc} \mathsf{Top} & \longrightarrow & \mathsf{dgVect}_{\mathbb{K}} \\ X & \longrightarrow & Chains_\bullet(X, \mathbb{K}) \end{array}$$

and the associated homology functor

$$\begin{array}{ccc} \mathsf{Top} & \longrightarrow & \mathsf{dgVect}_{\mathbb{K}} \\ X & \longrightarrow & H_\bullet(X, \mathbb{K}) \end{array}$$

are symmetric monoidal.

(vii) A *path* in a topological space X from a point $x_0 \in X$ to a point $x_1 \in X$ is a continuous map $u : [0, 1] \to X$ of the unit interval into X with $u(0) = x_0$ and $u_(1) = x_1$. Let $\pi(X)(x_0, x_1)$ stand for the set of of all homotopy classes of paths in X from x_0 to x_1. There is a natural composition map,

$$\pi(X)(x_0, x_1) \times \pi(X)(x_1, x_2) \longrightarrow \pi(X)(x_0, x_2),$$

which makes the set of points of X into a category, $\pi(X)$, in which the set of morphisms from an object $x_0 \in X$ to an object $x_1 \in X$ is identified with $\pi(X)(x_0, x_1)$. Every such a morphisms is an isomorphism so that $\pi(X)$ is in fact a groupoid called the *fundamental groupoid* of X. This construction gives us a monoidal functor,

$$\begin{array}{rccc} \pi: & \mathsf{Top} & \longrightarrow & \mathsf{CatG} \\ & X & \longrightarrow & \pi(X). \end{array} \tag{3.5}$$

Note that for any subset $A \subset X$ one can define a full subcategory of $\pi(X)$ whose objects are points of A; it is denoted by $\pi(X, A)$.

The set $\pi(X)(x_0, x_0) =: \pi_1(X, x_0)$ is a group called the *fundamental group of X base at x_0*. If Top_* stands for the category of *based topological spaces* then π_1 gives us a monoidal functor

$$\begin{array}{rccc} \pi_1: & \mathsf{Top}_* & \longrightarrow & \mathsf{Group} \\ & (X, x_0) & \longrightarrow & \pi_1(X, x_0) \end{array} \tag{3.6}$$

We shall use below both these functors to construct operads in the categories CatG and Group out of operads in the categories in Top and Top_* respectively.

(viii) Let G be a (discrete) group. For non-empty subsets $A, B \subset G$, let

$$[A,B] = \left\{[a,b] := aba^{-1}b^{-1} \mid a \in A,\ b \in B\right\}$$

stand for the subgroup of G generated by all commutators of elements of A with elements of B. Define inductively the *descending central series* of G as the following series of normal subgroups,

$$G_1 := G \ \supseteq\ G_2 = [G,G] \ \supseteq\ G_3 = [G_2,G] \ \supseteq \ldots \supseteq\ G_n = [G_{n-1},G] \supseteq\ G_{n+1} = [G_n,G] \ \supseteq \ldots$$

and let

$$\mathfrak{gr}_n(G) := G_n/G_{n+1}$$

be the associated n-th quotient. This is an Abelian group, i.e. a $\mathbb{Z}$-module, so that it makes sense to define the following positively graded vector space over a field $\mathbb{K}$,

$$\mathfrak{gr}_{\mathbb{K}}(G) := \bigoplus_{n=1}^{\infty} \mathfrak{gr}_n(G) \otimes_{\mathbb{Z}} \mathbb{K},$$

The commutator map

$$\begin{array}{rccc} [\ ,\ ]: & G \times G & \longrightarrow & G \\ & (a,b) & \longrightarrow & aba^{-1}b^{-1}, \end{array}$$

induces on $\mathfrak{gr}(G)_{\mathbb{K}}$ a linear skew-symmetric map,

$$[\ ,\ ] : \wedge^2 \mathfrak{gr}_{\mathbb{K}}(G) \longrightarrow \mathfrak{gr}_{\mathbb{K}}(G)$$

which satisfies the Jacobi identity. Hence this construction (which is natural in G) gives us a functor

$$\begin{array}{rccc} \widehat{\mathfrak{gr}}_{\mathbb{K}}: & \mathsf{Group} & \longrightarrow & \mathsf{Lie}_{\mathbb{K}} \\ & G & \longrightarrow & \widehat{\mathfrak{gr}}_{\mathbb{K}}(G) \end{array}$$

which is obviously symmetric monoidal. Here $\widehat{\mathfrak{gr}}_{\mathbb{K}}(G)$ is the completion, $\prod_{n=1}^{\infty} \mathfrak{gr}_n(G) \otimes_{\mathbb{Z}} \mathbb{K}$, of the filtered Lie algebra $\mathfrak{gr}_{\mathbb{K}}(G)$. It is called the (completed) *graded Lie algebra of G over $\mathbb{K}$*.

3.4.1. **A useful fact.** Let $f : G \to H$ be a morphism of groups, and assume that G is *residually nilpotent*, that is, its descending central series satisfies the condition

$$\bigcap_{i=1}^{\infty} G_n = 1.$$

If the associated morphism of graded Lie algebras,

$$\mathfrak{gr}^{\mathbb{Q}}(f) : \mathfrak{gr}^{\mathbb{Q}}(G) \longrightarrow \mathfrak{gr}^{\mathbb{Q}}(H)$$

is a monomorphism, then f is a monomorphism [**CW**].

3.5. Closed symmetric monoidal categories. A symmetric monoidal category C is called *closed* if for any object $A \in Ob(\mathsf{C})$ the functor

$$\begin{array}{rccc} \cdot \otimes_{\mathsf{C}} A & \mathsf{C} & \longrightarrow & \mathsf{C} \\ & B & \longrightarrow & B \otimes_{\mathsf{C}} A \end{array}$$

has a *right adjoint* functor,

$$\begin{array}{rccc} \mathrm{Hom}_{\mathsf{C}}(A,\ \cdot\ ) & \mathsf{C} & \longrightarrow & \mathsf{C} \\ & B & \longrightarrow & \mathrm{Hom}_{\mathsf{C}}(A,B) \end{array}$$

which, by definition, is a functor satisfying the condition,

$$\mathrm{Mor}_{\mathsf{C}}(B \otimes_{\mathsf{C}} A, C) \cong \mathrm{Mor}_{\mathsf{C}}(B, \mathrm{Hom}_{\mathsf{C}}(A,C)),$$

for any $A, B, C \in Ob(\mathsf{C})$. One can show using Yoneda lemma that this condition implies

$$\mathrm{Hom}_{\mathsf{C}}(B \otimes_{\mathsf{C}} A, C) \cong \mathrm{Hom}_{\mathsf{C}}(B, \mathrm{Hom}_{\mathsf{C}}(A,C)). \tag{3.7}$$

The object $\mathrm{Hom}_{\mathsf{C}}(A,B) \in Ob(\mathsf{C})$ is called the *internal hom* of A and B.

The categories Cat, Set and the category of finite-dimensional vector spaces are closed with $\mathrm{Hom}_{\mathsf{C}}(A,B) = \mathrm{Mor}_{\mathsf{C}}(A,B)$.

4. Operads

We shall mostly use in the subsequent sections the notion of operad at a rather elementary level — at the level of its basic properties which follow from its definition almost immediately. However, the definition itself (which is due to P. May [**May**]) is not, perhaps, very elementary, and we recommend a newcomer to pay attention not only to the formal definition(s) of operad given below, but also to examples illustrating that definition. Comprehensive expositions of the theory of operads can be found in the books [**LV, MSS**].

From now on S stands for the groupoid of finite sets and their bijections. This is a subcategory of Set.

4.1. Basic axioms for operadic compositions. An *operad* $\mathcal{O}$ in a symmetric monoidal category C is a collection of the following data [**T2**]:

1) a functor $\mathcal{O} : \mathsf{S} \to \mathsf{C}$ (which is often called an S-*module*);

2) for any finite sets I, J and an element $i \in I$ a morphism

$$\circ_i^{I,J} : \mathcal{O}(I) \otimes_{\mathsf{C}} \mathcal{O}(J) \longrightarrow \mathcal{O}(I \setminus \{i\} \sqcup J), \tag{4.1}$$

natural in i, I, J ("the insertion of $\mathcal{O}(J)$ into the i-th slot of $\mathcal{O}(I)$"), such that the following associativity conditions hold:

(a) for any finite sets I, J, K and any $i_1, i_2 \in I$, $i_1 \neq i_2$, the diagram

$$\begin{array}{ccccc}
\mathcal{O}(I) \otimes_{\mathsf{C}} \mathcal{O}(J) \otimes_{\mathsf{C}} \mathcal{O}(K) & \xrightarrow{\circ_{i_1}^{I,J} \otimes Id} & \mathcal{O}(I \setminus \{i_1\} \sqcup J) \otimes_{\mathsf{C}} \mathcal{O}(K) & \xrightarrow{\circ_{i_2}^{I \setminus \{i_1\} \sqcup J, K}} & \mathcal{O}(I \setminus \{i_1, i_2\} \sqcup J \sqcup K) \\
\downarrow{\scriptstyle Id \otimes \beta_{\mathcal{O}(J)\mathcal{O}(K)}} & & & & \| \\
\mathcal{O}(I) \otimes_{\mathsf{C}} \mathcal{O}(K) \otimes_{\mathsf{C}} \mathcal{O}(J) & \xrightarrow{\circ_{i_2}^{I,K} \otimes Id} & \mathcal{O}(I \setminus \{i_2\} \sqcup K) \otimes_{\mathsf{C}} \mathcal{O}(J) & \xrightarrow{\circ_{i_1}^{I \setminus \{i_2\} \sqcup K, J}} & \mathcal{O}(I \setminus \{i_1, i_2\} \sqcup K \sqcup J)
\end{array}$$

commutes;

(b) for any finite sets I, J, K and any $i \in I$ and $j \in J$ the diagram

$$\begin{array}{ccccc}
 & & \mathcal{O}(I) \otimes_{\mathsf{C}} \mathcal{O}(J \setminus \{j\} \sqcup K) & & \\
 & \nearrow{\scriptstyle Id \otimes \circ_j^{J,K}} & & \searrow{\scriptstyle \circ_i^{I, J \setminus \{j\} \sqcup K}} & \\
\mathcal{O}(I) \otimes_{\mathsf{C}} \mathcal{O}(J) \otimes_{\mathsf{C}} \mathcal{O}(K) & & & & \mathcal{O}(I \setminus \{i\} \sqcup J \setminus \{j\} \sqcup K) \\
 & \searrow{\scriptstyle \circ_i^{I,J} \otimes Id} & & \nearrow{\scriptstyle \circ_j^{I \setminus \{i\} \sqcup J, K}} & \\
 & & \mathcal{O}(I \setminus \{i\} \sqcup J) \otimes_{\mathsf{C}} \mathcal{O}(K) & &
\end{array}$$

commutes.

Sometimes we abbreviate compositions $\circ_i^{I,J}$ to $\circ_i$.

4.2. Axioms for a unit. An *operad with unit* is, by definition, an operad $\mathcal{O}$ equipped with a morphism,

$$e : \mathbb{1}_{\mathsf{C}} \longrightarrow \mathcal{O}(\bullet),$$

for any one-element set $\{\bullet\}$ which is natural in $\{\bullet\}$ and makes the following two compositions,

$$\mathcal{O}(I) \longrightarrow \mathcal{O}(I) \otimes_{\mathsf{C}} \mathbb{1}_{\mathsf{C}} \xrightarrow{Id \otimes e} \mathcal{O}(I) \otimes_{\mathsf{C}} \mathcal{O}(\bullet) \xrightarrow{\circ_i^{I,\bullet}} \mathcal{O}(I),$$

$$\mathcal{O}(I) \longrightarrow \mathbb{1}_{\mathsf{C}} \otimes_{\mathsf{C}} \mathcal{O}(I) \xrightarrow{e \otimes Id} \mathcal{O}(\bullet) \otimes_{\mathsf{C}} \mathcal{O}(I) \xrightarrow{\circ_\bullet^{\bullet,I}} \mathcal{O}(I),$$

into identity maps for any set I and any $i \in I$.

4.2.1. **Remark.** The skeleton of the groupoid S consists of the sets $[n]$, $n \geq 0$ (with $[0] := \emptyset$). If the category C admits small colimits (and all categories we work with in this paper do have this property), then a functor $\mathcal{O} : \mathsf{S} \to \mathsf{C}$ is equivalent to a collection of $\mathbb{S}_n$-modules, $\{\mathcal{O}(n) := \mathcal{O}([n]\}_{n\geq 0}$, which is often abbreviated to an $\mathbb{S}$-module in the category C. One can reformulate axioms of an operad in terms of the $\mathbb{S}$-module $\{\mathcal{O}(n) := \mathcal{O}([n]\}_{n\geq 0}$ and the operadic compositions,

$$\circ_i : \mathcal{O}(n) \otimes_{\mathsf{C}} \mathcal{O}(m) \longrightarrow \mathcal{O}([n] \setminus i \sqcup [m]) = \mathcal{O}(n+m-1), \quad \forall\, m, n \in \mathbb{N}, i \in [n],$$

where one first identifies

$$[n] \setminus i \sqcup [m] = \{1, 2, i-1, i+1, \ldots, n\} \sqcup \{1', 2', \ldots, m'\}$$

with the linearly ordered set,

$$\{1, 2, \ldots, i-1, 1', 2', \ldots, m', i+1, \ldots, n\},$$

and then identifies the latter with $[n+m-1] = \{1, 2, \ldots, i-1, i, i+1, \ldots, m+n-1\}$ as linearly ordered sets. In this way one recovers the original definition of an operad given in [**May**]. We shall often use this freedom below to describe an operad either as a functor $\mathcal{O} : \mathsf{S} \to \mathsf{C}$ or simply as an $\mathbb{S}$-module $\{\mathcal{O}(n) \in Ob(\mathsf{C})\}_{n\in\mathbb{N}}$.

4.2.2. **Non-$\mathbb{S}$ operads.** As noted just above an operad $\mathcal{O}$ in a monoidal category C can be defined as a collection of objects, $\mathcal{O} = \{\mathcal{O}(n) \in Ob(\mathsf{C})\}_{n\geq 0}$, such that each objects $\mathcal{O}(n)$ carries a representation of $\mathbb{S}_n$, and there are equivariant compositions,

$$\circ_i : \mathcal{O}(n) \otimes_{\mathsf{C}} \mathcal{O}(m) \longrightarrow \mathcal{O}(n+m-1) \quad \forall\, i \in [n],$$

which satisfy axioms (a) and (b) in §4.1.

If we forget in the definition above $\mathbb{S}_n$ actions on $\mathcal{O}(n)$, $n \geq 0$, and, correspondingly, omit the equivariance condition on the compositions $\circ_i$, then we get a notion of a *non-$\mathbb{S}$ operad*. More precisely, a *non-$\mathbb{S}$ operad* in a (semigroup) category C is an operad $\mathcal{O} = \{\mathcal{O}(n)\}$ in C such that the action of $\mathbb{S}_n$ on each object $\mathcal{O}(n)$ is trivial.

4.2.3. **Exercise.** Let $\mathcal{O}$ be an operad in a symmetric monoidal category C and $F : \mathsf{C} \to \mathsf{D}$ a symmetric monoidal functor to some other category. Show that the data

(i) the S-module structure, $F\mathcal{O} : \mathsf{S} \to \mathsf{D}$, given by the composition $\mathsf{S} \xrightarrow{\mathcal{O}} \mathsf{C} \xrightarrow{F} \mathsf{D}$, and
(ii) the operadic "insertions",

$$\bar{\circ}_i^{I,J} : F\mathcal{O}(I) \otimes_{\mathsf{C}} F\mathcal{O}(J) \longrightarrow F\mathcal{O}(I \setminus \{i\} \sqcup J),$$

given by the compositions

$$F\mathcal{O}(I) \otimes_{\mathsf{C}} F\mathcal{O}(J) \xrightarrow{\phi_{\mathcal{O}(I),\mathcal{O}(J)}} F(\mathcal{O}(I) \otimes_{\mathsf{C}} \mathcal{O}(J)) \xrightarrow{F(\circ_i^{I,J})} F(\mathcal{O}(I \setminus \{i\} \sqcup J)) = F\mathcal{O}(I \setminus \{i\} \sqcup J),$$

give us an operad $F\mathcal{O}$ in the symmetric monoidal category D. This fact is of an extreme importance in applications — starting with a "geometric" operad in the category, say, of topological spaces, and applying the chain or homology functor one arrives to an operad in the category of vector spaces. This particular property of operads is another manifestation of the *amazing unity of mathematics*.

4.2.4. **Exercise**. Introduce the notion of a *morphism*, $\rho : \mathcal{O}_1 \longrightarrow \mathcal{O}_2$, of operads in a symmetric monoidal category C.

4.2.5. **Exercise**. Introduce the notion of an *ideal* $\mathcal{I}$, of an operad $\mathcal{O}$ in the category $\mathsf{Vect}_{\mathbb{K}}$ and construct the *quotient* operad $\mathcal{O}/\mathcal{I}$.

4.3. Basic examples. Here we define a few operads which will be used in some constructions below.

4.3.1. **Endomorphism operad.** Let C be a closed symmetric monoidal category. An arbitrary object $A \in Ob(\mathsf{C})$ gives rise to an operad $\mathcal{E}nd_A$ with the $\mathbb{S}$-module,

$$\left\{\mathcal{E}nd_A(n) := \mathrm{Hom}_{\mathsf{C}}(A^{\otimes n}, A)\right\}$$

and with the compositions $\circ_i^{[n],[m]}$ given by the isomorphisms (use 3.7)),

$$\mathrm{Hom}_{\mathcal{C}}\left(A^{\otimes(i-1)} \otimes_{\mathsf{C}} \mathrm{Hom}_{\mathsf{C}}(A^{\otimes m}, A) \otimes_{\mathsf{C}} A^{\otimes(n-i+1)}, A\right) \cong \mathrm{Hom}_{\mathsf{C}}(A^{\otimes(n+m+1)}, A),$$

i.e. literally by the substitution of the object $\mathrm{Hom}_{\mathsf{C}}(A^{\otimes m}, A)$ into the i-th input slot of $\mathrm{Hom}_{\mathsf{C}}(A^{\otimes n}, A)$.

Let $\mathcal{O}$ be an operad in C. An *$\mathcal{O}$-algebra* structure on an object A is, by definition, a morphism of operads,

$$\rho : \mathcal{O} \longrightarrow \mathcal{E}nd_A,$$

which is also often called a *representation* of $\mathcal{O}$ in A.

4.3.2. **Operad of parenthesized permutations.** Consider an S-module in the category Set,

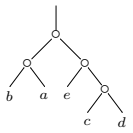
$$\begin{array}{rccl} \mathcal{P}a: & \mathsf{S} & \longrightarrow & \mathsf{Set} \\ & I & \longrightarrow & \mathcal{P}a(I) := \text{the set of all parenthesized permutations of } I \end{array}$$

One can equivalently define $\mathcal{P}a(I)$ as

- the set of all monomials built from elements of I using a non-commutative and non-associative product (using each element of I once),
- the set of all planar binary trees whose legs are labelled by elements of I (using each element once).

For example, for $I = \{a, b, c, d, e\}$,

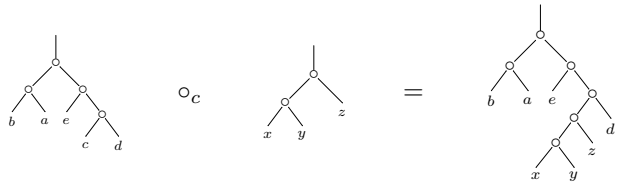
$$((ba)(e(cd)) \simeq$$



is an element of $\mathcal{P}a(I)$. The operadic compositions $\circ_i^{I,J} = \circ_i$ are given by the substitution of a monomial from $\mathcal{P}a(J)$ into the i-th letter of a monomial $\mathcal{P}a(I)$, e.g. for $I = \{a, b, c, d, e\}$ and $J = \{x, y, z\}$,

$$((ba)(e(cd)) \ \circ_c \ ((xy)z) = ((ba)(e(((xy)z)d)) \in \mathcal{P}a\left(\{I \setminus c\} \sqcup J\right)$$

or simply by grafting the root (which in our pictorial notation grows upward) of a planar binary tree from $\mathcal{P}a(J)$ into the i-labelled input leg of a tree from $\mathcal{P}a(I)$,



4.3.3. **Exercises**. (i) Show that there is a one-to-one correspondence between $\mathrm{Mor}_{\mathsf{Set}}(X \times X, X)$ and $\mathcal{P}a$-algebra structures on $X \in Ob(\mathsf{Set})$, that is, representations, $\rho : \mathcal{P}a \to \mathcal{E}nd_X$, of the operad $\mathcal{P}a$ in a set X.

(ii) Describe representations of the operad $Lin_{\mathbb{K}}(\mathcal{P}a)$ in a vector space V, where $Lin_{\mathbb{K}}$ is a monoidal functor $\mathsf{Set} \to \mathsf{Vect}_{\mathbb{K}}$ defined in §3.4(i).

(iii) Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be operads in a symmetric monoidal category C. Show that the S-module defined by,

$$\begin{array}{rccc} \mathcal{O}_1 \otimes_{\mathsf{C}} \mathcal{O}_2: & \mathsf{S} & \longrightarrow & \mathsf{C} \\ & I & \longrightarrow & \mathcal{O}_1(I) \otimes_{\mathsf{C}} \mathcal{O}_2(I), \end{array}$$

has an induced operadic structure. The resulting operad $\mathcal{O}_1 \otimes_{\mathsf{C}} \mathcal{O}_2$ is called the *tensor product* of operads $\mathcal{O}_1$ and $\mathcal{O}_2$.

4.3.4. **Graphs and rooted trees.** A *graph with legs (or hairs)* is a triple $\Gamma = (H(\Gamma), \sqcup, \tau)$, where

- $H(\Gamma)$ is a finite set whose elements are called *half-edges* (or *flags*),
- $\sqcup$ is a partition of $H(\Gamma)$ into a disjoint union of subsets,

$$H(\Gamma) = \coprod_{v \in V(\Gamma)} H(v),$$

parameterized by a set $V(\Gamma)$ which is called the *set of vertices of* Γ; the subset $H(v) \subset H(\Gamma)$ is called the *set of half-edges attached to the vertex* v; its cardinality, $\#H(v)$, is called the *valency* of v,

- $\tau : H(\Gamma) \to H(\Gamma)$ is an involution, that is, a map satisfying the condition $\tau^2 = Id$. This map defines a new partition of $H(\Gamma)$ into orbits $(h, \tau(h)) \subset H(\Gamma)$ which in general can have cardinality two ($h \neq \tau(h)$) or one ($h = \tau(h)$). Orbits of cardinality 2 are called *internal edges* or simply *edges* of the graph Γ; the set of (internal) edges is denoted by $E(\Gamma)$. Orbits of cardinality 1 are called *legs* (or *hairs*); the set of legs is denoted by $L(G)$.

Note that $L(\Gamma) = \emptyset$ if and only the involution τ has no fixed points. In this case $\Gamma = (H(\Gamma, \sqcup, \tau)$ is called simply a *graph*. We shall study operads of such graphs in §7 below.

It is convenient to think of a graph (with or without legs) Γ in terms of its *geometric realization* which is a topological space constructed as follows: (i) for each vertex v take $\#H(v)$ copies of the interval $[0, 1]$ labelled by elements of $H(v)$ and glue these intervals at the end-point 0, the result is a topological space (equipped with the quotient topology from $[0, 1]^{\#H(v)}$) which is called the *corolla of* v; (ii) then consider a union of all corollas and, for each internal edge $(h, \tau(h))$, identify the end-points 1 of the intervals $[0, 1]$ labelled by h and $\tau(h)$.

An *isomorphism* of graphs with legs $i : \Gamma_1 \to \Gamma_2$ is a bijection $i : H(\Gamma_1) \to H(\Gamma_2)$ which preserves partitions $\sqcup_1$ and $\sqcup_2$ (and hence induces a bijection of the sets of vertices $V(\Gamma_1) \to V(\Gamma_2)$) and commutes with involutions, $i \circ \tau_1 = \tau_2 \circ i$ (and hence induces bijections $E(\Gamma_1) \to E(\Gamma_2)$ and $L(\Gamma_1) \to L(\Gamma_2)$). The associated groupoid of graphs and their isomorphisms is denoted by $\mathsf{IsoGraph}$.

A graph (with legs) is called connected (resp., simply connected) if its geometric realization is a connected (resp., simply connected) topological space. A connected and simply connected graph is called a *tree*. A *rooted tree* is a tree T with $L(T) \neq \emptyset$ and with one one of the legs, say $r \in L(T)$, marked and called *the root*. Elements in $L(T) \setminus r$ are called *input legs* or *leaves* of the rooted tree T; the set of leaves is denoted by $Leaf(T)$. Each vertex on a geometric realization of a rooted tree is connected to a root by a unique path; this defines a flow on the tree and hence a partition,

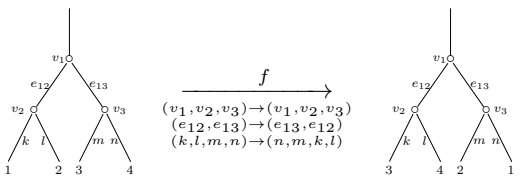
$$H(v) = In(v) \sqcup r_v.$$

of the set of half-edges at each vertex v into a disjoint union of a unique element r_v which lies on the path from v to the root and the subset of the remaining half-edges, $In(v)$, which are called *input (half) edges*. By abuse of language, the cardinality of $In(v)$ is called the *valency* of a vertex v of a *rooted* tree T (which is equal to the valency of v in T viewed as an unrooted tree minus one). The corolla of a vertex in a rooted tree can be visualized as follows (with flow directed from bottom to the top)

Let T_1 and T_2 be rooted trees. An isomorphism of graphs, $i : T_1 \to T_2$, which sends the root of T_1 into the root of T_2 is called an *isomorphism of rooted trees*. The associated groupoid of rooted trees and their isomorphisms is denoted by $\mathsf{IsoTree}$.

4.3.5. **Free operads.** Let $\mathsf{IsoTree}(I)$ be a category (in fact, a groupid) whose objects are rooted trees T equipped with an isomorphism $l_T : Leaf(T) \to I$ ("labeling of input legs with elements of I") and morphisms, $f : T_1 \to T_2$, are isomorphisms of rooted trees which respect the labelings, i.e. satisfy the condition $l_{T_1} = l_{T_2} \circ f$. An example of such a morphism is given by the following picture

(4.2)



where $V(T) = \{v_1, v_2, v_3\}$, $E(T) = \{e_{12}, e_{21}\}$ and $Leaf(T) = \{k, l, m, n\}$.

Given any $\mathbb{S}$-module $\mathcal{E} : \mathsf{S} \to \mathsf{C}$ in a symmetric monoidal category C (with small colimits) there is an associated *free* operad $\mathcal{F}ree\langle\mathcal{E}\rangle$ which has an obvious universal property. For simplicity, we shall explain the construction of $\mathcal{F}ree\langle\mathcal{E}\rangle$ in the case $\mathsf{C} = \mathsf{Vect}_{\mathbb{K}}$, the general case being essentially the same.

Given an $\mathbb{S}$-module, $\mathcal{E} : \mathsf{S} \to \mathsf{Vect}_{\mathbb{K}}$, to any I-labelled rooted tree $T \in Ob(\mathsf{IsoTree}(I))$ with, say, n vertices, one can associate a vector space ("unordered tensor product over the set of vertices of T"),

$$T\langle\mathcal{E}\rangle = \bigotimes_{v\in V(T)} \mathcal{E}(In_v) := \left(\bigoplus_{p:[n]\to V(T)} \mathcal{E}\left(In_{p(1)}\right) \otimes \ldots \otimes \mathcal{E}\left(In_{p(n)}\right) \right)_{\mathbb{S}_n}$$

(in a general symmetric monoidal category one has to take the equalizer over the set of isomorphisms $[n] \to V(T)$). It is not hard to see directly from this definition that the association $T \to T\langle\mathcal{E}\rangle$ defines a functor from the category $\mathsf{IsoTree}(I)$ to the category Vect_K, and hence it makes sense to consider its colimit

$$\operatorname*{colim}_{T\in\mathsf{IsoTree}(I)} T\langle\mathcal{E}\rangle,$$

which exists in Vect_K. As a vector space this colimit is (non-canonically) isomorphic to the direct sum,

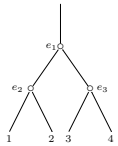
$$\operatorname*{colim}_{T\in\mathsf{IsoTree}(I)} T\langle\mathcal{E}\rangle \simeq \bigoplus_{T\in Ob(\mathsf{IsoTree}(I))/\sim} T\langle\mathcal{E}\rangle$$

where the summation runs over representatives T of isomorphism classes of I-labelled rooted trees from the category $\mathsf{IsoTree}(I)$. We use this colimit to define an S-module,

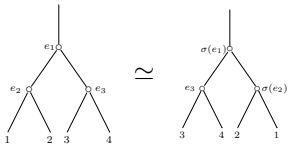
$$\begin{array}{rccc} \mathcal{F}ree\langle\mathcal{E}\rangle : & \mathsf{S} & \longrightarrow & \mathsf{Vect}_{\mathbb{K}} \\ & I & \longrightarrow & \mathcal{F}ree\langle\mathcal{E}\rangle(I) := \operatorname*{colim}_{T\in\mathsf{IsoTree}(I)} T\langle\mathcal{E}\rangle, \end{array}$$

and notice that it has a natural structure of the operad with respect to graftings of trees and unordered tensor products. This operad is called the *free operad generated by the* S*-module* E.

Elements of $\mathcal{F}ree\langle\mathcal{E}\rangle(I)$ can be visualized as linear combinations of $\mathcal{E}$*-decorated* I*-labelled rooted trees*. One can represent pictorially such a decorated tree as follows,

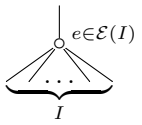


where e_i are vectors from $\mathcal{E}(In_{v_i})$, $i = 1, 2, 3$. Such a representation is not unique; for example, isomorphism (4.2) leads to the identification



where σ is the unique non-trivial automorphism of a two element set I_0 (say $I_0 = \{e_{12}, e_{13}\}$, $\{k, l\}$ or $\{m, n\}$, see (4.2)), and $\sigma(e_i)$ is the result of an action of this automorphism on the corresponding elements of $\mathcal{E}(I_0)$.

Every element in $\mathcal{F}ree\langle\mathcal{E}\rangle(I)$ is an iterated operadic composition of (isomorphism classes of) the following basic decorated graphs,



called the *generating corollas*.

4.3.6. **Exercises.** (i) Show that the operad of parenthesized permutations, $\mathcal{P}a$, is the free operad in the category Set generated by the following S-module,

$$\mathcal{E}(I) := \begin{cases} Bij(I \to [\#I]) & \text{if } \#I = 2 \\ 0 & \text{otherwise.} \end{cases}$$

where $Bij(I \to [\#I])$ is the set of all bijections $I \to [\#I]$.

(ii) Show that the operad $Lin_{\mathbb{K}}(\mathcal{P}a)$ (see § 4.3.3ii) is the free operad in the category Vect_K generated by the following $\mathbb{S}$-module,

$$\mathcal{E}(n) := \begin{cases} \mathbb{K}[\mathbb{S}_2] & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

4.3.7. **Operad of little disks.** Let

$$D_{a,\lambda} := \{z \in \mathbb{C} \mid |z - a| \leq \lambda\}$$

be the closed disk in $\mathbb{C}$ of radius λ and with center $a \in \mathbb{C}$. The *operad of little disks*, $\mathcal{D} = \{\mathcal{D}(n)\}_{n\geq 1}$, is an operad in the category, Top_{pc}, of path-connected topological spaces given by the following data,

- $\mathcal{D}(1)$ is the point (the unit in $\mathcal{D}$);
- $\mathcal{D}(n)$ for $n \geq 2$ is the space of configurations of n disjoint closed disks, $\{D_{a_i,\lambda_i}\}_{i\in[n]}$, lying inside the standard unital disk, $D_{0,1}$.
- the operadic composition, for any $k \in [n]$, is given by

$$\begin{array}{rccc} \circ_k : & \mathcal{D}(n) \times \mathcal{D}(m) & \longrightarrow & \mathcal{D}(n+m-1) \\ & \{D_{a_i,\lambda_i}\}_{i\in[n]} \times \{D_{b_j,\nu_i}\}_{i\in[m]} & \longrightarrow & \left\{\{D_{a_i,\lambda_i}\}_{i\in[n]\setminus k} \coprod \{D_{\lambda_k b_j + a_k, \lambda_k \nu_j}\}_{j\in[m]}\right\}. \end{array}$$

  Put another way, we replace the closed k-th disk D_{a_k,λ_k} in a configuration from $\mathcal{D}(n)$ with the λ_k-rescaled and a_k-translated configuration from $\mathcal{D}(m)$. [5]

The homology monoidal functor (see §3.4vi) sends the topological operad $\mathcal{D}$ into an operad, $H_\bullet(\mathcal{D})$, in the category $\mathsf{dgVect}_{\mathbb{K}}$. It was proven by F. Cohen that $H_\bullet(\mathcal{D})$ is precisely the operad of *Gerstenhaber algebras*, i.e. its representations $H_\bullet(\mathcal{D}) \to \mathcal{E}nd_V$ in a graded vector space V are equivalent to a *Gerstenhaber algebra* [6] structure on V.

The *fundamental groupoid* monoidal functor $\pi : \mathsf{Top}_* \to \mathsf{CatG}$ (see §3.4vii) sends $\mathcal{D}$ into an operad $\pi(\mathcal{D})$ in the category of groupoids.

4.4. Degree shift functor. Let $\mathcal{O}$ be an operad in the category $\mathsf{dgVect}_{\mathbb{K}}$. For any integer m one can uniquely associate to $\mathcal{O}$ an operad $\mathcal{O}\{m\}$ with the property that there is a 1-1 correspondence,

$$\left\{ \begin{array}{c} \text{reprersentations of } \mathcal{O} \\ \text{in a graded vector space } V \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{reprersentations of } \mathcal{O}\{m\} \\ \text{in a graded vector space } V[m] \end{array} \right\}.$$

From this property one easily reads the structure of $\mathcal{O}\{m\} = \{\mathcal{O}\{m\}(n)\}_{n\geq 1}$ as an $\mathbb{S}$-module,

$$\mathcal{O}\{m\}(n) = \mathcal{O}(n)[m(1-n)] \otimes sgn_n^{\otimes m}.$$

where sgn_n is the one-dimensional sign representation $\mathbb{S}_n$.

Put another way, $\mathcal{O}\{m\}$ is the tensor product of the operad $\mathcal{O}$ and the endomorphism operad $\mathcal{E}nd_{\mathbb{K}[-m]}$ of the 1-dimensional vector space concentrated in degree m.

4.5. Cosimplicial structures on operads. We shall often use below the following observation.

4.5.1. **Lemma.** *Let $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 1}$ be an operad in a symmetric monoidal category C. Assume there is an element $e \in \mathcal{O}(2)$ satisfying the condition*

$$e \circ_1 e = e \circ_2 e. \tag{4.3}$$

Then the family of maps,

$$\begin{array}{rccc} d_i : & \mathcal{O}(n) & \longrightarrow & \mathcal{O}(n+1) \\ & f & \longrightarrow & d_i f := \begin{cases} e \circ_2 f & \text{for } i = 0 \\ f \circ_i e & \text{for } i \in [n] \\ e \circ_1 f & \text{for } i = n+1 \end{cases} \end{array} \tag{4.4}$$

satisfies the equations

$$d_j d_i = d_i d_{j-1} \quad \text{for any} \quad i < j, \tag{4.5}$$

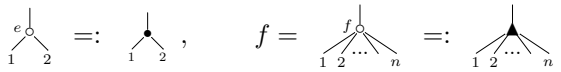
and hence makes the collection $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 1}$ into a cosimiplicial object in the category C.

[5] A pictorial description of this composition can be found at http://en.wikipedia.org/wiki/Operad_theory

[6] See http://en.wikipedia.org/wiki/Gerstenhaber_algebra for the definition of this notion.

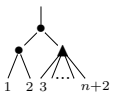
PROOF. Using the second definition of operads in terms of decorated trees, we shall identify elements $e \in \mathcal{O}(2)$ and $f \in \mathcal{O}(n)$ with decorated corollas,

$$e = \quad =: \quad , \qquad f = \quad =:$$



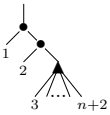
Then

$$d_1 d_0 f = d_1 \left(\right) =$$



On the other hand,

$$d_0 d_0 f = d_0 \left(\right) =$$



The condition (4.3) reads $\quad = \quad$ so that we get $d_1 d_0 f = d_0 d_0 f$ for any $f \in \mathcal{O}(n)$. Similar pictures establish identities (4.5) for all other values of $i \in [n+1]$ and $j \in [n+2]$ with $i < j$. We leave it to the reader to draw the details. □

If $\mathcal{O}$ is an operad in the category $\mathsf{Vect}_{\mathbb{K}}$, then under the conditions of the above Lemma one can associate to $\mathcal{O}$ a dg vector space,

$$\mathsf{Simp}^\bullet(\mathcal{O}) := \bigoplus_{n \geq 1} \mathcal{O}(n)[-n],$$

equipped with the differential

$$\begin{array}{ccccc} d & \mathcal{O}(n) & \longrightarrow & \mathcal{O}(n+1) \\ & f & \longrightarrow & df := \sum_{i=0}^{n+1} (-1)^i d_i f, \end{array}$$

whose cohomology is denoted by $H\mathsf{Simp}^\bullet(\mathcal{O})$ and is called the *cosimplicial* cohomology of $\mathcal{O}$. We shall see below that, for example, the cosimplicial cohomology of the operad of infinitesimal braids has much to do with the main topic of these lectures, the Grothendieck-Teichmüller group.

4.6. Lie algebras associated with operads. Let $\mathcal{O} = \{\mathcal{O}(n)\}_{n \geq 1}$ be an operad in the category $\mathsf{dgVect}_{\mathbb{K}}$ with operadic compositions $\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(m+n-1)$, $1 \leq i \leq n$. Consider a vector space

$$\mathcal{O}^{tot} := \bigoplus_{n \geq 1} \mathcal{O}(n)$$

Then the map

$$\begin{array}{ccccc} [\ ,\ ]: & \mathcal{O}^{tot} \otimes \mathcal{O}^{tot} & \longrightarrow & \mathcal{O}^{tot} \\ & (a \in \mathcal{O}(n), b \in \mathcal{O}(m)) & \longrightarrow & [a,b] := \sum_{i=1}^{n} a \circ_i b - (-1)^{|a||b|} \sum_{i=1}^{m} b \circ_i a \end{array}$$

makes $\mathcal{O}^{tot}$ into a dg Lie algebra [**KM**]. Moreover, the bracket descends to the space of coinvariants $(\mathcal{O}^{tot})_{\mathbb{S}} := \bigoplus_{n \geq 1} \mathcal{O}(n)_{\mathbb{S}_n}$ making this vector spaces into a dg Lie algebra as well. If $char(\mathbb{K}) = 0$, the space of coinvariants $(\mathcal{O}^{tot})_{\mathbb{S}}$ is isomorphic via the symmetrization map to the space of $\mathbb{S}$-invariants, $(\mathcal{O}^{tot})^{\mathbb{S}} := \bigoplus_{n \geq 1} \mathcal{O}(n)^{\mathbb{S}_n}$, so that one has an induced dg Lie algebra structure on $(\mathcal{O}^{tot})^{\mathbb{S}}$ with Lie bracket given by the above formula followed with the symmetrization.

5. Operad of parenthesized braids and $\widehat{GT}$

5.1. Geometric definition of the braid group. Let $C_n(\mathbb{C})$ be the configuration space of pairwise distinct points in the plane $\mathbb{C}$,

$$C_n(\mathbb{C}) = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C} \mid z_i \neq z_j \text{ for } i \neq j\}.$$

The group $\mathbb{S}_n$ acts on $C_n(\mathbb{C})$ by permuting the points,

$$\sigma : (z_1, z_2, \ldots, z_n) \longrightarrow \left(z_{\sigma(1)}, z_{\sigma(2)}, \ldots, z_{\sigma(n)}\right), \quad \sigma \in \mathbb{S}_n.$$

Let $C_n(\mathbb{C})/\mathbb{S}_n$ stand for the associated set of orbits equipped with the quotient topology. Let $p_n = (z_1^0, \ldots, z_n^0)$ be any point in $C_n(\mathbb{C})$ (say $p_n = \{(1, 2, \ldots, n) \in \mathbb{C}\}$) and let $\bar{p}_n$ be its image under the projection $C_n(\mathbb{C}) \to C_n(\mathbb{C})/\mathbb{S}_n$.

5.1.1. **Definitions.** (i) The fundamental group $\mathbb{B}_n := \pi_1(C_n(\mathbb{C})/\mathbb{S}_n, \bar{p}_n)$ is called the *group of braids*.
(ii) The fundamental group $\mathbb{PB}_n := \pi_1(C_n(\mathbb{C}), p_n))$ is called the *group of pure braids*.

5.1.2. **Remark.** It is often useful not to distinguish isomorphic groups. As topological spaces $C_n(\mathbb{C})$ and $C_n(\mathbb{C})/\mathbb{S}_n$ are path-connected, the groups $\pi_1(C_n(\mathbb{C}), p_n)$ and $\pi_1(C_n(\mathbb{C})/\mathbb{S}_n, \bar{p}_n)$ are isomorphic to each for different choices of base points. Therefore one should view $\mathbb{B}_n$ and $\mathbb{PB}_n$ as isomorphism classes, $\pi_1(C_n(\mathbb{C})/\mathbb{S}_n)$ and, respectively, $\pi_1(C_n(\mathbb{C}))$, of the fundamental groups $\pi_1(C_n(\mathbb{C})/\mathbb{S}_n, \bar{p}_n)$ and, respectively, $\pi_1(C_n(\mathbb{C}), p_n)$; this point of view makes the choice of the base points irrelevant.

The projection $C_n(\mathbb{C}) \to C_n(\mathbb{C})/\mathbb{S}_n$ is a regular $n!$-sheeted covering with $\mathbb{S}_n$ acting as covering transformations. Thus the subgroup $\mathbb{PB}_n \subset \mathbb{B}_n$ has index $n!$, and we have a short exact sequence of groups,

$$1 \longrightarrow \mathbb{PB}_n \longrightarrow \mathbb{B}_n \stackrel{p}{\longrightarrow} \mathbb{S}_n \longrightarrow 1.$$

Let b be an element in $\mathbb{B}_n$ and $\sigma_b := p(b) \in \mathbb{S}_n$ the associated permutation. One can visualize an element $b \in \mathbb{B}_n$ as (an isotopy classes of) n disjoint continuous curves (strands),

$$s_i : [0, 1] \longrightarrow \mathbb{C} \times [0, 1], \quad i = 1, 2, \ldots, n,$$

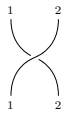
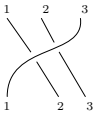
such that

$$s_i(0) = z_i^0, \quad s_i(1) = z_{\sigma_b(i)}^0,$$

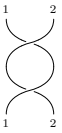
and the composition

$$[0, 1] \stackrel{s_i}{\longrightarrow} \mathbb{C} \times [0, 1] \stackrel{proj}{\longrightarrow} [0, 1]$$

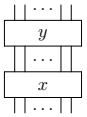
is a continuous monotonous function (with flow along the strands running up). For example,

$b_1 =$  and $b' =$ 

represent braids $b_1 \in \mathbb{B}_2$, $b' \in \mathbb{B}_3$ with $\sigma_{b_1} = (12)$ and $\sigma_{b'} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, while

$b'' =$ 

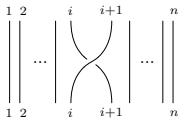
represents a pure braid from $\mathbb{PB}_2$. Multiplication of braids is represented by the concatenation of strands,

$xy =$  $\forall x, y \in \mathbb{B}_n$

For example, $b'' = (b_1)^2$.

5.1.3. **Remark.** $\mathbb{B}_2$ is the free group on one generator b_1, i.e. there is an isomorphism of groups $\mathbb{Z} \to \mathbb{B}_2$ which sends n into b_1^n. Similarly, $\mathbb{PB}_2$ is the free group on one generator b_1^2, i.e. its every element is of the form b_1^{2n} for some uniquely defined $n \in \mathbb{Z}$. If $\mathbb{K}$ is a field of characteristic zero, then the prounipotent completions, $\widehat{\mathbb{B}}_2(\mathbb{K})$ and $\widehat{\mathbb{PB}}_2(\mathbb{K})$, of these groups coincide with $\{b_1^{2\mu} \mid \mu \in \mathbb{K}\} \simeq \mathbb{K} = \widehat{F}_1(\mathbb{K})$, the prounipotent completion of the free group in one generator.

5.2. Algebraic definition of the (pure) braid group. According to M. Artin, the braid group $\mathbb{B}_n$ can be identified with a group of finite type with generators,

$b_i :=$ 

and (so called *braid*) relations,

$$\begin{aligned} b_i b_j &= b_j b_i \quad \text{for } |i-j|>1, \\ b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1}. \end{aligned}$$

Hence the prounipotent completion $\widehat{\mathbb{B}}_n(\mathbb{K})$ of $\mathbb{B}_n$ over a field $\mathbb{K}$ can be identified (see §2.1.1i) with $\exp \widehat{\mathfrak{b}}$, where $\widehat{\mathfrak{b}}$ is the quotient of the completed free Lie algebra $\widehat{\mathfrak{lie}}_n$ on letters $\{\gamma_i\}_{1\le i\le n}$ with respect to the Lie ideal generated by the following formal power series of Lie words,

$$\begin{aligned} \log(e^{\gamma_i} e^{\gamma_j} e^{-\gamma_i} e^{-\gamma_j}) &= 0 \quad \text{for } |i-j|>1, \\ \log(e^{\gamma_i} e^{\gamma_{i+1}} e^{\gamma_i} e^{-\gamma_{i+1}} e^{-\gamma_i} e^{-\gamma_{i+1}}) &= 0. \end{aligned}$$

Note that the permutation group $\mathbb{S}_n$ is a group of finite type with generators $\sigma_i = (i(i+1))$ and relations

$$\begin{aligned} \sigma_i^2 &= \mathbb{1} \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{for } |i-j|>1, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}. \end{aligned}$$

Hence the projection $p : \mathbb{B}_n \to \mathbb{S}_n$ is given on generators by $b_i \to \sigma_i$.

According to W. Burau and A. A. Markov, the pure braid group $\mathbb{PB}_n$ can be identified with a group of finite type with generators, $\{x_{ij}\}_{1\le i<j\le n}$, where

$$x_{ij} = (b_{j-1} b_{j-2} \cdots b_{i+1}) b_i^2 (b_{j-1} b_{j-2} \cdots b_{i+1})^{-1},$$

and relations,

$$\begin{aligned} x_{ij} x_{kl} &= x_{kl} x_{ij} \quad \text{for} \quad i<j<k<l \ \text{ and } \ i<k<l<j, \\ x_{ij} x_{ik} x_{jk} &= x_{ik} x_{jk} x_{ij} \quad \text{for } \ i<j<k \\ x_{ik} x_{jk} x_{ij} &= x_{jk} x_{ij} x_{ik} \quad \text{for } \ i<j<k \\ x_{ik} x_{jk} x_{jl} x_{jk}^{-1} &= x_{jk} x_{jl} x_{jk}^{-1} x_{ik} \quad \text{for } \ i<j<k<l. \end{aligned}$$

Hence its prounipotent completion $\widehat{\mathbb{PB}}_n$ can be described via the construction in §2.1.1(i) (see Exercise 6.1.4 below).

5.2.1. **Exercise.** (i) Show that $(b_1 b_2)^3 = (b_2 b_1)^3$.

(ii) Show that the element $(b_1 b_2)^3$ lies in $\mathbb{PB}_3 \subset \mathbb{B}_3$ and is central both in $\mathbb{PB}_3$ and in $\mathbb{B}_3$.

(iii) Show that

$$b_2 b_1^2 b_2^{-1} = (b_1 b_2)^3 b_1^{-2} b_2^{-2}, \quad (b_2 b_1) b_2^2 (b_2 b_1)^{-1} = b_1^2, \quad (b_1 b_2 b_1)^2 = (b_2 b_1 b_2)^2 = (b_1 b_2)^3$$

in $\mathbb{B}_3$.

5.3. Pure braids and semidirect products of free groups. For $n \ge 1$, the "forgetful" map

$$\begin{array}{rccc} f: & C_{n+1}(\mathbb{C}) & \longrightarrow & C_n(\mathbb{C}) \\ & (z_1, \dots, z_{n+1}) & \longrightarrow & (z_1, \dots, z_n) \end{array}$$

is a locally trivial fiber bundle [**FV**]. Its fiber is isomorphic to $\mathbb{C} \setminus \{z_1, \dots, z_n\}$ whose first homotopy group is the free group F_n on n generators while the second homotopy group is trivial, $\pi_2(\mathbb{C} \setminus \{z_1, \dots, z_n\}, *) = 1$. We can choose base points p_n in $C_n(\mathbb{C})$, $n \ge 1$, such that $f(p_{n+1}) = p_n$. Hence the associated long exact sequence of homotopy groups reads,

$$1 \longrightarrow \pi_2(C_{n+1}(\mathbb{C}), p_{n+1}) \longrightarrow \pi_2(C_n(\mathbb{C}), p_n) \longrightarrow F_n \longrightarrow \pi_1(C_{n+1}(\mathbb{C}), p_{n+1}) \longrightarrow \pi_1(C_n(\mathbb{C}), p_n) \longrightarrow 1, \quad n \ge 1.$$

As $\pi_2(C_2(\mathbb{C}),p_2)=\pi_2(\mathbb{C},p_1)=1$, one obtains by induction that $\pi_2(C_n(\mathbb{C}),p_n)=1$ for all n, and hence one gets an exact sequence of groups,

$$1\longrightarrow F_n\longrightarrow \mathbb{PB}_{n+1}\longrightarrow \mathbb{PB}_n\longrightarrow 1.$$

This exact sequence splits as the fiber bundle $f: C_{n+1}(\mathbb{C})\to C_n(\mathbb{C})$ admits a cross-section,

$$\begin{array}{rccc} s: & C_n(\mathbb{C}) & \longrightarrow & C_{n+1}(\mathbb{C}) \\ & (z_1,\dots,z_n) & \longrightarrow & (z_1,\dots,z_n,|z_1|+\dots+|z_n|+1). \end{array}$$

Hence we proved the following

5.3.1. **Theorem.** *For any $n\geq 1$ one has*

$$\begin{aligned} \mathbb{PB}_{n+1} &= F_n\rtimes \mathbb{PB}_n \\ &= F_n\rtimes(F_{n-1}\rtimes\dots\rtimes(F_3\rtimes(F_2\rtimes F_1))) \end{aligned}$$

5.3.2. **Lemma.** *Every element of $\mathbb{PB}_3$ can be uniquely represented as $f(b_1^2,b_2^2)(b_1b_2)^{3n}$ for some $n\in\mathbb{Z}$ and some element $f(x,y)$ in the free group F_2 on the generators x and y.*

Proof. The group $\mathbb{PB}_3$ is generated by $x_{12}=b_1^2$, $x_{23}=b_2^2$ and $x_{13}=b_2b_1^2b_2^{-1}$, or, equivalently, by b_1^2, b_2^2 and

$$x_{13}x_{23}x_{12}=b_2b_1^2b_2b_1^2=(b_2b_1)^3=(b_1b_2)^3,$$

where we used the braid relations $b_1b_2b_1=b_2b_1b_2$ in $\mathbb{B}_3$. As $(b_1b_2)^3$ is central in $\mathbb{PB}_3$ and there are no relations between the generators b_1^2 and b_2^2, the result follows. □

5.3.3. **Corollary.** $\mathbb{PB}_3=F_2\times\mathbb{Z}$.

This corollary can be seen also from the fact that the fibration $\mathbb{C}_3(\mathbb{C})\to\mathbb{C}_2(\mathbb{C})$ is trivial.

If $\mathbb{K}$ is a field of characteristic zero, then $\widehat{\mathbb{PB}}_3(\mathbb{K})=\widehat{F_2}(\mathbb{K})\times\mathbb{K}$, with $\widehat{F_2}(\mathbb{K})$ generated by $[b_1^2]^{\lambda_1}$, $[b_2^2]^{\lambda_2}$, and the factor $\mathbb{K}$ corresponding to $[(b_1b_2)^3]^{\lambda_3}$, $\lambda_i\in\mathbb{K}$, $i=1,2,3$. On the contrary, $\widehat{\mathbb{B}}_3(\mathbb{K})$ is a much simpler group.

5.3.4. **Proposition.** *If $char(\mathbb{K})=0$, then $\widehat{B}_3(\mathbb{K})\simeq\widehat{F}_1(\mathbb{K})\simeq\mathbb{K}$.*

Proof. Set $c:=(b_1b_2)^3$. As it is a central element in $\mathbb{B}_3$, c^λ commutes with any element in $\widehat{\mathbb{B}}_3(\mathbb{K})$ for any $\lambda\in\mathbb{K}$. Using the last equality in §5.2.1(iii), we obtain

$$b_1b_2b_1=b_2b_1b_2=c^{\frac{1}{2}}\quad\text{in}\quad\widehat{\mathbb{B}}_3(\mathbb{K})$$

so that $b_1b_2=b_2^{-1}c^{\frac{1}{2}}$, $b_2^{-1}c^{\frac{1}{2}}b_1=c^{\frac{1}{2}}$, and hence $b_1=b_2=c^{\frac{1}{6}}$ in $\widehat{\mathbb{B}}_3(\mathbb{K})$. □

5.3.5. **Corollary.** *Let $char(\mathbb{K})=0$. The injection $\mathbb{PB}_3\to\mathbb{B}_3$ induces a surjection $\widehat{\mathbb{PB}}_3(\mathbb{K})\to\widehat{\mathbb{B}}_3(\mathbb{K})$ whose kernel is the prounipotent completion of the free group generated by $\mathsf{b}_1=c^{-1/3}b_1^2$ and $\mathsf{b}_2:=c^{-1/3}b_2^2$.*

5.4. Non-$\mathbb{S}$ operad of pure braids. Consider a collection of groups,

$$\mathcal{PB}:=\{\pi_1(C_n(\mathbb{C}))\simeq\mathbb{PB}_n\}.$$

Recall that $\pi_1(C_n(\mathbb{C}))$ stands for the isomorphism class of the fundamental group $\pi_1(C_n(\mathbb{C}),p_n)$ defined for a particular choice of the base point p_n; as $C_n(\mathbb{C})$ is path connected, the class $\pi_1(C_n(\mathbb{C}))$ does not depend on such a choice. The collection $\mathcal{PB}$ can be made into a non-$\mathbb{S}$ operad with respect to the operadic composition,

$$\begin{array}{rccc} \circ_i: & \mathbb{PB}_n\times\mathbb{PB}_n & \longrightarrow & \mathbb{PB}_{n+m-1} \qquad \forall n,m\in\mathbb{N}, i\in[n] \\ & (b',b'') & \longrightarrow & b'\circ_i b'' \end{array}$$

which is defined by replacing the i-labelled strand in b' by the braid b'' made very thin.

5.4.1. **Cosimplicial structure on $\mathcal{PB}$.** Let e be the identity element in the group $\mathcal{PB}(2) = \mathbb{PB}_2$. Then it satisfies the condition

$$e \circ_1 e = e \circ_2 e$$

and hence gives rise to a pre-cosimplicial structure,

$$d_i : \mathbb{PB}_n \longrightarrow \mathbb{PB}_{n+1}, \quad i = 0, 1, \ldots, n+1,$$

on the collection $\mathcal{PB} = \{\mathbb{PB}_n\}_{n\geq 1}$ given by the explicit formulae in (4.4). For example, one has in $\mathcal{PB}_3$,

$$d_0(b_1^2) = (b_2)^2, \quad d_1(b_1^2) = (b_2 b_1)^2, \quad d_2(b_1^2) = (b_1 b_2)^2, \quad d_3(b_1^2) = (b_1)^2. \tag{5.1}$$

In fact, this structure can be completed to a *cosimplicial* structure on $\mathcal{PB}$ by defining the operators

$$s_i : \mathbb{PB}_{n+1} \longrightarrow \mathbb{PB}_n, \qquad i = 1, 2, \ldots, n+1,$$

where s_i erases the strand labelled by i.

5.5. An operad of parenthesized braids. If we were able to make the operad of little disks $\mathcal{D}$ into an operad in the category of *based* topological spaces, then, by applying the monoidal functor (3.6), we would have obtained an operad $\pi_1(\mathcal{D})$ in the category of groups. However this is impossible as there is *no* $\mathbb{S}_n$ invariant configuration of little disks in $\mathcal{D}(n)$ for every $n \geq 2$.

The fundamental groupoid functor (3.5) applied to $\mathcal{D}$ gives us an operad in the category of groupoids, CatG, with nontrivial action of permutations groups as endofunctors, $\sigma : \pi(\mathcal{D})(n) \to \pi(\mathcal{D})(n)$, $\sigma \in \mathbb{S}_n$, $n \geq 1$. However this operad is too big for our purposes, the category $\pi(\mathcal{D})(n)$ has too many objects (which are points in $\mathcal{D}(n)$). Let us construct by induction a "smallest possible" non-trivial suboperad, $\mathcal{Pa}\mathcal{B}$, of the operad $\pi(\mathcal{D})$ such that $\mathcal{Pa}\mathcal{B}(n)$ is a *full* subcategory of $\pi(\mathcal{D})(n)$ for each $n \geq 1$:

- $n = 1$: Set $\mathcal{Pa}\mathcal{B}(1)$ to a full subcategory of $\pi(\mathcal{D})(1)$ which has only one object, the configuration $D_{0,1} \in \pi(\mathcal{D})(1)$ (which is the unit in the operad $\mathcal{D}$).
- $n = 2$: Let $\mathbf{1}$ be any configuration of little disks in $\mathcal{D}(2)$, that is, a pair of disjoint disks inside the unit disk labeled by 1 and 2, and set $\mathbf{2} := \sigma(\mathbf{1})$, where $\sigma = (12) \in \mathbb{S}_2$. Set $\mathcal{Pa}\mathcal{B}(2)$ be the full subcategory of $\pi(\mathcal{D})(2)$ whose only objects are $\mathbf{1}$ and $\mathbf{2}$; the $\mathbb{S}_2$ action on $\pi(\mathcal{D})(2)$ leaves the subcategory $\mathcal{Pa}\mathcal{B}(2)$ invariant. It is useful to identify $Ob(\mathcal{Pa}\mathcal{B}(2))$ with $\mathcal{Pa}(2)$, the set of two planar binary corollas,

$$\mathbf{1} = \begin{array}{c} \circ \\ {\scriptstyle 1}\quad{\scriptstyle 2} \end{array} \qquad \mathbf{2} = \begin{array}{c} \circ \\ {\scriptstyle 2}\quad{\scriptstyle 1} \end{array}$$

- $n = 3$: Let $\circ_i : \mathcal{D}(2) \times \mathcal{D}(2) \to \mathcal{D}(3)$, $i = \{1, 2\}$, be the operadic compositions in $\mathcal{D}(n)$. Set $\mathcal{Pa}\mathcal{B}(3)$ be the full subcategory of $\pi(\mathcal{D})(3)$ whose only objects are $\mathbf{p} \circ_i \mathbf{q}$, where $\mathbf{p}, \mathbf{q} \in \{\mathbf{1}, \mathbf{2}\}$. It is clear that the set $Ob(\mathcal{Pa}\mathcal{B}(3))$ can be identified with $\mathcal{Pa}(3)$, the set of parenthesized permutations of $[3] = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ (see §4.3.2), or, equivalently, the set of [**3**]-labelled planar binary trees,

$$\mathbf{1} \circ_1 \mathbf{1} = (\mathbf{12})\mathbf{3}, \quad \mathbf{1} \circ_2 \mathbf{1} = \mathbf{1}(\mathbf{23}), \quad \mathbf{1} \circ_1 \mathbf{2} = (\mathbf{21})\mathbf{3}, \quad \mathbf{2} \circ_2 \mathbf{2} = \mathbf{3}(\mathbf{21}), \quad etc.$$

- $n \geq 3$: Let $Ob(\mathcal{Pa}\mathcal{B})$ be the suboperad of $Ob(\pi(\mathcal{D}))$ (in the category of sets) generated by the $\mathcal{S}$-module $E = \{E(n)\}$, where

$$E(n) := \begin{cases} Ob(\mathcal{Pa}\mathcal{B}(1)) = D_{0,1} & \text{for } n = 1, \\ Ob(\mathcal{Pa}\mathcal{B}(2)) = \{\mathbf{1}, \mathbf{2}\} & \text{for } n = 2, \\ \emptyset & \text{for } n \geq 2 \end{cases}$$

  It can be identified with the operad $\mathcal{Pa}$.

Let now $\mathcal{Pa}\mathcal{B}(n)$ be the full subcategory of the category $\pi(\mathcal{D}(n))$. By construction, the collection of groupoids,

$$\mathcal{Pa}\mathcal{B} := \{\mathcal{Pa}\mathcal{B}(n)\}_{n\geq 1},$$

is a suboperad of $\pi(\mathcal{D})$. It was first introduced by D. Bar-Natan [**B-N**]. The idea to use the language of operads in D. Bar-Natan's approach to Drinfeld associators and the Grothendieck-Teichmüller group is due to Tamarkin [**T2**] (see also the works of P. Severa and T. Willwacher [**Se, SW**]). This idea makes D. Bar-Natan's story short and transparent.

One can equivalently describe the underlying S-module of the operad $\mathcal{P}a\mathcal{B}$ as functor,

$$\begin{array}{rccl} \mathcal{P}a\mathcal{B}: & \mathsf{S} & \longrightarrow & \mathsf{CatG} \\ & I & \longrightarrow & \mathcal{P}a\mathcal{B}(I) = \left\{ \begin{array}{c} \text{objects,} \quad Ob(\mathcal{P}a\mathcal{B}(I)) \\ \text{sets of morphisms,} \quad \{\mathrm{Mor}(A_I, B_I)\}_{A_I, B_I \in Ob(\mathcal{P}a\mathcal{B}(I))} \end{array} \right. \end{array}$$

where $\mathcal{P}a\mathcal{B}(I)$ is the category given by the following data:

(i) the objects are parenthesized permutations of the set I, or, equivalently, planar I-labelled binary trees equipped with the operadic structure explained in §4.3.2; put another way, the "object" part of the operadic structure is given by the functor

$$\begin{array}{rccl} \mathcal{P}a: & \mathsf{S} & \longrightarrow & \mathsf{Set} \\ & I & \longrightarrow & \mathcal{P}a(I) =: Ob(\mathcal{P}a\mathcal{B}(I)) \end{array}$$

(ii) for any two objects A_I and B_I in $Ob(\mathcal{P}a\mathcal{B}(I))$, that is, for any two parenthesized permutations of the set I, the associated set of morphisms, $\mathrm{Mor}(A(I), B(I))$, is defined to be the set of braids whose strands connect the same elements of I (and hence have a non-ambiguous I-labelling). For example,

$$\beta_{1,2} := \quad \in \mathrm{Mor}\,(12, 21)\,, \qquad \mathbb{1}_{1,2,3} := \quad \in \mathrm{Mor}\,((12)3), 1(23))$$

The operadic composition $b' \circ_i b''$ on morphisms is defined by replacing the i-labelled strand of the braid b' with the braid b'' made thin.

5.5.1. **Exercise.** Show that the operad $\mathcal{P}a\mathcal{B}$ is generated (via operadic and categorical compositions) by the elements $\beta_{1,2}$, $\mathbb{1}_{1,2,3}$ and their inverses.

Find an explicit expression for the element $\quad \in \mathrm{Mor}\,(1(23), (23)1)$ in terms of the generators $\beta_{\bullet\bullet}$ and $\mathbb{1}_{\bullet,\bullet,\bullet}$.

5.5.2. **A family of functors on $\mathcal{P}a\mathcal{B}$.** Let $\mathbb{1}_{12}$ be the identity element in $\mathrm{Mor}(12, 12)$,

$$\mathbb{1}_{12} := \quad$$

and, for any $n \in \mathbb{N}_{\geq 1}$, define a family of functors,

$$d_i : \mathcal{P}a\mathcal{B}(n) \longrightarrow \mathcal{P}a\mathcal{B}(n+1), \qquad i = 0, 1, \ldots, n, n+1,$$

as follows:

(i) on objects (i.e. on parenthesized permutations of $[n]$), one has

$$\begin{array}{rccl} d_i: & Ob(\mathcal{P}a\mathcal{B}(n)) & \longrightarrow & Ob(\mathcal{P}a\mathcal{B}(n+1)) \\ & A_n & \longrightarrow & d_i A_n := \left\{ \begin{array}{ll} (12) \circ_2 A_n & \text{for } i = 0 \\ A_n \circ_i (12) & \text{for } i \in [n] \\ (12) \circ_1 A_n & \text{for } i = n+1 \end{array} \right. \end{array}$$

(ii) on morphisms (i.e. on braids connecting the same elements in a pair of parenthesized permutations), one has

$$\begin{array}{rccl} d_i: & \mathrm{Mor}(A_n, B_n) & \longrightarrow & \mathrm{Mor}(d_i A_n, d_i B_n) \\ & f_n & \longrightarrow & d_i f_n := \left\{ \begin{array}{ll} \mathbb{1}_{1,2} \circ_2 f_n & \text{for } i = 0 \\ f_n \circ_i \mathbb{1}_{1,2} & \text{for } i \in [n] \\ \mathbb{1}_{1,2} \circ_1 f_n & \text{for } i = n+1 \end{array} \right. \end{array}$$

Note that in the operad $Ob(\mathcal{P}a\mathcal{B})$,

$$(12) \circ_1 (12) \neq (12) \circ_2 (12)$$

so that the functors d_i do *not* make the collection of groupoids $\{\mathcal{P}a\mathcal{B}(n)\}_{n \geq 1}$ into a (pre)cosimplicial object in the category of small categories. However these functors will be very useful for us below for the the

following reason. Let $\mathcal{O}$ be an operad in the category of small categories with same set of objects as in $\mathcal{P}a\mathcal{B}$. Let $\mathbb{1}_{1,2}$ be the identity morphism from (12) to (12) in the category $\mathcal{O}(2)$. Then the formulae formally identical to the ones above give us a family of functors, $d_i : \mathcal{O}(n) \to \mathcal{O}(n+1)$; moreover, any morphism of operads $F : \mathcal{P}a\mathcal{B} \to \mathcal{O}$ *respects* these functors, i.e.

$$d_i \circ F(n) = F(n+1) \circ d_i,$$

for any $n \in \mathbb{N}$ and any $i \in \{0, 1, \dots n+1\}$.

5.5.3. **Remark.** The non-$\mathbb{S}$ operad $\mathcal{PB}$ also has a presimplicial structure given formally by the same formulae as above.

5.5.4. **Pentagon equation.** Consider the elements

$$d_0\mathbb{1}_{1,2,3} = \begin{array}{c} {\scriptstyle 1\ \ (2\ ((3\ 4))} \\ \vert\ \ \vert\ \ \vert\ \ \vert \\ {\scriptstyle 1\ \ ((2\ 3)\ \ 4)} \end{array}, \quad d_1\mathbb{1}_{1,2,3} = \begin{array}{c} {\scriptstyle (1\ \ 2)\ (3\ \ 4)} \\ \vert\ \ \vert\ \ \vert\ \ \vert \\ {\scriptstyle ((1\ \ 2)\ \ 3)\ \ 4} \end{array}, \quad d_2\mathbb{1}_{1,2,3} = \begin{array}{c} {\scriptstyle 1\ ((2\ 3)\ \ 4)} \\ \vert\ \ \vert\ \ \vert\ \ \vert \\ {\scriptstyle (1\ \ (2\ \ 3))\ 4} \end{array},$$

$$d_3\mathbb{1}_{1,2,3} = \begin{array}{c} {\scriptstyle 1\ ((2\ 3)\ \ 4)} \\ \vert\ \ \vert\ \ \vert\ \ \vert \\ {\scriptstyle (1\ \ 2)\ (3\ \ 4)} \end{array}, \quad d_4\mathbb{1}_{1,2,3} = \begin{array}{c} {\scriptstyle (1\ \ (2\ \ 3)\ \ 4} \\ \vert\ \ \vert\ \ \vert\ \ \vert \\ {\scriptstyle ((1\ \ 2)\ \ 3)\ \ 4} \end{array},$$

and check that they satisfy the pentagon equation (cf. (3.1))

$$d_1\mathbb{1}_{1,2,3} \circ d_3\mathbb{1}_{1,2,3} = d_4\mathbb{1}_{1,2,3} \circ d_2\mathbb{1}_{1,2,3} \circ d_0\mathbb{1}_{1,2,3} \tag{5.2}$$

Here $\circ$ denotes the categorical (rather than operadic) composition of morphisms.

Following a long tradition (see, e.g., [**Dr2, ES**]), the elements $d_0\mathbb{1}_{1,2,3}$, $d_1\mathbb{1}_{1,2,3}$, $d_1\mathbb{1}_{1,2,3}$, $d_3\mathbb{1}_{1,2,3}$ and $d_4\mathbb{1}_{1,2,3}$ should have been denoted, respectively, by $\mathbb{1}_{2,3,4}$, $\mathbb{1}_{12,3,4}$, $\mathbb{1}_{1,23,4}$, $\mathbb{1}_{1,2,34}$ and $\mathbb{1}_{1,2,3}$ so that the pentagon equation takes a familiar form (cf. equation (17.2) in the book [**ES**])

$$\mathbb{1}_{12,3,4} \circ \mathbb{1}_{1,23,4} = \mathbb{1}_{1,2,3} \circ \mathbb{1}_{1,23,4} \circ \mathbb{1}_{2,3,4}.$$

However, we shall mostly stick to the cosimplicial notation in this survey.

5.5.5. **Hexagon equations.** One has in $\mathcal{P}a\mathcal{B}(3)$,

$$d_0\beta_{1,2} = \begin{array}{c} {\scriptstyle 1\ \ (3\ \ \ 2)} \\ \vert\ \ \diagdown\!\diagup \\ {\scriptstyle 1\ \ (2\ \ \ 3)} \end{array}, \quad d_3\beta_{1,2} = \begin{array}{c} {\scriptstyle (2\ \ \ 1)\ 3} \\ \diagdown\!\diagup\ \ \vert \\ {\scriptstyle (1\ \ \ 2)\ 3} \end{array}, \quad d_1\beta_{1,2} = \begin{array}{c} {\scriptstyle 3\ \ \ (1\ 2)} \\ \diagdown\ \ \diagup \\ {\scriptstyle (1\ 2)\ \ \ 3} \end{array}$$

The permutation group $\mathbb{S}_n$ acts on $\mathcal{P}a\mathcal{B}(n)$ by relabelling of strands, for example,

$$(23)\cdot\mathbb{1}_{1,2,3} = \mathbb{1}_{1,3,2} = \begin{array}{c} {\scriptstyle 1\ \ (3\ \ 2)} \\ \vert\ \ \vert\ \ \vert \\ {\scriptstyle (1\ \ 3)\ \ 2} \end{array}, \quad (123)\cdot\mathbb{1}_{1,2,3} = \mathbb{1}_{3,1,2} = \begin{array}{c} {\scriptstyle 3\ \ (1\ \ 2)} \\ \vert\ \ \vert\ \ \vert \\ {\scriptstyle (3\ \ 1)\ \ 2} \end{array}, \quad (23)\cdot(d_3\beta_{1,2}) = \begin{array}{c} {\scriptstyle (3\ \ \ 1)\ 2} \\ \diagdown\!\diagup\ \ \vert \\ {\scriptstyle (1\ \ \ 3)\ 2} \end{array}$$

It is easy to see that the generators satisfy the following hexagon equation in $\mathcal{P}a\mathcal{B}(3)$,

$$d_1\beta_{1,2} = \mathbb{1}_{1,2,3} \ \circ\ d_0\beta_{1,2} \ \circ\ \left((23)\cdot\mathbb{1}_{1,2,3}^{-1}\right) \ \circ\ (23)\cdot d_3\beta_{1,2} \ \circ\ (123)\cdot\mathbb{1}_{1,2,3}. \tag{5.3}$$

Studying analogously

$$\beta_{2,1}^{-1} = (12)\cdot\beta_{1,2}^{-1} == \left((12)\cdot\beta_{1,2}\right)^{-1} = \begin{array}{c} {\scriptstyle 2\ \ \ 1} \\ \diagdown\!\diagup \\ {\scriptstyle 1\ \ \ 2} \end{array} \in \mathrm{Mor}\,(12, 21)$$

we get a second hexagon equation,

$$d_1\beta_{2,1}^{-1} = \mathbb{1}_{1,2,3} \ \circ\ d_0\beta_{2,1}^{-1} \ \circ\ \left((23)\cdot\mathbb{1}_{1,2,3}^{-1}\right) \ \circ\ (23)\cdot d_3\beta_{2,1}^{-1} \ \circ\ (123)\cdot\mathbb{1}_{1,2,3}. \tag{5.4}$$

for the generators $\beta_{1,2}$ and $\mathbb{1}_{1,2,3}$ of the operad $\mathcal{P}a\mathcal{B}$.

5.5.6. **Theorem.** *Let $\mathcal{O}$ be an operad in the small category of groupoids with same set of objects as in $\mathcal{P}a\mathcal{B}$. There is a one-to-one correspondence between morphisms of operads,*

$$F : \mathcal{P}a\mathcal{B} \longrightarrow \mathcal{O},$$

which are identical on objects, and elements $F(\beta_{1,2}) \in \mathrm{Mor}_{\mathcal{O}}(12, 21)$ and $F(\mathbb{1}_{1,2,3}) \in \mathrm{Mor}_{\mathcal{O}}((12)3, 1(23))$, which satisfy the pentagon equation,

$$d_1F(\mathbb{1}_{1,2,3}) \circ d_3F(\mathbb{1}_{1,2,3}) = d_4F(\mathbb{1}_{1,2,3}) \circ d_2F(\mathbb{1}_{1,2,3}) \circ d_0F(\mathbb{1}_{1,2,3}) \tag{5.5}$$

and the hexagon equations,

$$d_1F(\beta_{1,2}) = F(\mathbb{1}_{1,2,3}) \ \circ \ d_0F(\beta_{1,2}) \ \circ \ \left((23) \cdot F(\mathbb{1}_{1,2,3})^{-1}\right) \ \circ \ {}_{(23)}{\cdot}d_3F(\beta_{1,2}) \ \circ \ {}_{(321)}{\cdot}F(\mathbb{1}_{1,2,3}), \tag{5.6}$$

$$d_1F(\beta_{2,1})^{-1} = F(\mathbb{1}_{1,2,3}) \ \circ \ d_0(F(\beta_{2,1})^{-1}) \ \circ \ \left({}_{(23)}{\cdot}F(\mathbb{1}_{1,2,3})^{-1}\right) \ \circ \ {}_{(23)}{\cdot}d_3(F(\beta_{2,1})^{-1}) \ \circ \ {}_{(321)}{\cdot}F(\mathbb{1}_{1,2,3}). \tag{5.7}$$

Proof. As $\mathcal{P}a\mathcal{B}$ is generated by $\beta_{1,2}$ and $\mathbb{1}_{1,2,3}$, any morphism of operads $F : \mathcal{P}a\mathcal{B} \to \mathcal{O}$ is uniquely determined by its values on the generators which must satisfy the equations obtained by applying F to (5.2), (5.3) and (5.4).

The reverse statement follows from the Mac Lane coherence theorem for braided monoidal categories, i.e. from the fact that all the relations between iterations (via operadic and categorial compositions) of the generators follow from (5.2), (5.3) and (5.4). □

5.6. D. Bar-Natan's operad $\widehat{\mathcal{P}a\mathcal{B}}_{\mathbb{K}}$. Applying the functor $\Delta_{\mathbb{K}}$ (see §3.4(iii)) to the operad $\mathcal{P}a\mathcal{B}$ we get an operad,

$$\mathcal{P}a\mathcal{B}_{\mathbb{K}} := \Delta_{\mathbb{K}}(\mathcal{P}a\mathcal{B}),$$

in the symmetric monoidal category $\mathsf{Cat}(\mathsf{coAss}_{\mathbb{K}})$. For any finite set I the objects, A_I, of the category $\mathcal{P}a\mathcal{B}_{\mathbb{K}}(I)$ are the same as objects of $\mathcal{P}a\mathcal{B}$, that is, the parenthesized permutations of the set I, while morphisms are formal $\mathbb{K}$-linear combinations of braids,

$$\mathrm{Mor}_{\mathcal{P}a\mathcal{B}_{\mathbb{K}}}(A_I, B_I) = \left\{ \sum_i \lambda_i f_i \ | \lambda_i \in \mathbb{C}, f_i \in \mathrm{Mor}_{\mathcal{P}a\mathcal{B}}(A_I, B_I) \right\},$$

which connect the same letters in the words A_I and B_I. The coproduct structure on morphisms is uniquely defined by the condition that all f_i are group-like.

Every operad $\mathcal{O}$ in $\mathsf{Cat}(\mathsf{coAss}_{\mathbb{K}})$, in particular $\mathcal{P}a\mathcal{B}_{\mathbb{K}}$, comes equipped with a *diagonal* morphism of operads,

$$\Delta : \mathcal{O} \longrightarrow \mathcal{O} \otimes \mathcal{O}, \tag{5.8}$$

which is given on objects $A_I \in Ob(\mathcal{O}(I))$ by

$$\Delta(A_I) = A_I \times A_I,$$

and on morphisms Δ is given by the above coproduct in $\mathrm{Mor}_{\mathcal{O}}(A_I, B_I)$.

Let us define one more operad in this category,

$$\mathcal{P}a_{\mathbb{K}} : \mathsf{S} \longrightarrow \mathsf{Cat}(\mathsf{coAss}_{\mathbb{K}}),$$

whose objects are, by definition, identical to objects of the operad $\mathcal{P}a$ (and hence of $\mathcal{P}a\mathcal{B}_{\mathbb{K}}$) and whose morphisms spaces, $\mathrm{Mor}_{\mathcal{P}a_{\mathbb{K}}}(A_I, B_I)$, are identified with $\mathbb{K}$ equipped with the coproduct $\Delta(1) := 1 \otimes 1$. Speaking plainly, $\mathcal{P}a_{\mathbb{K}}$ is the operad $\mathcal{P}a\mathcal{B}_{\mathbb{K}}$ with "braiding" on morphisms forgotten.

There is a augmentation morphism of operads (cf. §2.4),

$$\varepsilon : \mathcal{P}a\mathcal{B}_{\mathbb{K}} \longrightarrow \mathcal{P}a_{\mathbb{K}},$$

which is identical on objects, and is given on morphisms by the formula,

$$\begin{array}{rccc} \varepsilon : & \mathrm{Mor}_{\mathcal{P}a\mathcal{B}_{\mathbb{K}}}(A_I, B_I) & \longrightarrow & \mathrm{Mor}_{\mathcal{P}a_{\mathbb{K}}}(A_I, B_I) \\ & \sum_i \lambda_i f_i & \longrightarrow & \sum_i \lambda_i \end{array}$$

The kernel, $\mathcal{I}_{\mathbb{K}} := \mathsf{Ker}\,\varepsilon$ is an operadic ideal in $\mathcal{P}a\mathcal{B}_{\mathbb{K}}$. One can define its power, $\mathcal{I}_{\mathbb{K}}^m$, $m \geq 1$, which is a suboperad of $\mathcal{I}_{\mathbb{K}}$ with the same objects as in $\mathcal{I}_{\mathbb{K}}$ but with morphisms given by the categorical composition

of at least m morphisms from $\mathcal{I}_{\mathbb{K}}$. In this way we get a family of ideals, $\{\mathcal{I}_{\mathbb{K}}^m\}_{m\geq 1}$, of the operad $\mathcal{P}a\mathcal{B}_{\mathbb{K}}$, and hence an associated inverse system of operads in the category $\mathsf{Cat}(\mathsf{coAss}_{\mathbb{K}})$,

$$\ldots \longrightarrow \mathcal{P}a\mathcal{B}_{\mathbb{K}}^{(m)} \longrightarrow \mathcal{P}a\mathcal{B}_{\mathbb{K}}^{(m-1)} \longrightarrow \ldots \longrightarrow \mathcal{P}a\mathcal{B}_{\mathbb{K}}^{(1)} \longrightarrow \mathcal{P}a\mathcal{B}_{\mathbb{K}}^{(0)},$$

where $\mathcal{P}a\mathcal{B}_{\mathbb{K}}^{(m)}$ is the quotient operad $\mathcal{P}a\mathcal{B}_{\mathbb{K}}/\mathcal{I}_{\mathbb{K}}^{m+1}$, $m \geq 0$. The inverse limit,

$$\widehat{\mathcal{P}a\mathcal{B}}_{\mathbb{K}} := \lim_{\longleftarrow} \mathcal{P}a\mathcal{B}_{\mathbb{K}}^{(m)}.$$

is an operad in the category $\widehat{\mathsf{Cat}}(\mathsf{coAss}_{\mathbb{K}})$.

5.7. Grothendieck-Teichmüller group $\widehat{GT}(\mathbb{K})$. The group of those automorphisms of the operad $\widehat{\mathcal{P}a\mathcal{B}}_{\mathbb{K}}$ which are identical on objects is called the *Grothendieck-Teichmüller group* and is denoted by $\widehat{GT}(\mathbb{K})$ or simply by $\widehat{GT}$ or even GT.[7]

A similar group of automorphisms of $\mathcal{P}a\mathcal{B}_{\mathbb{K}}^{(m)}$ is denoted by $GT^{(m)}(\mathbb{K})$. The group $\widehat{GT}(\mathbb{K})$ coincides with the inverse limit $\lim_{\longleftarrow} GT^{(m)}(\mathbb{K})$.

Let us next characterize elements of $\widehat{GT}(\mathbb{K})$ as solutions of certain explicit algebraic equations. This characterization proves that the above operadic definition of $\widehat{GT}(\mathbb{K})$ coincides with the one given by V. Drinfeld in his original paper [**Dr2**].

5.7.1. **Theorem** [**Dr2**]. *Let $\widehat{F}_2(\mathbb{K})$ be the pro-unipotent completion of the free group in two variables x and y. There is a one-to-one correspondence between elements of $\widehat{GT}(\mathbb{K})$ and pairs,*

$$\left(f \in \widehat{F}_2(\mathbb{K}), \lambda \in \mathbb{K}^*\right),$$

which satisfy the following equations,

$$f(x,y) = f(y,x)^{-1}, \tag{5.9}$$

$$f(x_3,x_1)x_3^{\mu} f(x_2,x_3) x_2^{\mu} f(x_1,x_2) x_1^{\mu} = 1 \quad \text{for } x_1x_2x_3 = 1 \text{ and } \mu := \frac{\lambda - 1}{2}, \tag{5.10}$$

and

$$f(x_{12},x_{23}x_{24})f(x_{13}x_{23},x_{34}) = f(x_{23},x_{34})f(x_{12}x_{13},x_{23}x_{34})f(x_{12},x_{23}) \quad \text{in} \quad \widehat{\mathbb{PB}}_4(\mathbb{K}). \tag{5.11}$$

The multiplication law in $\widehat{GT}(\mathbb{K})$ is given by,

$$(\lambda_1, f_1)\,(\lambda_2, f_2) = \left(\lambda_1\lambda_2, \quad f_2(f_1(x,y)x^{\lambda_1}f_1(x,y)^{-1}, y^{\lambda_1})f_1(x,y)\right)$$

Proof. By Theorem 5.5.6, every automorphism F of $\widehat{\mathcal{P}a\mathcal{B}}$ is uniquely determined by its values on the generators $\beta_{1,2}$ and $\mathbb{1}_{1,2,3}$ so that we have only to check that these values satisfy pentagon equation (5.5), two hexagon equations (5.6) and (5.7), and that F sends group-like elements in $\widehat{\mathcal{P}a\mathcal{B}}$ to group-like elements with respect to the diagonal Δ. The latter condition is equivalent to saying that $F(\beta_{1,2})$ and $F(\mathbb{1}_{1,2,3})$ are group-like in $\mathrm{Mor}_{\widehat{\mathcal{P}a\mathcal{B}}}(12,21)$ and $\mathrm{Mor}_{\widehat{\mathcal{P}a\mathcal{B}}}((12)3,1(23))$. Hence

$$F(\beta_{1,2}) \circ \beta_{1,2}^{-1} \text{ is grouplike in the coalgebra } \mathrm{Mor}_{\widehat{\mathcal{P}a\mathcal{B}}}(12,12)$$

and

$$F(\mathbb{1}_{1,2,3}) \circ \mathbb{1}_{1,2,3}^{-1} \text{ is grouplike in the coalgebra } \mathrm{Mor}_{\widehat{\mathcal{P}a\mathcal{B}}}((12)3,1(23))$$

which in turn imply (see §§5.3.3 and 5.1.3) that

$$F(\beta_{1,2}) \circ \beta_{1,2}^{-1} = [\beta_{1,2}^2]^{\mu} \in \widehat{\mathbb{PB}}_2(\mathbb{K})$$

for some $\mu \in \mathbb{K}$ with $\mu \neq -\frac{1}{2}$ (as otherwise F would not be invertible), and

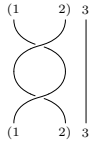
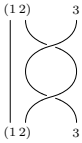
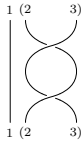
$$F(\mathbb{1}_{1,2,3}) \circ \mathbb{1}_{1,2,3}^{-1} = f(\mathsf{b}_2^2, \mathsf{b}_1^2)[(b_1b_2)^3]^{\eta} \in \widehat{\mathbb{PB}}_3(\mathbb{K})$$

[7] Since we understand $\widehat{\mathcal{P}a\mathcal{B}}_{\mathbb{K}}$ as an operad in $\widehat{\mathsf{Cat}}(\mathsf{coAss}_{\mathbb{K}})$, it is tacitly assumed that its every automorphism respects the diagonal Δ.

for some $f(x,y) \in \widehat{F_2}(\mathbb{K})$ and some $\eta \in \mathbb{K}$. Here b_1^2 and b_2^2 stand for the renormalized generators (cf. §5.3.5),

$$\mathsf{b}_1^2 := c^{-1/3} b_1^2, \qquad \mathsf{b}_2^2 := c^{-1/3} b_2^2,$$

and b_1^2, b_2^2 and $c := (b_1 b_2)^3$ are the grouplike elements of $\mathrm{Mor}_{\widehat{\mathcal{PaB}}}((12)3,(12)3)$ obtained from the corresponding braids in $\mathbb{PB}_3$ by attaching strands' inputs and outputs to one and the same object (12)3, for example,

$b_1^2 =$  $= d_3(\beta_{1,2}^2),$ $\quad b_2^2 :=$  $= \mathbb{1}_{1,2,3} \circ$  $\circ\, \mathbb{1}_{1,2,3}^{-1} = \mathbb{1}_{1,2,3} \circ d_0(\beta_{1,2}^2) \circ \mathbb{1}_{1,2,3}^{-1}$

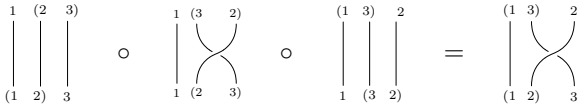
Using (2.4) we obtain

$$\begin{aligned} d_0 F(\beta_{1,2}) &= d_0([\beta_{1,2}^2]^\mu \circ \beta_{1,2}) \\ &= d_0([\beta_{1,2}^2]^\mu) \circ d_0(\beta_{1,2}) \\ &= \mathbb{1}_{1,2,3}^{-1} \circ [b_2^2]^\mu \circ \mathbb{1}_{1,2,3} \circ d_0\beta_{1,2} \end{aligned}$$

so that

$$F(\mathbb{1}_{1,2,3}) \ \circ\ d_0 F(\beta_{1,2}) = \left[f(\mathsf{b}_2^2, \mathsf{b}_1^2)(\mathsf{b}_2^2)^\mu c^{\eta + \frac{1}{3}\mu} \right] \circ \mathbb{1}_{1,2,3} \circ d_0\beta_{1,2}.$$

Denoting

$$g = \mathbb{1}_{1,2,3} \circ d_0\beta_{1,2} \circ (23)\mathbb{1}_{1,2,3}^{-1} =$$ 

we obtain, using the results of §5.2.1,

$$\begin{aligned} g \circ \left[(23)\cdot f(\mathsf{b}_2^2, \mathsf{b}_1^2) \right]^{-1} \circ g^{-1} &= f\left(g \circ (23)\cdot \mathsf{b}_2^2 \circ g^{-1}, g \circ (23)\cdot \mathsf{b}_1^2 \circ g^{-1} \right)^{-1} \\ &= f(\mathsf{b}_2^2, \mathsf{b}_1^{-2}\mathsf{b}_2^{-2})^{-1} \end{aligned}$$

so that the product of the first three terms on the r.h.s. of (5.6) now reads,

$$F(\mathbb{1}_{1,2,3}) \ \circ\ d_0 F(\beta_{1,2}) \circ \left((23)\cdot F(\mathbb{1}_{1,2,3})^{-1} \right) = \left[f(\mathsf{b}_2^2, \mathsf{b}_1^2)(\mathsf{b}_2^2)^\mu f(\mathsf{b}_2^2, \mathsf{b}_1^{-2}\mathsf{b}_2^{-2})^{-1} c^{\frac{1}{3}\mu} \right] \circ g.$$

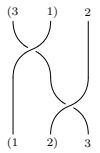
Next we have,

$$\begin{aligned} g \circ (23)\cdot d_3 F(\beta_{1,2}) \circ g^{-1} &= g \circ (23)\cdot [b_1^2]^\mu \circ g^{-1} \circ g \circ (23)\cdot d_3\beta_{1,2} \circ g^{-1} \\ &= \left[(\mathsf{b}_1^{-2}\mathsf{b}_2^{-2})^\mu c^{\frac{1}{3}\mu} \right] \circ g \circ (23)\cdot d_3\beta_{1,2} \circ g^{-1}, \end{aligned}$$

and hence

$$F(\mathbb{1}_{1,2,3}) \ \circ\ d_0 F(\beta_{1,2}) \circ \left((23)\cdot F(\mathbb{1}_{1,2,3})^{-1} \right) \circ (23)\cdot d_3 F(\beta_{1,2}) =$$
$$= \left[f(\mathsf{b}_2^2, \mathsf{b}_1^2)(\mathsf{b}_2^2)^\mu f(\mathsf{b}_2^2, \mathsf{b}_1^{-2}\mathsf{b}_2^{-2})^{-1} (\mathsf{b}_1^{-2}\mathsf{b}_2^{-2})^\mu c^{\frac{2}{3}\mu} \right] \circ h.$$

where

$$h := g \circ (23)\cdot d_3\beta_{1,2} =$$  $= d_1\beta_{1,2} \circ (123)\cdot \mathbb{1}_{1,2,3}^{-1}.$

As $(b_2 b_1)(\mathsf{b}_1^2)(b_2 b_1)^{-1} = \mathsf{b}_1^{-2}\mathsf{b}_2^{-2}$ and $(b_2 b_1)(\mathsf{b}_2^2)(b_2 b_1)^{-1} = \mathsf{b}_1^2$ in $\widehat{\mathbb{B}}_3(\mathbb{K})$, we have

$$h \circ (123)\cdot F(\mathbb{1}_{1,2,3}) \circ h^{-1} = [f(\mathsf{b}_1^2, \mathsf{b}_1^{-2}\mathsf{b}_2^{-2})] \circ h \circ (123)\cdot \mathbb{1}_{1,2,3} \circ h^{-1}$$

and hence

$$\text{r.h.s. of (5.6)} = \left[f(\mathsf{b}_2^2, \mathsf{b}_1^2)(\mathsf{b}_2^2)^\mu f(\mathsf{b}_2^2, \mathsf{b}_1^{-2}\mathsf{b}_2^{-2})^{-1} (\mathsf{b}_1^{-2}\mathsf{b}_2^{-2})^\mu f(\mathsf{b}_1^2, \mathsf{b}_1^{-2}\mathsf{b}_2^{-2}) c^{\eta + \frac{2}{3}\mu} \right] \circ h \circ (123)\cdot \mathbb{1}_{1,2,3}.$$

On the other hand, one has $d_1 b_1^2 = b_2 b_1^2 b_2 = (b_1 b_2)^3 b_1^{-2} = c^{2/3}\mathsf{b}_1^{-2}$ in $\widehat{\mathbb{B}}_3(\mathbb{K})$, and hence

$$\text{l.h.s. of (5.6)} = d_1 F(\beta_{1,2}) = [c^{\frac{2}{3}\mu}(\mathsf{b}_1^{-2})^\mu] \circ h \circ (123)\cdot \mathbb{1}_{1,2,3}.$$

Therefore, the first hexagon equation (5.6) is equivalent to the following equation in $\widehat{\mathbb{PB}}_3(\mathbb{K})$,

$$f(\mathsf{b}_2^2,\mathsf{b}_1^2)(\mathsf{b}_2^2)^\mu f(\mathsf{b}_2^2,\mathsf{b}_1^{-2}\mathsf{b}_2^{-2})^{-1}(\mathsf{b}_1^{-2}\mathsf{b}_2^{-2})^\mu f(\mathsf{b}_1^2,\mathsf{b}_1^{-2}\mathsf{b}_2^{-2})(\mathsf{b}_1^2)^\mu c^\eta = 1$$

Hence $\eta = 0$ and the element $f(x,y) \in \widehat{F_2}(\mathbb{K})$ satisfies the equation

$$f(x_3,x_1)x_3^\mu f(x_3,x_2)^{-1}x_2^\mu f(x_1,x_2)x_1^\mu = 1 \tag{5.12}$$

for $x_1x_2x_3 = 1$. Studying similarly the second hexagon equation (5.7), one obtains (using identities $b_2^{-1}\mathsf{b}_1^2b_2 = \mathsf{b}_2^{-2}\mathsf{b}_1^{-2}$, $(b_1b_2)^{-1}(\mathsf{b}_1^2)(b_1b_2) = \mathsf{b}_2^{-2}\mathsf{b}_1^{-2}$ and $(b_1b_2)^{-1}(\mathsf{b}_2^2)(b_1b_2) = \mathsf{b}_1^2$) the following equation,

$$(\mathsf{b}_1^2)^{-\mu}f(\mathsf{b}_2^2,\mathsf{b}_1^2)(\mathsf{b}_2^2)^{-\mu}f(\mathsf{b}_2^2,\mathsf{b}_2^{-2}\mathsf{b}_1^{-2})^{-1}(\mathsf{b}_2^{-2}\mathsf{b}_1^{-2})^{-\mu}f(\mathsf{b}_1^2,\mathsf{b}_2^{-2}\mathsf{b}_1^{-2}) = 1$$

which says that $f(x,y) \in \widehat{F_2}(\mathbb{K})$ satisfies

$$f(x_3,x_2)^{-1}x_2^\mu f(x_1,x_2)x_1^\mu f(x_1,x_3)^{-1}x_3^\mu = 1. \tag{5.13}$$

for $x_1x_2x_3 = 1$.

Equations (5.12) and (5.13) imply the first two claims, (5.9) and (5.10), of the Theorem.

Note that relation between the generators x_{ij} of $\mathbb{PB}_n$ are invariant under their renormalizations, $x_{ij} \to c^{\lambda_{ij}}x_{ij}$, by powers of the central element $c = (b_1b_2\cdots b_n)^n$. Therefore, when computing $d_i(f(\mathsf{b}_2^2,\mathsf{b}_1^2))$, $i = 0,1,2,3$, in (5.5) we can replace b_1^2 and b_2^2 with b_2^2 and b_2^2. As one has in $\mathbb{PB}_4$

$$d_1(b_2^2) = x_{34},\quad d_1(b_1^2) = x_{13}x_{23},\quad d_3(b_2^2) = x_{23}x_{24},\quad d_3(b_1^2) = x_{12},$$

the l.h.s. of the "pentagon" equation (5.5) reads,

$$d_1F(\mathbb{1}_{1,2,3}) \circ d_3F(\mathbb{1}_{1,2,3}) = f(x_{34},x_{13}x_{23})f(x_{23}x_{24},x_{12}).$$

Continuing in the same way we conclude that (5.5) is equivalent to the following equality in $\widehat{\mathbb{PB}}_4(\mathbb{K})$,

$$f(x_{34},x_{13}x_{23})f(x_{23}x_{24},x_{12}) = f(x_{23},x_{12})f(x_{24}x_{34},x_{12}x_{13})f(x_{34},x_{23}).$$

Taking inverses of both sides and using (5.9) we obtain from the latter equation the third claim, equation (5.11), of the Theorem.

The final claim about the formula for the group multiplication in $\widehat{GT}(\mathbb{K})$ is left as an exercise. □

5.7.2. **Definition.** There is a surjection of groups,

$$\begin{array}{ccc} \widehat{GT}(\mathbb{K}) & \longrightarrow & \mathbb{K}^* \\ (f,\lambda) & \longrightarrow & \lambda. \end{array}$$

Its kernel is a subgroup of $\widehat{GT}(\mathbb{K})$ often denoted by $\widehat{GT}_1(\mathbb{K})$.

5.8. H. Furusho's Theorem. A remarkable result of H. Furusho [**Fu1**] gives us a much shorter algebraic characterization of elements of the Grothendieck-Teichmüller group than the one given in the above Theorem 5.7.1: the pentagon equation (5.11) often implies in the hexagon equations (5.9) and (5.10). More precisely, one has

5.8.1. **Theorem** [**Fu1**]. *Let $\mathbb{K}$ be a field of characteristic 0 and $\overline{\mathbb{K}}$ be its algebraic closure. Suppose that an element $f = f(x,y) \in \widehat{F}_2(\mathbb{K})$ satisfies Drinfeld's pentagon equation (5.11). Then there exists an element $\lambda \in \overline{\mathbb{K}}$ (which is unique up to a sign) such that the pair (λ, f) satisfies hexagon equations (5.9) and (5.10). Moreover, this λ is equal to $\pm\sqrt{24c_2(f)+1}$, where $c_2(f)$ stands for the coefficient of XY in the formal power series $f(e^X,e^Y)$.*

Therefore, the group $\widehat{GT}(\overline{\mathbb{K}})$ can be identified with the set of pairs $(f = f(x,y) \in \widehat{F}_2(\overline{\mathbb{K}}), \pm\sqrt{24c_2(f)+1} \in \overline{\mathbb{K}} \setminus 0)$, where f is a solution of Drinfeld's pentagon equation with $c_2(f) \neq -\frac{1}{24}$.

6. Infinitesimal braids, GRT_1 and associators

6.1. Graded Lie algebra of $\mathbb{PB}_n$. Let $\mathfrak{gr}_{\mathbb{K}}$ be the functor associating to a discrete group the graded Lie algebra obtained from the descending central series of that group (see §3.4(viii) for details). It was proven in [**Ko**] that $\mathfrak{gr}_{\mathbb{K}}(\mathbb{PB}_n)$ can be identified with the quotient of the free Lie algebra $\mathfrak{lie}_{n(n-1)/2}$ generated by the symbols $\{t_{ij}=t_{ji}\}_{1\leq i<j\leq n}$ modulo the ideal generated by relations,

$$\begin{aligned}[t_{ij},t_{kl}] &= 0 \quad \text{if} \quad \#\{i,j,k,l\}=4\\ [t_{ij},t_{ik}+t_{kj}] &= 0 \quad \text{if} \quad \#\{i,j,k\}=3\end{aligned}$$

The Lie algebra $\mathfrak{t}_n=\mathfrak{gr}_{\mathbb{K}}(\mathbb{PB}_n)$ is called the *Lie algebra of infinitesimal braids*. This is a filtered Lie algebra whose completion is denoted by $\widehat{\mathfrak{t}}_n$ (which is equal, of course, to $\widehat{\mathfrak{gr}}_{\mathbb{K}}(\mathbb{PB}_n)$).

6.1.1. **Lemma.** *The element $c_n:=\sum_{i<j}t_{ij}$ is central in $\mathfrak{t}_n$.*

PROOF. $[t_{ij},c_n]=\sum_{k\neq i,j}[t_{ij},t_{ik}+t_{jk}]=0.$ □

6.1.2. **Lemma.** (i) *The Lie subalgebra of $\mathfrak{t}_n$ generated by t_{ij} with $i,j\in[n-1]$ can be identified with $\mathfrak{t}_{n-1}$.*

(ii) *The Lie subalgebra of $\mathfrak{t}_n$ generated by $t_{1,n},t_{1,n}\ldots,t_{n-1,n}$ is a Lie ideal and can be identified with the free Lie algebra $\mathfrak{lie}_n$.*

(iii) *There exists an isomorphism of Lie algebras,*

$$\mathfrak{t}_n\simeq\mathfrak{t}_{n-1}\ \oplus\ \mathfrak{lie}_n.$$

The statement (i) is obvious. The statement (ii) follows from the fact that, for any $i,j,k\in[n-1]$, $i\neq j$, one has

$$[t_{ij},t_{kn}]=\begin{cases}0 & \text{if } k\neq i,k\neq j\\ [t_{in},t_{jn}] & \text{if } k=i.\end{cases}$$

We skip the proof of (iii).

6.1.3. **Corollary.** *There is an isomorphism of Lie algebras,*

$$\mathfrak{t}_3=\mathbb{K}c_3\oplus\mathfrak{lie}_2,$$

where $\mathbb{K}c_3$ is the Abelian Lie algebra generated by $c_3=t_{12}+t_{13}+t_{23}$, and $\mathfrak{lie}_2$ is the free Lie algebra generated by t_{13} and t_{23} (or, equivalently, by t_{12} and t_{23}).

6.1.4. **Exercise.** As $\mathbb{PB}_n$ is a group of finite type, its prounipotent completion $\widehat{\mathbb{PB}}_n(\mathbb{K})$ over a field $\mathbb{K}$ can be identified with the group $\exp(\widehat{\mathfrak{pb}}_n(\mathbb{K}))$, where the complete Lie algebra $\widehat{\mathfrak{pb}}_n(\mathbb{K})$ is the quotient of the free Lie algebra $\widehat{\mathfrak{lie}}_{n(n-1)/2}$ on letters $\{\gamma_{ij}\}_{1\leq i<j\leq n}$ modulo the ideal generated by following formal power series of Lie words,

$$\begin{aligned}\log\left(e^{\gamma_{ij}}e^{\gamma_{kl}}e^{-\gamma_{ij}}e^{-\gamma_{kl}}\right) &= 0 \ \text{ for } \quad i<j<k<l \ \text{ and } \ i<k<l<j,\\ \log\left(e^{\gamma_{ij}}e^{\gamma_{ik}}e^{\gamma_{jk}}e^{-\gamma_{ik}}e^{-\gamma_{jk}}e^{-\gamma_{ij}}\right) &= 0 \ \text{ for } \ i<j<k\\ \log\left(e^{\gamma_{ik}}e^{\gamma_{jk}}e^{\gamma_{ij}}e^{-\gamma_{jk}}e^{-\gamma_{ij}}\gamma^{-\gamma_{ik}}\right) &= 0 \ \text{ for } \ i<j<k\\ \log\left(e^{\gamma_{ik}}e^{\gamma_{jk}}e^{\gamma_{jl}}e^{-\gamma_{jk}}e^{-\gamma_{jk}}e^{-\gamma_{jl}}e^{\gamma_{jk}}e^{-\gamma_{ik}}\right) &= 0 \ \text{ for } \ i<j<k<l.\end{aligned}$$

The Lie algebra $\widehat{\mathfrak{pb}}_n(\mathbb{K})$ has a natural filtration,

$$\widehat{\mathfrak{pb}}_n(\mathbb{K})=F_1(\widehat{\mathfrak{pb}}_n(\mathbb{K}))\supset F_2(\widehat{\mathfrak{pb}}_n(\mathbb{K}))\supset\ldots\supset F_p(\widehat{\mathfrak{pb}}_n(\mathbb{K}))\supset F_{p+1}(\widehat{\mathfrak{pb}}_n(\mathbb{K}))\supset\ldots,$$

where $F_p(\widehat{\mathfrak{pb}}_n(\mathbb{K}))$ is the Lie subalgebra generated by Lie words with at at least p letters. Let

$$gr(\widehat{\mathfrak{pb}}_n(\mathbb{K})) := \prod_{i=1}^{\infty} \frac{F_{p+1}(\widehat{\mathfrak{pb}}_n(\mathbb{K}))}{F_p(\widehat{\mathfrak{pb}}_n(\mathbb{K}))}$$

be the associated graded Lie algebra, and let $[\gamma_{ij}]$ be the image of the generators under the projection $gr(\widehat{\mathfrak{pb}}_n(\mathbb{K})) \to \widehat{\mathfrak{pb}}_n(\mathbb{K})$.

Show that the association $t_{ij} \longrightarrow [\gamma_{ij}]$ induces an isomorphism of Lie algebras,

$$\alpha : \widehat{\mathfrak{t}}_n \longrightarrow gr(\widehat{\mathfrak{pb}}_n(\mathbb{K}))$$

6.2. Operadic structure on Lie algebras of infinitesimal braids [T2]. Consider a functor from the groupoid of finite sets to the category of Lie algebras,

$$\begin{array}{rccc} \mathfrak{t}: & \mathsf{S} & \longrightarrow & \mathsf{Lie}_{\mathbb{K}} \\ & I & \longrightarrow & \mathfrak{t}(I) \end{array}$$

where $t(I)$ is the Lie algebra of infinitesimal braids on the set I, that is, the quotient of the free Lie algebra generated by symbols $t_{ij} = t_{ji}$, $i, j \in I$, $i \neq j$, modulo the Lie ideal generated by the relations $[t_{ij}, t_{kl}] = 0$ whenever i, j, k, l are four different elements of I, and $[t_{ij}, t_{ik} + t_{jk}] = 0$ whenever i, j, k are three different elements. We set $\mathfrak{t}(I) = 0$ if cardinality of I is 1.

6.2.1. **Exercises.** (i) Show that for any *injective* morphism of sets $f : I \to J$ there is an associated *push-forward* homomorphism of Lie algebras

$$\begin{array}{rccc} f_*: & \mathfrak{t}(I) & \longrightarrow & \mathfrak{t}(J) \\ & t_{ij} & \longrightarrow & f(t_{ij}) := t_{f(i)f(j)}. \end{array}$$

(ii) Show that for any morphism of sets $f : I \to J$ there is an associated *pull-back* Lie algebra homomorphism

$$\begin{array}{rccc} f^*: & \mathfrak{t}(J) & \longrightarrow & \mathfrak{t}(I) \\ & t_{\alpha\beta} & \longrightarrow & f^*(t_{\alpha\beta}) := \sum_{\substack{i,j \in I \\ f(i)=\alpha, f(j)=\beta}} t_{ij}. \end{array}$$

(iii) Let

$$I \xrightarrow{f} J \xrightarrow{g} K$$

be a sequence of sets and their morphisms such that $\mathsf{Im}\, f \circ g$ consists of at most one element. Show that the Lie subalgebras $f_*(I), g^*(K) \subset \mathfrak{t}(J)$ satisfy

$$[f_*(I), g^*(K)] = 0.$$

6.2.2. **Lemma.** *The* S*-module* $\mathfrak{t}$ *is an operad in the category* $\mathsf{Lie}_{\mathbb{K}}$ *with respect to the composition which is given on the generators as follows,*

$$\begin{array}{rccll} \circ_k^{I,J}: & \mathfrak{t}(I) \oplus \mathfrak{t}(J) & \longrightarrow & \mathfrak{t}(I \setminus k \sqcup J) & \forall I, J,\ k \in I \\ & (0, t_{\alpha\beta}) & \longrightarrow & t_{\alpha\beta} & \forall\ \alpha, \beta \in J \\ & (t_{ij}, 0) & \longrightarrow & \begin{cases} t_{ij} & \text{if } i, j \in I \setminus k \\ \sum_{p \in J} t_{jp} & \text{if } i = k. \end{cases} & \end{array}$$

Proof. Consider the maps of sets,

$$\begin{array}{rccc} f: & J & \longrightarrow & I \setminus k \sqcup J \\ & p & \longrightarrow & f(p) := p \end{array}$$

and

$$g: \quad I \setminus k \sqcup J \quad \longrightarrow \quad I$$
$$x \quad \longrightarrow \quad g(x) := \begin{cases} i & \text{if } \ x = i \in I \setminus k \\ k & \text{if } \ x = p \in J. \end{cases}$$

with f being an injection, and $\mathsf{Im}\, f \circ g$ being an subset of I of cardinality 1. Then the operadic composition $\circ_k^{I,J}$ can be identified with the map $f_* \oplus g^*$. Hence the Exercises just above imply that $\circ_k^{I,J}$ is a Lie algebra homomorphism. We leave it to the reader to check that the homomorphisms $\circ_k^{I,J}$ satisfy the axioms of a (non-unital) operad. □

The S-module $\widehat{\mathfrak{t}}$ of *completed* Lie algebras of infinitesimal braids is, of course, also an operad on the category $\mathsf{Lie}_{\mathbb{K}}$.

6.2.3. **Exercise.** Check that the operadic composition

$$\circ_2: \quad \mathfrak{t}(3) \oplus \mathfrak{t}(3) \longrightarrow \quad \mathfrak{t}(5)$$
$$([t_{12}, t_{23}], [t_{13}, t_{23}]) \quad \longrightarrow \quad [t_{12}, t_{23}] \circ_2 [t_{13}, t_{23}]$$

gives

$$[t_{12}, t_{23}] \circ_2 [t_{13}, t_{23}] = [t_{12} + t_{13} + t_{14}, t_{25} + t_{35} + t_{45}] + [t_{24}, t_{34}]$$

6.3. Cosimplicial complex of the operad of infinitesimal braids. Let $e := 0$ be the zero element in $\mathfrak{t}(2)$. It obviously satisfies the equation (4.3) and hence makes by Lemma §4.5.1 the collection $\mathfrak{t} = \{\mathfrak{t}(n)\}_{\geq 1}$ into the cosimplicial object in the category $\mathsf{Lie}_{\mathbb{K}}$. The cohomology of the associated cosimplicial dg Lie algebra,

$$\mathsf{Simp}^\bullet(\mathfrak{t}) = \left(\bigoplus_{n \geq 1} \mathfrak{t}_n[2-n], d = \sum_i (-1)^i d_i \right),$$

was studied by Thomas Willwacher in his breakthrough paper [**W1**]. Its relation to the Grothendieck-Teichmüller group is discussed below.

6.4. Operad of chord diagrams. Applying the universal enveloping monoidal functor $U : \mathsf{Lie}_{\mathbb{K}} \to \mathsf{Hopf}_{\mathbb{K}}$ we get an operad

$$\mathcal{CD}_{\mathbb{K}} := U(\mathfrak{t}) = \{\mathcal{CD}_{\mathbb{K}}(n) := U(\mathfrak{t}(n))\}$$

called in [**B-N**] the operad of *chord diagrams* because its elements, say, elements of $\mathcal{CD}_{\mathbb{K}}(n)$, can be pictorially presented as a collection of chords on n vertical strands (labelled by $1, 2, \ldots, n$ from left to right) e.g.

$\mathcal{CD}_{\mathbb{K}}(3) \ni t_{23}t_{12} \longleftrightarrow$

As an associative algebra $\mathcal{CD}_{\mathbb{K}}(n)$, $n \geq 2$, is the quotient of the free non-commutative algebra generated by symbols $t_{ij} = t_{ji}$, $i \neq j$ $i, j \in [n]$, modulo the ideal generated by the relations $[t_{ij}, t_{kl}] = 0$ if $\#\{i, j, k, l\} = 4$ and $[t_{ij}, t_{ik} + t_{kj}] = 0$ if $\#\{i, j, k\} = 3$. The coproduct in $\mathcal{CD}_{\mathbb{K}}(n)$ is given on the generators as follows,

$$\Delta(t_{ij}) = 1 \otimes t_{ij} + t_{ij} \otimes 1.$$

The operadic structure in $\mathcal{CD}_{\mathbb{K}}$ is completely determined by the one in $\mathfrak{t}$. For example, one has

$$\circ_2: \quad \mathcal{CD}_{\mathbb{K}}(3) \otimes_{\mathbb{K}} \mathcal{CD}_{\mathbb{K}}(3) \longrightarrow \quad \mathcal{CD}_{\mathbb{K}}(5)$$
$$(t_{12}t_{23}, t_{13}t_{23}) \quad \longrightarrow \quad (t_{12} + t_{13} + t_{14})(t_{25} + t_{35} + t_{45})t_{24}t_{34}.$$

The I-adic completion of the associative algebra $\mathcal{CD}_{\mathbb{K}}(n)$ with respect to the maximal ideal I is denoted by $\widehat{\mathcal{CD}}_{\mathbb{K}}(n)$. The collection $\widehat{\mathcal{CD}}_{\mathbb{K}} = \{\widehat{\mathcal{CD}}_{\mathbb{K}}(n)\}_{n \geq 1}$ is an operad in the category of

complete filtered Hopf algebras. For our purposes it is useful to view $\widehat{\mathcal{CD}}_{\mathbb{K}}$ instead as an operad in the category $\mathsf{Cat}(\mathsf{Coass}_{\mathbb{K}})$ as follows: each competed filtered Hopf algebra $\widehat{\mathcal{CD}}_{\mathbb{K}}(n) = \widehat{U}(\mathfrak{t}_n)$ is understood from now on as a category with one object $\bullet$ and with $Mor(\bullet,\bullet) := \widehat{\mathcal{CD}}_{\mathbb{K}}(n)$ equipped with its standard coalgebra structure.

6.5. Cosimplicial structure on $\widehat{\mathcal{CD}}_{\mathbb{K}}$. The element $e := 1$ in $\mathcal{CD}(2)$ satisfies equations (4.3) so that formulae (4.4) make the collection $\widehat{\mathcal{CD}}_{\mathbb{K}} = \{\widehat{U}(\mathfrak{t}_n)\}_{n\geq 1}$ into a cosimplicial object in the category $\mathsf{dgAss}_{\mathbb{K}}$.

6.5.1. **Exercise.** Let $t_{12}^2 t_{23} \in \widehat{U}(\mathfrak{t}_3)$. Check that

$$d_0\left(t_{12}^2 t_{23}\right) = t_{23}^2 t_{34}, \quad d_1\left(t_{12}^2 t_{23}\right) = (t_{12}+t_{13})^2\, t_{34}, \quad d_2\left(t_{12}^2 t_{23}\right) = (t_{12}+t_{13})^2\,(t_{24}+t_{34})\,.$$

6.6. D. Bar-Natan's operad of parenthesized chord diagrams. We have two operads in the symmetric monoidal category $\mathsf{Cat}(\mathsf{Coass}_{\mathbb{K}})$, $\mathcal{P}a_{\mathbb{K}}$ defined in §5.6 and $\widehat{\mathcal{CD}}_{\mathbb{K}}$ defined just above. Hence

$$\widehat{\mathcal{PaCD}}_{\mathbb{K}} := \mathcal{P}a_{\mathbb{K}} \otimes_{\mathsf{Cat}(\mathsf{Coass}_{\mathbb{K}})} \widehat{\mathcal{CD}}_{\mathbb{K}}$$

is an operad in the category $\mathsf{Cat}(\mathsf{Coass}_{\mathbb{K}})$ called in [**B-N**] the operad of *parenthesized chord diagrams*. Elements of $\mathcal{P}a_{\mathbb{K}}$ can be pictorially represented as elements of $\mathcal{P}a\mathcal{B}_{\mathbb{K}}$ but with braiding of strands forgotten so that we represent morphisms in the category $\widehat{\mathcal{PaCD}}_{\mathbb{K}}(n)$ as linear combinations, with coefficients in the algebra $\widehat{\mathcal{CD}}_{\mathbb{K}}(n)$, of non-braided strands connecting identical symbols in a pair of parenthesized permutations of $[n]$, for example

$$\mathbb{H}_{1,2} := t_{12} \cdot \begin{array}{c} {\scriptstyle 1\ \ 2} \\ \vert\ \ \vert \\ {\scriptstyle 1\ \ 2} \end{array} \in \mathrm{Mor}_{\widehat{\mathcal{PaCD}}_{\mathbb{K}}(2)}(12,12), \quad \mathbb{X}_{1,2} = 1 \cdot \begin{array}{c} {\scriptstyle 2\ \ 1} \\ \times \\ {\scriptstyle 1\ \ 2} \end{array} \in \mathrm{Mor}_{\widehat{\mathcal{PaCD}}_{\mathbb{K}}(2)}(12,21)$$

$$\mathbb{1}_{1,2,3} := \begin{array}{c} {\scriptstyle 1\ \ (2\ \ 3)} \\ \vert\ \ \vert\ \ \vert \\ {\scriptstyle (1\ \ 2)\ \ 3} \end{array} \in \mathrm{Mor}_{\widehat{\mathcal{PaCD}}_{\mathbb{K}}(3)}\left((12)3), 1(23)\right)$$

6.6.1. **Exercise.** Show that the operad $\widehat{\mathcal{PaCD}}_{\mathbb{K}}$ is generated by $\mathbb{H}_{1,2}$, $\mathbb{X}_{1,2}$ and $\mathbb{1}_{1,2,3}$.

6.6.2. **Exercise.** Let $\mathbb{1}_{12}$ be the identity element in $\mathrm{Mor}_{\widehat{\mathcal{PaCD}}_{\mathbb{K}}(2)}(12,12)$,

$$\mathbb{1}_{1,2} := \begin{array}{c} {\scriptstyle 1\ \ 2} \\ \vert\ \ \vert \\ {\scriptstyle 1\ \ 2} \end{array}$$

Show that, for any $n \in \mathbb{N}$, the family of functors,

$$d_i : \widehat{\mathcal{PaCD}}_{\mathbb{K}}(n) \longrightarrow \widehat{\mathcal{PaCD}}_{\mathbb{K}}(n+1), \quad i = 0, 1, \ldots, n, n+1,$$

defined on objects and morphisms of the category $\widehat{\mathcal{PaCD}}_{\mathbb{K}}(n)$ by formulae identical to the ones in §5.5.2, makes the collection of categories $\{\widehat{\mathcal{PaCD}}_{\mathbb{K}}(n)\}_{n\geq 1}$ a (pre)cosimplicial object in the category $\mathsf{Cat}(\mathsf{Coass}_{\mathbb{K}})$.

6.6.3. **Exercise.** Check that

$$d_0\mathbb{H}_{1,2} = \mathbb{1}_{1,2} \circ_2 \mathbb{H}_{1,2} = t_{23} \begin{array}{c} {\scriptstyle 1\ \ (2\ \ 3)} \\ \vert\ \ \vert\ \ \vert \\ {\scriptstyle 1\ \ (2\ \ 3)} \end{array}, \qquad d_1\mathbb{H}_{1,2} = \mathbb{H}_{1,2} \circ_1 \mathbb{1}_{1,2} = (t_{13}+t_{23}) \begin{array}{c} {\scriptstyle (1\ \ 2)\ \ 3} \\ \vert\ \ \vert\ \ \vert \\ {\scriptstyle (1\ \ 2)\ \ 3} \end{array},$$

$$d_2\mathbb{H}_{1,2} = \mathbb{H}_{1,2} \circ_2 \mathbb{1}_{1,2} = (t_{12}+t_{13}) \begin{array}{c} {\scriptstyle 1\ \ (2\ \ 3)} \\ \vert\ \ \vert\ \ \vert \\ {\scriptstyle 1\ \ (2\ \ 3)} \end{array}, \qquad d_3\mathbb{H}_{1,2} = \mathbb{1}_{1,2} \circ_1 \mathbb{H}_{1,2} = t_{12} \begin{array}{c} {\scriptstyle (1\ \ 2)\ \ 3} \\ \vert\ \ \vert\ \ \vert \\ {\scriptstyle (1\ \ 2)\ \ 3} \end{array}.$$

6.6.4. **Exercise.** Let

$$\Phi = \Phi(t_{12}, t_{23}) \mathbb{1}_{1,2,3} \in \mathrm{Mor}_{\widehat{\mathcal{PaCD}}_{\mathbb{K}}(3)} ((12)3), 1(23))$$

for some formal power series $\Phi(t_{12}, t_{23}) \in \mathbb{K}\langle\langle t_{12}, t_{23}\rangle\rangle \subset \widehat{U}(\mathfrak{t}_3)$. Check that

$$d_0\Phi = \mathbb{1}_{1,2} \circ_2 \Phi = \Phi(t_{23}, t_{34}) d_0 \mathbb{1}_{1,2,3}, \quad d_1\Phi = \Phi \circ_1 \mathbb{1}_{1,2} = \Phi(t_{12} + t_{13}, t_{34}) d_1 \mathbb{1}_{1,2,3}$$

$$d_2\Phi = \Phi \circ_2 \mathbb{1}_{1,2} = \Phi(t_{12} + t_{13}, t_{24} + t_{34}) d_2 \mathbb{1}_{1,2,3},$$

$$d_3\Phi = \Phi \circ_3 \mathbb{1}_{1,2} = \Phi(t_{12}, t_{23} + t_{24}) d_3 \mathbb{1}_{1,2,3}, \quad d_4\Phi = \mathbb{1}_{1,2} \circ_1 \Phi = \Phi(t_{12}, t_{23}) d_4 \mathbb{1}_{1,2,3}.$$

6.7. Grothendieck-Teichmüller group GRT_1 and its Lie algebra. The group of automorphisms of the operad $\widehat{\mathcal{PaCD}}_{\mathbb{K}}$ which preserve elements $\mathbb{H}_{1,2}$ and $\mathbb{X}_{1,2}$ is called the *graded Grothendieck-Tecihmüller group* and is denoted by GRT_1 [**Dr2, B-N, T2**].

6.7.1. **Theorem.** *There is a one-to-one correspondence between elements of GRT_1 and grouplike elements, $\Phi(x, y) = e^{\phi(x,y)}$, of the completed universal enveloping (Hopf) algebra $\widehat{U}(\mathfrak{lie}_2) \simeq \mathbb{K}\langle\langle x, y\rangle\rangle$ of the free Lie algebra on generators x and y which satisfy the following equations*

$$\Phi(x, y) = \Phi^{-1}(y, x), \tag{6.1}$$

$$\Phi(t_{12}, t_{23})\Phi(t_{13}, t_{23})^{-1}\Phi(t_{13}, t_{12}) = 1 \quad \textit{in } \widehat{U}(\mathfrak{t}_3), \tag{6.2}$$

$$\Phi(t_{13} + t_{23}, t_{34})\Phi(t_{12}, t_{23} + t_{24}) = \Phi(t_{12}, t_{23})\Phi(t_{12} + t_{13}, t_{24} + t_{34})\Phi(t_{23}, t_{34}) \quad \textit{in } \widehat{U}(\mathfrak{t}_4). \tag{6.3}$$

Proof. Any automorphism F of the operad $\widehat{\mathcal{PaCD}}_{\mathbb{K}}$ is uniquely determined by its values on the generators $\mathbb{H}_{1,2}$, $\mathbb{X}_{1,2}$ and $\mathbb{1}_{1,2,3}$. These values are not arbitrary, they must respect the relations between the generators. Mac Lane coherence theorem for symmetric monoidal categories implies that all the relations between the generators follow from the following two ones:

$$\text{Pentagon}: \quad d_1\mathbb{1}_{1,2,3} \circ d_3\mathbb{1}_{1,2,3} = d_4\mathbb{1}_{1,2,3} \circ d_2\mathbb{1}_{1,2,3} \circ d_0\mathbb{1}_{1,2,3}$$

$$\text{Hexagon}: \quad d_1\mathbb{X}_{1,2} = \mathbb{1}_{1,2,3} \circ d_0\mathbb{X}_{1,2} \circ \left((23)\cdot\mathbb{1}_{1,2,3}^{-1}\right) \circ {}_{(23)}\cdot d_3\mathbb{X}_{1,2} \circ {}_{(321)}\cdot\mathbb{1}_{1,2,3}.$$

As an element of GRT_1 preserves, by definition, the generators $\mathbb{H}_{1,2}$ and $\mathbb{X}_{1,2}$, we conclude that such an element F is uniquely determined by its value on $\mathbb{1}_{1,2,3}$,

$$F(\mathbb{1}_{1,2,3}) \in \mathrm{Mor}_{\widehat{\mathcal{PaCD}}_{\mathbb{K}}(3)} ((12)3, 1(23)) \simeq \widehat{U}(\mathfrak{t}_3),$$

which must satisfy the equations,

$$d_1F(\mathbb{1}_{1,2,3}) \circ d_3F(\mathbb{1}_{1,2,3}) = d_4F(\mathbb{1}_{1,2,3}) \circ d_2F(\mathbb{1}_{1,2,3}) \circ d_0F(\mathbb{1}_{1,2,3}) \tag{6.4}$$

and

$$d_1X_{1,2} = F(\mathbb{1}_{1,2,3}) \circ d_0X_{1,2} \circ \left((23)\cdot F(\mathbb{1}_{1,2,3})^{-1}\right) \circ {}_{(23)}\cdot d_3X_{1,2} \circ {}_{(321)}\cdot F(\mathbb{1}_{1,2,3}). \tag{6.5}$$

By Corollary §6.1.3

$$\widehat{U}(\mathfrak{t}_3) = \mathbb{K}[[c_3]] \times \mathbb{K}\langle\langle t_{12}, t_{23}\rangle\rangle$$

so that

$$F(\mathbb{1}_{1,2,3}) = f(c_3)\Phi(t_{12}, t_{23})$$

for some invertible formal power series,

$$f(c_3) = a_0 + a_1c_3 + a_2c_3^2 + \ldots, \quad a_0 \neq 0$$

in $c_3 = t_{12} + t_{13} + t_{23}$, and some invertible formal power series $\Phi \in \widehat{U}(\mathfrak{lie}_2)$. As $\mathbb{1}_{1,2,3}$ is grouplike in the coalgebra $\mathrm{Mor}_{\widehat{\mathcal{PaCD}}_{\mathbb{K}}(3)} ((12)3, 1(23))$, Φ is also grouplike, and hence it is of the form e^{ϕ} for some $\phi \in \widehat{\mathfrak{lie}}_2$.

The hexagon equation (6.5) implies,

$$\Phi(t_{12},t_{23})\Phi(t_{13},t_{23})^{-1}\Phi(t_{13},t_{12})=1,\quad f(c_3)=1.$$

The hexagon equation is $\mathbb{S}_3$-equivariant. Applying to that equation permutation (32), or, equivalently, applying this permutation to the above equation for Φ, we get,

$$\Phi(t_{13},t_{23})\Phi(t_{12},t_{23})^{-1}\Phi(t_{12},t_{13})=1$$

and hence

$$\Phi(t_{12},t_{13})\Phi(t_{13},t_{12})=1.$$

This proves claims (6.1) and (6.2). The last claim follows immediately from the pentagon equation for $F(\mathbb{1}_{1,2,3})$ and the definition of the functors d_i. We leave the details to the reader. □

The group GRT_1 is prounipotent. Hence it is of the form $\exp(\mathfrak{grt}_1)$ where the Lie algebra $\mathfrak{grt}_1$ of GRT_1 can be identified with elements $\phi(x,y)\in\widehat{\mathfrak{lie}}_2$ which satisfy the equations

$$\phi(t_{13}+t_{23},t_{34})+\phi(t_{12},t_{23}+t_{24})=\phi(t_{12},t_{23})+\phi(t_{12}+t_{13},t_{24}+t_{34})+\phi(t_{23},t_{34}) \tag{6.6}$$

in $\widehat{\mathfrak{t}}_4$, and

$$\phi(x,y)=-\phi(y,x) \tag{6.7}$$

$$\phi(x,y)+\phi(y,-x-y)+\phi(-x-y,x). \tag{6.8}$$

6.7.2. **Exercise.** Show that $\phi(x,y)=[x,y]$ satisfies equations (6.6) and (6.7), but not (6.8).

H. Furusho [**Fu1**] has proven that $\mathfrak{grt}_1$ can be identified with elements $\phi(x,y)\in\widehat{\mathfrak{lie}}_2$ which satisfy the pentagon equation (6.6) and do not contain $[x,y]$ as a summand. Put another, in this case hexagon equations (6.7) and (6.8) follow from (6.6).

6.7.3. **Proposition.** *Let* $\mathsf{Simp}^\bullet(\mathfrak{t})$ *be the cosimplicial complex of the operad of infinitesimal braids (see (6.3)). Then*

$$H^1(\mathsf{Simp}^\bullet(\mathfrak{t}))=\mathfrak{grt}_1\oplus\mathbb{K}$$

where the direct summand $\mathbb{K}$ *is generated by* $[t_{12},t_{23}]\in\mathfrak{t}_3$.

This is a reformulation of Proposition E.1 in [**W1**]. It gives a second proof of H. Furusho's result discussed just above.

6.8. Associators. We have operads $\widehat{\mathcal{P}a\mathcal{B}}_{\mathbb{K}}$ and $\widehat{\mathcal{P}a\mathcal{C}\mathcal{D}}_{\mathbb{K}}$ in the category $\mathsf{Cat}(\mathsf{coAss}_{\mathbb{K}})$. An isomorphism of operads (if it exists)

$$\mathcal{A}:\widehat{\mathcal{P}a\mathcal{B}}_{\mathbb{K}}\longrightarrow\widehat{\mathcal{P}a\mathcal{C}\mathcal{D}}_{\mathbb{K}}$$

is called an *associator* [**Dr2, B-N**]. It is not hard to show the following

6.8.1. **Theorem.** *There is a one-to-one correspondence between the set of associators and grouplike elements,* $\Phi(x,y)=e^{\phi(x,y)}$, *of the completed filtered Hopf algebra* $\widehat{U}(\mathfrak{lie}_2)$ *of the free Lie algebra on generators* x *and* y *which satisfy the following equations*

$$\Phi(x,y)=\Phi^{-1}(y,x), \tag{6.9}$$

$$\Phi(t_{12},t_{23})e^{-\frac{1}{2}t_{23}}\Phi(t_{13},t_{23})^{-1}e^{-\frac{1}{2}t_{13}}\Phi(t_{13},t_{12})e^{-\frac{1}{2}(t_{13}+t_{23})}=1\quad \textit{in } U(\widehat{\mathfrak{t}}_3), \tag{6.10}$$

$$\Phi(t_{13}+t_{23},t_{34})\Phi(t_{12},t_{23}+t_{24})=\Phi(t_{12},t_{23})\Phi(t_{12}+t_{13},t_{24}+t_{34})\Phi(t_{23},t_{34})\quad \textit{in } \widehat{U}(\mathfrak{t}_4). \tag{6.11}$$

It is much harder to show that for any field $\mathbb{K}$ of characteristic zero the set of associators,

$$\mathsf{As}_{\mathbb{K}}:=\{\widehat{\mathcal{P}a\mathcal{B}}_{\mathbb{K}}\to\widehat{\mathcal{P}a\mathcal{C}\mathcal{D}}_{\mathbb{K}}\}$$

is non-empty, see [**Dr2**] for the proof. The equations defining associators are algebraic, but the only two explicit solutions we know involve transcendental methods in their constructions.

6.8.2. **Example: Knizhnik-Zamolodchikov associator.** The first associator was constructed by V. Drinfeld with the help of the monodromy of the flat Knizhnik-Zamolodchikov connection and is known nowadays as the Knizhnik-Zamolodchikov associator $\Phi_{KZ}(x,y)$; this is a formal power series in two non-commutative formal variable x and y of the form,

$$\Phi_{KZ}(x,y) = 1 + \sum_{\substack{m,k_1,\dots,k_{m-1}\in\mathbb{N}_{\geq 1}\\ k_m\in\mathbb{N}_{\geq 2}}} (-1)^m \zeta(k_1,\dots,k_m) x^{k_m-1} y \cdots x^{k_1-1} y + \text{regularized terms},$$

where $\zeta(k_1,\dots,k_m)$ is the multiple zeta value, the real number defined by the following converging sum,

$$\zeta(k_1,\dots,k_m) = \sum_{0<n_1<\dots<n_m} \frac{1}{n_1^{k_1}\cdots n_m^{k_m}}.$$

For $m=1$ this gives us the Riemann zeta function $\zeta(k_1)$, $k_1>1$. There is an explicit iterative construction of the *regularized terms* in the above formula in terms of multiple zeta values so that *all* coefficients of Φ_{KZ} are rational linear combinations of that values.

6.8.3. **Example: Alekseev-Torossian associator.** The second explicit associator $\Phi_{AT}(x,y)$ was constructed by A. Alekseev and C. Torossian in [**AT1**] using Fulton-MacPherson's compactified configuration spaces of points in the complex plane and the integration theory of singular differential forms on semialgebraic chains. It was proved to be a Drinfeld associator in [**SW**]. H. Furusho found a method in [**Fu2**] of computing coefficients of the formal power series $\Phi_{AT}(x,y)$ in terms of rational linear combinations of iterated integrals of M. Kontsevich weight differential forms associated to Lie graphs (see also a nice paper [**BPP**] on the systematic computation of weights of the M. Kontsevich formality map).

7. Grothendieck-Teichmüller group, graph complexes and T.Willwacher theorems

7.1. Operads of graphs. In this section we consider only graphs Γ without hairs (see §4.3.4) which are called from now on simply *graphs*. Recall that the set of vertices of Γ is denoted by $V(\Gamma)$ and the set of edges by $E(\Gamma)$. A graph Γ is called *directed* if each edge $e=(h,\tau(h))\in E(\Gamma)$ comes equipped with a choice of a direction, that is, with a total ordering of its set $\{h,\tau(h)\}$ of half-edges. Here are a few examples of directed graphs

Let $G_{n,l}$ be the set of directed graphs Γ with n vertices and l edges such that some bijections $V(\Gamma)\to[n]$ and $E(\Gamma)\to[l]$ are fixed, i.e. every edge and every vertex of Γ is marked. The permutation group $\mathbb{S}_l$ acts on $G_{n,l}$ by relabeling the edges so that, for each any integer d, it makes sense to consider a collection of $\mathbb{S}_n$-modules,

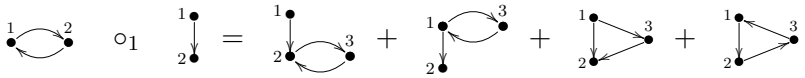
$$\mathcal{DGra}_d = \left\{ \mathcal{DGra}_d(n) := \prod_{l\geq 0} \mathbb{K}\langle G_{n,l}\rangle \otimes_{\mathbb{S}_l} sgn_l^{\otimes|d-1|}[l(d-1)] \right\}_{n\geq 1}$$

with the group $\mathbb{S}_n$ acting on $\mathcal{DGra}_d(n)$ by relabeling the vertices. This is a $\mathbb{Z}$-graded vector space obtained by assigning to each edge of a generating graph from $G_{n,l}$ the homological degree $1-d$. Note that if d is even, then each graph $\Gamma\in\mathcal{DGra}_d(n)$ is assumed to come equipped with a choice of ordering of edges up to an even permutation (an odd permutation acts as the multiplication by -1). In particular, the graph $\in\mathcal{DGra}_{d\in 2\mathbb{Z}}(2)$ vanishes identically as it admits an automorphism which changed the ordering of edges by an odd permutation, i.e. it is equal to minus itself.

This $\mathbb{S}$-module is an operad with respect to the following operadic composition,

$$\begin{array}{rccc} \circ_i: & \mathcal{DG}ra_d(n)\times\mathcal{DG}ra_d(m) & \longrightarrow & \mathcal{DG}ra_d(m+n-1),\ \ \forall\ i\in[n] \\ & (\Gamma_1,\Gamma_2) & \longrightarrow & \Gamma_1\circ_i\Gamma_2, \end{array}$$

where $\Gamma_1\circ_i\Gamma_2$ is defined by substituting the graph Γ_2 into the i-labeled vertex of Γ_1 and taking a sum over all possible re-attachments of dangling edges (attached before to v_i) to the vertices of Γ_2. For example,

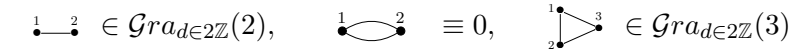


Note that for $d\in 2\mathbb{Z}$ one has to choose of an ordering of edges in each graph summand of the composition $\Gamma_1\circ_i\Gamma_2$, and there is a canonical way to do by, roughly speaking, placing all the edges of Γ_1 in front of the edges of Γ_2.

One can define an "undirected version" of the operad above by noticing that the group $\mathbb{S}_l\ltimes(\mathbb{S}_2)^l$ acts on set $G_{n,l}$ of directed labelled graphs by relabeling the edges and reversing the directions of the edges. Hence for each fixed integer $d\in\mathbb{Z}$, one can consider a collection of $\mathbb{S}_n$-modules,

$$\mathcal{G}ra_d(n):=\prod_{l\geq 0}\mathbb{K}\langle G_{n,l}\rangle\otimes_{\mathbb{S}_l\ltimes(\mathbb{S}_2)^l}sgn_l^{|d|}\otimes sgn_2^{\otimes l|d-1|}[l(d-1)]$$

where the group $\mathbb{S}_n$ acts by relabeling the vertices. For d even elements $\mathcal{G}ra_d(n)$ can be understood as *undirected* graphs, e.g.



while for d odd one has identifications of the type

7.1.1. **Exercise.** Show that graphs in $\mathcal{G}ra_{d\in 2\mathbb{Z}}$ which have multiple edges vanish identically.

7.1.2. **Remark.** The integer parameter d has sometimes a clear geometric meaning. With operads of graphs $\mathcal{DG}ra_d$ and $\mathcal{G}ra_d$ for $d\geq 2$ one can associate de Rham field theories on various compactified configuration spaces of points in (subspaces of) $\mathbb{R}^d$, the most prominent of which is the Kontsevich formality theory [**K2**] for $d=2$. Other examples of $d=2$ field theories have been studied in, e.g., [**Me2, Sh, W5**]; for an example of a highly non-trivial $d=3$ de Rham field theory we refer to [**MW5**]. It seems that to get *non-trivial* (in the sense that GRT_1 plays a classifying role, see §8) de Rham field theories for $d\geq 4$ one has to work with operads/complexes of *multi-oriented* graphs which are discussed below.

7.2. M. Kontsevich graph complexes and T. Willwacher theorems. As discussed in §4.6 above, for any operad $\mathcal{O}=\{\mathcal{O}(n)\}_{n\geq 1}$ in the category of graded vector spaces the linear map

$$\begin{array}{rccc} [\ ,\ ]: & \mathcal{O}^{tot}\otimes\mathcal{O}^{tot} & \longrightarrow & \mathcal{O}^{tot} \\ & (a\in\mathcal{P}(n), b\in\mathcal{P}(m)) & \longrightarrow & [a,b]:=\sum_{i=1}^n a\circ_i b-(-1)^{|a||b|}\sum_{i=1}^m b\circ_i a\ \in\mathcal{P}(m+n-1) \end{array}$$

makes a graded vector space $\mathcal{O}^{tot}:=\prod_{n\geq 1}\mathcal{P}(n)$ into a Lie algebra [**KM**]; moreover, these brackets induce a Lie algebra structure on the subspace of invariants $\mathcal{O}^{tot}_{\mathbb{S}}:=\prod_{n\geq 1}\mathcal{O}(n)^{\mathbb{S}_n}$. For any $d\in\mathbb{Z}$ one can consider a degree shifted operad $\mathcal{O}\{d\}$ (see §4.4) and conclude that for any operad $\mathcal{O}$ and any integer d the associated graded vector space

$$\mathcal{O}\{d\}^{tot}_{\mathbb{S}}\simeq\prod_{n\geq 1}\mathcal{O}(n)\otimes_{\mathbb{S}_n}sgn_n^{\otimes|d|}[d(1-n)]$$

is canonically a Lie algebra[8]. In particular, the graded vector spaces (the "directed full graph complex" and, respectively, the "full graph complex")

$$\mathsf{dfGC}_d := \prod_{n\geq 1} \mathcal{DG}ra_d(n) \otimes_{\mathbb{S}_n} sgn_n^{\otimes |d|}[d(1-n)]$$

and

$$\mathsf{fGC}_d := \prod_{n\geq 1} \mathcal{G}ra_d(n) \otimes_{\mathbb{S}_n} sgn_n^{\otimes |d|}[d(1-n)]$$

are Lie algebras with Lie brackets given by the substitution of a graph into a vertex of another graph as explained above. Note that the homological degree of graph Γ from dfGC_d or fGC_d is given by

$$|\Gamma| = d(\#V(\Gamma) - 1) + (1-d)\#E(\Gamma).$$

A generator $\Gamma \in \mathsf{dfGC}_d$ can be understood as a *directed* graph (with *no* labeling of vertices or edges) equipped with an orientation *or* which is, by definition, a unital vector in the following 1-dimensional Euclidian space

$$\mathbb{R}_\Gamma := \begin{cases} \wedge^{\#E(\Gamma)}\mathbb{K}[E(\Gamma)] & \text{if } d \in 2\mathbb{Z} \\ \wedge^{\#V(\Gamma)}\mathbb{R}[V(\Gamma)] & \text{if } d \in 2\mathbb{Z}+1 \end{cases}$$

Every directed graph has precisely two possible orientations, *or* and or^{opp}, and one identifies $(\Gamma, or) = -(\Gamma, or^{opp})$. We often abbreviate (Γ, or) to Γ.

A generator $\Gamma \in \mathsf{fGC}_d$ can be understood as an *undirected* graph (with *no* labeling of vertices or edges) equipped with an orientation *or* which for d even is the same as defined just above, while for d odd it is defined as a unital vector in the 1-dimensional Euclidian vector space

$$\mathbb{R}_\Gamma := \wedge^{\#V(\Gamma)}\mathbb{R}[V(\Gamma)] \otimes \bigotimes_{e=(h,\tau(h))\in E(\Gamma)} \wedge^2\mathbb{R}[h, \tau(h)]$$

where each edge is interpreted as the orbit consisting of two half-edges under the involution τ (see §4.3.4). Put another way, for d even an orientation *or* is a choice of ordering of edges (up to an even permutation), while for d odd *or* is a choice of ordering of vertices (up to an even permutation) and a choice of direction on each edge (up to a flip and multiplication by -1). Every undirected graph has precisely two possible orientations, *or* and or^{opp}, and one identifies $(\Gamma, or) = -(\Gamma, or^{opp})$.

7.2.1. **Exercise.** Show that the graph

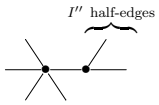
$$\gamma_0 := \quad \bullet\!\!-\!\!\!-\!\!\bullet \quad \in \mathsf{fGC}_d \quad \text{or} \quad \gamma_0 := \quad \bullet\!\!\longrightarrow\!\!\bullet \quad \in \mathsf{dfGC}$$

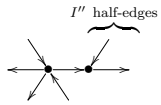
is a Maurer-Cartan element, i.e. it has degree 1 and satisfies $[\gamma_0, \gamma_0] = 0$

Hence Lie algebras fGC_d and, respectively, dfGC_d come equipped with a compatible differential

$$\delta\Gamma := [\Gamma, \gamma_0] = \sum_{v\in V(\Gamma)} \left(\delta'_v\Gamma - (-1)^{|\Gamma|}\delta''_v\Gamma\right)$$

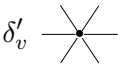
where δ'_v splits the vertex v of Γ into two vertices

δ'_v $=$ $\sum_{\substack{H(v)=I'\sqcup I'' \\ \#I',\#I''\geq 0}}$ 

and resp. δ'_v $:=$ $\sum_{\substack{H(v)=I'\sqcup I'' \\ \#I',\#I''\geq 0}}$ 

[8] In fact, this Lie algebra controls the deformation theory of the trivial morphism of operads $0 : \mathcal{L}ie\{1-d\} \to \mathcal{O}$, where $\mathcal{L}ie$ is the operad of Lie algebras. This observation gives us a very useful approach to the Kontsevich graph complexes and their ribbon graphs generalizations (see [**W1, MW2**]) which we skip in this survey.

connected by a (directed) edge and redistributes the attached half-edges to v (if any) along the two new vertices in all possible ways, while δ''_v attaches to v a new univalent vertex

δ'_v  $= 2$ and resp. δ'_v $:=$ $+$

Note that if the vertex v has at least one half-edge, then the two terms with univalent vertices in the sum $\delta'_v\Gamma$ cancel out with the two terms in $\delta''_v\Gamma$ so that in most cases one can ignore in the above formula for δ the second term δ''_v and assume in the first term δ'_v that the summation runs over decompositions $H(v) = I' \sqcup I''$ with $\#I' \geq 1$ and $\#I'' \geq 1$. For example, the graph γ_0 is a cycle, $\delta\gamma_0 = [\gamma_0, \gamma_0] = 0$, but the associated cohomology class is trivial, $\gamma_0 = -\delta(\bullet)$.

The dg Lie algebra (fGC_d, δ) was introduced by M.Kontsevich in [**K1**] in his attempt to calculate obstructions to universal quantizations of Poisson structures.

Both dg Lie algebras contain dg Lie subalgebras, fcGC_d and dfcGC_d, generated by *connected* graphs. There is an isomorphism of complexes,

$$\mathsf{fGC}_d = \odot^{\bullet \geq 1} (\mathsf{fcGC}_d[-d])\,[d], \quad \mathsf{dfGC}_d = \odot^{\bullet \geq 1} (\mathsf{dfcGC}_d[-d])\,[d]$$

so that at the cohomology level

$$H^\bullet(\mathsf{fGC}_d) = \odot^{\bullet \geq 1} (H^\bullet(\mathsf{fcGC}_d)[-d])\,[d], \quad H^\bullet(\mathsf{dfGC}_d) = \odot^{\bullet \geq 1} (H^\bullet(\mathsf{dfcGC}_d)[-d])\,[d]$$

Hence there is no loss of important information when working with connected graphs only.

Moreover, there is a monomorphism of graph complexes

$$\mathsf{fcGC}_d \longrightarrow \mathsf{dfcGC}_d \tag{7.1}$$

which sends an undirected graph Γ into a sum of graphs obtained by interpreting each edge as the sum of edges in both directions. It was proven in [**W1**](Appendix K) that this map is a *quasi-isomorphism*. Hence there is no need to study the complex dfGC_d by itself[9] so that we continue in this subsection discussing only the complex fcGC_d. That complex splits as a direct sum

$$\mathsf{fcGC}_d = \mathsf{fcGC}_d^{\leq 1} \oplus \mathsf{fcGC}_d^2 \oplus \mathsf{GC}_d^{\circlearrowleft}$$

where $\mathsf{fcGC}^{\leq 1}$ is spanned by connected graphs having at least one vertex of valency ≤ 1, fcGC_d^2 is spanned by connected graphs with no vertices of valency ≤ 1 but with at least one vertex of valency 2, and $\mathsf{GC}_d^{\circlearrowleft}$ is generated by graphs with each vertex of valency ≥ 3. It was proven in [**W1**] that the complex $\mathsf{fcGC}_d^{\leq 1}$ is acyclic while

$$H^\bullet(\mathsf{fcGC}_d^2) = \bigoplus_{\substack{j \geq 1 \\ j \equiv 2d+1 \mod 4}} \mathbb{K}[d-j],$$

where the summand $\mathbb{K}[d-j]$ is generated by the polytope with j vertices, that is, a connected graph with j bivalent vertices, e.g. for $j = 3$ (in fact this particular graph vanishes identically for d *even* as in this case it admits an automorphism reversing its orientation).

The complex $\mathsf{GC}_d^{\circlearrowleft}$ contains a subcomplex GC_d spanned by graphs with no *loops*, that is, edges attached to one and the same vertex, and the inclusion is a quasi-isomorphism [**W1**],

$$H^\bullet(\mathsf{GC}_d) = H^\bullet(\mathsf{GC}_d^{\circlearrowleft}).$$

One of the major results in [**W1**] is the following

[9] However the complex dfcGC_d contains two very important subcomplexes of *oriented* graphs and of *sourced* graphs which are of great significance in applications (see below).

7.2.2. **T.Willwacher Theorem.** *There is an isomorphism of Lie algebras,*

$$H^0(\mathsf{GC}_2) = \mathfrak{grt}_1,$$

where $\mathfrak{grt}_1$ *is the Lie algebra of the Grothendieck-Teichmüller group* GRT_1 *(see §6). Moreover,* $H^{\bullet<0}(\mathsf{GC}_2) = 0$.

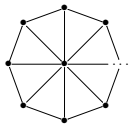
There is an explicit construction of infinitely many cohomology classes $\{[\mathfrak{w}_{2n+1}]\}_{n\geq 1}$ in $H^0(\mathsf{GC}_2)$, more precisely, of their cycle representatives $\{\mathfrak{w}_{2n+1}\}_{n\geq 1}$ in GC_2. The first two classes can be given explicitly as follows

$\mathfrak{w}_3 =$

$\mathfrak{w}_5 =$ $+ \frac{5}{2}$

For higher n there is an explicit transcendental formula for the cycles $\mathfrak{w}_{2n+1}$ given in [**RW**] which presents each cycle as a linear combination of graphs with $2n+2$ vertices and $4n+2$ edges

$$\mathfrak{w}_{2n+1} = \sum_{\Gamma\in G_{2n+2,4n+2}} c_\Gamma \Gamma = \quad \lambda_{2n+1} \quad + \ \ldots \tag{7.2}$$



where the weights c_Γ are given by explicit converging integrals over the configuration space of $2n$ different points in $\mathbb{C}\setminus\{0,1\}$ (see (23) in [**RW**]). In particular, the coefficients λ_{2n+1} of the wheel-type summands in $\mathfrak{w}_{2n+1}$ are equal to zeta values $\frac{\zeta(2n+1)}{(2\pi i)^{(2n+1)}}$ up to a non-zero rational factor, cf. [**G**]. (In fact, from the presence in $\mathfrak{w}_{2n+1}$ of such wheel-graphs one can conclude almost immediately that these cycles are *not* boundaries.)

7.2.3. **Deligne-Drinfeld-Ihara conjecture.** *The pro-nilpotent Lie algebra* $\mathfrak{grt}_1$ *is isomorphic as a* $\mathbb{Z}_{\geq 3}$ *graded Lie algebra to the degree completion of the free Lie algebra generated by formal variables of degrees* $2n+1$, $n\geq 1$.

It has been proved by F. Brown in [**B**] that $\mathfrak{grt}_1$ does contain indeed the above mentioned free Lie algebra.

Thus the graph cohomology classes $[\mathfrak{w}_{2n+1}]$ might generate the whole $\mathfrak{grt}_1$.

Denote by $\mathsf{GC}_d^{\geq 2}$ the dg Lie subalgebra of fcGC_d spanned by graphs with valency of each vertex ≥ 2 and with no loops. Then it follows from the results discussed above that

$$H^\bullet(\mathsf{fcGC}_d) = H^\bullet(\mathsf{GC}_d^{\geq 2}) = H^\bullet(\mathsf{GC}_d) + \bigoplus_{\substack{j\geq 1\\ j\equiv 2d+1 \mod 4}} \mathbb{K}[d-j].$$

There are other differentials on the Kontsevich graph complexes GC_d which have been studied in [**MW1, KWZ1**].

7.3. Oriented graph complexes. A graph Γ from the dg Lie algebra dfGC_d is called *oriented* if it contains no *wheels*, that is, directed paths of edges forming a closed circle. The subspace $\mathsf{fGC}_d^{or}\subset \mathsf{dfGC}_d$ spanned by oriented graphs is a dg Lie subalgebra. For example,

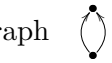
$\in \mathsf{fGC}_d^{or}$ while $\notin \mathsf{fGC}_d^{or}$.

Let dfcGC_d^{or} the subcomplex of fGC_d^{or} spanned by connected graphs, and $\mathsf{GC}_d^{\mathrm{or}}\subset \mathsf{dfcGC}_d^{or}$ the subcomplex spanned by graphs which each vertex at least bivalent and with no bivalent vertices of the form . Then [**W3**]

$$H^\bullet(\mathsf{fcGC}_d^{or}) = H^\bullet(\mathsf{GC}_d^{\mathrm{or}})$$

so that there no loss of important information when working with $\mathsf{GC}_d^{\mathrm{or}}$ solely.

7.3.1. **Theorem [W3].** $H^\bullet(\mathsf{GC}_{d+1}^{\mathrm{or}}) = H^\bullet(\mathsf{GC}_d^{\geq 2})$.

Hence one has a remarkable isomorphism of Lie algebras, $H^0(\mathsf{GC}_3^{\mathrm{or}}) = \mathfrak{grt}$. Moreover it follows from this theorem that $H^i(\mathsf{GC}_3^{\mathrm{or}}) = 0$ for $i \leq -2$ and $H^{-1}(\mathsf{GC}_3^{\mathrm{or}})$ is a 1-dimensional space generated by the graph  . This theorem tells us that in order to have some non-trivial de Rham field theory in dimension 3 (rather than two), one has to work with *oriented* graphs rather than with simply directed (or even undirected) graphs. We refer to [**MW5**] for an example of such a de Rham field theory providing us with an *explicit* universal formula for deformation quantization of Lie bialgebras.

The original argument in [**W2**] does not give us an explicit relation between the cohomology groups in the above Theorem. Let $\widetilde{\mathsf{GC}_{d+1}^{\mathrm{or}}}$ and $\widetilde{\mathsf{GC}_d^{\geq 2}}$ be the graph complexes dual to $\mathsf{GC}_{d+1}^{\mathrm{or}}$ and $\mathsf{GC}_d^{\geq 2}$ respectively. Then one also has $H^\bullet(\widetilde{\mathsf{GC}_{d+1}^{\mathrm{or}}}) = H^\bullet(\widetilde{\mathsf{GC}_d^{\geq 2}})$. An explicit construction sending a homology class in $H^\bullet(\widetilde{\mathsf{GC}_d^{\geq 2}})$ to a cohomology class in $H^\bullet(\widetilde{\mathsf{GC}_{d+1}^{\mathrm{or}}})$ has been found in [**Z1**]. It can be used to find an explicit 3-dimensional oriented incarnation $\mathfrak{w}_3^{or} \in H^0(\mathsf{GC}_3^{\mathrm{or}})$ of, for example, the tetrahedron cohomology class $\mathfrak{w}_3 \in H^0(\mathsf{GC}_2)$ discussed above; $\mathfrak{w}_3^{or}$ is a linear combination of oriented graphs with 7 vertices and 9 edges.

In fact, the complex $\mathsf{GC}_3^{\mathrm{or}}$ controls the deformation theory of the properad of Lie bialgebras so that GRT_1 is essentially a symmetry group of that properad [**MW4**]. In string topology and the theory of moduli spaces of algebraic curves it is important to work with *involutive* Lie bialgebras, and their deformation theory is controlled by the following deformation of the complex $\mathsf{GC}_3^{\mathrm{or}}$.

7.3.2. **A deformation of $\mathsf{GC}_3^{\mathbf{or}}$.** Consider a Lie algebra $(\mathsf{GC}_3^{\mathrm{or}}[[\hbar]], [\ ,\ ])$, where $\mathsf{GC}_3^{\mathrm{or}}[[\hbar]]$ is the (completed) topological vector space spanned by formal power series in a formal parameter $\hbar$ of homological degree 2, and $[\ ,\ ]$ are the Lie brackets obtained from the standard ones in $\mathsf{GC}_d^{\mathrm{or}}$ by the $\mathbb{K}[[\hbar]]$-linearity. It was shown in [**CMW**] that the formal power series

$$\gamma_\hbar := \sum_{k=1}^{\infty} \hbar^{k-1} \underbrace{\quad}_{k \text{ edges}}$$

is a Maurer-Cartan element in the Lie algebra $(\mathsf{fGC}_3^{or}[[\hbar]], [\ ,\ ])$ and hence makes the latter into a *differential* Lie algebra with the differential $\delta_\hbar \Gamma := [\Gamma, \gamma_\hbar]$. It was proven in [**CMW**] that $H^0(\mathsf{GC}_3^{or}[[\hbar]], \delta_\hbar) \simeq H^0(\mathsf{GC}_3^{or}, \delta) \simeq \mathfrak{grt}_1$ as Lie algebras. Moreover, $H^i(\mathsf{GC}_3^{or}[[\hbar]], \delta_\hbar) = 0$ for all $i \leq -2$ and $H^{-1}(\mathsf{GC}_3^{or}[[\hbar]], \delta_\hbar)$ is a 1-dimensional vector space class generated by the formal power series $\sum_{k=2}^{\infty}(k-1)\hbar^{k-2}$

.

7.4. Sourced graphs. A graphs Γ from the dg Lie algebra of connected oriented graphs dfcGC_d is called *sourced* if it contains at least one vertex v of valency ≥ 2 with no incoming edges, i.e. at least one vertex of the form

$$v = \overbrace{\quad\cdots\quad}^{\geq 2}$$

Such vertex is called a *source*. The subspace $\mathsf{sfGC}_d \subset \mathsf{dfcGC}_d$ spanned by sourced graphs is a dg Lie subalgebra. Consider a smaller subspace $\mathsf{sGC}_d \subset \mathsf{sfGC}_d$ spanned by sourced graphs with all vertices at least bivalent and with no bivalent vertices of the form . It was proven by in [**Z1**] that

$$H^\bullet(\mathsf{sfGC}_{d+1}) = H^\bullet(\mathsf{sGC}_{d+1}) = H^\bullet(\mathsf{GC}_d^{\geq 2})$$

so that one gets one more incarnation of the Grothendieck-Teichmüller Lie algebra in dimension 3

$$\mathfrak{grt}_1 = H^0(\mathsf{sGC}_3)$$

in terms of sourced graphs. Similarly one can work with directed graphs with at least one *target* vertex of the form .

7.5. Multi-oriented graph complexes. The Grothendieck-Teichmüller Lie algebra appears naturally as a single entity in dimension $d = 2$ as the cohomology group $H^0(\mathsf{GC}_2)$ of the "ordinary" graph complex. In dimension 3 it reappears as the cohomology group $H^0(\mathsf{GC}_3^{\rm or}) = H^0(\mathsf{sGC}_3)$ of the *oriented* or *sourced* graph complex, i.e. we have a sequence of isomorphism of Lie algebras,

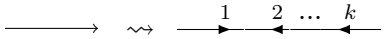
$$\mathfrak{grt}_1 = H^0(\mathsf{GC}_2) = H^0(\mathsf{GC}_3^{\rm or}) = H^0(\mathsf{sGC}_3).$$

What kind of graph complex $\mathsf{GC}_d^?$ gives us an incarnation of $\mathfrak{grt}_1$ in dimension $d = 4, 5, \ldots$,

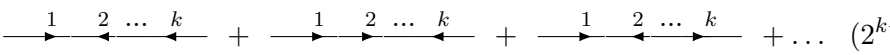
$$\mathfrak{grt}_1 = H^0(\mathsf{GC}_2) = H^0(\mathsf{GC}_3^{\rm or}) = H^0(\mathsf{sGC}_3) = H^0(\mathsf{GC}_4^?) = H^0(\mathsf{GC}_5^?) = \ldots ?$$

The answer to that question was found by Marko Živković in [**Z1, Z2**] who introduced and studied *multi-oriented sourced* graph complexes. Their algebro-geometric interpretation was found in [**Me4**] — they control quantizations and deformation theory of strong homotopy (even or odd) Lie bialgebra structures, in particular of Poisson structures, on infinite-dimensional spaces spaces *with branes*.

Let us go back for a moment to the Lie algebra of directed connected graphs dfcGC_d and define, for any integer $k \in \mathbb{N}_{\geq 1}$, its extension $\mathsf{d}^k\mathsf{fcGC}_d$ as a $\mathbb{Z}$-graded vector space spanned by directed graphs Γ with each edge decorated with $k-1$ new directions labelled by $2, 3, \ldots, k$, the value 1 being reserved for the original direction[10], and each new direction can take only two "values" — it can agree or disagree with the original direction,



Note that the Lie bracket in dfcGC_d does not change the number and the "internal structure" of edges in graphs so that exactly the same formula as in dfcGC_d gives us a Lie bracket in $\mathsf{d}^k\mathsf{fcGC}_d$ for any $k \geq 1$. It is not hard to see that the element

$\gamma_k :=$  $+ \ldots$ $(2^{k-1}$ terms$)$

where the summation runs over all possible way to decorate the directed edge with new directions $2, 3, \ldots k$, is a Maurer-Cartan element and hence makes $\mathsf{d}^k\mathsf{fcGC}_d$ into a *dg* Lie algebra with the differential $\delta\Gamma := [\Gamma, \gamma_k]$. Of course, the case $k = 1$ gives nothing new, $\mathsf{d}^1\mathsf{fcGC}_d \equiv \mathsf{dfcGC}_d$. In fact, the general case $k \in \mathbb{Z}_{\geq 2}$ also does not give us immediately anything really new as well: the natural monomorphisms which add to each edge of a graph one extra direction in all possible ways (cf. (7.1)),

$$\mathsf{fcGC}_d \longrightarrow \mathsf{d}^1\mathsf{fcGC}_d \longrightarrow \mathsf{d}^2\mathsf{fcGC}_d \longrightarrow \mathsf{d}^3\mathsf{fcGC}_d \longrightarrow \ldots$$

are quasi-isomorphisms. However we can consider now lots of interesting subcomplexes in the multidirected graph complex $(\mathsf{d}^k\mathsf{fcGC}_d, \delta)$. Given any two non-negative integers p and q with

[10] It is more suitable to understand new directions as *colored* ones, say the original direction 1 has black colour, the new direction 2 is red so on, because we do *not* assume that the set $[k]$ of all directions on each edge is *ordered*. Hence we call often the direction on an edge labeled by integer $l \in [k]$ the *l-coloured* direction. This new decoration of edges has no impact on the degree of graphs which is given by the standard formula $|\Gamma| = d(\#V(\Gamma) - 1) + (1-d)\#E(\Gamma)$, $\forall \Gamma \in \mathsf{d}^k\mathsf{fcGC}_d$.

$p+q \leq k$, and any two disjoint subsets I_p and I_q of the set $[k]$ of cardinalities p and q respectively (the particular choice of such subsets plays no role), one can define a subcomplex

$$\mathsf{s}^p\mathsf{o}^q\mathsf{d}^k\mathsf{fcGC}_d \subset \mathsf{d}^k\mathsf{fcGC}_d$$

of *k-directed p-sourced and q-oriented graphs* as the span of those multioriented graphs Γ which satisfy the conditions

(i) Γ is *sourced* with respect to every direction in the subset $I_p \subseteq [k]$, i.e. for each direction $c \in I_p$ there is at least one vertex $v \in V(\Gamma)$ which has valency ≥ 2 and no incoming edges with respect to c;

(ii) Γ is *oriented* with respect to every direction in the subset $I_q \subset [k]$, i.e. for each direction $c \in I_p$ the graph Γ has no *closed* directed paths of edges (*wheels*) with respect to c.

By analogy to the previous examples, there is no loss of generality when working with a subcomplex

$$\mathsf{s}^p\mathsf{o}^q\mathsf{d}^k\mathsf{GC}_d \subset \mathsf{s}^p\mathsf{o}^q\mathsf{d}^k\mathsf{fcGC}_d$$

generated by graphs whose vertices are at least bivalent as this inclusion is a quasi-isomorphism. The major result of [**Z2**] is the following

7.5.1. **M. Živković Theorem.** *For any integer $k \geq 1$ and any non-negative integers p and q with $p+q \leq k$ there is an isomorphism of cohomology groups*

$$H^\bullet(\mathsf{GC}_d^{\geq 2}) = H^\bullet(\mathsf{s}^p\mathsf{o}^q\mathsf{d}^k\mathsf{GC}_{d+p+q}).$$

Therefore extra directions which are neither oriented nor sourced can be omitted as irrelevant, i.e. one can set $k = p+q$ without loss of important information and work only with dg Lie algebras

$$\mathsf{s}^p\mathsf{o}^q\mathsf{GC}_{d+p+q} := \mathsf{s}^p\mathsf{o}^q\mathsf{d}^{p+q}\mathsf{GC}_{d+p+q}.$$

M. Živković Theorem gives us infinitely many graph incarnations of the Grothendieck-Teichmüller Lie algebra,

$$\mathfrak{grt}_1 = H^0(\mathsf{GC}_2) = H^0(\mathsf{s}^p\mathsf{o}^q\mathsf{GC}_{2+p+q}) \quad \forall\, p, q \in \mathbb{Z}_{\geq 0},$$

and hence a clue (cf. [**Me4**]) to what could be a non-trivial deformation quantization theory in dimension $d \geq 4$ in which the Grothendieck-Teichmüller group plays a classifying role (the cases $d = 2$ and $d = 3$ are well understood by now, see below).

7.5.2. **Remark on hairy graphs.** There is an important version of the graph complexes GC_d which is based on graphs with hairs [**KWZ2, W6, W4**]. These graph complexes control the rational homotopy groups of of long embeddings (modulo immersion) of $\mathbb{R}^m$ in $\mathbb{R}^n$. Very recently, their marked version has been used in the study of cohomology groups of moduli spaces $\mathcal{M}_{g,n}$ of genus g algebraic curves with n punctures [**CGP2**].

8. Some applications of the theory of Drinfeld's associators, GRT_1 and graph complexes

8.1. Universal quantizations of Lie bialgebras. A *Lie bialgebra* is a graded vector space V equipped with two linear maps,

$$[\ ,\ ] : \wedge^2 V \to V \quad , \quad \triangle : V \to V \wedge V,$$

such that the first map $[\ ,\ ]$ makes V into a Lie algebra, the second map $\triangle$ makes V into a Lie coalgebra, and the compatibility condition

$$\triangle\,[a,b] = \sum a_1 \otimes [a_2, b] + [a, b_1] \otimes b_2 - (-1)^{|a||b|}([b, a_1] \otimes a_2 + b_1 \otimes [b_2, a]),$$

holds for any $a, b \in V$ with $\triangle\ a =: \sum a_1 \otimes a_2$, $\triangle\ b =: \sum b_1 \otimes b_2$. This algebraic structure was introduced by V. Drinfeld [**Dr1**] in the context of studying universal deformations of the standard

Hopf algebra structure on universal enveloping algebras. More precisely, consider the symmetric tensor algebra $\odot^\bullet V$ equipped with the standard graded commutative and graded cocommutative bialgebra structure $(\cdot, \Delta_0)$, where $\cdot$ is the canonical multiplication on $\odot^\bullet V$ and Δ_0 is the coproduct on $\odot^\bullet V$ uniquely determined by the condition that the elements of $V \subset \odot^\bullet V$ are primitive. Assume the topological vector space $\odot^\bullet(V)[[\hbar]]$, $\hbar$ being a formal parameter of degree zero, has a continuous bialgebra structure $(\star_\hbar, \Delta_\hbar)$ of the form,

$$\begin{array}{rccc} \star_\hbar: & \odot^\bullet V \bigotimes \odot^\bullet V & \longrightarrow & \odot^\bullet V[[\hbar]] \\ & (f(x), g(x)) & \longrightarrow & f *_\hbar g = f \cdot g + \sum_{k\geq 1}^\infty \hbar^k P_k(f,g) \end{array}$$

$$\begin{array}{rccc} \Delta_\hbar: & \odot^\bullet V & \longrightarrow & \odot^\bullet V \bigotimes \odot^\bullet V[[\hbar]] \\ & f(x) & \longrightarrow & \Delta_\hbar f = \Delta_0 f + \sum_{k\geq 1}^\infty \hbar^k Q_k(f) \end{array}$$

where all operators P_k are bidifferential, and Q_k are co-bidifferential (that is, dual to bi-differential operators on polynomial functions, see, e.g., §4.2 in [**Me3**] for their explicit description). It is not hard to see that the bialgebra conditions on these operators modulo terms $O(\hbar^2)$ imply that the first order deformations P_1 and Q_1 are uniquely determined by some Lie bialgebra structure $(\triangle, [\ ,\ ])$ on V. In this case the data $(\star_\hbar, \Delta_\hbar)$ on $\odot^\bullet(V)[[\hbar]]$ is called a *deformation quantization* of that Lie bialgebra structure $(\triangle, [\ ,\ ])$ on V. Thus Lie bialgebra structures control infinitesimal deformations of the standard Hopf algebra structure on $\odot^\bullet V$.

The Drinfeld quantization problem: given any Lie bialgebra structure on a vector space V, does there always exist its deformation quantization $(\star_\hbar, \Delta_\hbar)$? Put another way, given infinitesimal deformations P_1 and Q_1 of the standard bialgebra on $\odot^\bullet V$, can we always find suitable operators $P_2, P_3, \ldots, Q_2, Q_3, \ldots$ which make $\odot^\bullet(V)[[\hbar]]$ into a (not necessarily commutative cocommutative) bialgebra?

Surprisingly enough, the problem can not be solve by a trivial inductive procedure[11]. It was proven by P. Etingof and D. Kazhdan in [**EK**] that, *given a choice of a Drinfeld associator, there does exist a corresponding universal* (in the sense, for *any* vector space V and *any* Lie bialgebra structure on V) *solution to the Drinfeld quantization problem.* It was proven in [**Me3**] that any solution of the Drinfeld quantization problem extends to a quasi-isomorphism — a *formality map* — of completed $\mathcal{L}ie_\infty$ algebras,

$$F_V : \widehat{\odot^{\bullet\geq 1}}(V[-1] \oplus V^*[-1]) \longrightarrow \prod_{m,n\geq 1} \mathrm{Hom}\left(\bigotimes^m(\odot^\bullet V), \bigotimes^n(\odot^\bullet V)\right)[2-m-n] \tag{8.1}$$

where the l.h.s. is equipped with the Poisson type Lie bracket induced by the standard paring $V \otimes V^* \to \mathbb{K}$, and the r.h.s. is the Gerstenhaber-Schack complex of the graded commutative cocommutative Hopf algebra $\odot^\bullet V$ equipped with the $\mathcal{L}ie_\infty$ structure uniquely determined by a choice of a minimal resolution $\mathcal{A}ss\mathcal{B}_\infty$ of the properad $\mathcal{A}ss\mathcal{B}$ controlling (associative) bialgebras. In the other direction: any quasi-isomorphism as above gives a solution of the the Drinfeld quantization problem.

Since we are interested in *universal* solutions, it makes sense to search for a reformulation of this problem which does not refer to a particular vector space at all. The theory of operads (see §4.1) is very effective in the study of the homotopy theory of algebraic operations on vector spaces with many inputs but only *one* output. It can not be applied to such algebraic structures as Lie bialgebras, but there is a very nice extension of that theory called the theory *of props, properads, and their wheeled versions.* We refer to [**V**] for an excellent introduction into the theory of props and properads. Roughly speaking, the theory prop(erad)s is based on (connected) decorated oriented

[11] The obstructions to such a universal iterative procedure belong to $H^1(\mathsf{GC}_2)$. A "folklore" conjecture says that $H^1(\mathsf{GC}_2) = 0$, but one hast to choose an associator to make every degree 1 cycle in GC_2 into a coboundary; thus an iterative procedure can exist but it can not be trivial.

graphs with legs. The deformation theory of *morphisms* of properads and props was developed in [**MV**]. There is a properad $\mathcal{L}ieb$ (see, e.g., [**MW3, MW5, V**] and references cited therein) whose representations in a vector space V are precisely Lie bialgebra structures on V and the homotopy theory of such structures is controlled by the minimal resolution $\mathcal{H}olieb$ of $\mathcal{L}ieb$. There is also a properad $\mathcal{A}ss\mathcal{B}$ whose representations in a vector space W are precisely bialgebra structures on W and the homotopy theory of such structures is controlled by a minimal resolution $\mathcal{A}ss\mathcal{B}_\infty$ of $\mathcal{A}ss\mathcal{B}$ (which exists but is not unique [**Ma**]). V. Drinfeld deformation quantization problem (in the extended *formality* version as explained above) can be reformulated as existence of a morphism of dg props [**MW3**]

$$F : \mathcal{A}ss\mathcal{B}_\infty \longrightarrow \mathcal{D}\widehat{\mathcal{H}olieb} \tag{8.2}$$

satisfying certain non-triviality conditions. Here $\widehat{\mathcal{H}olieb}$ is the genus completion of $\mathcal{H}olieb$ and

$$\mathcal{D} : \text{category of dg props} \longrightarrow \text{category of dg props}$$

a polydifferential (exact) endofunctor which creates out of any prop $\mathcal{P}$ another prop $\mathcal{D}\mathcal{P}$ with the property that there is a one-to-one correspondence between representations of $\mathcal{P}$ in a vector space V and representations of $\mathcal{D}\mathcal{P}$ in $\odot^\bullet V$ given in terms of polydifferential operators. The Etingof-Kazhdan theorem can be used to show that for any Drinfeld associator there exists a morphism F of dg props as above. To classify all solutions to the Drinfeld problem, i.e. to classify all possible homotopy non-trivial formality maps as in (8.2), one has to compute the cohomology of the deformation complex of any map F as above,

$$\mathsf{Def}\left(\mathcal{A}ss\mathcal{B}_\infty \stackrel{F}{\to} \mathcal{D}\widehat{\mathcal{H}olieb}\right).$$

This was done in [**MW3**] where it was proven that there is an isomorphism of cohomology groups

$$H^{\bullet+1}\left(\mathsf{Def}\left(\mathcal{A}ss\mathcal{B}_\infty \stackrel{F}{\to} \mathcal{D}\widehat{\mathcal{H}olieb}\right)\right) = H^\bullet(\mathsf{fGC}_2') \oplus \mathbb{K}$$

where fGC_2' is a version of the full graph complex fGC_2 to which we added by hand a new element $\emptyset$ concentrated in degree zero, "a graph with no vertices and edges". More precisely,

$$\mathsf{fGC}_2' := \widehat{\odot^\bullet}\left((\mathsf{GC}_2^{\geq 2} \oplus \mathbb{K})[-2]\right)[2],$$

the summands $\mathbb{K}$ being generated by $\emptyset$. The formal class $\emptyset$ takes care for (homotopy non-trivial) rescaling operation of the prop under considerations. The main point of this extension is that the Lie bracket of $\emptyset$ with elements Γ of $\mathsf{GC}_2^{\geq 2}$ is defined as the multiplication of Γ by twice the number of its loops.

This result implies in particular that

$$H^1\left(\mathsf{Def}\left(\mathcal{A}ss\mathcal{B}_\infty \stackrel{F}{\to} \mathcal{D}\widehat{\mathcal{H}olieb}\right)\right) = \mathfrak{grt}_1 \oplus \mathbb{K},$$

i.e. the Grothendieck-Teichmüller group $GRT = GRT_1 \rtimes \mathbb{K}^*$ acts faithfully and transitively on the set of solutions of the Drinfeld quantization problem which in turn implies that this set can be identified with the set of Drinfeld associators.

An explicit transcendental formula for a universal quantization of Lie bialgebras has been constructed in [**MW5**].

8.2. Universal quantizations of Poisson structures. Let $C^\infty(\mathbb{R}^n)$ be the commutative algebra of smooth functions in $\mathbb{R}^n$. A *star product* on $C^\infty(\mathbb{R}^n)$ is a continuous associative product on $C^\infty(\mathbb{R}^n)[[\hbar]]$, $\hbar$ being a formal parameter,

$$\begin{array}{rccc} *_\hbar : & C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n) & \longrightarrow & C^\infty(\mathbb{R}^n)[[\hbar]] \\ & (f(x), g(x)) & \longrightarrow & f *_\hbar g = fg + \sum_{k\geq 1}^\infty \hbar^k P_k(f,g) \end{array}$$

where all operators P_k are bidifferential. It is not hard to check that the associativity condition on $*_\hbar$ implies that $\pi(f,g) := B_1(f,g) - B_1(g,f)$ is a Poisson structure in $\mathbb{R}^n$; then $*_\hbar$ is called a *deformation quantization* of $\pi \in \mathcal{T}_{poly}(\mathbb{R}^n)$.

The *deformation quantization problem* addresses the question: given a Poisson structure π on $\mathbb{R}^n$, does there exist a star product $*_\hbar$ on $C^\infty(\mathbb{R}^n)$ which is a deformation quantization of π? A spectacular solution of this problem was given by Maxim Kontsevich [**K2**] in the form of a explicit map between the two sets

$$\left\{ \begin{array}{c} \text{Poisson} \\ \text{structures in } \mathbb{R}^n \end{array} \right\} \quad \stackrel{\stackrel{depends\ on}{\longrightarrow}}{associators} \quad \left\{ \begin{array}{c} \text{Star products} \\ *_\hbar \text{ in } C^\infty(\mathbb{R}^n)[[\hbar]] \end{array} \right\}$$

given by transcendental formulae. In fact a stronger statement was proven — the *formality theorem* which says that for any n there is an explicit $\mathcal{L}ie_\infty$ quasi-isomorphism of dg Lie algebras,

$$F_K : \mathcal{T}_{poly}(\mathbb{R}^n) \longrightarrow C^\bullet(C^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^n))$$

where $\mathcal{T}_{poly}(\mathbb{R}^n)$ is the Lie algebra of polyvector fields on $\mathbb{R}^n$ equipped with the Schouten-Nijenhuis bracket, and $C^\bullet(C^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^n))$ is the Hochschild complex of the algebra $C^\infty(\mathbb{R}^n)$, that is, the dg Lie algebra controlling deformations of $C^\infty(\mathbb{R}^n)$ as an associative (but not necessarily commutative) $\mathbb{R}$-algebra. Moreover, the formality theorem holds true for any manifold M, not necessarily for $\mathbb{R}^n$ [**K2**]. Later D. Tamarkin has proven the existence theorem for deformation quantizations which exhibited a key role of Drinfeld's associators [**T2**], and V. Dolgushev [**Do**] has proven that the set of homotopy classes of universal formality maps can be identified with the set of Drinfeld's associators so that GRT acts faithfully and transitively of homotopy classes of formality maps.

This classification result can be restated in a very short and compact form without any reference to any particular manifold as follows [**AnMe**]. Let $c\mathcal{A}ss_\infty$ be the dg free operad of strongly homotopy curved associative algebras, $\mathcal{H}olieb^{\circlearrowleft}_{1,0}$ the wheeled closure [**MMS**] of the prop $\mathcal{H}olieb_{1,0}$ of $(1,0)$ Lie bialgebras and

$$\mathcal{O} : \text{category of dg (wheeled) props} \ \longrightarrow \ \text{category of dg operads}$$

the polydifferential functor introduced in [**MW4**] (which is an operadic part of the functor $\mathcal{D}$ mentioned in the previous subsection). Then the Kontsevich formality map F_K can be understood as a morphism of dg operads

$$F_K : c\mathcal{A}ss_\infty \longrightarrow \mathcal{O}(\widehat{\mathcal{H}olieb}^{\circlearrowleft}_{1,0})$$

so that it makes sense to study a deformation complex of the Kontsevich map

$$\mathsf{Def}\left(c\mathcal{A}ss_\infty \stackrel{F_K}{\longrightarrow} \mathcal{O}(\widehat{\mathcal{H}olieb}^{\circlearrowleft}_{1,0})\right)$$

It was proven in [**AnMe**] that there exists isomorphism of cohomology groups

$$H^\bullet(\mathsf{fGC}'_2) = H^{\bullet+1}\left(\mathsf{Def}\left(c\mathcal{A}ss_\infty \stackrel{F_K}{\longrightarrow} \mathcal{O}(\mathcal{H}olieb^{\circlearrowleft}_{1,0})\right)\right)$$

implying

$$\mathfrak{grt} = H^1(\mathsf{fGC}'_2) = H^1\left(\mathsf{Def}\left(c\mathcal{A}ss_\infty \stackrel{F_K}{\longrightarrow} \mathcal{O}(\mathcal{H}olieb^{\circlearrowleft}_{1,0})\right)\right)$$

Hence the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_1$ controls *all* homotopy non-trivial infinitesimal deformations of the Kontsevich map F_K. Each such infinitesimal deformation can be exponentiated to a genuine deformation of F_K implying the fundamental classifying role of GRT_1 in the theory of universal deformation quantizations of Poisson structures. A concrete algorithm for computing weights in the M. Kontsevich formula for F_K was developed in [**BPP**] where it was proven that all such weights are rational linear combinations of multiple zeta values.

8.2.1. **Action of GRT on Lie bialgebras.** One can consider a degree shifted version of the notion of Lie bialgebra. For any integers $c, d \in \mathbb{Z}$ one defines [**MW4**] a *Lie (c,d)-bialgebra* as a graded vector space equipped with linear maps

$$[\ ,\ ] : \odot^2(V[d]) \to V[d+1] \quad , \quad \triangle \colon V[c] \to \odot^2(V[c])[1-2c],$$

making V into a degree shifted Lie algebra and degree shifted Lie coalgebra satisfying an analogue of the Drinfeld compatibility condition. There exist a prop(erad) $\mathcal{L}ieb_{c,d}$ which governs these structures and its minimal resolution $\mathcal{H}olieb_{c,d}$ which governs their homotopy theory [**MW4**]. The case $c = 1$, $d = 1$ gives us ordinary Lie bialgebras discussed above, while the case $c = 1$, $d = 0$ is important in the theory of Poisson structures on manifolds as there is a one-to-one correspondence [**Me1**] between representation of the pro(erad) $\mathcal{H}olieb_{1,0}$ on a graded vector space V and formal graded Poisson structures on V viewed as a formal manifold which vanish at the distinguished point $0 \in V$.

The deformation theory [**MV**] of the identity map $\mathrm{Id} : \widehat{\mathcal{H}olieb}_{c,d} \to \widehat{\mathcal{H}olieb}_{c,d}$ gives us a clue to the symmetry group of the genus completion $\widehat{\mathcal{H}olieb}_{c,d}$ of the dg prop $\mathcal{H}olieb_{c,d}$ (and hence of $\widehat{\mathcal{L}ieb_{c,d}}$). This problem was settled in [**MW5**] where it was proven that there is a quasi-isomorphism (up to one rescaling class) of complexes

$$\mathsf{GC}^{\mathrm{or}}_{c+d+1} \longrightarrow \mathsf{Def}\left(\widehat{\mathcal{H}olieb}_{c,d} \stackrel{\mathrm{Id}}{\to} \widehat{\mathcal{H}olieb}_{c,d}\right)[1]$$

implying

$$H^\bullet(\mathsf{GC}^{\geq 2}_{c+d}) \oplus \mathbb{K} = H^\bullet(\mathsf{GC}^{\mathrm{or}}_{c+d+1}) \oplus \mathbb{K} = H^{\bullet+1}\left(\widehat{\mathcal{H}olieb}_{c,d} \stackrel{\mathrm{Id}}{\to} \widehat{\mathcal{H}olieb}_{c,d}\right)$$

so that

$$\mathfrak{grt}_1 \oplus \mathbb{K} = H^1(\widehat{\mathcal{H}olieb} \stackrel{\mathrm{Id}}{\to} \widehat{\mathcal{H}olieb})$$

Hence the Grothendieck-Teichmüller group $GRT = GRT_1 \rtimes \mathbb{K}^*$ is essentially the automorphism group of the completed prop(erad) $\widehat{\mathcal{L}ieb}$.

There is a further generalization of the notion of Lie (c,d)-bialgebra to the *multioriented* case, and GRT acts on such structures for any $c, d \in \mathbb{N}$ with $c + d \geq 3$ [**An, Me4**].

8.3. Solutions of the Kashiwara-Vergne conjecture. Let $\widehat{\mathfrak{ass}}_2$ and $\widehat{\mathsf{lie}}_2$ be the completed free associative and, respectively, Lie algebra generated by formal variables x and y (see §2.2.1). Consider the quotient space

$$\widehat{Cyc}_2 := \widehat{\mathfrak{ass}}_2/\langle AB - BA \mid \forall\, A, B \in \widehat{\mathfrak{ass}}_2\rangle \equiv \widehat{\mathfrak{ass}}_2/[\widehat{\mathfrak{ass}}_2, \widehat{\mathfrak{ass}}_2]$$

the (completed) vector space spanned by cyclic words in two letters. There is a canonical projection $tr : \widehat{\mathfrak{ass}}_2 \to \widehat{Cyc}_2$. Note that every element $A \in \widehat{\mathfrak{ass}}_2$ has a unique decomposition,

$$A = A_0 + \partial_x(A)x + \partial_y(A)y$$

for some $A_0 \in \mathbb{K}$, $\partial_x(A), \partial_y(A) \in \widehat{\mathfrak{ass}}_2$.

Recall the Bernoulli power series series

$$\frac{x}{1+e^x} = 1 + \frac{x}{2} + \sum_{n\geq 1} \frac{b_{2n}}{2n!} x^{2n} =: 1 + \frac{x}{2} + \mathfrak{b}(x)$$

and the Baker-Cambell-Hausdorf power series $\mathfrak{bch}(x,y)$ defined in (2.2). A triple (A, B, g) consisting of two Lie series $A, B \in \widehat{\mathsf{lie}}_2$ and a formal power series $g(x) = \sum_{n\geq 2} g_n x^n$ is called a *solution of the (generalized) Kashiwara-Vergne problem* if they satisfy the equations

$$x + y - \mathfrak{bch}(x,y) = (1 - e^{-\mathrm{ad}_x})A + (e^{\mathrm{ad}_y} - 1)B \quad \text{ in } \widehat{\mathsf{lie}}_2$$

and

$$tr\left(\partial_x(A)x+\partial_y(B)y\right)=\frac{1}{2}tr\left(g(x)-g(\mathfrak{bch}(x,y))+g(y)\right).$$

Given a Lie group G with the Lie algebra $\mathfrak{g}$, if a solution of the KV problem exists, then the G-invariant harmonic analysis on $\mathfrak{g}$ is related with G-invariant harmonic analysis on the group G itself by the standard exponential map. Moreover, if a solution exists, then $g_{even}:=\sum_{n\geq 1}g_{2n}x^{2n}$ must be equal to the Bernoulli series $\mathfrak{b}(x)$.

Existence of solution of the KV problem was established in [**AlMe**]. An alternative solution of the KV problem which classified all such solutions in terms of so called KV *associators* was given in [**AT2**]. Any Drinfeld associator is a KV associator, and there is a conjecture which says that this association is one-to-one.

8.4. Formality theorem in the Goldman-Turaev theory. Let Σ be a Riemann surface of genus g with $N+1$ boundary components, $N\in\mathbb{N}$. The group algebra $\mathbb{K}\langle\pi_1(\Sigma)\rangle$ of the fundamental group $\pi_1(\Sigma)$ is canonically filtered by powers of the augmentation ideal (see §2.4) and hence admits a canonical completion $\widehat{\pi_1(\Sigma)}$. The associated (completed) vector space spanned by conjugacy classes in $\pi_1(\Sigma)$,

$$\widehat{\mathfrak{g}[\Sigma]}:=\frac{\widehat{\mathbb{K}\langle\pi_1(\Sigma)\rangle}}{[\widehat{\mathbb{K}\langle\pi_1(\Sigma)},\widehat{\mathbb{K}\langle\pi_1(\Sigma)\rangle}]}$$

that is, by free homotopy classes of loops in Σ, has a canonical *Goldman-Turaev Lie bialgebra structure*. It is filtered, and the associated graded vector space

$$\mathrm{gr}\widehat{\mathfrak{g}[\Sigma]}:=\frac{\widehat{\otimes}^\bullet H_1(\Sigma)}{[\widehat{\otimes}^\bullet H_1(\Sigma),\widehat{\otimes}^\bullet H_1(\Sigma)]},$$

where $H_1(\Sigma)$ is the first homology group of Σ over $\mathbb{K}$, has an induced Lie bialgebra structure which admits a rather simple combinatorial description. The *formality theorem* [**AKKN1, AKKN2, Mas**] (see also references cited therein) says that for any solution of the KV problem, in particular, for any Drinfeld associator there is an associated isomorphism

$$\Theta:\ \widehat{\mathfrak{g}[\Sigma]}\longrightarrow\ \ \mathrm{gr}\widehat{\mathfrak{g}[\Sigma]}$$

of Lie bialgebras. In the case $g=0$ and $\mathbb{K}=\mathbb{C}$ a nice explicit formula for such an isomorphism was constructed in [**AN**] with the help of the Knizhnik-Zamolodchikov connection. In particular, the group GRT_1 acts as automorphisms on $\mathrm{gr}\widehat{\mathfrak{g}[\Sigma]}$ for any Riemann surface Σ.

8.5. Cohomology of moduli spaces of algebraic curves and $\mathfrak{grt}_1$. Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g. The authors of [**CGP1**] constructed a remarkable monomorphism of cohomology groups

$$H^\bullet(\mathsf{GC}_2)\longrightarrow\prod_g H_c^{\bullet+2g}(\mathcal{M}_g;\mathbb{K})=\prod_g(H^{4g-6-\bullet}(\mathcal{M}_g;\mathbb{K}))^*$$

implying an injection of the Grothendieck-Teichmüller Lie algebra

$$\mathfrak{grt}_1\longrightarrow\prod_g H_c^{2g}(\mathcal{M}_g;\mathbb{K})$$

and hence the following estimation [**CGP1**] on the dimension of cohomology groups:

$$\dim H^{4g-6}(\mathcal{M}_g,\mathbb{Q})>\beta^g+\text{constant},$$

for any $\beta<\beta_0$, where $\beta_0\approx 1.3247\ldots$ is the real root of $t^3-t-1=0$.

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