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On SL_2 -actions of complexity one

I. V. Arzhantsev

Abstract. This paper is devoted to the actions of the group SL_2 with two-dimensional generic orbit on normal three-dimensional affine algebraic varieties. We obtain a complete classification of such actions whenever the stabilizer of general position is the subgroup of monomial matrices. We also prove some results for arbitrary actions of complexity one of reductive groups.

§ 1. Introduction

In this paper we study actions of the group $\mathrm{SL}_2(\mathbb{K}) = \mathrm{SL}_2$ on *normal* irreducible affine varieties. The ground field $\mathbb{K}$ is assumed to be algebraically closed and $\mathrm{char} \mathbb{K} = 0$.

The *complexity* $c(X, G)$ of an action $G : X$ of a reductive group G on an algebraic variety X is the codimension of a generic orbit for the induced action of a Borel subgroup $B : X$ or, by the Rosenlicht theorem,

$$c(X, G) = \mathrm{tr. deg} \mathbb{K}(X)^B.$$

Detailed information on this concept can be found in [8].

Actions of complexity zero are called *spherical*.

We consider some general properties of complexity of reductive group actions in § 2. It is proved there that the complexity of orbits is constant on the sheets and is an upper semicontinuous function on the variety.

Then we pass to the case $G = \mathrm{SL}_2$. For an affine variety X , we have $c(X, \mathrm{SL}_2) = \dim X - 2$ if the action is non-trivial.

Spherical actions for the group SL_2 can be easily described.

Lemma 1 ([2], § 4.1, Lemma 2). *Every algebraic subgroup of SL_2 is conjugate to one of the following:*

- 1) *a finite subgroup;*
- 2) *a Borel subgroup B ;*
- 3) *a maximal torus T ;*
- 4) *the normalizer N of a maximal torus;*

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5) a finite extension of a maximal unipotent subgroup

$$U_n = \left\{ \begin{pmatrix} \varepsilon & a \\ 0 & \varepsilon^{-1} \end{pmatrix} \middle| \begin{matrix} a, \varepsilon \in \mathbb{K} \\ \varepsilon^n = 1 \end{matrix} \right\}, \quad n = 1, 2, 3, \dots$$

It is known that the Levi subgroups of point stabilizers are conjugate on an open subset of X (see [4]). Hence only the unipotent radical of a generic point stabilizer can be deformed. It follows from Lemma 1 that for any subgroup of SL_2 the unipotent radical is either trivial or one-dimensional, and it cannot be deformed. This gives the following result.

Lemma 2. *For every action of SL_2 on X the stabilizer of general position (SGP) exists.*

The next lemma gives us important geometric information about the orbits of the actions under consideration.

Lemma 3 ([4], p. 224). *Let a reductive group G act on an affine variety X and the stabilizer of a point $x \in X$ contain a maximal torus of G . Then the orbit Gx is closed. In particular, for $G = \mathrm{SL}_2$ the orbits of type SL_2/T or SL_2/N are closed.*

So it is clear that if the action $\mathrm{SL}_2 : X$ is spherical, then X is either a homogeneous space isomorphic to SL_2/T or SL_2/N or the closure O_n of the highest vector orbit in the irreducible module V_n of SL_2 (see [7]).

Remark 1. We denote by V_n the $(n+1)$ -dimensional irreducible SL_2 -module defined uniquely up to isomorphism.

Further, it is natural to consider *actions of complexity 1* or, equivalently, SL_2 -actions on three-dimensional varieties. If the SGP for such an action is a finite group, then X contains a dense open orbit of SL_2 . These actions are completely classified in [1]. It is shown in [1] that a locally transitive SL_2 -action on a three-dimensional normal affine variety either is transitive or is specified by a pair of numbers from $\mathbb{N} \times (0, \frac{1}{2}]_{\mathbb{Q}}$. The first of these determines the order of a stabilizer of general position, which is cyclic in this case. The second is called the *height* and characterizes the algebra of U -invariants.

We consider below the cases where the SGP is equal to U_n , N and T respectively. In the last case we study only the actions on smooth varieties. Such SL_2 -varieties of complexity one will be called (S, U) -, (S, N) - and (S, T) -varieties respectively.

Remark 2. The case $\mathrm{SGP} = B$ is impossible. Indeed, $\mathrm{SL}_2/B \cong \mathbb{P}^1$, but the variety X is supposed to be affine.

By an isomorphism of two objects with given SL_2 -action we shall mean an SL_2 -equivariant isomorphism.

The aim of this paper is to show that the study of SL_2 -actions of complexity one with fixed SGP equal to H can be reduced to the study of actions of the group $N_{\mathrm{SL}_2}(H)$ on a lower-dimensional variety Y . For the reconstruction of the initial action of SL_2 it is necessary to fix an additional set of data on Y .

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§ 2. General statement of the problem

In this section the variety X is not supposed to be normal.

Proposition 1. *Let G be a reductive algebraic group and X a quasiprojective G -variety. Then the complexity of orbits (as homogeneous spaces of G) is constant along the sheets and is an upper semicontinuous function on the variety X .*

Remark 1. A *sheet* for an action of a group G on a variety X is an irreducible component of the locally closed subset $\{x \in X \mid \dim Gx = d\}$ for some fixed d (see [4]).

Proof. It is sufficient to prove that if the variety X is irreducible, then orbits of maximal dimension have the same complexity and the complexity can not increase for orbits of lower dimension. After that we can use induction on the dimension.

We claim that on any G -variety there is an open subset such that all orbits from this subset have the same complexity. Indeed, we can consider the set $G(U_G \cap U_B)$, where U_G (respectively U_B) consists of points whose G -orbits (respectively B -orbits) have the maximal dimension. Thus the complexity of an orbit in general position is defined, $\mathrm{COGP}(X, G) = c$. It is obvious that the complexity of any other G -orbit of maximal dimension can only be $\geq c$.

If the inequality is strict for some orbit O (of any dimension), then there are algebraically independent B -invariant rational functions $f_1, \dots, f_{(c+1)}$ on this orbit. Let us use the following lemma.

Lemma (Knop [9], § 2). *Let X be a quasiprojective G -variety. Then any rational B -semi-invariant function on a G -invariant subvariety can be extended to a B -semi-invariant function with the same weight on X .*

Hence the functions $f_1, \dots, f_{(c+1)}$ can be extended to B -invariant rational functions $F_1, \dots, F_{(c+1)}$ on the variety X . Let $\{\xi_j\}$ be a basis of the Lie algebra of the group G . The functions $f_1, \dots, f_{(c+1)}$ are algebraically independent, and the rank of the matrix $\left(\frac{\partial f_i}{\partial \xi_j}\right)$ equals $(c+1)$ along the orbit O . So the rank of the matrix $\left(\frac{\partial F_i}{\partial \xi_j}\right)$ equals $(c+1)$ in at least one point on the variety X and the functions $F_1, \dots, F_{(c+1)}$ are algebraically independent after restriction to an orbit of general position. This contradicts the assumption $\mathrm{COGP}(X, G) = c$.

Remark 2. This proposition was proved for the adjoint representation of a simple group in [11].

Remark 3. For different sheets of orbits of the same dimension, the complexity of orbits can be different. For example, consider the representation of the group $\mathrm{SL}_2 \times \mathrm{SL}_2$ in the module $V \oplus W$, where V is the space of 2×2 -matrices with the action $(g_1, g_2)m = g_1^{-1}mg_2$, and W is the representation of the first component SL_2 in the binary forms of degree 3 with the trivial action of the second component.

Then the orbit of the vector $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0\right)$ is spherical but the orbit of the vector $\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, x^2y\right)$ has complexity one.

Corollary 1. *If there is an open subset on X where all orbits are spherical, then all orbits on the variety X are spherical.*

Remark 4. Every action with spherical generic orbit has a rational section. In fact, it is known (see [4]) that for any action of a connected soluble group there exists a rational section. But in our situation, a rational section for a Borel subgroup B is also a G -section.

Corollary 2. *For every G -variety X the set of points with spherical G -orbits is a closed subvariety.*

For the adjoint representation of a simple group this subvariety is described in [11].

Let us consider actions of complexity one. We have two different possibilities: there is either an open G -orbit of complexity one on X or one-parameter family of spherical G -orbits of maximal dimension (for details see [8]).

The first possibility for affine homogeneous spaces was studied in [12]. In the following sections we consider the second possibility for the group SL_2 .

In the general situation, we have the following result.

Proposition 2. *Suppose that X is an affine variety with an action of a reductive group G , the complexity equals one and the quotient $X//G$ is one-dimensional. Then all irreducible components of the quotient morphism ϕ are equidimensional spherical varieties.*

Proof. Equidimensionality of the quotient morphism follows directly from a theorem on fibres of morphisms (see [5]). Let Y be a non-spherical component of some fibre of the morphism ϕ , and let $t \in \mathbb{K}[X]^G$ be a non-constant function. There is a function $w \in \mathbb{K}(Y)^B$ on Y which separates generic B -orbits. It follows by Knop's lemma that w can be extended to a function $W \in \mathbb{K}(X)^B = \mathbb{K}(X)^G$. Here we have $\mathrm{tr. deg.} \mathbb{K}(X)^B = 1$, the function t is constant on Y and there is an algebraic relation between t and W . So the function W is constant on Y too. We get a contradiction with the condition $W|_Y = w$. This completes the proof.

§ 3. (S, U) -varieties

We obtain the results of this section using general theory. Let a connected reductive group G act on an irreducible normal affine variety X , and let the stabilizer of a generic point contain a maximal unipotent subgroup $U \subset G$. It is known (see [3]) that

$$\mathbb{K}[X] \cong (\mathbb{K}[X]^U \otimes \mathbb{K}[G/U^-])^T,$$

where T is a maximal torus of the group G and U^- is the opposite unipotent subgroup with respect to U . The action of G on the algebra on the right-hand side is given by left multiplication on the second factor of the tensor product. The torus T acts naturally on $\mathbb{K}[X]^U$ and by right multiplication on $\mathbb{K}[G/U^-]$. Hence the G -variety X can be uniquely reconstructed if we have the T -variety $Y = X//U = \mathrm{Spec} \mathbb{K}[X]^U$. In this case, every weight of the T -module $\mathbb{K}[Y] = \mathbb{K}[X]^U$ is dominant.

Conversely, let Y be a variety with a non-trivial action of the torus T , and let all weights of T -module $\mathbb{K}[Y]$ be dominant. Then we can construct a G -variety X . Namely, $X = \mathrm{Spec}(\mathbb{K}[Y] \otimes \mathbb{K}[G/U^-])^T$. Using the Luna-Vust criterion [2], we claim that the normality of X is equivalent to the normality of $X//U$.

In the case $G = \mathrm{SL}_2$, the homogeneous space SL_2/U^- can be identified with $\mathbb{K}^2 \setminus \{0\}$ with the natural action of a one-dimensional torus. Then it follows that $\mathrm{Spec} \mathbb{K}[\mathrm{SL}_2/U^-] = \mathbb{K}^2$. We have proved the following theorem.

Theorem 1. *(S, U) -varieties correspond bijectively to normal affine surfaces Y with fixed non-trivial $\mathbb{Z}_+$ -grading on the algebra of regular functions $\mathbb{K}[Y]$. This grading defines the action of a one-dimensional torus T on Y . The corresponding (S, U) -variety is isomorphic to the variety*

$$X = (Y \times \mathbb{K}^2)//T.$$

In particular, $\mathbb{K}[X]^{\mathrm{SL}_2} \cong \mathbb{K}[Y]^T$.

Remark 1. If a generic orbit on X is isomorphic to SL_2/U_n , then any other orbit is either a fixed point or is isomorphic to SL_2/U_m , where n divides m .

Remark 2. One of the results of [10] is the following proposition. If X is unirational and the algebra $\mathbb{K}[X]$ is factorial and has no invertible non-constant elements, then:

- a) in the case $\mathbb{K}[X]^{\mathrm{SL}_2} = \mathbb{K}$, the surface Y is a complete intersection;
- b) in the case $\mathbb{K}[X]^{\mathrm{SL}_2} \neq \mathbb{K}$, the surface Y is the affine plane $\mathbb{A}^2$.

Remark 3. The smoothness of the surface Y does not guarantee the smoothness of the variety X . Using the etale slice theory (see some details in § 5), it is possible to show that smooth (S, U) -varieties correspond to surfaces that are locally trivial line bundles (in the Zariski topology) over some smooth affine curve. The torus acts naturally on the fibres: $(t, x) \rightarrow tx$, $t \in \mathbb{K}^*$, $x \in \mathbb{A}^1$. Therefore every smooth (S, U) -variety is a locally trivial vector bundle over a smooth affine curve with fibre V_1 .

For the (S, N) - and (S, T) -varieties considered below, the SL_2 -action is stable, and so we obtain the following result.

Proposition 3. *Every action of the group SL_2 on a smooth three-dimensional affine variety either is locally transitive or possess a non-constant invariant regular function.*

§ 4. (S, N) -varieties

Here and in the next section, we suppose that the ground field is the field $\mathbb{C}$ of complex numbers.

4.1. General properties of (S, N) -varieties. There is a one-parameter family of orbits of type SL_2/N on X . We already know that all such orbits are closed. The decomposition of the SL_2 -module $\mathbb{K}[X]$ into a direct sum of irreducible submodules can contain only irreducible SL_2 -modules that have a non-zero vector stabilized by N . These are exactly the modules V_{4n} . This observation shows that there is no orbit of type SL_2/T on X , and any closed orbit is either SL_2/N or a fixed point.

Lemma 4. *The set of N -fixed points X^N is a smooth irreducible affine curve isomorphic to $X//\mathrm{SL}_2$. The set of SL_2 -fixed points X^{SL_2} is a finite set of points on this curve.*

Proof. It is known (see [4]) that the quotient morphism separates closed orbits on X . Hence every fibre of the quotient morphism either is an orbit of type SL_2/N or contains exactly one fixed point. There is only one N -fixed point on an orbit of type SL_2/N . If we restrict the quotient morphism $\phi: X \rightarrow X//\mathrm{SL}_2$ to the subvariety X^N , then we get a bijective mapping. The normality of X implies the normality of $X//\mathrm{SL}_2$. For a curve, normality is equivalent to smoothness. Using the fact that a bijective morphism on a normal variety is an isomorphism, we get $X^N \cong X//\mathrm{SL}_2$. The set X^{SL_2} is a proper subset of the curve X^N .

Lemma 5. *Every fibre of the quotient morphism $\phi: X \rightarrow X//\mathrm{SL}_2$ is two-dimensional and irreducible.*

Proof. The fact that every fibre is two-dimensional follows immediately from a general theorem on fibres of morphisms. The irreducibility was first proved by me using an embedding of the variety X in some finite-dimensional SL_2 -module. Then Vinberg proposed the following invariant proof. Suppose that the fibre $\phi^{-1}(x)$ is reducible and $\phi^{-1}(x) = Y_1 \cup Y_2$, where Y_1 and Y_2 are closed subsets. We have

$$\mathbb{K}[Y_1] \cong \bigoplus_{i \in I} V_i, \quad \mathbb{K}[Y_2] \cong \bigoplus_{j \in J} V_j,$$

where I and J are subsemigroups in the semigroup $\mathbb{Z}_+$. If $i \in I$, $j \in J$, then the submodule isomorphic to V_{ij} appears in $\mathbb{K}[Y_1]$ and in $\mathbb{K}[Y_2]$. The components Y_1 and Y_2 have exactly one common point (the group SL_2 is connected, irreducible components of the fibre are SL_2 -invariant and the common point is the unique closed orbit in the fibre). So the G -modules $\mathbb{K}[\phi^{-1}(x)]$ and $\mathbb{K}[Y_1] \oplus \mathbb{K}[Y_2]$ differ only by one one-dimensional submodule. Then the submodule V_{ij} appears in $\mathbb{K}[\phi^{-1}(x)]$ with multiplicity > 1 .

On the other hand, a generic fibre of the quotient morphism is SL_2/N and the algebra of regular functions on this fibre is multiplicity-free because this variety is spherical (see [2]). Consider two functions f_λ^1 and f_λ^2 from $\mathbb{K}[X]$ that are the highest vectors with weight λ in the SL_2 -module $\mathbb{K}[X]$ with respect to the Borel subgroup B . If we restrict these functions to a generic fibre and use the multiplicity-free condition, we get that the rational function f_λ^1/f_λ^2 is a constant (possibly equal to ∞). So it determines a rational function on the quotient. A rational function on a smooth curve has a definite (possibly infinite) value at each point. As a result we see that the functions f_λ^1 and f_λ^2 are proportional under restriction to any fibre. Then the algebra of regular functions on any fibre of the quotient morphism is multiplicity-free. This contradiction concludes the proof.

Remark 1. In the proof of Lemma 5, we made essential use of the fact that the curve $C = X//\mathrm{SL}_2$ is normal. In fact, if we consider the curve

$$C = \{(u, v) \mid u^2 = v^2 + v^3\} \hookrightarrow V^N = \mathbb{K}^2 \hookrightarrow V_2 \oplus V_2 = V,$$

then the variety $\overline{\mathrm{SL}_2(C)}$ contains a unique fixed point, and this point belongs to the closures of two two-dimensional orbits $\mathrm{SL}_2(x^2, x^2)$ and $\mathrm{SL}_2(x^2, -x^2)$.

Remark 2. Lemma 5 enables us to determine a one-to-one correspondence between the set of fixed points X^{SL_2} and the set of non-closed orbits on X : every fibre of the quotient morphism either is an orbit of type SL_2/N or contains one fixed point and one non-closed orbit.

We shall later denote the curve $X//\mathrm{SL}_2$ by C .

4.2. Simple (S, N) -varieties.

Definition. We say that a normal (S, N) -variety X is *simple* if $X//\mathrm{SL}_2 = \mathbb{A}^1$ and $X^{\mathrm{SL}_2} = \{0\}$.

We denote a generator of the algebra of invariants for a simple (S, N) -variety X by z .

Example 1. a) Let us consider the action of the group $\mathbb{Z}_{2k}$ on the space V_2 by multiplication by $2k$ th roots of unity. We put $X^k \cong V_2//\mathbb{Z}_{2k}$. There arises an action $\mathrm{SL}_2 : X^k$ whose generic orbit is SL_2/N , the quotient $X^k//\mathrm{SL}_2$ is $\mathbb{A}^1$, $X^{\mathrm{SL}_2} = \{0\}$ and the non-closed orbit is $\mathrm{SL}_2//U_{4k}$. The variety X^k is normal, being the quotient of a normal variety.

b) Let $\phi : X^k \rightarrow \mathbb{A}^1$ be the quotient morphism. Consider the pullback corresponding to the morphism $z \rightarrow z^n$:

$$\begin{array}{ccc} X_n^k & \longrightarrow & X^k \\ \downarrow & & \downarrow \varphi \\ \mathbb{A}^1 & \xrightarrow{z \rightarrow z^n} & \mathbb{A}^1. \end{array}$$

So we get new SL_2 -varieties X_n^k . A point in the variety X_n^k can be realized as a pair (z, ω) , where ω is a binary quadratic form, which is defined up to multiplication by ϵ , $\epsilon^{2k} = 1$, and $z \in \mathbb{K}$, $z^n = (\det \omega)^k$.

Lemma 6. *The variety X_n^k is normal if and only if either $n = 1$ or $k = 1$.*

Proof. If $n = 1$, we get exactly the varieties of Example 1a), and they are normal.

By a, b, c we denote the standard coordinates in the three-dimensional space of binary forms of degree two, $V_2 = \{ax^2 + bxy + cy^2\}$.

The variety X_n^1 is the quotient of the hypersurface $z^n = b^2 - 4ac$ by the action of the group $\mathbb{Z}_2$. This hypersurface is singular only at the origin. It is well known that for a hypersurface normality is equivalent to smoothness in codimension one. So the variety X_n^1 is normal.

Let us assume that $n > 1$ and $k > 1$. The algebra of regular functions on X_n^k is generated by the monomials z and $a^i b^j c^l$ ($i + j + l = 2k$) with the relation $z^n = (b^2 - 4ac)^k$. It is easy to see that

$$\left(\frac{(b^2 - 4ac)^l b^{2k-2l}}{z} \right)^n = (b^2 - 4ac)^{nl-k} b^{2kn-2ln},$$

where $l = \lfloor \frac{k}{n} \rfloor + 1$. This relation shows that the algebra $\mathbb{K}[X_n^k]$ is not integrally closed.

Lemma 7. *Every simple (S, N) -variety is SL_2 -equivariantly homeomorphic in the $\mathbb{C}$ -topology to one of the varieties X_n^k .*

Proof. Consider an equivariant closed embedding of X in some finite-dimensional SL_2 -module:

$$\psi: X \hookrightarrow V = V_{4k_1} \oplus \cdots \oplus V_{4k_p}, \quad k_i \geq 0.$$

We have the embedding

$$\psi': X^N \cong \mathbb{A}^1 \hookrightarrow V^N = \langle x^{2k_1} y^{2k_1} \rangle \oplus \cdots \oplus \langle x^{2k_p} y^{2k_p} \rangle.$$

(Here we identify irreducible SL_2 -modules with the corresponding spaces of binary forms.)

We can assume that $\{0\} \in \mathbb{A}^1$ corresponds to the origin of the space V^N . Denote the coordinates in this space by $(\chi_1, \dots, \chi_p)$. Then the embedding ψ' is given by

$$\chi_i = a_i z^{n_i} + b_i z^{n_i+1} + \cdots, \quad a_i \neq 0 \quad (1)$$

(the dots stand for the terms of higher degree in z).

The embedding ψ can be defined on the open subset $\mathrm{SL}_2(X^N \setminus \{0\}) \subset X$ by the formula

$$gv \rightarrow ((a_i z^{n_i} + b_i z^{n_i+1} + \cdots) \omega^{2k_i})_{i=1}^p, \quad g \in \mathrm{SL}_2, \quad v \in X^N,$$

where $\omega = g(xy)$ and z is the coordinate of v on the curve $X^N \cong \mathbb{A}^1$.

There is only one non-closed orbit on a simple (S, N) -variety (by definition, we have only one fixed point), and this orbit is isomorphic to SL_2/U_{4k} for some k . The existence of this orbit means that for some sequence $\{\omega_j\}$ of binary quadratic forms with determinant 1 and for some sequence $\{z_j\}$, $z_j \rightarrow 0$, there exists a finite limit

$$\lim_{j \rightarrow \infty} a_i \left(z_j^{\frac{n_i}{2k_i}} \psi_j \right)^{2k_i} \quad \text{for any } i \text{ (the sequences } \{\psi_j\} \text{ and } \{z_j\} \text{ do not depend on } i).$$

This limit is non-zero for some i . This implies that the points of the non-closed orbit have non-zero coordinates only in those modules V_{4k_i} where the numbers $\tau_i = \frac{n_i}{k_i}$ are minimal. We may suppose that this minimum is achieved for $i = 1, \dots, l$. Then k divides the numbers $k_1, \dots, k_l$, and $(\frac{k_1}{k}, \dots, \frac{k_l}{k}) = 1$. Taking into account the equality $\frac{n_1}{(k_1/k)} = \cdots = \frac{n_l}{(k_l/k)}$, we conclude that the number $n = \frac{n_1 k}{k_1} = \cdots = \frac{n_l k}{k_l}$ is an integer. Define a rational mapping $X_n^k \rightarrow X$ by

$$(z, \omega) \rightarrow \left\{ (a_i + b_i z + \cdots) z^{n_i} \frac{\omega^{2k_i}}{(\det \omega)^{k_i}}, \quad i = 1, \dots, p \right\}.$$

This mapping defines an isomorphism on the open subset of orbits of type SL_2/N , since by construction the line $(X_n^k)^N$ is mapped isomorphically onto X^N . We have $\frac{n}{k} = \frac{n_i}{k_i}$ for $i = 1, \dots, l$ and $\frac{n}{k} > \frac{n_i}{k_i}$ for $i > l$, and therefore this rational mapping can be continuously extended to the points (z, ω) with $\det(\omega) = 0$ because $\frac{z^{n_i}}{(\det \omega)^{k_i}}$ tends to 1 for $i = 1, \dots, l$ and to 0 for $i > l$. Moreover, the condition $(\frac{k_1}{k}, \dots, \frac{k_l}{k}) = 1$ implies that the stabilizer of the image of a point from the

non-closed orbit under this mapping is U_{4k} . So the mapping is bijective on the non-closed orbit.

We have constructed a birational morphism $X_n^k \rightarrow X$, and this morphism can be continuously extended to a bijective mapping. Let us check the continuity of the inverse mapping. Proposition 5 below claims that all fibres of the quotient morphism $\phi: X \rightarrow X//\mathrm{SL}_2$ are normal. Hence the “null-fibre” is isomorphic to O_{4k} (see §1). The weight semigroup of the T -module $\mathbb{K}[O_{4k}]^U$ is $4k\mathbb{Z}_+$. On the other hand, the weight semigroup of the T -module $\mathbb{K}[\phi^{-1}(0)]^U$ is $4k_1\mathbb{Z}_+ \cup \dots \cup 4k_l\mathbb{Z}_+$. We get that one number from the set $k_1, \dots, k_l$ is equal to k . Let $k_1 = k$.

The image of X in every submodule V_{4k_i} lies in the set of binary forms of degree $4k_i$ that are $2k_i$ -powers of quadratic forms. For any $x \in X$, the quadratic form $\omega_i(x) = (\psi_i(x))^{\frac{1}{2k_i}}$ is defined up to multiplication by ϵ , $\epsilon^{2k_i} = 1$.

Let us define a mapping $X \rightarrow X_n^k$ by the formula

$$x \rightarrow \left(z(x), \frac{\omega_1(x)}{(a_1 + b_1 z(x) + \dots)^{\frac{1}{2k_1}}} \right).$$

This mapping is well defined for sufficiently small values of $z(x)$, and it extends the isomorphism of the open subsets $\mathrm{SL}_2 X^N$ and $\mathrm{SL}_2(X_n^k)^N$ to a continuous bijection in a neighbourhood of the “null-fibre”. Thus the desired homeomorphism is constructed.

In order to return from topological statements to algebraic ones, we need the following lemma.

Lemma 8. a) *Let X and Y be birationally isomorphic normal affine varieties. Suppose that there is an isomorphism of their open subsets U_X and U_Y which can be extended to a $\mathbb{C}$ -homeomorphism of X and Y . Then this $\mathbb{C}$ -homeomorphism is an isomorphism of algebraic varieties.*

b) *Under the conditions of a), if the variety X is not normal, then Y is the normalization of X and the normalization morphism is bijective.*

Proof. a) Consider the affine variety $X \times Y$ and the diagonal $U_{X \times Y}$ defined by the given isomorphism $U_X \cong U_Y$. Put $\Gamma = \overline{U_{X \times Y}}$. The closure of a quasi-affine variety in the Zarisky topology coincides with its closure in the $\mathbb{C}$ -topology. Hence the projection $\Gamma \rightarrow X$ is a homeomorphism. This is a bijective morphism and the normality of X implies the isomorphism $X \cong \Gamma$. Analogously, we get $Y \cong \Gamma$.

The statement b) follows from the proof of a).

We now study the normalization $\mathrm{Norm} X_n^k$ of the variety X_n^k for $n > 1$ and $k > 1$. Let $d = (k, n)$. Then there is a natural embedding $\mathbb{K}[X_n^k] \hookrightarrow \mathbb{K}\left[X_{\left(\frac{k}{d}\right)}^{\frac{k}{d}}\right]$, which is defined tautologically: $z \rightarrow z'$, $a^i b^j c^l \rightarrow (a')^i (b')^j (c')^l$. The algebra $\mathbb{K}\left[X_{\left(\frac{k}{d}\right)}^{\frac{k}{d}}\right]$ is integral over the subalgebra $\mathbb{K}[X_n^k]$, and the quotient fields of these algebras coincide because

$$\left((b')^{\frac{2k}{d}}\right)^d \in \mathbb{K}[X_n^k], \quad (b')^{\frac{2k}{d}} = \frac{(b^2 - 4ac)^{\frac{kd-k}{d}} b^{\frac{2k}{d}}}{z^{\frac{nd-n}{d}}} \in Qk[X_n^k],$$

and so on. Therefore the normalization X_n^k coincides with the normalization $X_{(\frac{n}{d})}^{\frac{k}{d}}$. Consequently, we can consider $\text{Norm}(X_n^k)$ only for coprime n and k .

Lemma 9. *The normalization morphism $\text{Norm}(X_n^k) \rightarrow X_n^k$ is a bijection if and only if k and n are coprime.*

Proof. Let us embed the algebra $\mathbb{K}[X_n^k] = \mathbb{K}[a^i b^j c^l (i + j + l = 2k), z \mid z^n = (b^2 - 4ac)^k]$ into the algebra $A_n^k = \mathbb{K}[a, b, c, z \mid z^n = (b^2 - 4ac)^k]$. The algebra A_n^k is integral over the algebra $\mathbb{K}[X_n^k]$. For coprime k and n , the algebra A_n^k has no zero divisors, and its integral closure is obtained by adding the element $\lambda = \frac{(b^2 - 4ac)^v}{z^u}$, where $(-u)k + vn = 1$ (we can always assume that $u > 0$ and $v > 0$). In fact, $\lambda^k = z$, and the algebra $A_n^k[\lambda] = \mathbb{K}[a, b, c, \lambda \mid \lambda^n = b^2 - 4ac]$ is integrally closed. We deduce that

$$\text{Norm}(\mathbb{K}[X_n^k]) = \mathbb{K}[X_n^k] \left[\frac{(b^2 - 4ac)^{tv} \xi}{z^{tu}} \right],$$

where ξ runs through all homogeneous forms of degree α in a, b, c such that $(\alpha + 2vt)$ is a multiple of $2k$, and t runs through the natural numbers. The element $\frac{(b^2 - 4ac)^{tv} \xi}{z^{tu}}$ raised to the power k is equal to $z^t \xi^k$. When one restricts regular functions to the fibre $z = 0$, every added element becomes nilpotent, and the reduced “null-fibre” of the quotient morphism for the variety $\text{Norm}(X_n^k)$ is isomorphic to the corresponding fibre for the variety X_n^k . The normalization morphism is surjective and hence determines a bijection on the “null-fibre”. The variety X_n^k is smooth outside the “null-fibre”, so the normalization morphism is an isomorphism. This completes the proof for coprime k and n .

In the case $d = (k, n) > 1$, the “null-fibre” of the quotient morphism for the variety $\text{Norm}(X_n^k)$ contains a non-closed orbit of type $\text{SL}_2 / U_{\frac{4k}{d}}$, and the normalization morphism is a d -sheeted covering on this fibre. This covering is ramified exactly over the fixed point.

Comparing the statements of Lemma 7, Lemma 8b) and Lemma 9, one obtains the following result.

Proposition 4. *Every simple (S, N) -variety is isomorphic to one of the varieties $\text{Norm}(X_n^k)$ for coprime k and n .*

Remark 3. The varieties $\text{Norm}(X_n^k)$ and $\text{Norm}(X_{n'}^{k'})$ with different pairs of coprime numbers (k, n) and (k', n') are not isomorphic. Indeed, if $k \neq k'$, then these varieties have different non-closed orbits. The number n for $\text{Norm}(X_n^k)$ is the order of the group of those equivariant automorphisms of this variety that act identically on the “null-fibre” of the quotient morphism. (Such automorphisms act identically on the functions $a^i b^j c^l (i + j + l = 2k)$ and multiply z by an n th root of unity.)

4.3. The classification of (S, N) -varieties. Let X be an arbitrary (S, N) -variety. We fix a local analytic parameter z in the neighbourhood of a point $x \in X^{\text{SL}_2}$ on the smooth curve X^N . Then, for an embedding $X \hookrightarrow V$, the embedding $X^N \hookrightarrow V^N$ is defined by formulae of type (1), where we now have series (not necessarily polynomials) on the right. However, this does not affect the subsequent

reasoning, and one obtains that the variety X in the neighbourhood of x is SL_2 -equivariantly homeomorphic to the variety X_n^k for some coprime numbers k and n . We shall call these numbers the *marks* of the point $x \in X^{\mathrm{SL}_2}$.

Let us remark that a fibre of general position for the quotient morphism $\phi: X \rightarrow X//\mathrm{SL}_2$ is isomorphic to SL_2/N , and $\mathrm{Aut}_{\mathrm{SL}_2} \mathrm{SL}_2/N \cong N_{\mathrm{SL}_2}(N)/N = \{e\}$. Therefore $X \setminus \phi^{-1}(\phi(X^{\mathrm{SL}_2})) \cong (C \setminus \phi(X^{\mathrm{SL}_2})) \times \mathrm{SL}_2/N$. This implies that an (S, N) -variety is uniquely determined by the following set of data: a smooth curve C , a set of points X^{SL_2} , and the pairs of marks at the points from X^{SL_2} .

We shall now prove that to any data there corresponds an (S, N) -variety. We need the following standard lemma from the theory of algebraic curves.

Lemma 10. *For an irreducible smooth affine algebraic curve C and a fixed point x on this curve there exists a regular morphism $\phi: C \rightarrow \mathbb{A}^1$ such that x is not a ramification point of ϕ .*

Given a point $x_i \in X^{\mathrm{SL}_2}$ with marks (k_i, n_i) on C , we remove from C all ramification points of ϕ and all the points of X^{SL_2} except x_i . We get an open subset $C_i \hookrightarrow C$. Consider the variety $\mathrm{Norm}(X_{n_i}^{k_i})$, the quotient morphism $\phi_i: \mathrm{Norm}(X_{n_i}^{k_i}) \rightarrow \mathbb{A}^1$ and the induced fibering Z_i over C_i :

$$\begin{array}{ccc} Z_i & \longrightarrow & \mathrm{Norm}(X_{n_i}^{k_i}) \\ \downarrow \psi_i & & \downarrow \varphi_i \\ C_i & \longrightarrow & \mathbb{A}^1. \end{array}$$

It is easy to see that Z_i is normal and has an open subset $(C_i \setminus \{x_i\}) \times \mathrm{SL}_2/N$. One can consider the projection

$$\psi: Z = (C \setminus X^{\mathrm{SL}_2}) \times \mathrm{SL}_2/N \rightarrow (C \setminus X^{\mathrm{SL}_2}).$$

The varieties Z and Z_i (over all i) can be uniquely glued to give an algebraic variety X (a fibre of general position is SL_2/N , this fibre has only the identity equivariant automorphism, and one can identify two fibres in a unique way) and a mapping $X \rightarrow C$ which is compatible with all morphisms ψ and ψ_i .

Lemma (on the affine morphism [6]). *Let $f: X \rightarrow Y$ be a morphism of algebraic varieties such that there exists an open covering $\{V_i\}$ of the variety Y and $f^{-1}(V_i)$ is affine for every i . Then for every open affine subset $V \hookrightarrow Y$ the variety $f^{-1}(V)$ is affine.*

So the variety X is affine. Normality is a local property of algebraic varieties, and X is locally isomorphic to either Z or Z_i . The desired variety X is now constructed. These arguments prove the following theorem.

Theorem 2. *(S, N) -varieties are in one-to-one correspondence with the following data:*

- 1) C , a smooth irreducible affine algebraic curve;
- 2) K , a finite collection of m points on this curve (possibly $m = 0$);
- 3) W , a set of pairs of coprime numbers $(k_1, n_1), \dots, (k_m, n_m)$, which are the marks of the points from K .

Remark 4. An (S, N) -variety is smooth if and only if $X \cong C \times \mathrm{SL}_2/N$. (In terms of Theorem 2 this means that the set K is empty.) In other cases the set of singular points of the variety X is X^{SL_2} . In fact, the tangent space at every fixed point is an SL_2 -module which contains an irreducible submodule of dimension ≥ 5 . But the variety X is three-dimensional. Other points on X are regular because their orbits are two-dimensional and a normal variety is smooth in codimension 1.

We would now like to establish some general facts about SL_2 -actions on three-dimensional normal affine varieties.

Proposition 5. *If the group SL_2 acts on an irreducible three-dimensional normal affine variety X , then all fibres of the quotient morphism are irreducible and normal.*

We may assume that $\dim X//\mathrm{SL}_2 = 1$, since otherwise a fibre of the quotient morphism is the whole variety X .

The passage to the variety $\mathrm{Spec} \mathbb{K}[X]^U$ shows that this proposition is equivalent to the following.

Proposition 5'. *Suppose that a one-dimensional torus T acts faithfully on a normal affine surface Y , the T -module $\mathbb{K}[Y]$ has only non-negative weights of the torus T , and the quotient $Y//T$ is one-dimensional. Then every fibre of the quotient morphism ϕ is isomorphic to the affine line.*

Proof. It was proved in [14], Chapter 4 that the surface Y is *toroidal*. This means that each point on Y has a T -invariant neighbourhood which is obtained from a T -invariant neighbourhood on an affine toric variety for a two-dimensional torus $T \times \mathbb{K}^*$ by means of an étale morphism. Under the conditions of Proposition 5', the irreducibility and normality of every fibre of the quotient morphism is easily checked for such a toric variety. The unique normal affine toric variety for a one-dimensional torus with non-negative weights is the affine line. So the proof is complete.

Proposition 6. *Let the group SL_2 act on an irreducible three-dimensional normal affine variety X . Suppose that one of the following conditions is fulfilled:*

- 1) *the action $\mathrm{SL}_2 : X$ is locally transitive;*
- 2) *$\mathbb{K}[X]^{\mathrm{SL}_2} \neq \mathbb{K}$.*

Then all singularities of the variety X are rational.

Proof. In case 1) this was proved by Popov (see [3]). In the case $\mathbb{K}[X]^{\mathrm{SL}_2} \neq \mathbb{K}$ we have the quotient morphism $X \rightarrow X//\mathrm{SL}_2$ to the smooth curve $X//\mathrm{SL}_2$. This morphism is flat and each of its fibres is a normal spherical variety. It is shown in [3] that such fibres have only rational singularities. To complete the proof we use the following fact. Suppose that $f: X \rightarrow S$ is a flat morphism, $x \in X$, the point $s = f(x)$ is a rational singularity on S , and x is a rational singularity on the fibre $f^{-1}(s)$. Then x is a rational singularity on X (see [13]).

Remark 5. This result can also be reformulated for actions of a one-dimensional torus on a normal surface.

Remark 6. The case of (S, U) -varieties with $\mathbb{K}[X]^{\mathrm{SL}_2} = \mathbb{K}$ is the only case not covered by Proposition 6. In this case, only one singular point is possible on X ,

the one-point orbit. The singularity at this point can be non-rational. In fact, in terms of Theorem 1 one can consider $Y = \{(x, y, z) \mid x^3 + y^3 + z^3 = 0\}$ with the natural torus action $t(x, y, z) = (tx, ty, tz)$.

§ 5. Smooth (S, T) -varieties

By definition, the orbit of general position on X is the homogeneous space SL_2/T . Also there can be finitely many orbits of types SL_2/N , SL_2/U_{2k} and $\{\text{point}\}$ on X .

We have an easy example of a smooth (S, T) -variety: if C is a smooth irreducible affine curve, then put $X = (\mathrm{SL}_2/T) \times C$. Let us call such a variety *a variety of trivial type*. (The triviality is equivalent to the condition that the curve X^T is not connected.) All other smooth (S, T) -varieties will be called varieties of non-trivial type.

Using the etale slice theory due to Luna, one can prove that the smoothness of X implies the smoothness of the curve X^T (see [4], p. 212). Therefore the curve X^T is irreducible for varieties of non-trivial type.

The group $\mathbb{Z}_2 = N(T)/T$ acts naturally on the curve X^T , and if all orbits on the variety X are closed, then

$$X = \mathrm{SL}_2/T *_{{\mathbb{Z}_2}} X^T.$$

We now study the structure of X in a neighbourhood of a non-closed orbit SL_2/U_{2k} . The reasoning about the tangent space at a fixed point from the previous section shows that smoothness of X implies $k = 1$. Using etale slice theory [4], p. 213, we obtain that some complex neighbourhood of a non-closed orbit in X is analytically equivariantly isomorphic to an invariant neighbourhood of zero in the three-dimensional SL_2 -module $T_x X$. This implies that $T_x X$ is isomorphic to the module V_2 .

So we have the following data: a smooth irreducible affine curve X^T with a given non-trivial algebraic action of the group $N(T)/T = \mathbb{Z}_2$ and a finite collection of points X^{SL_2} on X^N . Let us show how to construct the (S, T) -variety corresponding to such data.

Let C be a smooth irreducible affine curve with non-trivial action of the group $\mathbb{Z}_2$. We fix a set $K \subset C^{\mathbb{Z}_2}$, which will play the role of X^{SL_2} . Let x be a point of the set K . Consider an etale slice S for the action $\mathbb{Z}_2 : C$ at the point x . One of Luna's theorems [4], p. 212 shows that there exists an excellent morphism $\phi : S \rightarrow \mathbb{A}^1$ which is $\mathbb{Z}_2$ -equivariant with respect to the natural action of the group $\mathbb{Z}_2$ on the line and satisfies $\phi(x) = 0$. This morphism induces a morphism of quotients $\phi/\mathbb{Z}_2 : C' \rightarrow \mathbb{A}^1$. On the other hand, there is a morphism of quotients $V_2 \rightarrow \mathbb{A}^1$. Considering the pullback of the space V_2 for the morphism $\phi/\mathbb{Z}_2$, we obtain some SL_2 -variety. If we “glue” all these varieties (for all points $x \in K$) and the variety $\mathrm{SL}_2 *_N (C \setminus K)$ over the common open subset $\mathrm{SL}_2 *_N (C \setminus C^{\mathbb{Z}_2})$, we obtain the desired smooth (S, T) -variety X .

By definition of an excellent morphism, the etale slice of the variety X at a fixed point x is isomorphic to the corresponding pullback of the space V_2 . Using the formula $N_{\mathrm{SL}_2}(T)/T \cong \mathbb{Z}_2$ and the fact that a complete complex curve without

finitely many points is connected (this guarantees the uniqueness of the “gluing”), we obtain the uniqueness of the corresponding (S, T) -variety.

Thus we have proved the following theorem.

Theorem 3. *Smooth (S, T) -varieties of non-trivial type are in one-to-one correspondence with the following data:*

- 1) C , a smooth irreducible affine curve with a fixed non-trivial algebraic action of the group $\mathbb{Z}_2$;
- 2) K , a subset (possibly empty) of the set of fixed points $C^{\mathbb{Z}_2}$ (possibly empty).

Remark. The fact that all orbits on X are isomorphic to the homogeneous space SL_2/T does not imply that the variety X is of trivial type. For example, one can consider the variety $\mathrm{SL}_2 *_N C$, where $C = \{(x, y) \mid xy = 1\}$ and the involution is defined by $(x, y) \rightarrow (-x, -y)$.

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