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On the Factoriality of Cox rings

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Abstract—The generalized Cox construction associates with an algebraic variety a remarkable invariant — its total coordinate ring, or Cox ring. In this note, we give a new proof of the factoriality of the Cox ring when the divisor class group of the variety is finitely generated and free. The proof is based on the notion of graded factoriality. We show that if the divisor class group has torsion, then the Cox ring is again factorially graded, but factoriality may be lost.

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1. INTRODUCTION

Let X be an irreducible normal algebraic variety over an algebraically closed field $\mathbb{K}$ with a free finitely generated divisor class group $\mathrm{Cl}(X)$. Denote by $\mathrm{WDiv}(X)$ the group of Weil divisors on X and fix a sublattice $K \subset \mathrm{WDiv}(X)$, which maps onto $\mathrm{Cl}(X)$ isomorphically. Following D. Cox's famous construction [1] from toric geometry, define the Cox ring of the variety X as

$$R(X) = \bigoplus_{D \in K} \mathscr{O}(X, D), \qquad \text{where} \quad \mathscr{O}(X, D) = \{f \in \mathbb{K}(X) \mid \mathrm{div}(f) + D \geq 0\}.$$

Multiplication on graded components of $R(X)$ coincides with the multiplication in the field $\mathbb{K}(X)$ of rational functions, and extends to other elements by distributivity. It can be easily checked that the ring $R(X)$ depends on the choice of the lattice K only up to isomorphism (for a more general statement, see Proposition K 3.2). An important property of $R(X)$ is that it is a factorial ring; see [2], [3]. Here we give a new proof of this result.

Theorem 1.1. *The ring $R(X)$ is factorial.*

Our main aim is to show that the factoriality of $R(X)$ reflects the fact that any effective Weil divisor on X is a unique nonnegative integral combination of prime divisors. This observation immediately implies that the multigraded ring $R(X)$ is "factorial on the set of homogeneous elements" (we call this a priori weaker property *graded factoriality*). Further, we prove that factoriality follows from graded factoriality. This approach is realized in Sec. 2.

In Sec. 3, the Cox ring $R(X)$ is defined in the case when the divisor class group has torsion. Following [2] and [4], we check that $R(X)$ is well defined. Here the ring $R(X)$ is also factorially graded, but factoriality may be lost. The corresponding examples are given in Sec. 4, where we describe the Cox ring of a homogeneous space of an affine algebraic group.

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2. PROOF OF THEOREM 1.1

We start with some elementary properties of multigraded algebras. Let R be a commutative associative algebra with unit over a field $\mathbb{K}$. Assume that R is graded by the lattice $\mathbb{Z}^n$, i.e.,

$$R = \bigoplus_{u \in \mathbb{Z}^n} R_u, \qquad R_u R_v \subseteq R_{u+v}.$$

Denote by $R^\times$ (respectively, R^+) the multiplicative semigroup of invertible (respectively, homogeneous) elements of R.

Lemma 2.1. (i) *Suppose that for any $a, b \in R^+$ the condition $ab = 0$ implies $a = 0$ or $b = 0$. Then R has no zero-divisors.*

(ii) *If R has no zero-divisors and for any $a, b \in R^+$ the condition $ab = 1$ implies $a, b \in R_0$, then $R^\times = R_0^\times$.*

(iii) *Suppose that R has no zero-divisors. If $a \in R^+$ and $a = bc$, then $b, c \in R^+$.*

Proof. Let us fix the lexicographic order on the lattice $\mathbb{Z}^n$. With any element $a \in R$ one associates two homogeneous elements $L(a)$ and $l(a)$, namely, its leading and lowest terms. Clearly, $L(ab) = L(a)L(b)$ and $l(ab) = l(a)l(b)$. Now the statements of the lemma follow easily. □

Corollary 2.2. (i) *The Cox ring $R(X)$ has no zero-divisors.*

(ii) *The semigroup $R(X)^\times$ coincides with $\mathcal{O}(X)^\times$, where $\mathcal{O}(X)$ is the algebra of regular functions on the variety X.*

Proof. Statement (i) follows from Lemma 2.1, (i), because $\mathbb{K}(X)$ has no zero-divisors. To prove (ii), note that if $f_1 \in \mathcal{O}(X, D_1)$, $f_2 \in \mathcal{O}(X, D_2)$, and $f_1 f_2 = 1$, then

$$D_1 + D_2 = 0, \qquad \mathrm{div}(f_1) + \mathrm{div}(f_2) = 0,$$

and the sum of the effective divisors $\mathrm{div}(f_1) + D_1$ and $\mathrm{div}(f_2) + D_2$ equals zero. Therefore

$$\mathrm{div}(f_i) + D_i = 0, \quad i = 1, 2, \qquad \text{and} \qquad D_i = 0.$$

Now it remains to use Lemma 2.1, (ii) and the equality $R(X)_0 = \mathcal{O}(X)$. □

Definition 2.3. Let A be a finitely generated Abelian group, and let $R = \bigoplus_{u \in A} R_u$ be an A-graded algebra.

- A nonzero element $a \in R^+ \setminus R^\times$ is called *h-irreducible* if the condition $a = bc$, $b, c \in R^+$, implies that either b or c is invertible.
- An A-graded algebra R is said to be *factorially graded* if any its nonzero noninvertible homogeneous element may be expressed as a product of h-irreducible elements, and such an expression is unique up to association and renumbering.

Remark 2.4. Assume that $R = \bigoplus_{u \in \mathbb{Z}^n} R_u$ has no zero-divisors. It follows from Lemma 2.1, (iii) that if R is factorial, then it is factorially graded.

Proposition 2.5. *The Cox ring $R(X) = \bigoplus_{D \in K} \mathcal{O}(X, D)$ is factorially graded.*

Proof. Effective Weil divisors on X are in one-to-one correspondence with classes of associated elements of $R(X)^+$, and the product of homogeneous elements corresponds to the sum of divisors. This shows that classes of h-irreducible elements of the ring $R(X)$ correspond to prime divisors. Since any effective Weil divisor is a unique nonnegative integral combination of prime divisors, the ring $R(X)$ is factorially graded. □

Below we shall need the following well-known lemma. For the convenience of the reader, we give a short proof; cf. [5, Proposition 17.1].

Lemma 2.6. *Let T be an algebraic torus and Z a normal algebraic variety with a regular T-action. Then any Weil divisor on Z is linearly equivalent to a T-invariant Weil divisor.*

Proof. By normality, the singular locus of Z has codimension ≥ 2, and one may assume that Z is smooth. Then any Weil divisor D on Z is Cartier. We may assume that D is effective. There exists a T-linearization of the corresponding line bundle, which defines the structure of a rational T-module on $\mathscr{O}(Z, D)$ [6, Sec. 2.4]. Any T-eigenvector in $\mathscr{O}(Z, D)$ represents a T-invariant divisor equivalent to D. $\square$

Again by normality, the passage from X to its smooth locus does not change the Cox ring $R(X)$. Further we shall assume that X is smooth. Following [2], let us introduce the "universal torsor" $\widehat{X} \to X$ over X. Consider the K-graded sheaf of $\mathscr{O}_X$-algebras

$$\mathscr{R}_X = \bigoplus_{D \in K} \mathscr{O}(D), \qquad \text{where} \quad \mathscr{O}(U, D) = \{f \in \mathbb{K}(X) \mid [\operatorname{div}(f) + D]|_U \geq 0\},$$

and the relative spectrum $\widehat{X} = \operatorname{Spec}_X(\mathscr{R}_X)$ of this sheaf over X. Clearly,

$$R(X) = H^0(X, \mathscr{R}_X) = \mathscr{O}(\widehat{X}).$$

The K-grading on $\mathscr{R}_X$ defines a regular action of the torus $T = \operatorname{Spec}(\mathbb{K}[K])$ on $\widehat{X}$, and the canonical affine morphism $p\colon \widehat{X} \to X$ given by the inclusion $\mathscr{O}_X \subset \mathscr{R}_X$ is T-invariant. Since all divisors on X are Cartier, p is a locally trivial fibration with T as a fiber. In particular, $\widehat{X}$ is smooth. Fix an open affine covering $\{U_i\}$ of X. Each divisor $D_i = X \setminus U_i$ corresponds to an element $f_i \in R(X)^+$. The section f_i of the sheaf $\mathscr{R}_X$ is invertible exactly over U_i. Therefore, $p^{-1}(U_i)$ coincides with

$$\widehat{X}_{f_i} = \{z \in \widehat{X} : f_i(z) \neq 0\}.$$

We get an open affine covering $\{\widehat{X}_{f_i}\}$ of $\widehat{X}$, where $f_i \in \mathscr{O}(\widehat{X})$. It is easy to deduce from this that $\widehat{X}$ is a quasi-affine variety, see [7, Chap. 2, Appendix, Lemma 8].

We are ready to finish the proof of Theorem 1.1. Assume that the ring $R(X)$ is not factorial. Then $R(X)$ contains a nonprincipal prime ideal of height one. Since $R(X) = \mathscr{O}(\widehat{X})$, it is a Krull ring, and its prime ideals of height one are in bijection with essential discrete valuations of the ring $R(X)$, see [8, Sec. 1.3]. On the other hand, prime divisors on a normal quasi-affine variety are in bijection with essential discrete valuations of the ring of regular functions. Consequently, there is a nonprincipal divisor on $\widehat{X}$. By Lemma 2.6, one may assume that there is a prime T-invariant nonprincipal divisor on $\widehat{X}$. The corresponding ideal $I \lhd R(X)$ contains a homogeneous, and even an h-irreducible element. Suppose that for some $a, b \in R(X)$ the product ab is divisible by p, but any homogeneous component of both a and b is not divisible by p. Since $R(X)$ is factorially graded, by considering the product of leading components of a and b, we come to a contradiction. Thus, the ideal $(p) \lhd R(X)$ is prime. The inclusion $(p) \subseteq I$ implies the equality $(p) = I$. This contradiction completes the proof of Theorem 1.1.

We finish this section with the following observation.

Proposition 2.7. *Let $R = \bigoplus_{u \in \mathbb{Z}^n} R_u$ be a multigraded finitely generated $\mathbb{K}$-algebra without zero divisors. Assume that R is factorially graded. Then R is factorial.*

Proof. Let us show that R is integrally closed. The multigrading defines actions of an n-dimensional torus T on the algebra R and its quotient field $\operatorname{Quot}(R)$ by automorphisms. It is known that the integral closure $\overline{R}$ of R in $\operatorname{Quot}(R)$ is a T-invariant subalgebra, and the T-action defines a structure of a rational T-module on $\overline{R}$. In particular, $\overline{R}$ is $\mathbb{Z}^n$-graded, and R is its homogeneous subalgebra. Any homogeneous element $r \in \overline{R}$ may be expressed as $r = r_1/r_2$, $r_1, r_2 \in R^+$. Since $R(X)$ is factorially graded and r_1/r_2 is integral over R, one has $r_1/r_2 \in R^+$ and $\overline{R} = R$.

It is well known that R is factorial if and only if $\operatorname{Cl}(Z) = 0$ for the normal affine variety $Z = \operatorname{Spec}(R)$. The arguments given above show that any prime T-invariant divisor on Z is principal, and Lemma 2.6 completes the proof. $\square$

Remark 2.8. As in the proof of Proposition 2.7, one may show that in characteristic zero a factorially graded finitely generated algebra $R = \bigoplus_{u \in A} R_u$ without zero divisors is integrally closed for any finitely generated Abelian group A.

3. TORSION IN THE DIVISOR CLASS GROUP

In this section, we define the Cox ring $R(X)$ for a variety X with arbitrary finitely generated divisor class group and check that $R(X)$ is well defined; cf. [2], [4].

Let $S \subset \mathrm{WDiv}(X)$ be a finitely generated subgroup that projects to $\mathrm{Cl}(X)$ surjectively. Consider a ring

$$T_S(X) = \bigoplus_{D \in S} \mathscr{O}(X, D).$$

Let $S^0 \subset S$ be the kernel of the projection $S \to \mathrm{Cl}(X)$. Take compatible bases $D_1, \dots, D_s$ in S and $D_1^0 = d_1 D_1, \dots, D_r^0 = d_r D_r$ in S^0, $r \le s$. We call a family of rational functions

$$\mathscr{F} = \{F_D \in \mathbb{K}(X)^\times : D \in S^0\}$$

coherent (the term "shifting family" was used in [2] for a similar notion) if $\mathrm{div}(F_D) = D$ and $F_{D+D'} = F_D F_{D'}$. Obviously, the family $\{F_D\}$ is defined by $F_{D_i^0}$, $i = 1, \dots, r$: if $D = a_1 D_1^0 + \dots + a_r D_r^0$, then $F_D = F_{D_1^0}^{a_1} \dots F_{D_r^0}^{a_r}$. Let us fix a coherent family $\mathscr{F}$.

Let $D_1, D_2 \in S$ and $D_1 - D_2 \in S^0$. A map $f \to F_{D_1 - D_2} f$ is an isomorphism between vector spaces $\mathscr{O}(X, D_1)$ and $\mathscr{O}(X, D_2)$. One easily checks, that the linear span of elements $f - F_{D_1 - D_2} f$ over all D_1, D_2 with $D_1 - D_2 \in S^0$ and all $f \in \mathscr{O}(X, D_1)$ is an ideal $I(S, \mathscr{F}) \lhd T_S(X)$. Define the Cox ring of the variety X as

$$R_{S,\mathscr{F}}(X) = T_S(X)/I(S, \mathscr{F}).$$

Since $D/D^0 \cong \mathrm{Cl}(X)$, the ring $R_{S,\mathscr{F}}(X)$ carries a natural $\mathrm{Cl}(X)$-grading.

Lemma 3.1. *Assume that $\mathscr{O}(X)^\times = \mathbb{K}^\times$. Then the ring $R_{S,\mathscr{F}}(X)$ does not depend on a choice of $\mathscr{F}$ up to isomorphism.*

Proof. The set of functions $F_{D_1^0}, \dots, F_{D_r^0}$ is defined up to transformations $F_{D_i^0} \to \gamma_i F_{D_i^0}$, $\gamma_i \in \mathbb{K}^\times$. Fix elements $\alpha_i \in \mathbb{K}^\times$, $i = 1, \dots, r$, such that $\alpha_i^{d_i} = \gamma_i$ and set $\alpha_{r+1} = \dots = \alpha_s = 1$. Then the desired isomorphism of the quotient rings is induced by an automorphism $T_S(X) \to T_S(X)$, which acts on the component $\mathscr{O}(X, D)$, $D = a_1 D_1 + \dots + a_s D_s$, via multiplication by $\alpha_1^{a_1} \dots \alpha_s^{a_s}$. □

Proposition 3.2. *Assume that $\mathscr{O}(X)^\times = \mathbb{K}^\times$. Then the ring $R_S(X)$ does not depend, up to isomorphism, on the choice of S.*

Proof. Let $M \subset \mathrm{WDiv}(X)$ be another finitely generated subgroup that projects to $\mathrm{Cl}(X)$ surjectively. One may assume that $\mathrm{rk}(S) \ge \mathrm{rk}(M)$.

For the rest of the proof, we need the following lemma.

Lemma 3.3. *There is a surjective homomorphism $\phi \colon S \to M$ such that the following diagram is commutative*:

$$\begin{array}{ccc} S & \xrightarrow{\ \phi\ } & M \\ & \searrow \quad \swarrow & \\ & \mathrm{Cl}(X) & \end{array}$$

Proof. Take compatible bases $D_1, \dots, D_s$ in S and $d_1 D_1, \dots, d_r D_r$ in S^0, and also $M_1, \dots M_k$ in M and $m_1 M_1, \dots, m_p M_p$ in M^0 such that d_i (respectively, m_i) is divisible by d_{i+1} (respectively, m_{i+1}). The condition

$$S/S^0 \cong \mathrm{Cl}(X) \cong M/M^0$$

implies that $s - r = k - p$ and the final sequences $(d_1, \dots, d_r)$ and $(m_1, \dots, m_p)$ may differ only in the group of ones at the end of these sequences. It remains to put $\phi(D_i) = M_i$ for $i = 1, \dots, p$, $\phi(D_i) = M_{i-r+p}$ for $i = r+1, \dots, s$, and $\phi(D_i) = 0$ for the other i. □

We return to the proof of Proposition 3.2. Fix a coherent family $\mathscr{F}$ for the subgroup

$$(S + M)^0 \subset S + M$$

and define a surjective homomorphism $\Phi\colon T_S(X) \to T_M(X)$ that sends a homogeneous component $\mathscr{O}(X, D) \to \mathscr{O}(X, \phi(D))$ as $\Phi(f) = F_{D-\phi(D)} f$. The kernel of Φ is contained in the ideal $I(S, \mathscr{F}|_{S^0})$, and the ideal $I(S, \mathscr{F}|_{S^0})$ itself maps surjectively onto $I(M, \mathscr{F}|_{M^0})$. This shows that Φ defines a homomorphism $R_S(X) \to R_M(X)$ that is, in fact, an isomorphism. □

Note that homogeneous elements of the ring $T_S(X)$ from different components corresponding to D_1 and D_2 with $D_1 - D_2 \in S^0$ may define the same effective divisor on X. However, after factorization by the ideal $I(S, \mathscr{F})$ effective Weil divisors on X are again in bijection with association classes in $R(X)^+$.

Proposition 3.4. *The ring $R(X)$ is factorially graded.*

Proof. The statement follows from the proof of Proposition 2.5. □

Remark 3.5. By [3, Corollary 1.2], the ring $T_S(X)$ is factorial. As we shall see in the next section, this property may be lost for the ring $R(X)$ if the group $\mathrm{Cl}(X)$ has torsion.

4. HOMOGENEOUS SPACES OF AN ALGEBRAIC GROUP

In this section, we assume that the ground field $\mathbb{K}$ has characteristic zero. Let G be a connected affine algebraic group with $\mathrm{Cl}(G) = 0$ and without nontrivial characters. Note that the first condition may be achieved by passing to a finite covering of the given group G [6, Proposition 4.6]. Denote by $\mathbb{X}(G)$ the group of characters of an algebraic group G. By Rosenlicht's Theorem, the condition $\mathbb{X}(G) = 0$ is equivalent to $\mathscr{O}(G)^\times = \mathbb{K}^\times$.

Let H be a closed subgroup of G. The homogeneous space G/H admits the canonical structure of a smooth quasi-projective algebraic G-variety. In [9], it was proved that

$$\mathrm{Cl}(G/H) \cong \mathrm{Pic}(G/H) \cong \mathbb{X}(H).$$

Let us recall how to establish the last isomorphism. Any character $\chi \in \mathbb{X}(H)$ defines a one-dimensional H-module $\mathbb{K}_\chi$. Consider the homogeneous fiber bundle

$$L_\chi = G \times_H \mathbb{K}_\chi := (G \times \mathbb{K}_\chi)/H, \qquad h \cdot (g, a) := (gh^{-1}, \chi(h)a).$$

It is shown in [9] that the projection $L_\chi \to G/H$ is a G-linearized line bundle over G/H, and

$$L_{\chi_1} \otimes L_{\chi_2} \cong L_{\chi_1 + \chi_2}.$$

Moreover, the map $\chi \to L_\chi$ defines an isomorphism between $\mathbb{X}(H)$ and $\mathrm{Pic}(G/H)$.

Since $\mathrm{Cl}(G) = 0$, the pull-back of the line bundle L_χ with respect to the projection $G \to G/H$ is a trivial G-linearized line bundle on G. This allows to identify the space of sections $H^0(X, L_\chi)$ with the following subspace of $\mathscr{O}(G)$:

$$\mathscr{O}(G)^{(H)}_\chi := \{f \in \mathscr{O}(G) : f(gh^{-1}) = \chi(h) f(g) \text{ for all } h \in H,\ g \in G\}.$$

The tensor product of sections corresponds to the product in $\mathscr{O}(G)$.

Consider the subgroup

$$H_1 = \bigcap_{\chi \in \mathbb{X}(H)} \operatorname{Ker}(\chi).$$

The next theorem gives an effective description of the Cox ring of the homogeneous space G/H; see also [10, Lemma 3.14].

Theorem 4.1. *Let G be a connected affine algebraic group with $\mathbb{X}(G) = 0$ and $\operatorname{Cl}(X) = 0$, and let H be a closed subgroup of G. Then*

$$R(G/H) \cong \mathscr{O}(G/H_1).$$

Proof. The diagonalizable group $Q = H/H_1$ acts on G/H by right multiplication. Since $\mathbb{X}(H) = \mathbb{X}(Q)$ and any rational Q-module is a direct sum of one-dimensional submodules, we obtain

$$\mathscr{O}(G/H_1) = \bigoplus_{\chi \in \mathbb{X}(H)} \mathscr{O}(G)^{(H)}_{\chi}.$$

Weil divisors on G/H are in bijection with lines generated by the Q-semiinvariants

$$f \in \operatorname{Quot}(\mathscr{O}(G/H_1)) \subseteq \mathbb{K}(G).$$

Further, effective divisors correspond to semiinvariants $f \in \mathscr{O}(G/H_1)$. Let us choose a multiplicative finitely generated group of semiinvariants in $\operatorname{Quot}(\mathscr{O}(G/H_1))$ whose weights run through the whole group $\mathbb{X}(Q)$. Let us identify this subgroup with $S \subset \operatorname{WDiv}(G/H)$; then we can easily check that the Cox ring associated with S is isomorphic to $\mathscr{O}(G/H_1)$. □

For a connected H, the group Q is a torus and $\mathbb{X}(H)$ is free. Moreover, in this case $\mathbb{X}(H_1) = 0$; thus, $\operatorname{Cl}(G/H_1) = 0$ and the ring $R(G/H)$ is factorial (this also follows from Theorem 1.1). For a disconnected H, the character group $\mathbb{X}(H_1)$ may be nontrivial.

Example 4.2. Let $G = \mathrm{SL}(2)$ and H be the normalizer N of a maximal torus $T \subset \mathrm{SL}(2)$. Here $\mathbb{X}(H)$ is isomorphic to the cyclic group $\mathbb{Z}_2$ of order 2, $H_1 = T$ and $\mathbb{X}(H_1) \cong \mathbb{Z}$. This shows that the Cox ring

$$R(\mathrm{SL}(2)/N) \cong \mathscr{O}(\mathrm{SL}(2)/T)$$

is not factorial. The space $\mathrm{SL}(2)/N$ is a smooth affine surface X with $\operatorname{Cl}(X) \cong \mathbb{Z}_2$, the ring $R(X)$ is isomorphic to $\mathbb{K}[x_1, x_2, x_3]/(x_2^2 - x_1x_3 - 1)$, and the $\mathbb{Z}_2$-grading on $R(X)$ is given by

$$\deg(x_1) = \deg(x_2) = \deg(x_3) = \overline{1}.$$

There are several ways to associate a factorial ring with the surface X. First, consider the subgroup $S \subset \operatorname{WDiv}(X)$ generated by a nonprincipal divisor. The ring $T_S(X)$ is isomorphic to

$$(\mathbb{K}[x_1, x_2, x_3, t, t^{-1}]/(x_2^2 - x_1x_3 - 1))^{\mathbb{Z}_2},$$

where $\mathbb{Z}_2$ acts on the variables x_1, x_2, x_3, t, and t^{-1} via multiplication by -1.

Second, the space $\mathrm{SL}(2)/N$ admits a wonderful (the term is due to D. Luna) $\mathrm{SL}(2)$-equivariant embedding in $\mathbb{P}^2$, and

$$R(\mathbb{P}^2) \cong \mathbb{K}[x_1, x_2, x_3].$$

The Cox ring of a wonderful embedding of any spherical homogeneous space is described in [11].

Example 4.3. Using the construction of Example 4.2, one may find a smooth affine variety X with $\operatorname{Cl}(X) \cong A$ and a nonfactorial Cox ring for any nonfree finitely generated Abelian group A. Indeed, let

$$A \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_s} \oplus \mathbb{Z}^n.$$

Put $X = G/H$, where G is a direct product of $s + n$ simple groups,

$$\mathrm{SL}(d_1) \times \cdots \times \mathrm{SL}(d_s) \times \mathrm{SL}(2) \times \cdots \times \mathrm{SL}(2)$$

and

$$H = H(1) \times \cdots \times H(s) \times T \times \cdots \times T,$$

where $H(i)$ is an extension of a maximal torus of the group $\mathrm{SL}(d_i)$ by elements of its normalizer that act as degrees of one d_i-cycle. In this case, $\mathbb{X}(H) \cong A$ and

$$\mathbb{X}(H_1) \cong \mathbb{Z}^{d_1+\cdots+d_s-s}.$$

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