OPEN HURWITZ NUMBERS AND THE MKP HIERARCHY

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Abstract. We give a natural definition of open Hurwitz numbers, where the weight of each ramified covering includes an integer parameter N taken to the power that is equal to the number of boundary components of a Riemann surface with boundary mapping to $\mathbb{CP}^1$. We prove that the resulting sequence of partition functions, depending on $N \in \mathbb{Z}$, is a tau-sequence of the mKP hierarchy, or in other words it is a sequence of tau-functions of the KP hierarchy where each tau-function is obtained from the previous one by a Bäcklund–Darboux transformation. Our result is motivated by a previous observation of Alexandrov and the first two authors that the refined intersection numbers on the moduli spaces of Riemann surfaces with boundary give a tau-sequence of the mKP hierarchy.

1. Introduction

The appearance of integrable systems in the study of curve-counting invariants is a well-known phenomenon, which was first manifested by Witten's conjecture [Wit91], claiming that the generating series F^c of the intersection numbers of psi-classes on the moduli spaces of stable algebraic curves $\overline{\mathcal{M}}_{g,n}$ is the logarithm of a tau-function of the KP hierarchy. Witten's conjecture was proved by Kontsevich [Kon92], and it inspired many similar results for Gromov–Witten invariants (see, e.g., [DZ04, OP06b, MST16]), Fan–Jarvis–Ruan–Witten invariants (see, e.g., [Wit93, FJR13]), and more generally for the correlators of cohomological field theories (see, e.g., [Kaz09, Bur15, LYZZ21, DZ00, BS12, BS24, LWZ25]). A new direction was opened by Okounkov [Oko00], who proved that the generating series H^c of simple Hurwitz numbers is the logarithm of a tau-function of the KP hierarchy. This motivated an extensive study of the integrability of Hurwitz numbers of various types (see, e.g., [KL15, Har16, BDKS22].

More recently, studying the integrability of curve-counting invariants for Riemann surfaces with boundary became very active. The paper [PST24] (published in arXiv in 2014) initiated a study of the intersection theory on a compactification $\overline{\mathcal{M}}_{g,k,l}$ of the moduli space of Riemann surfaces with (nonempty) boundary $\mathcal{M}_{g,k,l}$, where g is the genus of the double a Riemann surface with boundary, k is the number of boundary marked points, and l is the number of internal marked points. The authors of [PST24] proposed an open (this adjective is often used to indicate that Riemann surfaces with boundary are considered) analog of Witten's conjecture, claiming that the exponent e^{F^o} of the generating series F^o of the intersection numbers on $\overline{\mathcal{M}}_{g,k,l}$ satisfies a certain system of PDEs, which was shown in [Bur16] to be equivalent to the system of equations for the wave function of the KP hierarchy. Based on the work [Tes23] (published in arXiv in 2015), the Pandharipande–Solomon–Tessler conjecture was proved in [BT17]. An important clarification of the structure of e^{F^o} was obtained in [Ale15], where the author proved that the Pandharipande–Solomon–Tessler conjecture implies that $e^{F^c+F^o}$ is a tau-function of the KP hierarchy, which is related to e^{F^c} by a Bäcklund–Darboux transformation.

In the paper [ABT17], the authors considered a refinement of the open intersection numbers by distinguishing contributions from surfaces with different numbers of boundary components. To be more precise, there is a decomposition $\overline{\mathcal{M}}_{g,k,l} = \bigsqcup_{1\le b\le g+1} \overline{\mathcal{M}}_{g,k,l,b}$, where $\overline{\mathcal{M}}_{g,k,l,b}$ parameterizes surfaces with boundary with b boundary components. In [ABT17], the authors considered the intersection numbers on each $\overline{\mathcal{M}}_{g,k,l,b}$, assigned the weight N^b to them, and

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formed the generating series denoted by $F^{o,N}$, $N \in \mathbb{Z}$. Then the authors of [ABT17] conjectured that $e^{F^c+F^{o,N}}$ coincides with the tau-function given by the Kontsevich–Penner matrix model, which, according to [Ale15], gives a tau-sequence of the modified KP (mKP) hierarchy. The recent paper [Wan25] claims to prove this conjecture.

Our motivation was to give a Hurwitz-type analog of the observation from [ABT17], i.e., to propose a natural definition of open Hurwitz numbers, counting maps from Riemann surfaces with boundary to $\mathbb{CP}^1$, that would give a tau-sequence of the mKP hierarchy including the KP tau-function e^{H^c}. This is done in this paper, which is organized as follows. After a brief review, in Section 2, of simple Hurwitz numbers, which we call closed Hurwitz numbers, we move on, in Section 3, to our definition of open Hurwitz numbers counting certain maps from Riemann surfaces with boundary to $\mathbb{CP}^1$. Very similarly to what was done in [ABT17], the weight of each map includes an integer parameter N taken to the power that is equal to the number of boundary components of a Riemann surface with boundary mapping to $\mathbb{CP}^1$. Then we prove Theorem 3.2 claiming that, after a simple rescaling, the sequence of the exponents τ^o_N of the generating series of open Hurwitz numbers is a tau-sequence of the mKP hierarchy. Also, in Theorem 3.7, we describe explicitly the Bäcklund–Darboux transformation relating τ^o_0 and τ^o_1. In order to make the paper more self-contained, we review in Appendix A and Appendix B the necessary facts about the KP hierarchy and the Bäcklund–Darboux transformations.

Note that the Gromov–Witten invariants of $\mathbb{CP}^1$ are closely related to Hurwitz numbers of certain type [OP06a]. In the recent paper [BNPT22], the authors proposed a construction of open Gromov–Witten invariants of $\mathbb{CP}^1$. We plan to present an open extension of the Gromov–Witten/Hurwitz correspondence from [OP06a] in a separate work.

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2. Closed Hurwitz numbers

For $n \in \mathbb{Z}_{\geq 0}$, we denote by $\mathcal{P}_n$ the set of all partitions λ of n. For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$, $\lambda_1 \geq \dots \geq \lambda_l \geq 1$, we denote $l(\lambda) := l$.

Definition 2.1. Let $d \in \mathbb{Z}_{\geq 1}$, $m \in \mathbb{Z}_{\geq 0}$, $\lambda \in \mathcal{P}_d$. Let us fix pairwise distinct points $x_1, \dots, x_m \in \mathbb{C} \subset \mathbb{CP}^1$.

(1) Consider a connected closed Riemann surface C. A *closed simple ramified covering* of degree d is a nonconstant holomorphic map $f\colon C \to \mathbb{CP}^1$ such that $x_1, \dots, x_m$ are simple critical values of f, the ramification profile over ∞ is given by the partition λ, and the map f is not ramified over $\mathbb{C}\backslash\{x_1, \dots, x_m\}$.

(2) A *closed Hurwitz number* $h^c(\lambda, m)$ is the number of isomorphism classes $[f]$ of closed simple ramified coverings counted with the weight $\frac{1}{\sharp \operatorname{Aut}(f)}$, where $\operatorname{Aut}(f)$ is the automorphism group of the covering.

Introduce formal variables $p_1, p_2, \dots$, β, q, and consider the generating series

$$H^c(\mathbf{p}, \beta, q) := \sum_{\substack{d \geq 1 \\ m \geq 0}} \sum_{\lambda \in \mathcal{P}_d} h^c(\lambda, m) \frac{\beta^m}{m!} q^d p_\lambda \in \mathbb{C}[[p_1, p_2, \dots, \beta, q]],$$

where $p_\lambda := \prod_{i=1}^{l(\lambda)} p_{\lambda_i}$ and we denote by $\mathbf{p}$ the sequence of variables $(p_1, p_2, \dots)$. Define a *closed partition function* τ^c by

$$\tau^c := e^{H^c}.$$

It is well known that τ^c is a tau-function of the KP hierarchy (this was proved in [Oko00], but see also [KL07] for a nice presentation), where the variables p_i are related to the times t_i of the KP hierarchy (see Appendix A for a brief review) by $p_i = it_i$. Moreover, the tau-function can be explicitly described using the infinite wedge formalism in the following way:

(2.1)
$$\tau^c = \Psi_0\Big(\bigwedge_{i\geq 1} f_i^{\mathrm{H}}(z,\beta,q)\Big), \quad \text{where} \quad f_i^{\mathrm{H}}(z,\beta,q) := z^{-i}\left(\sum_{l\geq 0} e^{\beta\frac{(l-i+\frac{1}{2})^2-(i-\frac{1}{2})^2}{2}}\frac{(qz)^l}{l!}\right), \quad i\geq 1.$$

Remark 2.2. In literature, tau-functions of the KP hierarchy are usually elements of the algebra of formal power series $\mathbb{C}[[\mathbf{t}]] := \mathbb{C}[[t_1, t_2, \ldots]]$. However, in many constructions and results, $\mathbb{C}$ can be replaced by an arbitrary commutative associative $\mathbb{C}$-algebra K (see Remark A.9 for more details). In this case, we will sometimes say that the *base algebra* is K. So, τ^c can be considered as a family of KP tau-functions depending on the complex parameters β and q or as a tau-function of the KP hierarchy where the base algebra is $\mathbb{C}[[\beta, q]]$ (a bit more precisely, we have $\tau^c \in \mathbb{C}[e^{\beta}, e^{-\beta}, q][[\mathbf{t}]]$).

3. Open Hurwitz numbers

3.1. **Definition.** We view S^1 as the unit circle in the complex plane $\mathbb{C}$. Let us define the *north* and the *south* hemisphere of the complex projective line $\mathbb{CP}^1 = \mathbb{C}\cup\{\infty\}$ by
$$\mathbb{CP}^1_+ := \{z\in\mathbb{C}\,||z|<1\}, \qquad \mathbb{CP}^1_- := \{z\in\mathbb{C}\,||z|>1\}\cup\{\infty\}.$$
Denote $\overline{\mathbb{CP}^1_+} := \{z\in\mathbb{C}\,||z|\leq 1\}$ and $\overline{\mathbb{CP}^1_-} := \{z\in\mathbb{C}\,||z|\geq 1\}\cup\{\infty\}$. We will identify the relative homology group $H_2(\mathbb{CP}^1, S^1;\mathbb{Z})$ with $\mathbb{Z}\oplus\mathbb{Z}$, the generators being $[\overline{\mathbb{CP}^1_+}]$ and $[\overline{\mathbb{CP}^1_-}]$.

Definition 3.1. Let $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ and $(d_1, d_2) \in \mathbb{Z}^2_{\geq 0}\backslash\{(0,0)\}$. Let us fix m_1 pairwise distinct points in $\mathbb{CP}^1_+$ and m_2 pairwise distinct points in $\mathbb{CP}^1_-\backslash\{\infty\}$ and denote the union of these points by $\mathrm{Cr}^s \subset \mathbb{CP}^1$. Let us fix a partition $\lambda\in\mathcal{P}_{d_2}$.

(1) Consider a connected compact Riemann surface with nonempty boundary $(C, \partial C)$. An *open simple ramified covering* of bidegree (d_1, d_2) is a nonconstant holomorphic map
$$f\colon (C,\partial C)\to(\mathbb{CP}^1, S^1)$$
satisfying the following properties:
   (a) The points from the set Cr^s are simple critical values of f.
   (b) The ramification profile of the map f over ∞ is given by the partition λ.
   (c) The map f is not ramified over $\mathbb{C}\backslash\mathrm{Cr}^s$.
   (d) We have $f_*([C]) = (d_1, d_2) \in \mathbb{Z}\oplus\mathbb{Z} = H_2(\mathbb{CP}^1, S^1;\mathbb{Z})$, where $[C]\in H_2(C,\partial C;\mathbb{Z})$ is the fundamental class of C.

   We will say that a component of the boundary ∂C is *negative* (respectively, *positive*) if the image of a small neighbourhood of this component belongs to the south (respectively, north) hemisphere. See an illustration in Fig. 1.

(2) Let $N\in\mathbb{Z}$. An *open Hurwitz number* $h^o_N(\lambda, m_1, m_2, d_1)$ is the number of isomorphism classes $[f]$ of open simple ramified coverings counted with the weight
$$\frac{(-1)^{\sharp\left\{\substack{\text{negative}\\ \text{components of }\partial C}\right\}} N^{\sharp\{\text{components of }\partial C\}}}{\sharp\,\mathrm{Aut}(f)}.$$

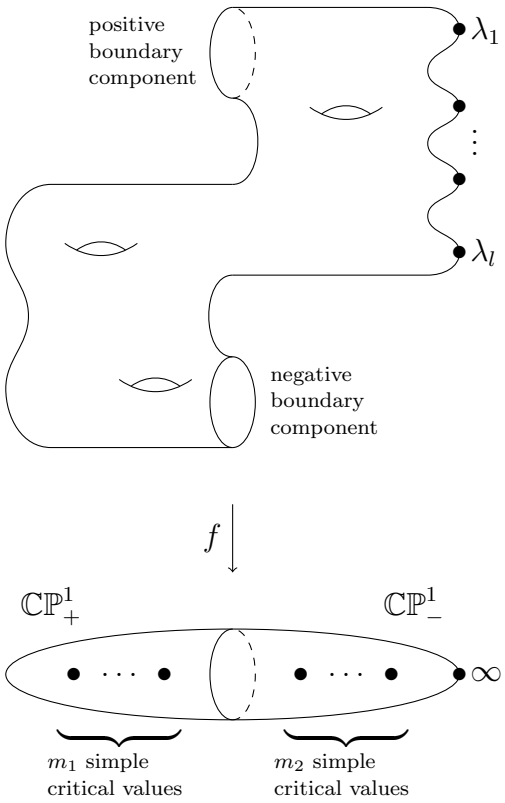

FIGURE 1. Open simple ramified covering

3.2. **Open partition function as an mKP tau-sequence.** Let us introduce formal variables $q_1, q_2, \beta_1, \beta_2$ and form the following generating series:

$$H^o_N(\mathbf{p}, \beta_1, \beta_2, q_1, q_2) := \sum_{\substack{(d_1,d_2)\in\mathbb{Z}^2_{\geq 0}\setminus\{0\} \\ m_1,m_2\geq 0}} \sum_{\lambda\in\mathcal{P}_{d_2}} h^o_N(\lambda, m_1, m_2, d_1) \frac{\beta_1^{m_1}}{m_1!}\frac{\beta_2^{m_2}}{m_2!} q_1^{d_1} q_2^{d_2} p_\lambda.$$

Define an *open partition function* τ^o_N by

$$\tau^o_N(\mathbf{p}, \beta_1, \beta_2, q_1, q_2) := e^{H^o_N(\mathbf{p},\beta_1,\beta_2,q_1,q_2)+H^c(\mathbf{p},\beta_1+\beta_2,q_1q_2)}.$$

Note that

$$\tau^o_0(\mathbf{p}, \beta_1, \beta_2, q_1, q_2) = \tau^c(\mathbf{p}, \beta_1 + \beta_2, q_1q_2). \tag{3.1}$$

Theorem 3.2. *Define* $\widetilde{\tau}_N := \tau^o_N|_{q_2\mapsto e^{N\beta_2}q_2}$. *Then the sequence* $(\widetilde{\tau}_N)_{N\in\mathbb{Z}}$ *is a tau-sequence of the mKP hierarchy.*

In other words, for each $N \in \mathbb{Z}$ we have the following:

(1) $\widetilde{\tau}_N$ is a tau-function of the KP hierarchy.
(2) $\widetilde{\tau}_{N+1}$ is related to $\widetilde{\tau}_N$ by a forward Bäcklund–Darboux transformation (see Appendix B for a brief review).

Proof. The proof will proceed in four steps.

(1) Lemma 3.3 gives a formula for $\tau^o_N|_{\beta_2=0}$ in terms of τ^c.
(2) Proposition 3.4 for $\tau = \tau^c$ implies that $(\tau^o_N|_{\beta_2=0})_{N\in\mathbb{Z}}$ is an mKP tau-sequence.
(3) By Lemma 3.5, the full series τ^o_N can be recovered as $\tau^o_N = e^{\beta_2\mathcal{A}}(\tau^o_N|_{\beta_2=0})$, where $\mathcal{A}$ is the classical cut-and-join operator for closed Hurwitz numbers.
(4) Finally, Proposition 3.6 implies that $(\widetilde{\tau}_N)_{N\in\mathbb{Z}}$ is an mKP tau-sequence, which completes the proof.

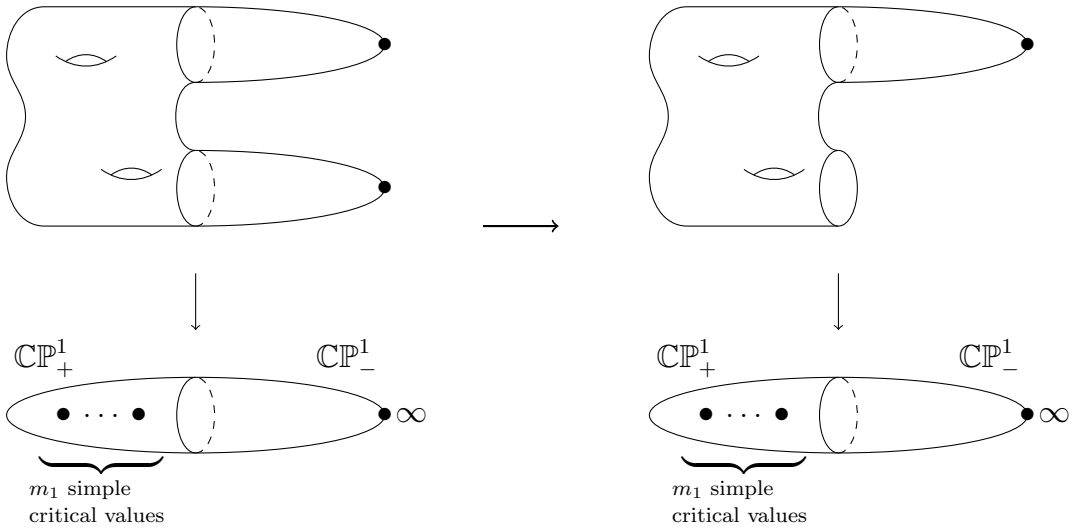

FIGURE 2. Cutting procedure

Lemma 3.3. *We have*

$$\tau_N^o|_{\beta_2=0} = \tau^c(\mathbf{p} - N[q_2^{-1}], \beta_1, q_1q_2)e^{N\sum_{n\geq 1}\frac{p_n}{n}q_2^n}, \tag{3.2}$$

where $\mathbf{p} - N[q_2^{-1}] := \left(p_1 - \frac{N}{q_2}, p_2 - \frac{N}{q_2^2}, \dots\right)$.

Proof. Suppose that $f\colon C \to \mathbb{CP}^1$ is a closed simple ramified covering of degree d and without critical values in $\mathbb{CP}^1_-\backslash\{\infty\}$. Let $f^{-1}(\infty) = \{p_1, \dots, p_l\}$ and the multiplicity of f at p_i is equal to λ_i. Denote by C_i, $1 \leq i \leq l$, the connected component of $f^{-1}(\mathbb{CP}^1_-)$ containing p_i. Since the map f doesn't have critical values in $\mathbb{CP}^1_-\backslash\{\infty\}$, we have $C_i \cap C_j = \emptyset$ for $i \neq j$, and there exist biholomorphisms $\phi_i\colon \mathbb{CP}^1_- \to C_i$ such that $(f \circ \phi_i)(z) = z^{\lambda_i}$. One can remove some of the sets $C_1, \dots, C_l$, say $C_{i_1}, \dots, C_{i_k}$, and obtain an open simple ramified covering of bidegree $(d, d - \sum_{j=1}^k \lambda_{i_j})$ with only negative boundary components. Let us call this procedure the *cutting procedure*, see Fig. 2. Conversely, if $f\colon (C, \partial C) \to (\mathbb{CP}^1, S^1)$ is an open simple ramified covering with only negative boundary components, there is a unique way to glue to C closed disks along the components of ∂C to obtain a closed simple ramified covering $\widetilde{f}\colon \widetilde{C} \to \mathbb{CP}^1$ such that f is obtained from $\widetilde{f}$ by the cutting procedure.

Consider an open simple ramified covering $f\colon (C, \partial C) \to (\mathbb{CP}^1, S^1)$ with $m_2 = 0$. There are two cases.

Case 1: $f^{-1}(\mathbb{CP}^1_+) = \emptyset$. Then one can easily see that $l(\lambda) = 1$ and f is isomorphic to the open simple ramified covering $(\overline{\mathbb{CP}^1_-}, S^1) \to (\overline{\mathbb{CP}^1_-}, S^1)$ mapping z to z^{λ_1}. Therefore, $h_N^o(\lambda_1, 0, 0, 0) = \frac{N}{\lambda_1}$. This gives the term $e^{N\sum_{n\geq 1}\frac{p_n}{n}q_2^n}$ on the right-hand side of (3.2).

Case 2: $f^{-1}(\mathbb{CP}^1_+) \neq \emptyset$. Then C has only negative boundary components and is obtained from some closed simple ramified covering by the cutting procedure. This gives the term $\tau^c(\mathbf{p} - N[q_2^{-1}], \beta_1, q_1q_2)$ on the right-hand side of (3.2). □

Proposition 3.4. *Consider an arbitrary tau-function $\tau(\mathbf{t})$ of the KP hierarchy, with a base algebra K. Let $\tau_N(\mathbf{t}, x) := \tau(\mathbf{t} - N[x^{-1}])e^{N\xi(\mathbf{t},x)}$, $N \in \mathbb{Z}$. Then $(\tau_N(\mathbf{t}, x))_{N\in\mathbb{Z}}$ is a tau-sequence of the mKP hierarchy, with the base algebra $K((x^{-1}))$.*

Proof. Let us fix $N \geq 1$. Introduce a *shift operator* G_x by $G_x(f) := f|_{\mathbf{t}\mapsto\mathbf{t}-[x^{-1}]}$. Consider the group $K[[x_1^{\pm}, \dots, x_N^{\pm}]]$ and denote by K_N its subgroup defined as the sum of the subgroups of the form

$$K[[x_i^{-1}]]_{i\in I}[x_j]_{j\in[N]\backslash I}, \quad I \subset [N] := \{1, \dots, N\}.$$

Clearly, the product of the elements from K_N is well defined, so K_N is an algebra. We have the inclusion of algebras

$$K_N \subset \underbrace{K((x_N^{-1}))((x_{N-1}^{-1}))\dots((x_1^{-1}))}_{\widetilde{K}_N :=}.$$

Note that for any element $f \in K_N$ the substitution $f|_{x_i \mapsto x}$ is well defined giving an element from $K((x^{-1}))$.

We have $G_{x_1}G_{x_2}\dots G_{x_N}\tau \times \prod_{1\le i\le N} e^{\xi(\mathbf{t},x_i)} \in K_N[[\mathbf{t}]]$ and

$$\tau_N = \tau(\mathbf{t} - N[x^{-1}])e^{N\xi(\mathbf{t},x)} = \Big(G_{x_1}G_{x_2}\dots G_{x_N}\tau \times \prod_{1\le i\le N} e^{\xi(\mathbf{t},x_i)}\Big)\Big|_{x_i\mapsto x}.$$

Also, we have $G_{x_1}e^{\xi(\mathbf{t},x_2)} = \frac{x_1-x_2}{x_1}e^{\xi(\mathbf{t},x_2)} \in K((x_2^{-1}))((x_1^{-1}))[[\mathbf{t}]] \subset \widetilde{K}_N[[\mathbf{t}]]$. Therefore,

$$G_{x_1}G_{x_2}\dots G_{x_N}\tau \times \prod_{1\le i\le N} e^{\xi(\mathbf{t},x_i)} = \tag{3.3}$$
$$= \underbrace{G_{x_1}\big(G_{x_2}\big(\dots\big(G_{x_N}\tau\cdot e^{\xi(\mathbf{t},x_N)}\big)\dots\big)\cdot e^{\xi(\mathbf{t},x_2)}\big)\cdot e^{\xi(\mathbf{t},x_1)}}_{\in \widetilde{K}_N[[\mathbf{t}]]} \times \underbrace{\prod_{1\le i<j\le N}\frac{x_i}{x_i-x_j}}_{\in \widetilde{K}_N}.$$

Without loss of generality, we can assume that $\tau(0) = 1$. We know that our tau-function τ has a fermionic representation

$$\tau(\mathbf{t}) = \Psi_0\Big(\bigwedge_{i\le -1} f_i(z)\Big), \quad f_i(z) = z^i\Big(1 + \sum_{j\ge 1} f_{i,j}z^j\Big), \quad f_{i,j} \in K.$$

By Proposition A.8, the right-hand side of equation (3.3) can be represented as

$$\Psi_N\Big(x_1^{-N+1}\theta(x_1)\,x_2^{-N+2}\theta(x_2)\dots\theta(x_N)\bigwedge f_i\Big) \times \prod_{1\le i<j\le N}\frac{x_i}{x_i-x_j} =$$
$$= \Psi_N\Big(\theta(x_1)\theta(x_2)\dots\theta(x_N)\bigwedge f_i\Big) \times \prod_{1\le i<j\le N}\frac{1}{x_i-x_j},$$

where we view $\theta(x_1)\theta(x_2)\dots\theta(x_N)\bigwedge f_i$ as an element of the Fock space $\mathcal{F}^{[N]}_{\widetilde{K}_N}$.

Consider integers $a_1 > a_2 > \dots > a_N$. The coefficient of $\theta_{a_1}\theta_{a_2}\dots\theta_{a_N}$ in the operator $\theta(x_1)\theta(x_2)\dots\theta(x_N)\prod_{1\le i<j\le N}\frac{1}{x_i-x_j}$ is equal to the Laurent polynomial

$$s_\lambda(x_1,\dots,x_N) = \frac{\det\left(x_i^{a_j}\right)_{i,j=1}^N}{\prod_{1\le i<j\le N}(x_i-x_j)}, \quad \lambda_i = a_i + i - N, \quad i = 1,\dots,N.$$

The specialization to $x_i = x$ is found by the following standard computation:

$$s_\lambda(x,\dots,x) = x^{|\lambda|}\,s_\lambda(1,q,\dots,q^{N-1})\big|_{q=1} = x^{|\lambda|}\,\frac{\det\left(q^{(i-1)a_j}\right)}{\prod_{i<j}(q^{i-1}-q^{j-1})}\bigg|_{q=1} =$$
$$= x^{|\lambda|}\prod_{i<j}\frac{q^{a_j}-q^{a_i}}{q^{i-1}-q^{j-1}}\bigg|_{q=1} = x^{|\lambda|}\prod_{i<j}\frac{a_i-a_j}{j-i} = x^{|\lambda|}\frac{\det\left(a_i^{N-j}\right)}{\prod_{i=1}^N(i-1)!}.$$

This is the coefficient of $\theta_{a_1}\theta_{a_2}\dots\theta_{a_N}$ in

$$\prod_{i=1}^N\frac{x^{-i+1}}{(i-1)!} \times \big((x\partial_x)^{N-1}\theta(x)\big)\circ\dots\circ\big((x\partial_x)\theta(x)\big)\circ\theta(x).$$

We conclude that

$$\tau_N = \Psi_N\left(\left[\prod_{i=1}^{N}\frac{x^{-i+1}}{(i-1)!}\times\big((x\partial_x)^{N-1}\theta(x)\big)\circ\cdots\circ\big((x\partial_x)\theta(x)\big)\circ\theta(x)\right]\bigwedge_{i\le -1} f_i\right) = \\ = \Psi_N\left(h_{N-1}\wedge h_{N-2}\wedge\ldots\wedge h_0\wedge\bigwedge_{i\le -1} f_i\right),$$

where $h_j = \sum_{a\in\mathbb{Z}}\frac{(a+j)^j}{j!}x^a z^{a+j}$, $0\le j\le N-1$.

We can decompose $h_0 = \sum_{a\in\mathbb{Z}} x^a z^a$ in the following way:

$$h_0 = \sum_{i\le -1} c_{0,i} f_i + \widetilde{h}_0, \quad \widetilde{h}_0\in K((x^{-1}))[[z]], \quad c_{0,i}\in x^i\mathbb{C}[[x^{-1}]].$$

Therefore, $\tau_1 = \Psi_1\left(\widetilde{h}_0\wedge\bigwedge_{i\le -1} f_i\right)$, and we see that $\operatorname{Coef}_{z^0}\widetilde{h}_0 = \tau(-[x^{-1}])\in K[[x^{-1}]]$. Continuing this process, we construct a sequence of elements $\widetilde{h}_i\in z^i K((x^{-1}))[[z]]$, $i\ge 0$, such that

$$h_i = \sum_{j\le -1} c_{i,j} f_j + \sum_{k=0}^{i-1} d_{i,k}\widetilde{h}_k + \widetilde{h}_i, \quad c_{i,j}\in x^{j-i}K[[x^{-1}]], \quad d_{i,k}\in K((x^{-1})),$$

and moreover $\operatorname{Coef}_{z^i}\widetilde{h}_i = \frac{\tau(-(i+1)[x^{-1}])}{\tau(-i[x^{-1}])}$.

We have $\tau_M = \Psi_M\left(\widetilde{h}_{M-1}\wedge\widetilde{h}_{M-2}\wedge\ldots\wedge\widetilde{h}_0\wedge\bigwedge_{i\le -1} f_i\right)$, $M\ge 1$, and therefore τ_M is obtained from τ_{M-1} by a forward Bäcklund–Darboux transformation. Since $\tau_{-M} = ((\tau^*)_M)^*$, we see that τ_{-M} is obtained from $\tau_{-(M-1)}$ by a backward Bäcklund–Darboux transformation, and thus $\tau_{-(M-1)}$ is obtained from τ_{-M} by a forward Bäcklund–Darboux transformation, which completes the proof of the proposition. □

Lemma 3.5. *Define the (classical)* cut-and-join operator $\mathcal{A}$ *as:*

$$\mathcal{A} := \frac{1}{2}\sum_{i,j\ge 1}\left((i+j)p_i p_j\frac{\partial}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial^2}{\partial p_i\partial p_j}\right).$$

Then we have $\frac{\partial\tau_N^o}{\partial\beta_2} = \mathcal{A}\tau_N^o$.

Proof. Taking into account (3.1), for $N=0$ this becomes the famous cut-and-join equation for the closed Hurwitz numbers, see the paper [GJ97], where the authors gave a proof using the interpretation of the closed Hurwitz numbers as the number of factorisations of a permutation into transpositions. The cut-and-join equation for the closed Hurwitz numbers has also a geometric proof, see e.g. [EMS11, Section 3]. One can easily see that exactly the same arguments work in the case of arbitrary N. □

Proposition 3.6. *Let* $\mathbf{f}(z,q) = (f_k(z,q))_{k\in\mathbb{Z}}$ *be a sequence of Laurent series of the form*

$$f_k(z,q) = z^k\left(1+\sum_{j\ge 1} f_{k,j}(qz)^j\right)\in\mathbb{C}[[q]]((z)), \quad f_{k,j}\in\mathbb{C}.$$

Let $(\tau_N)_{N\in\mathbb{Z}}$ *be the corresponding mKP tau-sequence given by* (B.4)*. Then the sequence* $(\widetilde{\tau}_N(\beta))_{N\in\mathbb{Z}}$*, where* $\widetilde{\tau}_N(\beta) := (e^{\beta\mathcal{A}}\tau_N)\big|_{q\mapsto e^{N\beta}q}$*, is also an mKP tau-sequence.*

Proof. By Proposition A.10, we have $\widetilde{\tau}_N(\beta) = \Psi_N\left(\bigwedge_{i\geqslant 1} \tilde{f}^{(N)}_{N-i}(z,q,\beta)\right)$, where

$$\tilde{f}^{(N)}_k(z,q,\beta) := z^k\Big(1+\sum_{j\geq 1} e^{\beta \frac{(j+k-N+\frac{1}{2})^2-(k-N+\frac{1}{2})^2}{2}} e^{\beta jN} f_{k,j}(qz)^j\Big).$$

We observe that $\tilde{f}^{(N)}_k(z,q,\beta)$ does not depend on N by transforming the exponent of e^β: $\frac{(j+k-N+\frac{1}{2})^2-(k-N+\frac{1}{2})^2}{2} + jN = \frac{(j+k+\frac{1}{2})^2-(k+\frac{1}{2})^2}{2}$, and therefore, by formula (B.4), $(\widetilde{\tau}_N(\beta))_{N\in\mathbb{Z}}$ is an mKP tau-sequence, as desired. □

□

The next theorem describes explicitly the Bäcklund–Darboux transformation relating $\widetilde{\tau}_0 = \tau^c(\mathbf{p}, \beta_1+\beta_2, q_1q_2)$ and $\widetilde{\tau}_1 = \tau^o_1|_{q_2\mapsto e^{\beta_2}q_2}$.

Theorem 3.7. *We have*

$$\tau^o_1 = \mathrm{Coef}_{x^0}\left[D(x,\beta_1,\beta_2,q_1,q_2)\tau^c(\mathbf{p}-[x^{-1}],\beta_1+\beta_2,e^{-\beta_2}q_1q_2)e^{\sum_{n\geq 1}\frac{p_n}{n}x^n}\right],$$

where

$$D(z,\beta_1,\beta_2,q_1,q_2) = \sum_{l\geq 0}\left(1+\sum_{k\geq 1} e^{\beta_1\frac{l^2+l-k^2+k}{2}}\frac{(-1)^k q_1^{k+l}}{(l+k)(k-1)!l!}\right)e^{\beta_2\frac{l^2-l}{2}}\left(\frac{q_2}{z}\right)^l.$$

Proof. Using Propositions A.8, A.10, and formula (2.1), we compute

$$\begin{aligned}
\tau^o_1 =& e^{\beta_2\mathcal{A}}\left(\tau^c(\mathbf{p}-[q_2^{-1}],\beta_1,q_1q_2)e^{\sum_{n\geq 1}\frac{p_n}{n}q_2^n}\right) = \\
=& e^{\beta_2\mathcal{A}}\Psi_1\left(\sum_{i\in\mathbb{Z}} q_2^i z^i \wedge \bigwedge_{j\geq 1} f^{\mathrm{H}}_j(z,\beta_1,q_1q_2)\right) = \\
=& e^{(\beta_1+\beta_2)\mathcal{A}}\Psi_1\left(\sum_{i\in\mathbb{Z}} q_2^i e^{-\beta_1\frac{i^2-i}{2}} z^i \wedge \bigwedge_{j\geq 1} f^{\mathrm{H}}_j(z,0,e^{\beta_1}q_1q_2)\right) = \\
=& e^{(\beta_1+\beta_2)\mathcal{A}}\Psi_1\left(\sum_{i\in\mathbb{Z}} q_2^i e^{-\beta_1\frac{i^2-i}{2}} z^i \wedge \bigwedge_{j\geq 1} z^{-j}e^{\widetilde{q}_1q_2z}\right),
\end{aligned}$$

where $\widetilde{q}_1 := e^{\beta_1}q_1$

Lemma 3.8. *For $k\geq 1$, we have*

$$z^{-k}\wedge\bigwedge_{j\geq 1} z^{-j}e^{qz} = (-q)^k\left(\sum_{l\geq 0}\frac{q^l}{(l+k)(k-1)!l!}z^l\right)\wedge\bigwedge_{j\geq 1} z^{-j}e^{qz}.$$

Proof. We have

$$z^{-k} = e^{-qz}(e^{qz}z^{-k}) = \sum_{m\geq 0}\frac{(-q)^m}{m!}\cdot e^{qz}z^{m-k} =$$

$$= \sum_{0\leq m<k}\frac{(-q)^m}{m!}\cdot e^{qz}z^{m-k} + (-q)^k\sum_{m\geq 0}\frac{(-q)^m}{(m+k)!}e^{qz}z^m.$$

The first sum vanishes in the wedge product, and we transform the second sum:

$$\sum_{m\geq 0}\frac{(-q)^m}{(m+k)!}e^{qz}z^m = \sum_{m,n\geq 0}\frac{(-q)^m}{(m+k!)}\cdot\frac{q^n}{n!}z^{m+n} = \sum_{l\geq 0}\left(\sum_{0\leq m\leq l}\frac{(-1)^m}{(k+m)!(l-m)!}\right)q^lz^l.$$

The sum in the parenthesis can be simplified via the binomial formula:

$$\sum_{0\le m\le l}\frac{(-1)^m}{(k+m)!(l-m)!}=\frac{1}{(l+k)!}\sum_{0\le m\le l}(-1)^m\binom{l+k}{k+m}=$$
$$=\frac{1}{(l+k)!}\binom{l+k-1}{k-1}=\frac{1}{(l+k)(k-1)!l!},$$

which completes the proof of the lemma. □

So we obtain

$$\tau_1^o=e^{(\beta_1+\beta_2)\mathcal{A}}\Psi_1\left(\sum_{l\ge0}\left(e^{-\beta_1\frac{l^2-l}{2}}+\sum_{k\ge1}e^{-\beta_1\frac{k^2+k}{2}}\frac{(-1)^k\widetilde{q}_1^{k+l}}{(l+k)(k-1)!l!}\right)(q_2z)^l\wedge\bigwedge_{j\ge1}z^{-j}e^{\widetilde{q}_1q_2z}\right)=$$
$$=\Psi_1\left(\sum_{l\ge0}\left(1+\sum_{k\ge1}e^{\beta_1\frac{l^2+l-k^2+k}{2}}\frac{(-1)^kq_1^{k+l}}{(l+k)(k-1)!l!}\right)e^{\beta_2\frac{l^2-l}{2}}(q_2z)^l\wedge\bigwedge_{j\ge1}f_j^{\mathrm{H}}(z,\beta_1+\beta_2,e^{-\beta_2}q_1q_2)\right),$$

which completes the proof of the theorem. □

Appendix A. KP hierarchy

A.1. **KP hierarchy in the Lax form.** In this section, we recall the definition of the Kadomtsev–Petviashvili (KP) hierarchy in terms of formal pseudo-differential operators. For more detailed information, see, e.g., [Dic03].

Let $\mathbf{t}:=(t_1,t_2,\dots)$ be an infinite sequence of variables and $\mathbb{C}[[\mathbf{t}]]:=\mathbb{C}[[t_1,t_2,\dots]]$. Let $\partial=\partial_1:=\frac{\partial}{\partial t_1}$ and $\partial_n:=\frac{\partial}{\partial t_n}$ for $n>1$ be the partial derivatives acting on the algebra $\mathbb{C}[[\mathbf{t}]]$. A *formal pseudo-differential operator* is an expression of the form

$$A=\sum_{i=-N}^{\infty}a_i\partial^{-i},\quad a_i\in\mathbb{C}[[\mathbf{t}]].$$

We define multiplication of two such expressions using the following generalization of the Leibniz rule: $\partial^n\circ a:=\sum_{i=0}^{\infty}\binom{n}{i}(\partial^i a)\circ\partial^{n-i}$, n, $a\in\mathbb{C}[[\mathbf{t}]]$, where $\binom{n}{i}:=\frac{1}{i!}\cdot n(n-1)\cdots(n-i+1)$ for any $n\in\mathbb{Z}$. For a pseudo-differential operator A as above, we define the residue $\operatorname{res}A:=a_{-1}$ and the decomposition $A=A_++A_-$: $A_+:=\sum_{i\le0}a_i\partial^{-i}$, $A_-:=\sum_{i\ge1}a_i\partial^{-i}$. We use the shorthand $A_+^n:=(A^n)_+$, and similarly, $A_-^n:=(A^n)_-$.

For a sequence $(u_1(\mathbf{t}),u_2(\mathbf{t}),\dots)$, $u_i(\mathbf{t})\in\mathbb{C}[[\mathbf{t}]]$, let the pseudo-differential operator $L(\mathbf{t})$ be given by

$$L(\mathbf{t})=\partial+\sum_{i=1}^{\infty}u_i(\mathbf{t})\partial^{-i}.$$

Definition A.1. A sequence $(u_i(\mathbf{t}))$, or the corresponding pseudo-differential operator $L(\mathbf{t})$, is a solution of the *KP hierarchy* if, for each $n>1$, the operator $L(\mathbf{t})$ satisfies

$$\partial_nL(\mathbf{t})=[(L(\mathbf{t})^n)_+,\,L(\mathbf{t})],$$

where derivatives ∂_n act on $L(\mathbf{t})$ coefficient-wise in ∂.

Proposition A.2. *For each solution $L=L(\mathbf{t})$ of the KP hierarchy, there exists a pseudo-differential operator P, called a* dressing operator*, of the form $P=1+\sum_{i=1}^{\infty}w_i(\mathbf{t})\partial^{-i}$, such that $L=P\circ\partial\circ P^{-1}$ and $\partial_nP=-L_-^n\circ P$. Such operator P is defined up to* gauge transformations *of the form $P\longmapsto P\circ(1+\sum_{i=1}^{\infty}c_i\partial^{-i})$, where (c_i) is an arbitrary sequence of constant coefficients.*

Let $\xi(\mathbf{t},x):=\sum_{i\geq 1}t_ix^i$. To each solution L of the KP hierarchy and its dressing operator P, we associate the *wave function* $\psi(\mathbf{t},x):=Pe^{\xi(\mathbf{t},x)}$. Explicitly, $\psi(\mathbf{t},x)=(1+\sum_{i=1}^{\infty}w_i(\mathbf{t})z^{-i})\,e^{\xi(\mathbf{t},x)}$.

Proposition A.3. *The wave function satisfies the following equations for all $n\geq 1$: $\partial_n\psi=L_+^n\psi$.*

It will be convenient to introduce additional dressing operators and wave functions *of level* $N\in\mathbb{Z}$ by the following formulas: $P_N:=P\circ\partial^N$, $\psi_N(\mathbf{t},x):=x^N\psi(\mathbf{t},x)$.

The KP hierarchy has the following remarkable property: even though *a priori* its solution is given in terms of an infinite sequence of unknown functions $u_i(\mathbf{t})$, in fact, each solution may be described by a single function of $\mathbf{t}$, called the *tau-function* $\tau(\mathbf{t})$. Let $\mathbf{t}-[x^{-1}]$ be the shifted sequence of variables $\left(t_1-x^{-1},\,t_2-\frac{1}{2}x^{-2},\,t_3-\frac{1}{3}x^{-3},\dots\right)$.

Theorem A.4. *For each solution $L(\mathbf{t})$ of the KP hierarchy and its associated wave function $\psi(\mathbf{t},x)$, there exists a function $\tau(\mathbf{t})$ that satisfies*

$$\psi(\mathbf{t},x)=\frac{\tau(\mathbf{t}-[x^{-1}])}{\tau(\mathbf{t})}e^{\xi(\mathbf{t},x)},\quad \tau(0)\neq 0.$$

The function τ is unique up to the multiplication $\tau\mapsto\lambda\tau$, $\lambda\neq 0$. It is called a tau-function *corresponding to the solution $L(\mathbf{t})$.*

The action of gauge transformations of the dressing operator on the wave and tau-function can be written as follows:

$$P\longmapsto P\circ\exp\left(\sum_{i\geq 1}d_i\partial^{-i}\right),\quad \psi\longmapsto\psi\exp\left(\sum_{i\geq 1}d_ix^{-i}\right),\quad \tau\longmapsto\tau\exp\left(-\sum_{i\geq 1}id_it_i\right),$$

where $\sum d_ix^i=\log(1+\sum c_ix^i)$.

A.2. **Adjoint operators.** For a pseudo-differential operator $A=\sum a_i\partial^i$, we define its formal adjoint by $A^\dagger:=\sum(-\partial)^i\circ a_i$. Adjoint operators satisfy the following identities:

(1) $(A\circ B)^\dagger=B^\dagger\circ A^\dagger$, hence $[A,B]^\dagger=-[A^\dagger,B^\dagger]$;
(2) $(A_+)^\dagger=(A^\dagger)_+$ and $(A_-)^\dagger=(A^\dagger)_-$;
(3) $(A^\dagger)^\dagger=A$.

Given a KP solution L with a dressing operator P, we define its *adjoint wave function* $\psi^\dagger(\mathbf{t},x):=(P^\dagger)^{-1}e^{-\xi(\mathbf{t},x)}$. It satisfies a system of linear equations analogous to those in Proposition A.3: $\partial_n\psi^\dagger=-(L_+^n)^\dagger\psi^\dagger$. The adjoint wave function can be expressed in terms of the tau-function as

$$\psi^\dagger(\mathbf{t},x)=\frac{\tau(\mathbf{t}+[x^{-1}])}{\tau(\mathbf{t})}e^{-\xi(\mathbf{t},x)},$$

where $\mathbf{t}+[x^{-1}]:=(t_1+x^{-1},t_2+\frac{1}{2}x^{-2},t_3+\frac{1}{3}x^{-3},\dots)$. The adjoint wave function at level N is introduced as $\psi_N^\dagger(\mathbf{t},x):=x^{-N}\psi^\dagger(\mathbf{t},x)$.

For a pseudo-differential operator $A=\sum a_i(\mathbf{t})\partial^i$, define $A^!:=\sum a_i(-\mathbf{t})(-\partial)^i$. This transformation possesses the following properties:

$$(A\circ B)^!=A^!\circ B^!,\qquad (A_\pm)^!=A^!_\pm,\qquad (A^!)^!=A,\qquad (A^\dagger)^!=(A^!)^\dagger.$$

Let $L=\partial+\sum u_i(\mathbf{t})\partial^{-i}$ be a solution of the KP hierarchy. Define

$$L^*(\mathbf{t}):=(L^!)^\dagger.$$

Then L^* also satisfies the KP equations. This operation defines an involution on the set of all KP solutions, which we call the *adjunction involution*. It acts on the dressing operator and on the wave and tau-function by the following formulas:

$$P^* = ((P^\dagger)^{-1})^!, \qquad \psi^*(\mathbf{t}, x) = \psi^\dagger(-\mathbf{t}, x), \qquad \tau^*(\mathbf{t}) = \tau(-\mathbf{t}).$$

A.3. **Fock space formalism.** We begin by recalling the notion of the fermionic Fock space, also known as the semi-infinite wedge space $\Lambda^{\frac{\infty}{2}}\mathbb{C}((z))$.

A sequence of integers $\mathbf{a} = (a_1, a_2, \dots)$ is called *admissible of level* N if it satisfies $a_i = -i+N$ for all sufficiently large i. The *degree* of such a sequence is defined as $\deg(\mathbf{a}) := \sum_i (a_i + i - N)$. To any admissible sequence $\mathbf{a}$, we associate the infinite Grassmann monomial $z^{\mathbf{a}} := z^{a_1} \wedge z^{a_2} \wedge \cdots$, subject to the following identification: $z^{\mathbf{a}} = (-1)^{|\sigma|} z^{\sigma(\mathbf{a})}$ for any finitely supported permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$. In particular, if $a_i = a_j$ for some $i \neq j$, then $z^{\mathbf{a}} = 0$. It follows that if $\deg(\mathbf{a}) < 0$, then $z^{\mathbf{a}} = 0$.

Let $\mathcal{F}_d^{[N]}$ denote the vector space spanned by all monomials $z^{\mathbf{a}}$ of level N and degree d. The *fermionic Fock space of level* N, denoted $\mathcal{F}^{[N]}$, consists of all formal sums $w_0 + w_1 + \cdots$, where $w_d \in \mathcal{F}_d^{[N]}$. The *full fermionic Fock space* is then defined as the direct sum $\mathcal{F} := \bigoplus_{N\in\mathbb{Z}} \mathcal{F}^{[N]}$.

A *partition* λ is a non-increasing sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \dots$, where $\lambda_i = 0$ for $i \gg 0$. The space $\mathcal{F}_d^{[N]}$ has a basis $\{v_\lambda^{[N]}\}$, $v_\lambda^{[N]} := z^{\lambda_1 - 1 + N} \wedge z^{\lambda_2 - 2 + N} \wedge \dots$, where λ runs over all partitions of weight d. Denote the monomial $v_0^{[N]} = z^{N-1} \wedge z^{N-2} \wedge \dots$ by $|N\rangle$ (the *vacuum vector of level* N). It is the unique monomial of the lowest possible degree 0 in $\mathcal{F}^{[N]}$. We define the dot product $\langle \cdot \mid \cdot \rangle$ on $\mathcal{F}_d^{[N]}$ by declaring $\{v_\lambda^{[N]}\}$ to be an orthonormal basis.

For $a \in \mathbb{Z}$, define linear operators $\theta_a\colon \mathcal{F}_d^{[N]} \to \mathcal{F}_{d+a-N}^{[N+1]}$ and $\theta_a^\dagger\colon \mathcal{F}_d^{[N]} \to \mathcal{F}_{d-a+N-1}^{[N-1]}$ as follows:

$$\theta_a(z^{a_1} \wedge z^{a_2} \wedge \dots) := z^a \wedge z^{a_1} \wedge z^{a_2} \wedge \dots,$$
$$\theta_a^\dagger(z^{a_1} \wedge z^{a_2} \wedge \dots) := \sum_{j\geq 1} \delta_{a,a_j} (-1)^{j-1} z^{a_1} \wedge \dots \wedge z^{a_{j-1}} \wedge \widehat{z^{a_j}} \wedge z^{a_{j+1}} \wedge \dots.$$

We also define the generating series of these operators: $\theta(x) := \sum_{i\in\mathbb{Z}} \theta_i x^i$, $\qquad \theta^\dagger(x) := \sum_{i\in\mathbb{Z}} \theta_i^\dagger x^{-i}$.

For $0 \neq n \in \mathbb{Z}$, define $\alpha_n\colon \mathcal{F}_d^{[N]} \to \mathcal{F}_{d-n}^{[N]}$ as $\alpha_n := \sum_{i\in\mathbb{Z}} \theta_i \circ \theta_{i+n}^\dagger$. A direct computation shows that α_n act on the monomials $z^{\mathbf{a}}$ as follows:

$$\alpha_n(z^{a_1} \wedge z^{a_2} \wedge \dots) = \sum_{j\geq 1} z^{a_1} \wedge \dots \wedge z^{a_j - n} \wedge \dots.$$

Define $\Gamma(\mathbf{t}) := \sum_{n\geq 1} t_n \alpha_n$.

The *bosonic Fock space* is the space of formal power series $\mathbb{C}[[\mathbf{t}]]$. The *bosonic-fermionic correspondence at level* N is a linear operator $\Psi_N : \mathcal{F}^{[N]} \to \mathbb{C}[[\mathbf{t}]]$ defined by the following formula:

$$\Psi_N(u) = \left\langle N \mid e^{\Gamma(\mathbf{t})} u \right\rangle, \quad u \in \mathcal{F}^{[N]}.$$

The operator Ψ_N is an isomorphism between $\mathcal{F}^{[N]}$ and $\mathbb{C}[[\mathbf{t}]]$.

The bosonic-fermionic correspondence will be used in Theorem A.6 to give a geometric intepretation of KP tau-functions in terms of decomposable Fock space vectors, or, equivalently, elements of the infinite-dimensional Grassmannian.

A.4. **Sato Grassmannian.** Let $\mathbb{C}((z))$ be the field of formal Laurent series. As a vector space over $\mathbb{C}$, it admits a direct sum decomposition $\mathbb{C}((z)) = \mathbb{C}[[z]] \oplus z^{-1}\mathbb{C}[z^{-1}]$. Let $\pi\colon \mathbb{C}((z)) \to z^{-1}\mathbb{C}[z^{-1}]$ be the corresponding projection along $\mathbb{C}[[z]]$.

Definition A.5. The *Sato Grassmannian* Gr is the set of all $\mathbb{C}$-linear subspaces H in $\mathbb{C}((z))$ such that the restriction $\pi|_H\colon H \to z^{-1}\mathbb{C}[z^{-1}]$ has finite-dimensional kernel and cokernel.

Gr is the disjoint union of its components Gr_N, defined as follows:

$$\mathrm{Gr}_N := \{H \in \mathrm{Gr} \mid \dim \ker \pi|_H - \dim \operatorname{coker} \pi|_H = N\}.$$

The *big cell* Gr_N^0 is the set of subspaces H such that the projection $\pi_N|_H$ from H to $z^{N-1}\mathbb{C}[z^{-1}]$ along $z^N\mathbb{C}[[z]]$ is an isomorphism. Each subspace $H \in \mathrm{Gr}_N^0$ has an *adapted basis* $(f_{N-1}, f_{N-2}, \dots)$ of the following form: $f_k(z) = (\pi_N|_H)^{-1}(z^k) = z^k\Big(1 + \sum_{j\geq 0} f_{k,j} z^j\Big)$, $k < N$.

Given an adapted basis of $H \in \mathrm{Gr}_N^0$, we can construct a decomposable vector $\bigwedge_i f_{N-i}(z)$ in the Fock space $\mathcal{F}^{[N]}$. The relation of decomposable vectors to the KP hierarchy is revealed in the following theorem.

Theorem A.6 ([DKJM83, MJD00])**.** *For some $N \in \mathbb{Z}$, consider a sequence of Laurent series $\mathbf{f} = (f_{N-1}(z), f_{N-2}(z), \dots)$, such that for all $k < N$, $f_k(z) \in \mathbb{C}((z))$ has the following form:*

$$f_k(z) = z^k\Big(1 + \sum_{j\geq 1} f_{k,j} z^j\Big), \quad f_{k,j} \in \mathbb{C}.$$

Then the formal power series

$$\tau_{\mathbf{f}} := \Psi_N\Big(\bigwedge_{i\geq 1} f_{N-i}(z)\Big) \in \mathbb{C}[[\mathbf{t}]] \tag{A.1}$$

is a tau-function of the KP hierarchy. Moreover, an arbitrary tau-function τ of the KP hierarchy satisfying $\tau(0) = 1$ can be represented in the form (A.1).

Thus, via the boson-fermion correspondence, from each subspace $H \in \mathrm{Gr}_N^0$ we obtain a tau-function of the KP hierarchy $\tau_H(\mathbf{t})$.

Let $(-,-)$ be an inner product on $\mathbb{C}((z))$ given by the formula $(f(z), g(z)) = \mathrm{res}_{z=0} f(z)g(z)$. Then, for any $H \in \mathrm{Gr}$, its orthogonal subspace $H^\perp$ is also an element of Gr. Denote by ι the resulting map $\mathrm{Gr} \to \mathrm{Gr}, H \mapsto H^\perp$. The map ι is an involution and satisfies $\iota(\mathrm{Gr}_N) = \mathrm{Gr}_{-N}$ and $\iota(\mathrm{Gr}_N^0) = \mathrm{Gr}_{-N}^0$.

Proposition A.7 ([HB21, Section 4.14])**.** *Let $H \in \mathrm{Gr}_N^0$ and $H^\perp = \iota(H) \in \mathrm{Gr}_{-N}^0$. Then the corresponding tau-functions are related as follows: $\tau_{H^\perp}(\mathbf{t}) = \tau_H(-\mathbf{t})$.*

Thus, the orthogonality involution ι acts on KP tau-functions as the adjunction involution introduced in A.2: $\tau_{H^\perp} = (\tau_H)^*$.

The wave and the adjoint wave functions also admit fermionic representations, given in the following proposition.

Proposition A.8 ([MJD00])**.** *Let $\tau_{\mathbf{f}} = \Psi_N\Big(\bigwedge_{i\geq 1} f_{N-i}(z)\Big)$. Then the wave function $\psi_N(\mathbf{t}, x)$ and the adjoint wave function $\psi_N^\dagger(\mathbf{t}, x)$ at level N, associated to $\tau_{\mathbf{f}}$, are given by*

$$\psi_N(\mathbf{t}, x) = \frac{\Psi_{N+1}\Big(\theta(x)\bigwedge_{i\geq 1} f_{N-i}(z)\Big)}{\Psi_N\Big(\bigwedge_{i\geq 1} f_{N-i}(z)\Big)}, \quad \psi_N^\dagger(\mathbf{t}, x) = x^{-1}\frac{\Psi_{N-1}\Big(\theta^\dagger(x)\bigwedge_{i\geq 1} f_{N-i}(z)\Big)}{\Psi_N\Big(\bigwedge_{i\geq 1} f_{N-i}(z)\Big)}. \tag{A.2}$$

In other words, the following equations hold:

$$\tau_{\mathbf{f}}(\mathbf{t}-[x^{-1}])e^{\xi(\mathbf{t},x)} = x^{-N}\Psi_{N+1}\Big(\theta(x)\bigwedge f_{N-i}(z)\Big),$$
$$\tau_{\mathbf{f}}(\mathbf{t}+[x^{-1}])e^{-\xi(\mathbf{t},x)} = x^{N-1}\Psi_{N-1}\Big(\theta^{\dagger}(x)\bigwedge f_{N-i}(z)\Big).$$

Remark A.9. All the results presented above have a direct generalization to the case when we start from the pseudo-differential operators $A=\sum_{i=-N}^{\infty}a_i\partial^{-i}$ with $a_i\in K[[\mathbf{t}]]=K[[t_1,t_2,\dots]]$, where K is an arbitrary commutative associative $\mathbb{C}$-algebra. In this generalization, one should require in Theorem A.4 that $\tau(0)\in K$ and $\lambda\in K$ are invertible and also, in the Fock space formalism, one should consider K-modules instead of vector spaces over $\mathbb{C}$. In the case, when K is different from $\mathbb{C}$, we will sometimes say that the *base algebra* is K. The Fock space with the coefficients from K will be denoted by $\mathcal{F}_K$.

A.5. **Cut-and-join operator.** The *cut-and-join operator* is defined by

$$\mathcal{A}:=\frac{1}{2}\sum_{i,j\geq 1}\left(ijt_it_j\frac{\partial}{\partial t_{i+j}}+(i+j)t_{i+j}\frac{\partial^2}{\partial t_i\partial t_j}\right).$$

Proposition A.10 (see, e.g., [KL07])**.** *Let $\tau\in\mathbb{C}[[\mathbf{t}]]$ be an arbitrary tau-function of the KP hierarchy.*

(1) *The formal power series $e^{\beta\mathcal{A}}\tau\in\mathbb{C}[[\beta,\mathbf{t}]]$ is also a tau-function of the KP hierarchy, with the base algebra $\mathbb{C}[[\beta]]$.*

(2) *If τ is given by formula (A.1) from Theorem A.6, then $e^{\beta\mathcal{A}}\tau=\tau_{\mathbf{f}(\beta)}$, where the sequence $\mathbf{f}(\beta)=(f_{N-1}(z,\beta),f_{N-2}(z,\beta),\dots)$ is given by*

$$f_k(z,\beta):=z^k\Big(1+\sum_{j\geq 1}e^{\beta\frac{(k+j-N+\frac{1}{2})^2-(k-N+\frac{1}{2})^2}{2}}f_{k,j}z^j\Big).$$

Example A.11. The famous cut-and-join equation for the closed Hurwitz numbers (see Section 2) claims that $\frac{\partial\tau^c}{\partial\beta}=\mathcal{A}\tau^c$. Since $\tau^c|_{\beta=0}=e^{t_1}=\Psi_0(\bigwedge_{i\geq 1}f_{-i})$, where $f_{-i}(z)=e^zz^{-i}$, by Proposition A.10, τ^c is a tau-function of the KP hierarchy, and moreover τ^c is given by formula (2.1).

Appendix B. Bäcklund–Darboux transformations and mKP hierarchy

B.1. **Bäcklund–Darboux transformations in the Lax form.** Here we discuss Bäcklund–Darboux transformations and their properties, following the results presented in [CSY92, HvdL01, KvdL18]. In order to make the paper more self-contained, we will give short proofs of some statements.

Let $L(\mathbf{t})$ be a solution of the KP hierarchy, and let $D(\mathbf{t})$ be an invertible pseudo-differential operator. We formulate conditions under which $\widetilde{L}=D\circ L\circ D^{-1}$ is also a solution of the KP hierarchy.

Lemma B.1. *If D satisfies the following equations for all $n\geq 1$: $\partial_nD=\widetilde{L}_+^n\circ D-D\circ L_+^n$, then $\widetilde{L}=D\circ L\circ D^{-1}$ is a solution of the KP hierarchy.*

Definition B.2. Let $L(\mathbf{t})$ be a KP solution. An associated *eigenfunction* of L is a formal power series $\Phi(\mathbf{t})\in A$ that satisfies the following equations for all $n\geq 1$: $\partial_n\Phi=L_+^n\Phi$.

Proposition B.3. *Let L be a KP solution, and let Φ be its associated eigenfunction satisfying $\Phi(0) \neq 0$. Define $D := \Phi \circ \partial \circ \Phi^{-1} = \partial - \partial \log \Phi$. Then D satisfies the conditions of Lemma B.1, and so $\widehat{L} := D \circ L \circ D^{-1}$ is again a KP solution.*

Proof. Let us check that $\partial_j D = \widehat{L}_+^j \circ D - D \circ L_+^j$. For this, we transform the right-hand side as follows:

$$\widehat{L}_+^j \circ D - D \circ L_+^j = (D \circ L^j \circ D^{-1})_+ \circ D - D \circ L_+^j = \\ = (D \circ L_+^j \circ D^{-1})_+ \circ D - D \circ L_+^j = -(D \circ L_+^j \circ D^{-1})_- \circ D.$$

We see that the pseudo-differential operator under consideration is actually a function, which is equal to

$$-\operatorname{res}(D \circ L_+^j \circ D^{-1}) = -\operatorname{res}(\Phi \circ \partial \circ \Phi^{-1} \circ L_+^j \circ \Phi \circ \partial^{-1} \circ \Phi^{-1}) = \\ = -\operatorname{res}(\partial \circ \Phi^{-1} \circ L_+^j \circ \Phi \circ \partial^{-1}) = -\partial \left(\Phi^{-1} L_+^j \Phi\right) = -\partial(\partial_j \log \Phi) = \partial_j D,$$

as required. □

Definition B.4. Given L and $\widehat{L}$ as in Proposition B.3, we say that $\widehat{L}$ is related to L by a *(forward) Bäcklund–Darboux transformation.*

Under a forward Bäcklund–Darboux transformation, the dressing operator and the wave function are transformed as follows: $\widehat{P}_{N+1} = D \circ P_N$, $\widehat{\psi}_{N+1} = D\psi_N$.

Proposition A.3 implies that any infinite linear combination of the coefficients of x^i of the wave function $\psi(\mathbf{t}, x)$ is an eigenfunction. The converse statement is also true.

Proposition B.5. *Consider an arbitrary solution L of the KP hierarchy.*

(1) *For any $k \in \mathbb{Z}$ and any formal power series $C(x) = \sum_{i \geq 0} c_i x^{-i}$ with constant coefficients, the formal power series*

$$\Phi(\mathbf{t}) = \operatorname{Coef}_{x^k} \left(C(x)\psi(\mathbf{t}, x)\right), \tag{B.1}$$

is an eigenfunction.

(2) *An arbitrary eigenfunction Φ has the form (B.1) with $k = 0$. Moreover, the power series $C(x)$ is determined by an eigenfunction uniquely.*

Proof. 1. As we already mentioned above, this immediately follows from Proposition A.3.

2. It suffices to check that any initial condition $\Phi|_{t_{\geq 2}=0} =: \sum_{i \geq 0} \phi_i t_1^i$ can be obtained as $\operatorname{Coef}_{x^0} \left(C(x)\psi(\mathbf{t}, x)\right)|_{t_{\geq 2}=0}$. Denote $1 + \sum_{i \geq 1} p_i(t_1)\partial^{-i} := P|_{t_{\geq 2}=0}$. Then

$$\operatorname{Coef}_{x^0} \left(C(x)\psi(\mathbf{t}, x)\right)|_{t_{\geq 2}=0} = \operatorname{Coef}_{x^0} \left(\left(\sum_{i \geq 0} c_i x^{-i} \right) \left(1 + \sum_{j \geq 1} p_j(t_1) x^{-j} \right) e^{x t_1} \right) = \\ = \sum_{j \geq 0} \frac{c_j t_1^j}{j!} + \sum_{i \geq 1, j \geq 0} \frac{p_i(t_1) c_j t_1^{i+j}}{(i+j)!},$$

and we recursively choose c_k so that $\frac{c_k}{k!} = \phi_k - \operatorname{Coef}_{t_1^k} \left(\sum_{i \geq 1, 0 \leq j \leq k-i} \frac{p_i(t_1) c_j t_1^{i+j}}{(i+j)!} \right)$, $k \geq 0$. □

Note that the eigenfunction given by (B.1) with $k = 0$ satisfies the property $\Phi(0) \neq 0$ if and only if $c_0 \neq 0$.

Proposition B.6. *Under the Bäcklund–Darboux transformation generated by an eigenfunction Φ satisfying $\Phi(0) \neq 0$, the tau-function is transformed as $\widehat{\tau} = \Phi\tau$.*

Proof. We have to check that $\widehat{\psi} = \frac{\Phi(\mathbf{t}-[x^{-1}])}{\Phi(\mathbf{t})}\psi$, which is equivalent to the equation

$$x^{-1}(\partial\psi - \partial\log\Phi\cdot\psi) = \frac{\Phi(\mathbf{t}-[x^{-1}])}{\Phi(\mathbf{t})}\psi \quad \Leftrightarrow \quad \Phi(\mathbf{t}-[x^{-1}]) + x^{-1}\partial\Phi = x^{-1}\partial\log\psi\cdot\Phi.$$

Since Φ has the form $\Phi(\mathbf{t}) = \mathrm{Coef}_{z^0}\left(C(z)\psi(\mathbf{t},z)\right)$, the last equation follows from the equation

$$\psi(\mathbf{t}-[x^{-1}],z) + x^{-1}\partial\psi(\mathbf{t},z) = x^{-1}\partial\log\psi(\mathbf{t},x)\cdot\psi(\mathbf{t},z) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{\psi(\mathbf{t}-[x^{-1}],z)}{\psi(\mathbf{t},z)} = x^{-1}\left(\partial\log\psi(\mathbf{t},x) - \partial\log\psi(\mathbf{t},z)\right) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{\psi(\mathbf{t}-[x^{-1}],z)}{\psi(\mathbf{t},z)} = x^{-1}\partial\log\frac{\psi(\mathbf{t},x)}{\psi(\mathbf{t},z)}.$$

In terms of tau-functions, the last equation is equivalent to

$$\frac{\tau(\mathbf{t}-[x^{-1}]-[z^{-1}])\tau(\mathbf{t})}{\tau(\mathbf{t}-[x^{-1}])\tau(\mathbf{t}-[z^{-1}])}(1-zx^{-1}) = x^{-1}\left(\partial\log\left(\frac{\tau(\mathbf{t}-[x^{-1}])}{\tau(\mathbf{t}-[z^{-1}])}\right) + x - z\right).$$

Finally, this is equivalent to

$$(x-z)\left(\tau(\mathbf{t}-[x^{-1}]-[z^{-1}])\tau(\mathbf{t}) - \tau(\mathbf{t}-[x^{-1}])\tau(\mathbf{t}-[z^{-1}])\right) =$$

$$= \partial\tau(\mathbf{t}-[x^{-1}])\cdot\tau(\mathbf{t}-[z^{-1}]) - \partial\tau(\mathbf{t}-[z^{-1}])\cdot\tau(\mathbf{t}-[x^{-1}]),$$

where we recognize the differential Fay identity [Dic03, Section 6.4.13] (one should substitute $x = s_1^{-1}$ and $z = s_2^{-1}$). □

Combining Propositions B.5 and B.6, we obtain the following formula:

$$\widehat{\tau}(\mathbf{t}) = \mathrm{Coef}_{x^0}\left(C(x)\,\psi(\mathbf{t},x)\tau(\mathbf{t})\right) = \mathrm{Coef}_{x^0}\left(C(x)\,\tau(\mathbf{t}-[x^{-1}])\,e^{\xi(\mathbf{t},x)}\right). \tag{B.2}$$

There is a dual notion of backward Bäcklund–Darboux transformations.

Definition B.7. An *adjoint eigenfunction* $\bar{\Phi}(\mathbf{t})$ for a KP solution $L(\mathbf{t})$ is a function which satisfies the following equations for all $n \geq 1$: $\partial_n\bar{\Phi} = -(L_+^n)^\dagger\bar{\Phi}$.

Analogously to Proposition B.5, we have the representation of adjoint eigenfunctions in terms of the adjoint wave function:

$$\bar{\Phi}(\mathbf{t}) = \mathrm{Coef}_{x^k}\left(C(x)\psi^\dagger(\mathbf{t},x)\right), \quad k\in\mathbb{Z}, \tag{B.3}$$

where $C(x) = \sum_{i\geq 0} c_i x^{-i}$ is a power series with constant coefficients.

Proposition B.8. *Let L be a KP solution, and let $\bar{\Phi}$ be an associated adjoint eigenfunction satisfying $\bar{\Phi}(0)\neq 0$. Define $\bar{D} := \bar{\Phi}^{-1}\circ\partial^{-1}\circ\bar{\Phi}$. Then we have the following:*

1. *$\bar{D}$ satisfies the conditions of Lemma B.1, and so $\check{L} := \bar{D}\circ L\circ\bar{D}^{-1}$ is again a KP solution. We say that $\check{L}$ is related to L by a* backward Bäcklund–Darboux transformation.
2. *The action on the dressing operator, the wave function, and the tau-function is given by $\check{P}_{N-1} = \bar{D}\circ P_N$, $\check{\psi}_{N-1} = \bar{D}\psi_N$, $\check{\tau} = \bar{\Phi}\tau$.*

Proof. 1. The conditions of Lemma B.1 are equivalent to $\partial_n(\bar{D}^{-1}) = L_+^n\circ\bar{D}^{-1} - \bar{D}^{-1}\circ\check{L}_+^n$, which can be checked analogously to the proof of Proposition B.3.

2. The formulas for $\check{P}_{N-1}$ and $\check{\psi}_{N-1}$ are easily proved. The formula for $\check{\tau}$ is equivalent to

$$\check{\psi} = \frac{\bar{\Phi}(\mathbf{t}-[x^{-1}])}{\bar{\Phi}(\mathbf{t})}\psi \;\Leftrightarrow\; x\left(\bar{\Phi}^{-1}\circ\partial^{-1}\circ\bar{\Phi}\right)\psi = \frac{\bar{\Phi}(\mathbf{t}-[x^{-1}])}{\bar{\Phi}(\mathbf{t})}\psi \;\Leftrightarrow\; x\bar{\Phi}\psi = \partial\left(\bar{\Phi}(\mathbf{t}-[x^{-1}])\psi\right),$$

which, using (B.3), follows from the equation $x\psi^\dagger(\mathbf{t},z)\psi(\mathbf{t},x) = \partial\left(\psi^\dagger(\mathbf{t}-[x^{-1}],z)\psi(\mathbf{t},x)\right)$. Expressing ψ and $\psi^\dagger$ in terms of the tau-function, this equation is derived from the differential Fay identity by a direct computation. □

It follows that the backward Bäcklund–Darboux transformation of the tau-function can be expressed as

$$\check{\tau}(\mathbf{t}) = \mathrm{Coef}_{x^0}\left(C(x)\,\tau(\mathbf{t}+[x^{-1}])\,e^{-\xi(\mathbf{t},x)}\right).$$

Forward and backward Bäcklund–Darboux transformations are inverse to each other.

Proposition B.9. *Suppose $\widehat{L}$ is related to L by the forward Bäcklund–Darboux transformation generated by an eigenfunction Φ. Then Φ^{-1} is an adjoint eigenfunction for $\widehat{L}$, and L is related to $\widehat{L}$ by the backward Bäcklund–Darboux transformation generated by Φ^{-1}.*

Proof. We only have to prove that Φ^{-1} is an adjoint eigenfunction for $\widehat{L}$, the rest is elementary. So we have to prove that $\partial_n\Phi = \Phi^2(\widehat{L}_+^n)^\dagger(\Phi^{-1})$, which is done by the following computation:

$$\begin{aligned}
\Phi^2(\widehat{L}_+^n)^\dagger(\Phi^{-1}) =& \mathrm{res}\left(\Phi^2\circ(\widehat{L}_+^n)^\dagger\circ\Phi^{-1}\circ\partial^{-1}\right) = \mathrm{res}\left(\Phi\circ(\widehat{L}_+^n)^\dagger\circ\Phi^{-1}\circ\partial^{-1}\circ\Phi\right) = \\
=& \mathrm{res}\Big(\underbrace{\Phi\circ\partial^{-1}\circ\Phi^{-1}}_{=D^{-1}}\circ\widehat{L}_+^n\circ\Phi\Big) = \mathrm{res}\left((L_+^n\circ D^{-1} + D^{-1}\circ\partial_n D\circ D^{-1})\circ\Phi\right) = \\
=& \mathrm{res}\left(L_+^n\circ D^{-1}\circ\Phi\right) = \mathrm{res}\left(L_+^n\circ\Phi\circ\partial^{-1}\right) = L_+^n\Phi = \\
=& \partial_n\Phi.
\end{aligned}$$

□

The adjunction involution interchanges forward and backward Bäcklund–Darboux transformations in the following sense.

Proposition B.10. *Suppose L is a solution of the KP hierarchy, and M is obtained from L by a forward Bäcklund–Darboux transformation. Then M^* is related to L^* by a backward Bäcklund–Darboux transformation.*

Proof. Elementary computation. □

B.2. **Example: multi-soliton solutions.** To give a concrete example, we will recall the definition of N-soliton solutions of the KP hierarchy and show that they are obtained from the trivial solution by N successive Bäcklund–Darboux transformations.

Let N be a natural number. Choose some constants α_i, β_i, a_i for $i = 1,\dots,N$. Let $y_i(\mathbf{t}) := e^{\xi(\mathbf{t},\alpha_i)} + a_i e^{\xi(\mathbf{t},\beta_i)}$. For $k = 1,\dots,N$, define the Wronskian determinants

$$\Delta_k := \det\begin{pmatrix} y_1 & \dots & y_k \\ y_1' & \dots & y_k' \\ \dots & \dots & \dots \\ y_1^{(k-1)} & \dots & y_k^{(k-1)} \end{pmatrix},$$

where $y_j^{(i)} := \partial^i y_j$.

Proposition B.11 (see, e.g., Section 6.3 in [Dic03])**.** *If the constants α_i, β_i, a_i satisfy $\Delta_N(0) \neq 0$, then $\Delta_N(\mathbf{t})$ is a tau-function of the KP hierarchy. The KP solution defined by Δ_N is called the N-soliton solution.*

Suppose that $\Delta_k(0) \neq 0$ for all $1 \leq k \leq N$.

Proposition B.12. *For all $1 \leq k \leq N$, Δ_k is a KP tau-function, obtained from Δ_{k-1} by a Bäcklund–Darboux transformation.*

Proof. As shown in [Dic03, Section 6.3], we have

$$\Delta_{k-1}(\mathbf{t}-[x^{-1}]) = \det \begin{pmatrix} y_1 & \dots & y_{k-1} & x^{-k+1} \\ y_1' & \dots & y_{k-1}' & x^{-k+2} \\ \dots & \dots & \dots & \dots \\ y_1^{(k-1)} & \dots & y_{k-1}^{(k-1)} & 1 \end{pmatrix}.$$

By direct computation, it follows that $\Delta_k(\mathbf{t}) = \mathrm{Coef}_{x^0}\left(\gamma_k(x)\,\Delta_{k-1}(\mathbf{t}-[x^{-1}])\,x^{k-1}e^{\xi(\mathbf{t},x)}\right)$, where $\gamma_k(x) = \frac{1}{1-\alpha_k x^{-1}} + \frac{a_k}{1-\beta_k x^{-1}}$, which completes the proof. □

B.3. **Bäcklund–Darboux transformations in fermionic form.** Consider a tau-function of the KP hierarchy given by $\tau = \Psi_N\Big(\bigwedge_{i\geq 1} f_{N-i}(z)\Big)$, and let $C(x) = \sum_{i\geq N} c_i x^{-i}$, where $c_i \in \mathbb{C}$ and $c_N \neq 0$. Combining formula (B.2) for a Bäcklund–Darboux transformation of a tau-function and the fermionic representation (A.2) for the wave function, we obtain the following formula:

$$\begin{aligned}\widehat{\tau}(\mathbf{t}) = \mathrm{Coef}_{x^0}\left[C(x)\psi_N(\mathbf{t},x)\tau(\mathbf{t})\right] = \mathrm{Coef}_{x^0}\left[\Psi_{N+1}\Big(\sum_{j\geq N} c_j x^{-j}\sum_{k\in\mathbb{Z}}\theta_k x^k \bigwedge_{i\geq 1} f_{N-i}(z)\Big)\right] = \\ = \Psi_{N+1}\Big(\sum_{j\geq N} c_j\theta_j \bigwedge_{i\geq 1} f_{N-i}(z)\Big) = \Psi_{N+1}\Big(\bigwedge_{i\geq 1} f_{N+1-i}(z)\Big),\end{aligned}$$

where we set $f_N(z) = C(z^{-1})$. Therefore, if $\tau = \tau_H$ for $H \in \mathrm{Gr}_N^0$, then $\widehat{\tau} = \tau_{\widehat{H}}$, where $H \subset \widehat{H} := H \oplus \mathrm{span}(f_N) \in \mathrm{Gr}_{N+1}^0$.

In other words, in the Sato Grassmannian formalism, a forward Bäcklund–Darboux transformation corresponds to an extension of the infinite-dimensional space H by some one-dimensional subspace.

Dually, since a backward Bäcklund–Darboux transformation can be represented using the adjoint involution as $\check{\tau} = (\widehat{\tau^*})^*$, we obtain $\widetilde{\tau_H} = \tau_{\check{H}}$, where $H \supset \check{H} := H \cap \ell^\perp$ for some one-dimensional subspace $\ell \subset \mathbb{C}((z))$. Thus, a backward Bäcklund–Darboux transformation restricts the subspace H to its codimension one subspace.

It follows that two KP solutions τ_H and $\tau_{H'}$ are related by some sequence of forward and backward Bäcklund–Darboux transformations if and only if for the corresponding subspaces $H, H' \subset \mathbb{C}((z))$, their intersection $H \cap H'$ has finite codimension in both H and H'.

Remark B.13.

- Similarly to Remark A.9, the theory of Bäcklund–Darboux transformations has a direct generalization to the case of arbitrary base algebra K. The necessary changes are minimal: in Propositions B.3 and B.6 one should require that $\Phi(0)$ is invertible, and in Proposition B.8 one should require that $\bar{\Phi}(0)$ is invertible.
- Consider an arbitrary tau-function τ of the KP hierarchy, with the base algebra K. By Proposition A.3, the wave function $\psi(\mathbf{t},x)$ can be viewed as an eigenfunction of the corresponding solution, if we extend the base algebra to $K((x^{-1}))$. Note that $\psi(0,x) = \frac{\tau(-[x^{-1}])}{\tau(0)} \in K((x^{-1}))$ is an invertible element. Therefore, $\tau(\mathbf{t})\psi(\mathbf{t},x) = \tau(\mathbf{t}-[x^{-1}])e^{\xi(\mathbf{t},x)}$ is a KP tau-function, which is obtained from τ by a Bäcklund–Darboux transformation.

B.4. **MKP hierarchy.**

Definition B.14. A *tau-sequence of the mKP (modified KP) hierarchy* is a sequence $(\tau_N)_{N\in\mathbb{Z}}$ of KP tau-functions such that for each $N \in \mathbb{Z}$, the tau-function τ_{N+1} is obtained from τ_N via a forward Bäcklund–Darboux transformation.

It follows from the discussion in Section B.3 that an mKP tau-sequence (τ_N) can be represented by a point $H_\bullet$ in the big cell of flag manifold Fl^0, that is, an infinite sequence of subspaces $(\ldots \subset H_{-1} \subset H_0 \subset H_1 \subset \ldots)$ with $H_N \in \mathrm{Gr}_N^0$.

Equivalently, given a sequence of Laurent series $(f_N)_{N\in\mathbb{Z}}$ of the form

$$f_N(z) = z^N\Big(1 + \sum_{j\geq 1} f_{N,j} z^j\Big), \quad f_{N,j} \in \mathbb{C},$$

we define the corresponding an mKP tau-sequence $(\tau_N)_{N\in\mathbb{Z}}$ by

$$\tau_N := \Psi_N\Big(\bigwedge_{i\geq 1} f_{N-i}(z)\Big) \in \mathbb{C}[[\mathbf{t}]]. \tag{B.4}$$

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