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Nonsingular Flows with a Twisted Saddle Orbit on Orientable 3-Manifolds

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Abstract—In this paper we consider nonsingular Morse – Smale flows on closed orientable 3-manifolds, under the assumption that among the periodic orbits of the flow there is only one saddle orbit and it is twisted. It is found that any manifold admitting such flows is either a lens space, or a connected sum of a lens space with a projective space, or Seifert manifolds with base sphere and three special layers. A complete topological classification of the described flows is obtained and the number of their equivalence classes on each admissible manifold is calculated.

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1. INTRODUCTION AND FORMULATION OF RESULTS

In this paper we consider *nonsingular* (without fixed points) Morse – Smale flows (*NMS flows*) f^t defined on closed connected orientable 3-manifolds M^3. The nonwandering set of such a flow consists of a finite number of periodic hyperbolic orbits. In a neighborhood of a hyperbolic periodic orbit $\mathcal{O}$, the flow admits a simple description (up to topological equivalence), namely, there exists its tubular neighborhood $V_{\mathcal{O}}$ homeomorphic to the solid torus $\mathbb{V} = \mathbb{D}^2 \times \mathbb{S}^1$, in which the flow is topologically equivalent to the suspension over some linear diffeomorphism of the plane given by a matrix with positive determinant and real eigenvalues modulo different from unity (see, e. g., [6]). If both eigenvalues are modulo greater than (less than) one, then the corresponding periodic orbit is *repelling* (*attracting*), otherwise it is a *saddle*. A saddle orbit is called *twisted* if both eigenvalues are negative, otherwise it is called *untwisted*.

The study of nonsingular Morse – Smale (NMS) flows on 3-manifolds has a rich history, with several foundational contributions. M. Wada [15] provided a complete classification of the links formed by closed orbits of NMS flows on the 3-sphere $\mathbb{S}^3$, showing how such links correspond to specific topological configurations of periodic trajectories. Building on the structural aspects of these flows, Azimov [1] described the decomposition of the ambient manifold into so-called round handles. In particular, this handlebody decomposition helps capture the global topology induced by the flow.

A key result in the classification of NMS flows with a minimal number of periodic orbits — specifically, two (one attracting and one repelling, which are present in every NMS flow) — is that such flows can exist only on lens spaces. This was shown in [9], where it was further established that each lens space admits exactly two equivalence classes of such flows, with the exceptions of the 3-sphere $\mathbb{S}^3$ and the projective space $\mathbb{RP}^3$, for which the equivalence class is unique.

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When the number of periodic orbits increases, the range of manifolds that support NMS flows broadens significantly. For instance, the construction in [9] (specifically dealing with three untwisted periodic orbits) demonstrates that NMS flows can be realized on small Seifert fibered spaces. However, the classification becomes more intricate when considering flows with twisted periodic orbits — an area that has not been thoroughly explored in the literature.

Morgan [8] conducted a more detailed study of the topological types of compact orientable 3-manifolds that admit NMS flows, contributing to our understanding of how the qualitative dynamics of such flows interact with the manifold's topology. However, even in Morgan's work, flows with twisted orbits have not been systematically addressed.

To clarify the topology of ambient manifolds for Morse – Smale flows (including nonsingular ones) and to find conditions for their topological classification, various methods have been proposed, see, e. g., [1, 8, 10, 16]. We develop a new approach that allows us to see how the dynamics are related to the topology. One of the ways to see such a relationship is to find invariants for such flows that allow us to describe the topology of the ambient manifold.

In the present paper we obtain an exhaustive classification for the set $G_1^-(M^3)$ of NMS flows $f^t\colon M^3 \to M^3$ with a single saddle orbit, under the assumption that it is twisted.

Since the ambient manifold of the flow $f^t \in G_1^-(M^3)$ is the union of stable (unstable) manifolds of all its periodic orbits [14], the flow must have at least one attracting and at least one repelling orbit. In the present paper the following fact is established.

Lemma 1. *The nonwandering set of any flow $f^t \in G_1^-(M^3)$ consists of exactly three periodic orbits S, A, R, saddle, attracting and repelling, respectively.*

Due to the topological equivalence of the flow f^t in the interior of tubular neighborhood $V_{\mathcal{O}}$ of a periodic orbit to a suspension over a linear diffeomorphism, the unstable and stable manifolds of these orbits have the following topology:

- $W^u_S \cong W^s_S \cong \mathbb{R}\tilde{\times}\mathbb{S}^1$ (open Möbius band);
- $W^s_A \cong W^u_R \cong \mathbb{R}^2 \times \mathbb{S}^1$;
- $W^u_A \cong W^s_R \cong \mathbb{S}^1$.

Using these properties of the topology of invariant manifolds of periodic orbits and Lemma 1, we obtain the following result on the representation of the ambient manifold M^3. Lemma 2 is similar to the result of Azimov on round-handle decompostion for all dimensions except 3.

Lemma 2. *The ambient manifold M^3 of any flow $f^t \in G_1^-(M^3)$ is represented as a union of three solid tori*

$$M^3 = V_A \cup V_S \cup V_R$$

with nonintersecting interior (glued at the boundaries), which are tubular neighborhoods of the orbits A, S, R, respectively, with the following properties:

- *the torus $T_S = \partial V_S$ is the union of compact tubular neighborhoods K_A, K_R of knots $\gamma_A = W^u_S \cap T_S$, $\gamma_R = W^s_S \cap T_S$, respectively, such that $K_A \cap K_R = \partial K_A \cap \partial K_R$ (see Fig. 1);*
- *the torus $T_A = \partial V_A$ is the union of the annulus K_A and the compact surface $K = T_A \setminus \operatorname{int} K_A$,*
- *the torus $T_R = \partial V_R$ is the union of the annulus K_R and the compact surface $K = T_R \setminus \operatorname{int} K_R$.*

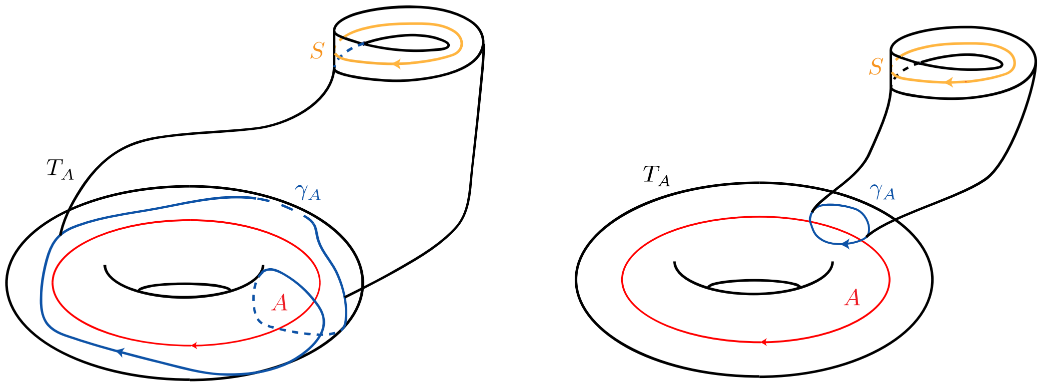

Fig. 1. Knot γ_A on boundary ∂V_A of canonical neighborhood.

For $\mathcal{O} \in \{A, S, R\}$ we choose a *parallel* $L_{\mathcal{O}}$ on the torus $T_{\mathcal{O}}$ (a curve homologous in $V_{\mathcal{O}}$ to the orbit $\mathcal{O}$) and a *meridian* $M_{\mathcal{O}}$ (a curve, homotopic to zero on $V_{\mathcal{O}}$ and essential on $T_{\mathcal{O}}$) such that the ordered pair of curves $L_{\mathcal{O}}$, $M_{\mathcal{O}}$ defines the outer side of the solid torus $V_{\mathcal{O}}$.

Let γ_S be the connected component of the set ∂K oriented coherently with the saddle orbit S. By virtue of the equivalence of the flow $f^t|_{V_S}$ to the suspension, the meridian M_S can be chosen such that γ_S intersects the meridian M_S at exactly two points (see Section 3.1 for details). Then the generators L_S, M_S can be chosen so that with respect to them the knot γ_S has homotopic type

$$\langle \gamma_S \rangle = \langle l_S, m_S \rangle = \langle 2, 1 \rangle.$$

We orient the knots γ_R, γ_A consistent with the knot γ_S. Let us write down the homotopy type of the knot γ_R with respect to the L_R, M_R

$$\langle \gamma_R \rangle = \langle l_R, m_R \rangle$$

and the homotopy type of the knot γ_A with respect to the L_A, M_A

$$\langle \gamma_A \rangle = \langle l_A, m_A \rangle.$$

Since $T_R \setminus \gamma_R$ is homeomorphic by flow-map to $T_A \setminus \gamma_A$, then

$$(l_R, m_R) = (0, 0) \iff (l_A, m_A) = (0, 0).$$

If $(l_R, m_R) = (0, 0)$, then let us write the homotopy type of the meridian $M_R \subset K$ with respect to the formers L_A, M_A

$$\langle M_R \rangle = \langle p_A, q_A \rangle.$$

If $(l_R, m_R) \neq (0, 0)$, then choose a knot $\sigma_S \subset T_S$ such that

$$\langle \sigma_S \rangle = \langle 1, 1 \rangle.$$

and σ_S intersects with each component of the ∂K connectivity at exactly one point (this can be done since the intersection index of the knots γ_S and σ_S is 1).

Let us choose knots $\sigma_R \subset T_R$, $\sigma_A \subset T_A$ coinciding with each other on the annulus K and such that $\sigma_S = (\sigma_R \cup \sigma_A) \cap T_S$. Let us write their homotopy types with respect to generators

$$\langle \sigma_R \rangle = \langle b_R, c_R \rangle, \ \langle \sigma_A \rangle = \langle b_A, c_A \rangle.$$

Definition 1. By the flow $f^t \in G_1^-(M^3)$, we define the *set*

$$C_{f^t} = (l_1,\, b_1,\, l_2,\, b_2)$$

as follows:

- $(l_1,\, b_1,\, l_2,\, b_2) = (l_R,\, b_R,\, l_A,\, b_A)$ if $(l_R,\, m_R) \neq (0,\, 0)$;

- $(l_1, l_2, b_1, b_2) = (0, 2, p_A, q_A)$ if $(l_R, m_R) = (0, 0)$ and the 2-disk bounded by knot γ_R remains on the left when moving along the knot;
- $(l_1, b_1, l_2, b_2) = (0, -2, -p_A, -q_A)$ if $(l_R, m_R) = (0, 0)$ and the 2-disk bounded by the knot γ_R remains on the right when moving along the knot.

Note that the set C_{f^t} of the flow $f^t \in G_1^-(M^3)$ is admissible in the sense of the following definition.

Definition 2. The set of integers $C = (l_1, b_1, l_2, b_2)$ is called *admissible* if

- $(l_1, b_1) = (0, \pm 2)$ or $\gcd(l_1, b_1) = 1$;
- $\gcd(l_2, b_2) = 1$.

Definition 3. We call the admissible sets $C = (l_1, b_1, l_2, b_2)$, $C' = (l_1', b_1', l_2', b_2')$ *consistent* $(C \sim C')$ if:

- $l_i = l_i',\ i = 1, 2,$

and there exists $\delta \in \{-1, 1\}$ such that

- $b_i \equiv \delta b_i' \pmod{l_i}$;
- $l_1 l_2\big(2l_2(b_1 - \delta b_1') + 2l_1(b_2 - \delta b_2') + l_1 l_2(1 - \delta)\big) = 0.$

In the present work, the following classification result is established.

Theorem 1. *The flows f^t, $f'^t \in G_3^-(M^3)$ are topologically equivalent if and only if $C_{f^t} \sim C_{f'^t}$. Moreover, for any admissible set C there exists a flow $f^t \in G_3^-(M^3)$ such that $C \sim C_{f^t}$.*

We also managed to construct a correspondence between invariants and ambient manifolds of flows of the class considered.

Theorem 2. *Flows of class $G_1^-(M^3)$ admit all lens spaces $L_{p,q}$, all connected sums of the form $L_{p,q}\#\mathbb{R}\mathrm{P}^3$ and all Seifert manifolds of the form $M\big(\mathbb{S}^2, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (2, 1)\big)$. More precisely, let the flow $f^t \in G_1^-(M^3)$ have the invariant $C_{f^t} = (l_1, b_1, l_2, b_2)$. Then*

- *If $(|l_1| - 1)(|l_2| - 1) = 0$, then $M^3 \cong L_{p,q}$, thus:*
  - *if $l_1 l_2 = 0$, then* $M^3 \cong \mathbb{R}\mathrm{P}^3$;
  - *if $C_{f^t} = (\pm 1, b_1, l_2, b_2),\ l_2 \neq 0$, then* $M^3 \cong L_{l_2 - 2b_2, b_2}$;
  - *if $C_{f^t} = (l_1, b_1, \pm 1, b_2),\ l_1 \neq 0$, then* $M^3 \cong L_{l_1 - 2b_1, b_1}$.
- *If $l_1 l_2 = 0$ and $(|l_1| - 1)(|l_2| - 1) \neq 0$, then $M^3 \cong L_{p,q}\#\,\mathbb{R}\mathrm{P}^3$, thus:*
  - *if $C_{f^t} = (0, b_1, l_2, b_2)$, then* $M^3 \cong L_{l_2, b_2}\#\,\mathbb{R}\mathrm{P}^3$;
  - *if $C_{f^t} = (l_1, b_1, 0, \pm 1)$, $_1l \neq 0$, then $M^3 \cong L_{l_1, b_1}\#\,\mathbb{R}\mathrm{P}^3$.*
- *If $C_{f^t} = (l_1, b_1, l_2, b_2),\ |l_1| > 1,\ |l_2| > 1$, then $M^3 \cong M(\mathbb{S}^2, (l_1, b_1), (l_2, b_2), (2, 1))$.*

Due to the fact that the topological equivalence class and the topology of a manifold are defined using the same invariant, it becomes possible to compute the number of topological equivalence classes on each admissible manifold.

Theorem 3. *The set $G_1^-(L_{p,q})$, $|p|| \neq 2$, decomposes into a countable number of equivalence classes, whereas the sets $G_1^-(\mathbb{R}\mathrm{P}^3)$, $G_1^-(L_{p,q}\#\,\mathbb{R}\mathrm{P}^3)$, $G_1^-\Big(M\big(\mathbb{S}^2,\ (\alpha_1,\beta_1),\ (\alpha_2,\beta_2),\ (2,1)\big)\Big)$ consist of a finite number of classes.*

Moreover, we explicitly give the correspondence between the invariants and ambient manifolds in Section 7 after the proof of Theorem 3.

Note that, on the three-dimensional sphere $\mathbb{S}^3$, the list of flows representing equivalence classes of the set $G_1^-(\mathbb{S}^3)$ listed in Theorem 3 is exactly the same as that obtained in Bin Yu's paper (see Proposition 7.4 in [17]).

2. TOPOLOGY OF 3-MANIFOLDS

2.1. Lens Spaces

Further, we will assume that the constituents of the homotopy types of knots on the boundary $\partial\mathbb{V}$ of the standard fullness $\mathbb{V} = \mathbb{D}^2 \times \mathbb{S}^1$ are the meridian $\mathbb{M} = (\partial\mathbb{D}^2) \times \mathbb{S}^1$ with homotopy type $\langle 0, 1\rangle$ and the parallel $\mathbb{L} = \{x\} \times \mathbb{S}^1$, $x \in \partial\mathbb{D}^2$ with homotopy type $\langle 1, 0\rangle$.

A three-dimensional manifold $L_{p,q} = V_1 \cup_j V_2$ resulting from gluing two copies of a solid torus $V_1 = \mathbb{V}$ is called *a lens space*, $V_2 = \mathbb{V}$ by some homeomorphism $j\colon \partial V_1 \to \partial V_2$ such that $j_*(\langle 0, 1\rangle) = \langle p, q\rangle$.

Proposition 1 ([5]). *Two lens spaces $L_{p,q}$, $L_{p',q'}$ are homeomorphic if and only if $|p| = |p'|$, $q \equiv \pm q' \pmod p$ or $qq' \equiv \pm 1 \pmod p$. Also,*

$$L_{0,1} \cong \mathbb{S}^2 \times \mathbb{S}^1,\ L_{1,0} \cong \mathbb{S}^3,\ L_{2,1} \cong \mathbb{R}P^3.$$

2.2. Dehn Surgery Along the Knots and Links

In this section, we provide some facts and notations regarding Dene's surgery. The reader is referred to [11] for further details.

Suppose we are given

- a closed 3-manifold M;
- knot $\gamma \subset M$;
- tubular neighborhood U_γ of knot γ, which is a solid torus with standard generators on ∂U_γ — meridian M_γ and parallel L_γ;
- homeomorphism $h\colon \partial\mathbb{V} \to \partial U_\gamma$, which induces an isomorphism in the given generators such that $h_*(\langle 0, 1\rangle) = \langle \beta, \alpha\rangle$.

The manifold

$$M_\gamma = (M \setminus \mathrm{int}\ U_\gamma) \cup_h .\mathbb{V}$$

is called the *manifold obtained from the manifold M by Dehn surgery along the knot γ with equipment β, α.*

Let $p_\gamma\colon (M \setminus \mathrm{int}\ U_\gamma) \sqcup \mathbb{V} \to M_\gamma$ denote the natural projection. Put $\tilde\gamma = p_\gamma(\{0\} \times \mathbb{S}^1)$, $U_{\tilde\gamma} = p_\gamma(\mathbb{V})$, $\tilde h = p_\gamma h^{-1}\colon \partial U_\gamma \to \partial U_{\tilde\gamma}$. Then the manifold M is recovered from M_γ by the following inverse surgery.

Proposition 2 ([11]). *Let* $\gamma \subset M$ *be a knot with* β, α *and* $\tilde{\gamma}$ *be a knot with* $-\beta, \xi$, *where* $\alpha\xi \equiv 1 \pmod{\beta}$. *Then*

$$M \cong (M_\gamma)_{\tilde{\gamma}}.$$

The Dehn surgery naturally generalizes to the case where $\gamma = \gamma_1 \sqcup \dots \sqcup \gamma_r \subset M$ is the disjunctive union (link) of equipped knots. The resulting manifold M_γ in this case is called *the manifold obtained from the manifold* M^3 *by Dehn surgery along the equipped link* γ. A link $\gamma = \gamma_1 \sqcup \dots \sqcup \gamma_r \subset M$ is called *is trivial* if the knots $\gamma_1, \dots, \gamma_r$ bound the pairwise nonoverlapping 2-discs $d_1, \dots, d_r \subset M$.

Proposition 3 ([11]). *Let* $\gamma = \gamma_1 \sqcup \dots \sqcup \gamma_r \subset M$ *be a trivial link with equipment* $\beta_1, \alpha_1; \dots ; \beta_r, \alpha_r$. *Then*

$$M_\gamma \cong M \# L_{\alpha_1,\beta_1} \# \dots L_{\alpha_r,\beta_r}.$$

2.3. Seifert Fiber Spaces

A solid torus $\mathbb{V}$ partitioned into fibers of the form $\{x\} \times \mathbb{S}^1$ is called a *trivially fibered solid torus*. Consider the solid torus $\mathbb{V} = \mathbb{D}^2 \times \mathbb{S}^1$ as a solid cylinder $\mathbb{D}^2 \times [0, 1]$ with bases glued together by virtue of rotation by an angle $2\pi\nu/\alpha$ for coprime integers $\alpha, \nu, \quad \alpha > 1$. The partitioning of a solid cylinder into segments of the form $\{x\} \times [0, 1]$ determines the partitioning of this solid cylinder into circles called *fibers*. The segment $\{0\} \times [0, 1]$ generates a fiber called *special*, all other (*nonspecial*) fibers of the solid torus are wrapped α times around the special layer and ν times around the meridian of the solid torus. The number α is called the *multiplicity* of the singular fiber. A solid torus with such a partition into fibers is called a *nontrivially fibered solid torus* with *orbital invariants* (α, ν).

A *Seifert fiber space* is a compact, orientable 3-manifold M, partitioned into nonintersecting simple closed curves (fibers) such that each fiber has a tubular neighborhood entirely composed of fibers, layer-by-layer homeomorphic to the fibered solid torus. Such a partitioning is called *Seifert fibration*. Fibers that under some such homeomorphism pass to the center of a nontrivially fibered solid torus are called *special*.

The *base* of a Seifert fiber space M is a compact surface $\Sigma = M/_\sim$, where $\sim$ is an equivalence relation such that $x \sim y$ if and only if x and y belong to the same layer.

The base of any Seifert fiber space is a compact surface which is closed if and only if the manifold M is closed; in particular, the base of any fibered solid torus is a disk (see, e. g., [7]). Thus, any Seifert fibration M with a given base Σ and orbital invariants $(\alpha_1, \nu_1), \dots, (\alpha_r, \nu_r)$, $r \in \mathbb{N}$) is obtained from the manifold $\Sigma \times \mathbb{S}^1$ by Dehn surgery along the link $\gamma = \bigsqcup\limits_{i=1}^{r} \gamma_i$, where $\gamma_i = \{s_i\} \times \mathbb{S}^1$, $s_i \in \Sigma$ is a knot with equipment β_i, α_i, $\nu_i\beta_i \equiv 1 \pmod{\alpha_i}$. Therefore, the generally accepted notation of such a Seifert fibration is as follows:

$$M\big(\Sigma, (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)\big).$$

Thus, *the orientation on the fibers* of the Seifert fibration is uniquely determined by the orientation of the circle $\mathbb{S}^1$ in the manifold $\Sigma \times \mathbb{S}^1$.

Two Seifert fibrations M, M' are called *isomorphic* if there exists a homeomorphism $h \colon M \to M'$ which maps the fibers of one fibration into the fibers of the other while preserving the orientation of the fibers. The homeomorphism h in this case is called the *isomorphism of Seifert fibrations*. It is not difficult to see that the fibrations on solid tori with orbital invariants (α, ν) and α', ν' are isomorphic if and only if $\alpha = \alpha'$ and $\nu \equiv \delta\nu' \pmod{\alpha}$ $(\delta = \pm 1)$, and if $\delta = +1$, then the isomorphism preserves the orientation of the solid torus, otherwise it changes.

The following statement, which gives a criterion for the isomorphism of two Seifert stratifications by their invariants, was proposed by Herbert Seifert in [12]. An exposition of this statement, in notations closer to those given in this section, but only for orientation-preserving isomorphisms, can be found in the notes by Allen Hatcher [5] and the textbook by Sergey Matveev and Anatoly Fomenko [7].

Proposition 4. *The Seifert fibration* $M\big(\Sigma,(\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r)\big)$ *and* $M'\big(\Sigma',(\alpha'_1,\beta'_1),\ldots,(\alpha'_{r'},\beta'_{r'})\big)$ *are isomorphic if and only if* $r=r'$ *and the following conditions are satisfied for* $\delta=\pm1$ *and the permutation* $\sigma\colon\{1,2,\ldots,r\}\to\{1,2,\ldots,r\}$*:*

- Σ *is homeomorphic to* Σ'*;*
- $\alpha_i=\alpha'_{\sigma(i)}$; $\beta_i\equiv\delta\beta'_{\sigma(i)} \pmod{\alpha_i}$ *for* $i\in\{1,\ldots,r\}$*;*
- *if the surface of* Σ *is closed, then* $\sum\limits_{i=1}^{r}\frac{\beta_i}{\alpha_i}=\delta\sum\limits_{i=1}^{r}\frac{\beta'_i}{\alpha'_i}$.

Thus, if $\delta=+1$*, the isomorphism is orientation-preserving, and if* $\delta=-1$*, the isomorphism is orientation-reversing.*

Note that some manifolds admit nonisomorphic Seifert fibrations. All such manifolds are well known (see, for example, [5]) and, as can be seen from the following statement, such manifolds include, for example, lens spaces.

Proposition 5 ([3]). 3*-manifolds admits a Seifert fibration with a base homeomorphic to a sphere and at most two special fibers if and only if it is homeomorphic to a lens space. Thus, the list of all Seifert fibrations on lens spaces is as follows:*

- *only the manifold* $\mathbb{S}^2\times\mathbb{S}^1$ *admits fibrations without special fibers;*
- $M\big(\mathbb{S}^2,(\alpha,\beta)\big)\cong L_{\beta,\alpha}$*;*
- $M\big(\mathbb{S}^2,(\alpha_1,\beta_1),(\alpha_2,\beta_2)\big)\cong L_{p,q}$*, where* $p=\beta_1\alpha_2+\alpha_1\beta_2$, $q=\beta_1\nu_2+\alpha_1\xi_2$ *and* $\alpha_2\xi_2-\nu_2\beta_2=1$.

The following statement, on the contrary, allows us to infer the nonhomeomorphism of ambient manifolds from the nonisomorphism of Seifert stratifications.

Proposition 6 ([5], Theorem 2.3). *If two Seifert fibrations with three special fibers and a base sphere are not isomorphic, then the manifolds on which they are defined are not homeomorphic.*

3. DYNAMICS OF FLOWS OF CLASS $G_1^-(M^3)$

In this section we prove Lemmas 1 and 2 in the Introduction.

Let us start with Lemma 1:

Proof. The basis of the proof is the following representation of the ambient manifold M^3 of the NMS flow f^t with a set of periodic orbits Per_{f^t} (see, e. g., [14])

$$M^3=\bigcup_{\mathcal{O}\in Per_{f^t}}W^u_{\mathcal{O}}=\bigcup_{\mathcal{O}\in Per_{f^t}}W^s_{\mathcal{O}},\tag{3.1}$$

as well as the asymptotic behavior of invariant manifolds

$$cl(W^u_{\mathcal{O}})\setminus W^u_{\mathcal{O}}=\bigcup_{\tilde{\mathcal{O}}\in Per_{f^t}\colon W^u_{\mathcal{O}}\cap W^s_{\tilde{\mathcal{O}}}\neq\varnothing}W^u_{\tilde{\mathcal{O}}},$$

$$cl(W^s_{\mathcal{O}})\setminus W^s_{\mathcal{O}}=\bigcup_{\tilde{\mathcal{O}}\in Per_{f^t}\colon W^s_{\mathcal{O}}\cap W^u_{\tilde{\mathcal{O}}}\neq\varnothing}W^s_{\tilde{\mathcal{O}}}.$$

In particular, it follows from the above relations that any NMS flow has at least one attracting orbit and at least one repelling orbit. Moreover, if the NMS flow has a saddle periodic orbit, then the basin of any attracting orbit has a nonempty intersection with the unstable manifold of at least one saddle orbit (see Proposition 2.1.3 [4]) and the same situation with the basin of the repelling orbit.

Let now $f^t\in G_1^-(M^3)$ and S be its only saddle orbit. It follows from relation (3.1) that $W^u_S\setminus S$ intersects only with basins of attracting orbits. Since the set $W^u_S\setminus S$ is connected and the basins of attracting orbits are open, W^u_S intersects exactly one such basin. Let A denote the corresponding attracting orbit. Since the saddle orbit is unique, the attracting orbit is unique. Similar reasoning for W^s_S leads to the existence of a single repelling orbit R. □

3.1. Canonical Neighborhoods of Periodic Orbits

The flows admit a simple description (up to topological equivalence) in the neighborhood of a hyperbolic periodic orbit, namely, they are suspensions over linear diffeomorphisms of the plane.

Let us recall the definition of suspension. Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism. Let us define the diffeomorphism $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ by the formula

$$\Phi(x_1, x_2, x_3) = \big(\phi(x_1, x_2), x_3 - 1\big).$$

Then the group $\{\Phi^n\} \cong \mathbb{Z}$ acts freely and discontinuously on $\mathbb{R}^3$, by virtue of which the orbit space $\Pi_\phi = \mathbb{R}^3/\Phi$ is a smooth 3-manifold, and the natural projection $v_\phi\colon \mathbb{R}^3 \to \Pi_\phi$ is a covering. In this case, the flow $\xi^t\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by the formula

$$\xi^t(x_1, x_2, x_3) = (x_1, x_2, x_3 + t)$$

induces a flow $[\phi]^t = v_\phi \xi^t v_\phi^{-1} : \Pi_\phi \to \Pi_\phi$, called *suspension*.

We define the diffeomorphisms $a_{\pm 1}, a_2, a_0\colon \mathbb{R}^2 \to \mathbb{R}^2$ by the formulas

$$a_{\pm 1}(x_1, x_2) = (\pm 2x_1, \pm x_2/2),\ a_2(x_1, x_2) = (2x_1, 2x_2),\ a_0 = a_2^{-1}.$$

Suppose

$$V_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ 4^{x_3} x_1^2 + 4^{x_3} x_2^2 \leqslant 1\},$$

$$V_{\pm 1} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ 4^{-x_3} x_1^2 + 4^{x_3} x_2^2 \leqslant 1\},$$

$$V_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ 4^{-x_3} x_1^2 + 4^{-x_3} x_2^2 \leqslant 1\},$$

For $i \in \{0, -1, +1, 2\}$, let

$$T_i = \partial V_i,\ \mathbb{V}_i = v_{a_i}(V_i),\ \mathbb{T}_i = \partial \mathbb{V}_i,\ \mathbb{O}_i = v_{a_i}(Ox_3).$$

The following fact asserts canonical neighborhoods at hyperbolic periodic orbits.

Proposition 7 ([6]). *For any hyperbolic periodic orbit $\mathcal{O}$ of a flow $f^t\colon M^3 \to M^3$ defined on a closed orientable manifold M^3, there exists a tubular neighborhood $V_\mathcal{O}$ of the orbit $\mathcal{O}$ and a number $i_\mathcal{O} \in \{0, -1, +1, 2\}$ such that the flow $f^t\big|_{V_\mathcal{O}}$ is topologically equivalent, via some homeomorphism $H_\mathcal{O}$, to the flow $[a_{i_\mathcal{O}}]^t|_{\mathbb{V}_{i_\mathcal{O}}}$.*

Let us call the neighborhood $V_\mathcal{O} = H_\mathcal{O}(\mathbb{V}_{i_\mathcal{O}})$ the *canonical neighborhood* of the periodic orbit of $\mathcal{O}$.

On the torus $\mathbb{T}_i$ we choose *longitude* $\mathbb{L}_i$ (a curve homologous in $\mathbb{V}_i$ to the orbit of $\mathbb{O}_i$) and *meridian* $\mathbb{M}_i$ (a curve homotopic to zero on $\mathbb{V}_i$ and essential on $\mathbb{T}_i$) such that the ordered pair of curves $\mathbb{L}_i$, $\mathbb{M}_i$ defines the outer side of the solid torus $\mathbb{V}_i$.

In the proof of topological equivalence we will use the following fact, which follows from the proof of Theorem 4 and Lemma 4 in [9] and can also be found in [16, Theorem 1.1].

Proposition 8. *The homeomorphism $h\colon \mathbb{T}_i \to \mathbb{T}_i$ for $i \in \{0, 2\}$ continues up to the homeomorphism $H\colon \mathbb{V}_i \to \mathbb{V}_i$, realizing the equivalence of the flow $[a_i]^t$ with itself, if and only if the induced isomorphism is of the form[1)] $h_* = \begin{pmatrix} 1 & k \\ 0 & \delta \end{pmatrix}$, where $\delta \in \{-1, 1\}$, $k \in \mathbb{Z}$.*

[1)] Throughout the paper, we assume that the string (l, m) is multiplied by the matrix on the left and the first element of the basis is the parallel of the torus.

The boundary of the canonical neighborhood of a saddle orbit, in contrast to an attracting or repulsing orbit, contains curves tangent to the suspension trajectories. Precisely, we denote by $\mathcal{O}_{x_1,x_2}$ the flow trajectory ξ^t intersecting the plane Ox_1x_2 at a point with coordinates $(x_1, x_2, 0)$. It is directly verified that the trajectory $\mathcal{O}_{x_1,x_2}$ intersects the surface $T_{\pm 1}$ if and only if $|x_1x_2| \leqslant \frac{1}{2}$ and $(x_1, x_2) \neq (0, 0)$. The trajectories touch the surface at one point if $|x_1x_2| = \frac{1}{2}$, transversally intersect the surface at one point if $x_1x_2 = 0$, and otherwise transversally intersect the surface at two points

$$\mathcal{O}_{x_1,x_2} \cap T_{\pm 1} = \{(x_1, x_2, x_3^s), (x_1, x_2, x_3^u)\}, \ x_3^s < x_3^u.$$

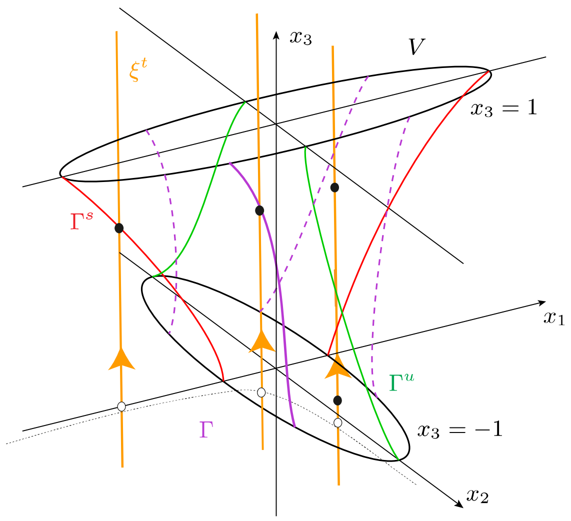

Fig. 2. Cylinder $T_{\pm 1}$ and orbits of the flow ξ^t.

Let $\Gamma = \{(x_1, x_2, x_3) \in T_{\pm 1} : |x_1x_2| = \frac{1}{2}\}$, $\Gamma^u = Ox_1x_3 \cap T_{\pm 1}$ and $\Gamma^s = Ox_2x_3 \cap T_{\pm 1}$. The sets Γ^u, Γ^s consist of two curves by construction, the set Γ consists of four curves dividing $T_{\pm 1}$ into four connected components. The closure T^u of two of these components contains Γ^u, the closure T^s of two other contains Γ^s (see Fig. 2). We assume that Γ^u and Γ^s are oriented in ascending order of coordinate x_3. For $i \in \{-1, 1\}$, let us put

$$\mathbb{T}_i^s = v_{a_i}(T^s), \ \mathbb{T}_i^u = v_{a_i}(T^u), \ \Gamma_i = v_{a_i}(\Gamma), \ \Gamma_i^s = v_{a_i}(\Gamma^s), \ \Gamma_i^u = v_{a_i}(\Gamma^u).$$

So the longitude $L_S = H_S(\Gamma_i^s)$ and meridian $M_S = H_S(Ox_1x_2 \cap T_{-1})$ are chosen.

3.2. Trajectory Mappings

In this section we prove Lemma 2:

Proof. Without loss of generality we will assume that the neighborhoods $\mathcal{V}_A = H_A(\mathbb{V}_0)$, $V_S = H_S(\mathbb{V}_{-1})$, $\mathcal{V}_R = H_A(\mathbb{V}_2)$ of orbits A, S, R are pairwise disjoint.

Note that the knots

$$\gamma_A = W_S^u \cap T_S = H_S(\Gamma_{-1}^u), \ \gamma_R = W_S^s \cap T_S = H_S(\Gamma_{-1}^s)$$

have tubular neighborhoods

$$K_A = H_S(\mathbb{T}_{-1}^u), \ K_R = H_S(\mathbb{T}_{-1}^s),$$

respectively, which are homeomorphic to annuli with a common boundary

$$\Gamma_S = H_S(\Gamma_{-1}).$$

Next, we “blow up” the solid tori $\mathcal{V}_A$ and $\mathcal{V}_R$ along the trajectories so that they become “adjacent” to each other and to $\mathcal{V}_S$. For this purpose we introduce the following notations:

- let $\mathcal{T}_A = \partial\mathcal{V}_A, \quad \mathcal{T}_R = \partial\mathcal{V}_R$,

$$K^s_R = \left(\bigcup_{t>0,\, w\in cl(K_R)} f^{-t}(w)\right) \cap \mathcal{T}_R, \quad K^u_R = \mathcal{T}_R \setminus K^s_R,$$

$$K^u_A = \left(\bigcup_{t>0,\, w\in cl(K_A)} f^{t}(w)\right) \cap \mathcal{T}_A, \quad K^s_A = \mathcal{T}_A \setminus K^u_A;$$

- define a continuous function $\tau_R \colon \mathcal{T}_R \to \mathbb{R}^+$ such that $f^{\tau_R(r)}(r) \in K_R$ for $r \in K_R$ and the set $K = \bigcup\limits_{r\in cl(K^u_R)} f^{\tau_R(r)}(r)$ disjoint with torus $\mathcal{T}_A$, let $T_R = K \cup K_R$ and define homeomorphism $\psi_R \colon \mathcal{T}_R \to T_R$ by the formula $\psi_R(r) = f^{\tau_R(r)}(r)$. Also, let V_R denote the connected component of $M^3 \setminus T_R$ which contains R;

- define a continuous function $\tau_A \colon \mathcal{T}_A \to \mathbb{R}^+$ such that $f^{-\tau_A(a)}(a) \in K_A$ for $a \in K^u_A$ and $f^{-\tau_A(a)}(a) \in K$ for $a \in K^s_A$, let $T_A = K \cup K_A$ $\psi_A \colon \mathcal{T}_A \to T_A$ and define homeomorphism $\psi_A(a) = f^{-\tau_A(a)}(a)$. Also, let V_A denote the connected component of $M^3 \setminus T_A$ which contains A;

- define a continuous function $\tau_{RA} \colon K_R \setminus \gamma_R \to \mathbb{R}^+$ such that $f^{\tau_{RA}(w)}(w) \in K_A \setminus \gamma_A$, define homeomorphism $\psi \colon T_R \setminus \gamma_R \to T_A \setminus \gamma_A$ by the formula

$$\psi(w) = \begin{cases} f^{\tau_{RA}(w)}(w), & w \in (K_R \setminus \gamma_R) \\ w, & w \in (T_R \setminus K_R). \end{cases}$$

Thus, the constructed solid tori V_A, V_S, V_R satisfy the conditions of the lemma. □

4. TOPOLOGICAL CLASSIFICATION OF FLOWS $f^t \in G^-_1(M^3)$

Let us prove the first statement of Theorem 1: flows f^t, $f'^t \in G^-_1(M^3)$ are topologically equivalent if and only if their sets $C_{f^t} = (l_1, b_1, l_2, b_2)$, $C_{f'^t} = (l'_1, b'_1, l'_2, b'_2)$ are consistent.

Proof. Recall that for a periodic orbit $\mathcal{O} \in \{A, S, R\}$ of the flow $f^t \in G^-_1(M^3)$ we denote by $V_\mathcal{O}$ its canonical neighborhood with boundary $T_\mathcal{O}$. In this case, the ambient manifold M^3 of the flow f^t is represented as a union of three solid tori $M^3 = V_A \cup V_S \cup V_R$ with nonintersecting interior, torus T_S is the union of compact tubular neighborhoods of K_A, K_R, the knots $\gamma_A = W^u_S \cap T_S$, $\gamma_R = W^s_S \cap T_S$, $K = T_R \setminus \operatorname{int} K_A = T_A \setminus \operatorname{int} K_A$ and the knot γ_S is the connected component of the boundary of the annulus K.

On the torus $T_{\{\mathcal{O}}$, $\mathcal{O} \in \{S, R, A\}$ we have chosen the longitude $L_\mathcal{O}$ (a curve homologous in $V_\mathcal{O}$ to the orbit of $\mathcal{O}$) and the meridian $M_\mathcal{O}$ (a curve homotopic to zero on $V_\mathcal{O}$ and essential on $T_\mathcal{O}$) such that the ordered pair of curves $L_\mathcal{O}$, $M_\mathcal{O}$ defines the outer side of the solid torus of $V_\mathcal{O}$. The knot $\gamma_\mathcal{O}$ is oriented coherently with the saddle orbit S and has homotopy type $\langle\gamma_\mathcal{O}\rangle = \langle l_\mathcal{O}, m_\mathcal{O}\rangle$ with respect to the generators $L_\mathcal{O}, M_\mathcal{O}$.

If $(l_R, m_R) = (0, 0)$, then we have written the homotopy type of the meridian $M_R \subset K$ with respect to the generators L_A, M_A

$$\langle M_R\rangle = \langle p_A, q_A\rangle.$$

If $(l_R, m_R) \neq (0, 0)$, then any knot $\sigma_\mathcal{O} \subset T_\mathcal{O}$ having homotopy type $\langle\sigma_\mathcal{O}\rangle = \langle b_\mathcal{O}, c_\mathcal{O}\rangle$ and the intersection index 1 with knot $\gamma_\mathcal{O}$ has the following property:

$$l_\mathcal{O} c_\mathcal{O} - m_\mathcal{O} b_\mathcal{O} = 1. \tag{4.1}$$

Let $\Sigma^{+1}_{\mathcal{O}}$ ($\Sigma^{-1}_{\mathcal{O}}$) denote the set of all knots on $T_{\mathcal{O}}$, having intersection index +1 (-1) with the knot $\gamma_{\mathcal{O}}$. Then

$$\tilde{\sigma}_{\mathcal{O}} \in \Sigma^{\pm 1}_{\mathcal{O}} \iff \langle \tilde{\sigma}_{\mathcal{O}} \rangle = \langle \pm b_{\mathcal{O}} + n_{\mathcal{O}} l_{\mathcal{O}}, \pm c_{\mathcal{O}} + n_{\mathcal{O}} m_{\mathcal{O}} \rangle,\ n_{\mathcal{O}} \in \mathbb{Z}. \tag{4.2}$$

It is easily verified that the intersection index of knots $\sigma_{\mathcal{O}}$, $\tilde{\sigma}_{\mathcal{O}}$ is $-n_{\mathcal{O}}$. Then, if $\tilde{\sigma}_S = (\tilde{\sigma}_R \cup \tilde{\sigma}_A) \cap T_S$, then

$$n_A + n_R + n_S = 0. \tag{4.3}$$

Also recall that the generators L_S, M_S are chosen such that with respect to them the knot γ_S has homotopy type

$$\langle \gamma_S \rangle = \langle l_S, m_S \rangle = \langle 2, 1 \rangle. \tag{4.4}$$

If $(l_R, m_R) \neq (0,0)$ let the knot $\sigma_S \subset T_S$ be chosen such that

$$\langle \sigma_S \rangle = \langle b_S, c_S \rangle = \langle 1, 1 \rangle \tag{4.5}$$

and σ_S intersects with each connected component of the ∂K at exactly one point (this can be done since the intersection index of the knots γ_S and σ_S is 1). Let the knots $\sigma_R \subset T_R$, $\sigma_A \subset T_A$ be chosen such that

$$\sigma_S = (\sigma_R \cup \sigma_A) \cap T_S. \tag{4.6}$$

By definition, $C_{f^t} = (l_1, b_1, l_2, b_2)$, where

- $(l_1,\, b_1,\, l_2,\, b_2) = (l_R,\, b_R,\, l_A,\, b_A)$, if $(l_R,\, m_R) \neq (0,\, 0)$;
- $(l_1,\, b_1,\, l_2,\, b_2) = (0,\, 2,\, p_A,\, q_A)$, if $(l_R,\, m_R) = (0,\, 0)$ and 2-ball, bounded by the knot γ_R remains to the left when traveling along the knot;
- $(l_1,\, b_1,\, l_2,\, b_2) = (0,\, -2,\, -p_A,\, -q_A)$, if $(l_R,\, m_R) = (0,\, 0)$ and 2-ball, bounded by the knot γ_R remains to the right when traveling along the knot.

Similar equalities with primes hold for the flow f'^t.

Let us prove separately the necessity and sufficiency of the conditions of Theorem 1.

Necessity. Let the flows f^t and f'^t with periodic orbits A, R, S and A', R', S' be topologically equivalent via the homeomorphism $h\colon M^3 \to M^3$. For $\mathcal{O} \in \{A, S, R\}$, without reducing generality, let $V_{\mathcal{O}'} = h(V_{\mathcal{O}})$. Let $h_{\mathcal{O}} = h\big|_{T_{\mathcal{O}}}\colon T_{\mathcal{O}} \to T_{\mathcal{O}'}$.

Since $h_{\mathcal{O}}$ is a restriction of a homeomorphism of a solid torus, the action of the homeomorphism $h_{\mathcal{O}}$ in the fundamental group $\pi_1(T_{\mathcal{O}})$ in the generators $L_{\mathcal{O}}$, $M_{\mathcal{O}}$ is given by a matrix:

$$h_{\mathcal{O}*} = \begin{pmatrix} 1 & k_{\mathcal{O}} \\ 0 & \delta_{\mathcal{O}} \end{pmatrix},\ k_{\mathcal{O}} \in \mathbb{Z},\ \delta_{\mathcal{O}} \in \{-1, +1\}. \tag{4.7}$$

Thus, since the tori $T_A,\ T_S,\ T_R$ are pairwise intersecting two-dimensional manifolds, all numbers $\delta_A, \delta_S, \delta_R$ have the same sign, let

$$\delta_A = \delta_S = \delta_R = \delta_R = \delta.$$

From the properties of the conjugating homeomorphism it follows that $h_{\mathcal{O}}(\gamma_{\mathcal{O}}) = \gamma_{\mathcal{O}'}$, $\mathcal{O} \in \{S, A, R\}$, whence

$$l_{\mathcal{O}} = l_{\mathcal{O}'} \tag{4.8}$$

and

$$(l_R, m_R) = (0,0) \iff (l_{R'}, m_{R'}) = (0,0). \tag{4.9}$$

Let us prove that the consistency condition of the sets C_{f^t}, $C_{f'^t}$ holds separately for two cases: I) $(l_R, m_R) = (0,0)$, II) $(l_R, m_R) \neq (0,0)$.

In case I), it follows from the definition of the sets C_{f^t}, $C_{f'^t}$ that $l_1 = l_2 = 0$, $|b_2| = |b_2'| = 2$. Since the homeomorphism h_R maps the 2-disc bounded by the knot γ_R into the 2-disc bounded by the knot $\gamma_{R'}$ with the direction of knots preserved, we have $b_2 = \delta b_2'$.

It follows from Eqs. (4.7) that $h_{R*}(\langle 0, 1\rangle) = \langle 0, \delta\rangle$. Thus, $h_{A*}(\langle p_A, q_A\rangle) = \langle \delta p_{A'}, \delta q_{A'}\rangle$. Since $(l_2, b_2) = (\pm p_A, \pm q_A)$, we have $(l_2', b_2') = (\delta(\pm p_{A'}), \delta(\pm q_A'))$ which implies $h_{A*}(\langle l_2, b_2\rangle) = \langle l_2', b_2'\rangle$. Also, it follows from Eqs. (4.7) that $h_{A*}(\langle l_2, b_2\rangle) = \langle l_2, \delta b_2 + k_A l_2\rangle$, whence $l_2 = l_2'$ and $b_2 \equiv \delta b_2'$ (mod l_2).

In case II), equality (4.8) is equivalent to the equality $l_i = l_i'$, $i = 1, 2$. Let $\tilde\sigma_{\mathcal{O}'} = h_{\mathcal{O}}(\sigma_{\mathcal{O}})$ and denote by

$$\langle \tilde\sigma_{\mathcal{O}'}\rangle = \langle \tilde b_{\mathcal{O}'}, \tilde c_{\mathcal{O}'}\rangle$$

the homotopy type of the knot $\tilde\sigma_{\mathcal{O}'}$ with respect to the generators $L_{\mathcal{O}'}, M_{\mathcal{O}'}$. Then it follows from the formula (4.7) that

$$\tilde b_{\mathcal{O}'} = b_{\mathcal{O}}. \tag{4.10}$$

Since the determinant of the matrix $h_{\mathcal{O}*}$ equals δ and $h_{\mathcal{O}}(\gamma_{\mathcal{O}}) = \gamma_{\mathcal{O}'}$, it follows that $\tilde\sigma_{\mathcal{O}'} \in \Sigma^{\delta}_{\mathcal{O}'}$. Then from the formula (4.2) we obtain

$$\tilde b_{\mathcal{O}'} = \delta b_{\mathcal{O}'} + n_{\mathcal{O}'} l_{\mathcal{O}'}, \tilde c_{\mathcal{O}} = \delta c_{\mathcal{O}'} + n_{\mathcal{O}'} m_{\mathcal{O}'}. \tag{4.11}$$

Whence, taking into account equalities (4.8) and (4.10), we find that

$$b_{\mathcal{O}} = \delta b_{\mathcal{O}'} + n_{\mathcal{O}'} l_{\mathcal{O}}, \tag{4.12}$$

so

$$b_{\mathcal{O}} \equiv \delta b_{\mathcal{O}'} \pmod{l_{\mathcal{O}}}. \tag{4.13}$$

By construction, $\tilde\sigma_S = (\tilde\sigma_R \cup \tilde\sigma_A) \cap T_S$, which, given equality (4.3), entails the equality

$$n_{A'} + n_{R'} + n_{S'} = 0. \tag{4.14}$$

If $l_A l_R \neq 0$, then by expressing $n_{\mathcal{O}'}$ from equality (4.12) and substituting into equality (4.14), given that $l_S = 2,\ b_S = b_{S'} = 1$, we arrive at

$$2l_R(b_A - \delta b_{A'}) + 2l_A(b_R - \delta b_{R'}) + l_A l_R(1 - \delta) = 0,$$

which is equivalent to

$$l_A l_R\big(2l_R(b_A - \delta b_{A'}) + 2l_A(b_R - \delta b_{R'}) + l_A l_R(1 - \delta)\big) = 0,$$

which holds when $l_A l_R = 0$.

Sufficiency. Let the sets $C_{f^t} = (l_1, b_1, l_2, b_2)$, $C_{f'^t} = (l_1', b_1', l_2', b_2')$ of flows f^t, f'^t be consistent via the parameter $\delta \in \{-1, 1\}$. We define the homeomorphism $Q_\delta\colon \mathbb{V}_{-1} \to \mathbb{V}_{-1}$ by the formula

$$Q_\delta = v_{a_{-1}} \bar Q_\delta v_{a_{-1}}^{-1},\ \text{ where } \bar Q_\delta(x_1, x_2, x_3) = (\delta x_1, x_2, x_3)\colon V_{-1} \to V_{-1}.$$

We check directly that the constructed homeomorphism Q_δ realizes the equivalence of the flow $[a_{-1}]^t$ with itself. Let

$$h_S = H_{S'} Q_\delta H_S^{-1} : V_S \to V_{S'}.$$

We show that the homeomorphism $h_S|_{K_A}$ can be extended to a homeomorphism $h_A\colon T_A \to T_{A'}$ inducing an isomorphism $h_{A*} = \begin{pmatrix} 1 & k_A \\ 0 & \delta \end{pmatrix}$ for some $k_A \in \mathbb{Z}$ and the homeomorphism $h_S|_{K_R}$ can be extended to the homeomorphism $h_R : T_R \to T_{R'}$ inducing isomorphism $h_{R*} = \begin{pmatrix} 1 & k_R \\ 0 & \delta \end{pmatrix}$ for some

$k_R \in \mathbb{Z}$ such that $h_A|_K = h_R|_K$. Then, by virtue of Proposition 8, the homeomorphisms h_A, h_R can be extended to homeomorphisms $h_A\colon V_A \to V_{A'}$, $h_R\colon V_R \to V_{R'}$ realizing the equivalence of the flows $f^t\big|_{V_A}$ with $f'^t\big|_{V_{A'}}$ and $f^t\big|_{V_R}$ with $f'^t\big|_{V_{R'}}$, respectively, and the desired homeomorphism $h\colon M^3 \to M^3$ realizing the equivalence of f^t, f'^t flows coincides with $h_{\mathcal{O}}$ on $V_{\mathcal{O}}$ for $\mathcal{O} \in \{S, A, R\}$.

Let us consider the cases separately: I) $(l_1, b_1) = (0, \pm 2)$, II) $(l_1, b_1) \neq (0, \pm 2)$.

In case I), it follows from the consistency condition of the sets C_{f^t}, $C_{f'^t}$ that $b_1 = \delta b_1'$, $l_2 = l_2'$ and $b_2' = \delta b_2 + k_A l_2$ for some $k_A \in \mathbb{Z}$. Since the annuli K_A, $K_{A'}$ are contractible on tori T_A, $T_{A'}$, the homeomorphism $h_S|_{K_A}$ can be extended to the homeomorphism $h_A\colon T_A \to T_{A'}$ inducing an isomorphism

$$h_{A*} = \begin{pmatrix} 1 & k_A \\ 0 & \delta \end{pmatrix}.$$

Let us define the homeomorphism $h_R\colon T_R \to T_{R'}$ by the formula

$$h_R(x) = \begin{cases} h_S(x), & x \in K_R \\ h_A(x), & x \in K \end{cases}.$$

Since $h_{A*}(\langle l_2, b_2\rangle) = \langle l_2', b_2'\rangle$, it follows that $h_{R*}(\langle 0, 1\rangle) = \langle 0, \delta\rangle$, so

$$h_{R*} = \begin{pmatrix} 1 & k_R \\ 0 & \delta \end{pmatrix}$$

for some $k_R \in \mathbb{Z}$.

In case II), it follows from the consistency condition of the sets C_{f^t}, $C_{f'^t}$ that $l_i' = l_i$, $b_i' \equiv \delta b_i \pmod{l_i}$, $i = 1, 2$. So,

$$l_{R'} = l_R,\ b_R' \equiv \delta b_R \pmod{l_R};\ l_{A'} = l_A,\ b_A' \equiv \delta b_A \pmod{l_A}. \tag{4.15}$$

Next, we consider separately cases IIa) $l_A l_R = 0$, IIb) $l_A l_R \neq 0$.

In case IIa) we assume without loss of generality that $l_R = 0$ (in case $l_A = 0$ the reasoning is similar). It follows from (4.15) and (4.1) that $m_{A'} = \delta m_A + k_A l_A$ for some $k_A \in \mathbb{Z}$. Then the homeomorphism $h_S|_{K_A}$ continues to a homeomorphism $h_A\colon T_A \to T_{A'}$ inducing an isomorphism

$$h_{A*} = \begin{pmatrix} 1 & k_A \\ 0 & \delta \end{pmatrix}.$$

Let us define the homeomorphism $h_R\colon T_R \to T_{R'}$ by the formula

$$h_R(x) = \begin{cases} h_S(x), & x \in K_R \\ h_A(x), & x \in K \end{cases}.$$

Since $l_R = 0$, we have $m_R = \pm 1$, $m_{R'} = \pm\delta$ and hence $h_{R*}(\langle 0, \pm 1\rangle) = \langle 0, \pm\delta\rangle$. Then

$$h_{R*} = \begin{pmatrix} 1 & k_R \\ 0 & \delta \end{pmatrix}$$

for some $k_R \in \mathbb{Z}$.

In case IIb), the homeomorphisms $h_A\colon T_A \to T_{A'}$ and $h_R\colon T_R \to T_{R'}$ are constructed as in case IIa). Let us show that $h_{R*} = \begin{pmatrix} 1 & k_R \\ 0 & \delta \end{pmatrix}$, where $m_{R'} = \delta m_R + k_R l_R$.

From equality (4.11) we find that the knot $h_S(\sigma_S)$ has the intersection index

$$n_{S'} = \frac{1-\delta}{2}$$

with the knot $\sigma_{S'}$, and the knot $h_A(\sigma_A)$ has the intersection index

$$n_{A'} = \frac{b_A - \delta b_{A'}}{l_A}$$

with knot $\sigma_{A'}$. According to equality (4.14), knot $h_R(\sigma_R)$ has intersection index $n_{R'} = -(n_{S'} + n_{A'})$. Then from the consistency condition of the sets, we find that

$$n_{R'} = \frac{b_R - \delta b_{R'}}{l_R}.$$

Whence $h_{R*}(\langle b_R, c_R\rangle) = \langle b_R, \tilde{c}_R\rangle$. Since $h_{R*}(\langle l_R, m_R\rangle) = \langle l_R, m_{R'}\rangle$, we have

$$h_{R*} = \begin{pmatrix} 1 & k_R \\ 0 & \delta \end{pmatrix}.$$

□

5. REALIZATION OF FLOWS $f^t \in G_1^-(M^3)$

In this section we prove the second part of Theorem 1: for any admissible invariant C there exists a flow $f^t \in G_1^-(M^3)$. Recall that an invariant C is called admissible if:

- $(l_1, b_1) = (0, \pm 2)$ or $\gcd(l_1, b_1) = 1$;
- $\gcd(l_2, b_2) = 1$.

Proof. Let $C = (l_1,\ b_1,\ l_1,\ l_1,\ b_1)$. We construct the three-dimensional manifold M^3 and the flow $f^t \in G_1^-(M^3)$ such that $C_{f^t} = C$ separately for the cases: I) $(l_1, b_1) = (0, \pm 2)$, II) $(l_1, b_1) \neq (0, \pm 2)$.

In case I), for $(l_1,\ b_1) = (0, \pm 2)$, let $(p, q) = (\pm l_2, \pm b_2)$. Let us define a homeomorphism $\psi\colon \mathbb{T}_2 \to \mathbb{T}_0$ inducing an isomorphism defined by the integer matrix $h_* = \begin{pmatrix} r & s \\ p & q \end{pmatrix}$ with the determinant equal to -1. On the torus $\mathbb{T}_2$, we choose a γ_R essential knot with a tubular neighborhood $K_R \subset (\mathbb{T}_2 \setminus \mathbb{M}_2)$ and orient it so that the 2-disk bounded by it remains on the left in the case $b_1 = +2$ and $-$ on the right in the case $b_1 = -2$. Let $\gamma_A = \psi(\gamma_R)$ and $K_A = \psi(K_R)$.

Let $\psi_R\colon K_R \to \mathbb{T}^s_{-1}$,
$psi_A\colon K_A \to \mathbb{T}^u_{-1}$ be homeomorphisms such that $\psi_A^{-1}\psi_R|_{\partial K_R} = \psi|_{\partial K_R},\ \psi_R(\gamma_R) = \Gamma^s,\ \psi_A(\gamma_A) = \Gamma^u$. Let $\sim$ be the minimal equivalence relation on $\mathbb{V}_0 \sqcup \mathbb{V}_{-1} \sqcup \mathbb{V}_2$ for which $x \sim \psi(x), x \in (\mathbb{T}_2 \setminus \operatorname{int} K_A),\ x \sim \psi_A(x), x \in K_A,\ x \sim \psi_R(x), x \in K_R$. Then

$$M^3 = (\mathbb{V}_0 \sqcup \mathbb{V}_{-1} \sqcup \mathbb{V}_2)/\sim.$$

We denote by $\pi\colon \mathbb{V}_0 \sqcup \mathbb{V}_{-1} \sqcup \mathbb{V}_2 \to M^3$ the natural projection. Let the flow $f^t\colon M^3 \to M^3$ be given by the formula

$$f^t(x) = \begin{cases} \pi\Big([a_0]^t\big(\pi^{-1}(x)\big)\Big), & x \in \pi(\mathbb{V}_0) \\ \pi\Big([a_{-1}]^t\big(\pi^{-1}(x)\big)\Big), & x \in \pi(\mathbb{V}_{-1}) \\ \pi\Big([a_2]^t\big(\pi^{-1}(x)\big), & x \in \pi(\mathbb{V}_2)\Big) \end{cases}$$

By construction, $C_{f^t} = C$.

In case II), we represent the sphere $\mathbb{S}^2$ as a union of three two-dimensional disks D_A, D_S, D_R with centers O_A, O_S, O_R, glued along the boundary as depicted in Fig. 3 (glued segments are marked with the same color). Then the manifold $\mathbb{S}^2\times\mathbb{S}^1$ is represented as a union of three solid tori $V_A=D_A\times\mathbb{S}^1$, $V_S=D_S\times\mathbb{S}^1$, $V_R=D_R\times\mathbb{S}^1$, which are tubular neighborhoods of the knots $\ell_A=O_A\times\mathbb{S}^1$, $\ell_S=O_S\times\mathbb{S}^1$, $\ell_R=O_R\times\mathbb{S}^1$, glued along the boundaries $T_A=\partial V_A$, $T_S=\partial V_S$, $T_R=\partial V_R$, along the annuli $K_A=T_A\cap T_S$, $K_R=T_R\cap T_S$, $K=T_A\cap T_R$, $K=T_A\cap T_R$, respectively.

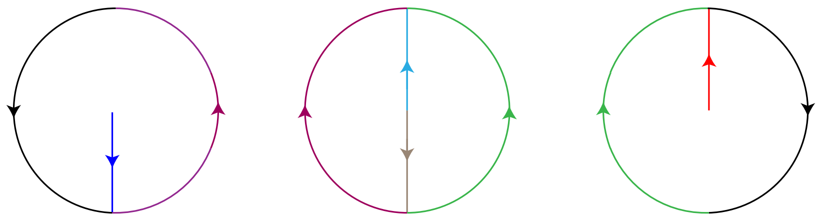

Fig. 3. Disks D_A, D_S, D_R. The color and arrows show identification of boundary subsets.

Let $C=(l_1,b_1,l_2,b_2)$. Then the sought manifold M^3 is obtained by the Dehn surgery along the link $\ell_A\sqcup\ell_S\sqcup\ell_R$ with equipment $(-b_2,l_2),(-1,2),(-b_1,l_1)$. Moreover, the homeomorphisms of surgery $h_A\colon\mathbb{V}_0\to V_A$, $h_S\colon\mathbb{V}_{-1}\to V_S$, $h_R:\mathbb{V}_{-1}\to V_S$, $h_R\colon\mathbb{V}_2\to V_R$ are chosen such that $h_S(\mathbb{T}^u_{-1})=K_A$, $h_S(\mathbb{T}^s_{-1})=K_R$. Denote by $\pi\colon\mathbb{V}_0\sqcup\mathbb{V}_{-1}\sqcup\mathbb{V}_2\to M^3$ the natural projection. Let $f^t\colon M^3\to M^3$ be defined by the formula

$$f^t(x)=\begin{cases}\pi\Big([a_0]^t\big(\pi^{-1}(x)\big)\Big), & x\in\pi(\mathbb{V}_0)\\ \pi\Big([a_{-1}]^t\big(\pi^{-1}(x)\big)\Big), & x\in\pi(\mathbb{V}_{-1})\\ \pi\Big([a_2]^t\big(\pi^{-1}(x)\big)\Big), & x\in\pi(\mathbb{V}_2)\end{cases}$$

By construction, $C_{f^t}=C$. □

6. TOPOLOGY OF AMBIENT MANIFOLDS OF FLOWS $f^t\in G_1^-(M^3)$

In this section we prove Theorem 2.

Proof. Let us prove the theorem separately for the cases: I) $l_1l_2=0$, II) $l_1l_2\neq0$.

In case I), we denote by M^3_S the manifold obtained by Dehn surgery along knot S with equipment $(1,1)$ in the generators L_S,M_S. Let $v_S\colon(M^3\setminus\text{int }V_S)\sqcup\mathbb{V}\to M^3_S$ be the natural projection. For simplicity, we keep the labels of all objects on $v_S(M^3\setminus\text{int }V_S)$ the same as they were on $M^3\setminus\text{int }V_S$ and put $\tilde S=v_S(\{0\}\times\mathbb{S}^1)$, $V_{\tilde S}=v_S(\mathbb{V})$. Then $\tilde V_R=V_R\cup V_{\tilde S}$ is a solid torus with boundary $\tilde T_R$ and there exists an isotopy $\zeta_t\colon V_R\to\tilde V_R$, $t\in[0,1]$ such that $\zeta_0=id|_{V_A}$, $\zeta_t|_K=id|_K$, $t\in[0,1]$, $\zeta_1(V_R)=\tilde V_R$, $\zeta_1(\sigma_S\cap K_R)=\sigma_S\cap K_A$. For any curve $c\subset T_R$, let us put $\tilde c=\zeta_1(c)\subset\tilde T_R$. Then the isomorphism ζ_{1*} is identical in the generators L_R,M_R; $\tilde L_R,\tilde M_R$ and

$$M^3_S=\tilde V_R\cup_\psi V_A,\tag{6.1}$$

where $\psi\colon\partial\tilde V_R\to\partial V_A$ is a homeomorphism inducing the isomorphism in generators $\tilde L_R,\tilde M_R$; L_A,M_A and $\psi_*=\begin{pmatrix}r&s\\p&q\end{pmatrix}$. Hence, $M^3_S\cong L_{p,q}$. From Statement 2, we find that $M^3\cong(L_{p,q})_{\tilde S}$, where $\tilde S$ is a knot with equipment $(-1,2)$. Since the knot $\tilde S$ bounds a 2-ball on at least one of the tori $\tilde V_A,V_R$, by virtue of Statement 3 $(L_{p,q})_{\tilde S}\cong L_{p,q}\#L_{2,-1}$. Whence, by virtue of Statement 1,

$$M^3\cong L_{p,q}\#\mathbb{R}P^3.\tag{6.2}$$

Let us show how the proof of 2) and 1i) follows from the deductions made.

If $(l_1, b_1) = (0, \pm 2)$, it follows from the definition of the set C_{f^t} that $\psi_*(\langle 0, 1\rangle) = \langle \pm l_2, \pm b_2\rangle$. By virtue of Statement 3,

$$2\text{i)}\ M^3 \cong L_{l_2,b_2} \# \overset{3}{\mathbb{R}\mathrm{P}},\ |l_2| \neq 1;$$

$$1\text{i)}\ M^3 \cong \overset{3}{\mathbb{R}\mathrm{P}},\ |l_2| = 1.$$

If $(l_1, b_1) \neq (0, \pm 2)$. It follows from the definition of the set C_{f^t} that $\psi_*(\langle 0, 1\rangle) = \langle \pm l_2, \pm m_A\rangle$, $\psi_*(\langle 1, \pm c_R\rangle) = \langle \pm b_2, \pm c_A\rangle$. By direct calculation we see that $\psi_*^{-1}(\langle 0, 1\rangle) = \langle \pm l, b\rangle$, where $|l| = |l_2|$, $|b| \equiv |b_2| \pmod{l_2}$. By virtue of Statement 3,

$$2\text{i)}\ M^3 \cong L_{l_2,b_2} \# \overset{3}{\mathbb{R}\mathrm{P}},\ |l_2| \neq 1;$$

$$1\text{i)}\ M^3 \cong \overset{3}{\mathbb{R}\mathrm{P}},\ |l_2| = 1.$$

By reasoning analogous to the above, we find that

$$2\text{ii)}\ M^3 \cong L_{l_1,b_1} \# \overset{3}{\mathbb{R}\mathrm{P}},\ |l_1| \neq 1;$$

$$1\text{i)}\ M^3 \cong \overset{3}{\mathbb{R}\mathrm{P}},\ |l_1| = 1.$$

In case II), consider first the subcase $|l_1| = 1$. Then $\tilde{V}_S = V_S \cup V_R$ is a filled torus with boundary $\tilde{T}_S$ and there exists an isotopy $\zeta_t\colon V_S \to \tilde{V}_S$, $t \in [0, 1]$ such that $\zeta_0 = id|_{V_S}$, $\zeta_t|_{K_A} = id|_{K_A}$, $t \in [0, 1]$, $\zeta_1(V_S) = \tilde{V}_S$, $\zeta_1(\sigma_S \cap K_R) = \sigma_R \cap K$. For any curve $c \subset T_S$, let $\tilde{c} = \zeta_1(c) \subset \tilde{T}_S$. Then the isomorphism ζ_{1*} is identical in the generators L_S, M_S; $\tilde{L}_S, \tilde{M}_S$ and

$$M^3_S = \tilde{V}_S \cup_\psi V_A, \tag{6.3}$$

where $\psi\colon \partial\tilde{V}_R \to \partial V_A$ is a homeomorphism, inducing in the generators $\tilde{L}_S, \tilde{M}_S$; L_A, M_A isomorphism $\psi_* = \begin{pmatrix} r & s \\ p & q \end{pmatrix}$. Hence, $M^3_S \cong L_{p,q}$. From the definition of the set C_{f^t}, it follows that $\psi_*(\langle 2, 1\rangle) = \langle \pm l_2, m_A\rangle$, $\psi_*(\langle 1, 1\rangle) = \langle \pm b_2, \pm c_A\rangle$. By direct calculation, we see that $\psi_*^{-1}(\langle 0, 1\rangle) = \langle \pm l, b\rangle$, where $|l| = ||l_2 - 2b_2|$, $|b| \equiv |b_2| \pmod{l_2}$. By virtue of Statement 3,

$$1\text{ii)}\ M^3 \cong L_{l_2-2b_2,b_2}.$$

In the case $|l_2| = 1$, by similar reasoning, we find that

$$1\text{iii)}\ M^3 \cong L_{l_1-2b_1,b_1}.$$

In the case $|l_1| > 1$, $|l_2| > 1$, it follows from the procedure for realizing a flow over an admissible set (see the proof of the second part of Theorem 1 in case II)) that M^3 is a Seifert fiber space with base sphere with three special fibers

$$M^3 \cong M\big(\mathbb{S}^2, (l_1, b_1), (l_2, b_2), (2, 1)\big).$$

This completes the proof of Theorem 2. □

7. COUNTING THE NUMBER OF TOPOLOGICAL EQUIVALENCE CLASSES

In this section we give a proof of Theorem 3. To do this, recall that for any pair p, q of integer prime numbers, we put $\bar{p} = |p|$ and denote by $\bar{q}$ the smallest nonnegative of the numbers q' satisfying the condition $q \equiv \pm q' \pmod{p}$, and by $\tilde{q}$ the smallest nonnegative of the numbers q' satisfying the condition $qq' \equiv \pm 1 \pmod{p}$.

Proof. By virtue of Theorem 2, flows of class $G_1^-(M^3)$ admit three types of manifolds 1) $L_{p,q}$; 2) $L_{p,q} \# \mathbb{R}\mathrm{P}^3$; 3) $M\big(\mathbb{S}^2, (l_1, b_1), (l_2, b_2), (2, 1)\big)$. Let us prove the proof separately for each of these cases.

1) According to Statement 1, two lens spaces $L_{p,q}$, $L_{p',q'}$ are homeomorphic if and only if $\bar{p} = \bar{p}'$ and either $\bar{q} = \bar{q}'$ or $\bar{q} = \tilde{q}'$. Whence it follows that $L_{p,q} \cong L_{\bar{p},\bar{q}}$ and, $L_{p',q'} \cong L_{\bar{p},\bar{q}}$ if and only if at least one of the following conditions for $k \in \mathbb{Z}$ is satisfied:

$$p' = \bar{p},\ q' = \bar{q} + k\bar{p}; \tag{7.1}$$

$$p' = -\bar{p},\ q' = \bar{q} + k\bar{p}; \tag{7.2}$$

$$p' = \bar{p},\ q' = -\bar{q} + k\bar{p}; \tag{7.3}$$

$$p' = -\bar{p},\ q' = -\bar{q} + k\bar{p}; \tag{7.4}$$

$$p' = \bar{p},\ q' = \tilde{q} + k\bar{p}; \tag{7.5}$$

$$p' = -\bar{p},\ q' = \tilde{q} + k\bar{p}; \tag{7.6}$$

$$p' = \bar{p},\ q' = -\tilde{q} + k\bar{p}; \tag{7.7}$$

$$p' = -\bar{p},\ q' = -\tilde{q} + k\bar{p}; \tag{7.8}$$

By virtue of Theorem 2, the lens $L_{\bar{p},\bar{q}}$, $\bar{p} \neq 2$ is an ambient manifold for flows with invariants

$$(\pm 1, n, p' + 2q', q'); \tag{7.9}$$

$$(p' + 2q', q', \pm 1, n), \tag{7.10}$$

where $n \in \mathbb{Z}$. Substituting condition (7.1) into (7.9), we obtain sets of the form

$$(\pm 1, b_1, \bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}).$$

From the definition of consistency, it follows that two sets of

$$\big(\pm 1, n_1, \bar{p} + 2(\bar{q} + k_1\bar{p}), \bar{q} + k_1\bar{p}\big), \big(\pm 1, n_2, \bar{p} + 2(\bar{q} + k_2\bar{p}), \bar{q} + k_2\bar{p}\big), \bar{q} + k_2\bar{p}$$

are consistent if and only if $k_1 = k_2$, $n_1 = n_2$. Thus, each representation of the lens $L_{p,q}$ in the form (7.1) gives rise to the family $(\pm 1, n, \bar{p} + 2(\bar{q} + k + k\bar{p}), \bar{q} + k\bar{p})$, $n, k \in \mathbb{Z}$ of pairwise nonconsistent sets corresponding, by virtue of Theorem 2, to pairwise nonequivalent flows. If $|p| > 2$, then similar families are obtained from each of the representations (7.2)–(7.4). It is directly verified that the sets of all four families are not pairwise equivalent. Finally, if $\bar{q} \neq \tilde{q}$ (equivalent to $q^2 \not\equiv \pm 1 \pmod{p}$), we obtain four more families of pairwise nonequivalent sets corresponding to the representations (7.5)–(7.7). Adding sets of type (7.10) to the list of sets, we obtain a list of eight more pairwise nonequivalent sets, from which the result of the theorem follows directly in cases 1a) and 1b).

In cases 1c) and 1d), by directly substituting the pairs $\bar{p} = 0, \bar{q} = 1$; $\bar{p} = 1, \bar{q} = 0$ into the sets 1b), respectively, we obtain the announced lists of pairwise nonequivalent pairs.

In the case $|p| = 2$, the lens $L_{p,q}$ is an ambient manifold for the flows with invariants

$$(0, c, \pm 1, n); \tag{7.11}$$

$$(\pm 1, n, 0, d), \tag{7.12}$$

where $n \in \mathbb{Z}$, $c \in \{-2, -1, 1, 2\}$, $d \in \{-1, 1\}$. From the definition of consistency, it follows that the two sets

$$(0, c_1, \pm 1, n_1), (0, c_2, \pm 1, n_2)$$

are consistent if and only if $c_1 = \pm c_2$, $n_1 \equiv \pm n_2 \pmod 1$. A similar statement is true for sets of the form (7.12), resulting in the announced list 1e).

2) By virtue of Theorem 2, the manifold $L_{p,q} \# \mathbb{R}\mathrm{P}^3$, $|p|| \neq 1$ is an ambient manifold for flows with invariants

$$(0, c, p', q'); \tag{7.13}$$

$$(p', q', 0, d), \tag{7.14}$$

where $n \in \mathbb{Z}$, $c \in \{-2, -1, 1, 2\}$, $d \in \{-1, 1\}$. It follows from the definition of consistency that for $|p| > 2$ the two sets of

$$(0, c_1, \bar{p}, \bar{q} + k_1 \bar{p}), (0, c_2, \bar{p}, \bar{q} + k_2 \bar{p})$$

are consistent if and only if $c_1 = c_2$, $k_1 \equiv k_2 \pmod 1$. Thus, each representation of the lens $L_{p,q}$ in the form (7.1) gives rise to a family $(0, c, \bar{p}, \bar{q})$ of pairwise nonconsistent sets corresponding, by virtue of Theorem 2, to pairwise nonequivalent flows. Similar families are obtained from each of the representations (7.2), (7.3), (7.4), (7.5), (7.7), (7.8) if $\bar{q} \neq \tilde{q}$. Adding the sets of type (7.14) to the list of sets, we obtain the list of sets announced in 2a) and 2b) of this theorem.

In cases 2c) and 2d), by directly substituting into sets 2b) the pairs $\bar{p} = 0, \bar{q} = 1$; $\bar{p} = 2, \bar{q} = 1$, respectively, we obtain the announced lists of pairwise nonequivalent pairs.

3) By virtue of Theorem 2, the manifold $M\big(\mathbb{S}^2,\ (\alpha_1, \beta_1),\ (\alpha_2, \beta_2),\ (2, 1)\big)$ is an ambient manifold for flows with invariants

$$(\alpha_1, \beta_1, \alpha_2, \beta_2). \tag{7.15}$$

By virtue of Statement 4, $M\big(\mathbb{S}^2,\ (\alpha_1, \beta_1),\ (\alpha_2, \beta_2),\ (2, 1)\big) \cong M\big(\mathbb{S}^2,\ (\alpha'_1, \beta'_1),\ (\alpha'_2, \beta'_2),\ (2, 1)\big)$ if and only if at least one of the following conditions is met:

$$\alpha'_1 = \alpha_1,\ \beta'_1 = \beta_1 + k_1 \alpha_1,\ \alpha'_2 = \alpha_2,\ \beta'_2 = \beta_2 + k_2 \alpha_2; \tag{7.16}$$

$$\alpha'_1 = \alpha_2,\ \beta'_1 = \beta_2 + k_1 \alpha_2,\ \alpha'_2 = \alpha_1,\ \beta'_2 = \beta_1 + k_2 \alpha_1, \tag{7.17}$$

where $k_1, k_2 \in \mathbb{Z}$. By virtue of Theorem 2, $M\big(\mathbb{S}^2,\ (\alpha_1, \beta_1),\ (\alpha_2, \beta_2),\ (2, 1)\big)$ is an ambient manifold for flows with invariants

$$(\alpha'_1, \beta'_1, \alpha'_2, \beta'_2). \tag{7.18}$$

Substituting (7.16) into (7.18), we obtain sets of the form

$$(\alpha_1, \beta_1 + k_1 \alpha_1, \alpha_2, \beta_2 + k_2 \alpha_2).$$

It follows from the definition of consistency that all such sets are equivalent to the set $(\alpha_1, \beta_1, \alpha_2, \beta_2)$. A similar situation is obtained with relation (7.17). This gives us the sets announced in 3a) and 3b). □

As the proof of Theorem 3 suggests, the correspondence between the invariants and ambient manifolds of flows $f^t \in G_1^-(M^3)$ is:

- equivalence classes of the set $G_1^-(L_{p,q})$ depending on p, q are represented by flows with the following invariants:

a) $|p| > 2$, $q^2 \not\equiv \pm 1 \pmod p$, $n, k \in \mathbb{Z}$

$$(\pm 1, n, \bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}), (\pm 1, n, -\bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}),$$
$$(\pm 1, n, \bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}), (\pm 1, n, -\bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}),$$
$$(\pm 1, n, \bar{p} + 2(\tilde{q} + k\bar{p}), \tilde{q} + k\bar{p}), (\pm 1, n, -\bar{p} + 2(\tilde{q} + k\bar{p}), \tilde{q} + k\bar{p}),$$
$$(\pm 1, n, \bar{p} + 2(-\tilde{q} + k\bar{p}), -\tilde{q} + k\bar{p}) (\pm 1, n, -\bar{p} + 2(-\tilde{q} + k\bar{p}), -\tilde{q} + k\bar{p}),$$
$$(\bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}, \pm 1, n), (-\bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}, \pm 1, n,),$$
$$(\bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}, \pm 1, n), (-\bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}, \pm 1, n),$$
$$(\bar{p} + 2(\tilde{q} + k\bar{p}), \tilde{q} + k\bar{p}, \pm 1, n), (-\bar{p} + 2(\tilde{q} + k\bar{p}), \tilde{q} + k\bar{p}, \pm 1, n),$$
$$(\bar{p} + 2(-\tilde{q} + k\bar{p}), -\tilde{q} + k\bar{p}, \pm 1, n) (-\bar{p} + 2(-\tilde{q} + k\bar{p}), -\tilde{q} + k\bar{p}, \pm 1, n,);$$

b) $|p| > 2$, $q^2 \equiv \pm 1 \pmod p$, $n, k \in \mathbb{Z}$

$$(\pm 1, n, \bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}), (\pm 1, n, -\bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}),$$
$$(\pm 1, n, \bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}), (\pm 1, n, -\bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}),$$
$$(\bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}, \pm 1, n), (-\bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}, \pm 1, n,),$$
$$(\bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}, \pm 1, n), (-\bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}, \pm 1, n);$$

c) $p = 0$, $n \in \mathbb{Z}$

$$(\pm 1, n, 2, 1), (\pm 1, n, -2, -1), (2, 1, \pm 1, n), (-2, -1, \pm 1, n);$$

d) $|p| = 1$, n, $k \in \mathbb{Z}$

$$(\pm 1, n, 1 + 2k, k), (1 + 2k, k, \pm 1, n);$$

e) $|p| = 2$

$$(\pm 1, 0, 0, 1), (0, 1, \pm 1, 0), (0, 2, \pm 1, 0);$$

- equivalence classes of the set $G_1^-(L_{p,q} \# \mathbb{R}\mathrm{P}^3)$ depending on p, q are represented by flows with the following invariants:

a) $|p| > 2$, $q^2 \not\equiv \pm 1 \pmod p$

$$(0, 2, \bar{p}, \pm\bar{q}), (0, -2, \bar{p}, \pm\bar{q}), (0, -2, \bar{p}, \pm\bar{q}), (0, -2, -\bar{p}, \pm\bar{q}),$$
$$(0, 2, \bar{p}, \pm\tilde{q}), (0, 2, -\bar{p}, \pm\tilde{q}), (0, -2, \bar{p}, \pm\tilde{q}), (0, -2, -\bar{p}, \pm\tilde{q}),$$
$$(0, 1, \bar{p}, \pm\bar{q}), (0, 1, -\bar{p}, \pm\bar{q}), (0, -1, \bar{p}, \pm\bar{q}), (0, -1, -\bar{p}, \pm\bar{q}),$$
$$(0, 1, \bar{p}, \pm\tilde{q}), (0, 1, -\bar{p}, \pm\tilde{q}), (0, -1, \bar{p}, \pm\tilde{q}), (0, -1, -\bar{p}, \pm\tilde{q}),$$
$$(\bar{p}, \pm\bar{q}, 0, 1), (-\bar{p}, \pm\bar{q}, 0, 1), (\bar{p}, \pm\bar{q}, 0, -1), (-\bar{p}, \pm\bar{q}, 0, -1),$$
$$(\bar{p}, \pm\tilde{q}, 0, 1), (-\bar{p}, \pm\tilde{q}, 0, 1), (\bar{p}, \pm\tilde{q}, 0, -1), (-\bar{p}, \pm\tilde{q}, 0, -1);$$

b) $|p| > 2$, $q^2 \equiv \pm 1 \pmod p$

$$(0, 2, \bar{p}, \pm\bar{q}), (0, 2, -\bar{p}, \pm\bar{q}), (0, -2, \bar{p}, \pm\bar{q}), (0, -2, -\bar{p}, \pm\bar{q}),$$
$$(0, 1, \bar{p}, \pm\bar{q}), (0, 1, -\bar{p}, \pm\bar{q}), (0, -1, \bar{p}, \pm\bar{q}), (0, -1, -\bar{p}, \pm\bar{q}),$$
$$(\bar{p}, \pm\bar{q}, 0, 1), (-\bar{p}, \pm\bar{q}, 0, 1), (\bar{p}, \pm\bar{q}, 0, -1), (-\bar{p}, \pm\bar{q}, 0, -1);$$

c) $p = 0$

$$(0,2,0,\pm 1)\,,(0,1,0,\pm 1)\,;$$

d) $|p| = 2$

$$(0,2,2,\pm 1)\,,(0,2,-2,\pm 1)\,,(0,1,2,\pm 1)\,,(0,1,-2,\pm 1)\,,$$
$$(2,\pm 1,0,1)\,,(-2,\pm 1,0,1)\,;$$

- equivalence classes of the set $G_1^-\Big(M\big(\mathbb{S}^2,\ (\alpha_1,\beta_1),\ (\alpha_2,\beta_2),\ (2,1)\big)\Big)$ depending on $\alpha_1,\beta_1,\alpha_2,\beta_2$ are represented by flows with the following invariants:

a) $\alpha_1 = \alpha_2 = \alpha,\ \beta_1 = \beta_2 = \beta$

$$(\alpha,\beta,\alpha,\beta)\,.$$

b) $|\alpha_1 - \alpha_2| + |\beta_1 - \beta_2| > 0$

$$(\alpha_1,\beta_1,\alpha_2,\beta_2)\,,(\alpha_2,\beta_2,\alpha_1,\beta_1)\,.$$

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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