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On actions of reductive groups with one-parameter family of spherical orbits

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Abstract. Actions of reductive groups on normal algebraic varieties with one-parameter families of spherical orbits of maximal dimension are studied under the assumption that the categorical quotient for the action is one-dimensional. As an application, the classification of the actions of the group SL_2 on three-dimensional normal affine varieties is completed. The ground field K is assumed to be algebraically closed and of characteristic zero.

Bibliography: 15 titles.

1. Introduction

By the *complexity* $c(X,G)$ of an action $G:X$ of a linear algebraic group G on an algebraic variety X we mean the codimension of a generic orbit for the induced action of a Borel subgroup $B:X$ or, which is the same,

$$c(X,G) = \operatorname{tr.deg} K(X)^B.$$

The concept of complexity is discussed in greater detail in [1].

Actions of complexity zero are said to be *spherical*. For actions of complexity one, there are two fundamentally different possibilities:

(1) there is an open orbit of G in X having complexity one;

(2) G acts on X with one-parameter family of spherical orbits of maximal dimension; in this case we call X a *qs*-variety (a *quasi-spherical variety*) with respect to the action of G.

We shall consider only the second case in this paper.

In what follows, unless otherwise stated, we assume that G is a connected reductive group acting regularly on an irreducible *normal* affine variety X.

Assume that a generic orbit of this action is spherical. Then all the orbits of G on X are spherical (see [2]). We prove that in this case, under the assumption that there exists a generic stabilizer, almost all closures of generic orbits are isomorphic, that is, a 'generic closure' is well defined.

After that we proceed to actions of complexity one with spherical orbits. Examples of such actions are provided by embeddings of homogeneous spaces $K^* \times G/H$,

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where H is an algebraic spherical subgroup of G (the homogeneous space $K^*\times G/H$ is spherical with respect to the group $K^*\times G$ and has complexity one under the natural action of G). Such embeddings can be classified in the framework of the general Luna–Vust theory.

A normal affine qs-variety such that $\dim X/\!/G=1$ is called a $qs1$-variety. If the action $G:X$ is stable and, in particular, if H contains a maximal torus of G, then the condition $\dim X/\!/G=1$ holds automatically. We claim that any birationally trivial $qs1$-variety X can be obtained by means of the operation of gluing (defined below) from spherical embeddings of the homogeneous space $K^*\times G/H$. Here 'birational triviality' means that X contains an open G-invariant subset isomorphic to $G/H\times U$, where U is a smooth affine curve with trivial action of G. If the group of equivariant automorphisms of the homogeneous space G/H is finite (this group is isomorphic to $N_G(H)/H$), then the gluing is unambiguously defined, and therefore the study of this class of actions is reduced to the well-developed theory of spherical embeddings. For a birationally non-trivial action we prove that it can be obtained as a quotient, by an action of a finite group, from a birationally trivial action with the same generic G-orbit. In particular, this means that the singularities of $qs1$-varieties are rational. Furthermore, we consider the class of G-varieties obtained by gluing over an arbitrary smooth algebraic curve, which need not be affine. It turns out that this class of G-varieties is characterized by the existence of a 'good quotient' for the action of G in the sense of Mumford. Thereupon, we shall study the properties of the fibres of the factorization morphism for $qs1$-varieties.

The results so obtained will make it possible to complete the classification of the actions of SL_2 on normal irreducible three-dimensional affine algebraic varieties. We recall the necessary information.

Lemma 1.1 (see, for instance, [3], Subsection 4.1, Lemma 2). *Each algebraic subgroup of SL_2 is conjugate to one of the following subgroups*:

(1) *a finite subgroup*;
(2) *a Borel subgroup B*;
(3) *a maximal torus T*;
(4) *the normalizer N of a maximal torus*;
(5) *a finite extension of the maximal unipotent subgroup*

$$U_n=\left\{\begin{pmatrix}\varepsilon & a\\ 0 & \varepsilon^{-1}\end{pmatrix}\;\middle|\; a,\varepsilon\in K,\ \varepsilon^n=1\right\},\qquad n=1,2,3\dots.$$

Lemma 1.2. *For each action of SL_2 there exists a generic stabilizer* (GS).

Let $SL_2:X$ be an action in which X is a three-dimensional variety. If the GS of this action is a finite subgroup, then X contains a dense orbit of SL_2. Such actions are classified in [4] (see also [3]). It is shown there that a locally transitive action of SL_2 on a three-dimensional affine variety is either transitive or can be characterized by a pair of numbers in $\mathbb{N}\times(0,\frac{1}{2}]_{\mathbb{Q}}$. The first of these numbers defines the order of the generic stabilizer, which is a cyclic subgroup in this case. The second number is called the *height* and characterizes the algebra of U-invariants.

The three remaining possibilities are as follows:

(1) $\mathrm{GS}=U_n$. We call such SL_2-varieties (S,U)-*varieties.*

(2) GS $= N$. We call such SL_2-varieties (S,N)-*varieties.*
(3) GS $= T$. We call such SL_2-varieties (S,T)-*varieties.*

Remark. The case GS $= B$ cannot occur because SL_2/B is a projective line.

Each (S,U)-variety can be uniquely recovered from the spectrum of the algebra of U-invariants, which is a normal surface with natural action of the maximal torus of SL_2. To be more precise, we have the following result.

Theorem 1 [2]**.** *The (S,U)-varieties are in one-to-one correspondence with normal affine surfaces Y with a fixed non-trivial $\mathbb{Z}_+$-grading of the algebra $K[Y]$ of regular functions. This grading defines an action on Y of the one-dimensional torus T, and the corresponding (S,U)-variety is isomorphic to the variety*

$$X = (Y \times K^2)//T.$$

Here $K[X]^{SL_2} \cong K[Y]^T$.

This theorem reduces the classification of (S,U)-varieties to that of actions of a one-dimensional torus on normal surfaces. As regards the latter, see [5].

We now consider the case of (S,N)-varieties. Any such variety is birationally trivial. An embedding of a spherical homogeneous space $K^* \times SL_2/N$ is defined by a pair of coprime positive integers, and there exists a unique way of gluing such embeddings over a smooth affine curve. Therefore, to define an (S,N)-variety we must define a smooth irreducible affine curve and to place marks (pairs of coprime positive integers) at finitely many points on this curve; see Theorem 3 below. In [2], this result was obtained over the complex field by topological methods, without referring to the Luna–Vust theory.

Birationally trivial (S,T)-varieties can be described in a perfectly similar way to (S,N)-varieties. It remains to consider the case of birationally non-trivial (S,T)-varieties. Each of these can be obtained as a quotient, by an action of $\mathbb{Z}_2$, of a birationally trivial (S,T)-variety. Hence a birationally non-trivial variety is defined by a smooth irreducible affine curve with a fixed non-trivial action of $\mathbb{Z}_2$ and with $\mathbb{Z}_2$-invariant system of marks; see Theorem 4.

Thus, we have described all non-trivial actions of SL_2 on normal affine three-dimensional algebraic varieties. Smooth (S,N)-varieties and (S,T)-varieties are described in [2].

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An approach to the classification of the actions of complexity one in terms of the Luna–Vust theory is developed in the recent paper [6]. The comparison of those results with ours enabled us to correct some inaccuracies. The author thanks D. A. Timashev for the critical reading of the preliminary version of this text.

2. Actions with generic spherical orbit

In the following proposition, it is not essential that X be normal.

Proposition 1. *Let X be a G-variety with all G-orbits spherical, assume that there exists a generic stabilizer, and let* $\mathrm{GS}(G : X) = H$*. Then almost all the closures of orbits of type G/H (for an open dense subset of orbits of type G/H) are isomorphic to some embedding Y of the spherical homogeneous space G/H.*

Proof. Let $\Gamma(X)$ be the semigroup of highest weights of G that correspond to the irreducible G-modules occurring in the decomposition of the algebra $K[X]$ into irreducible G-modules. This semigroup is finitely generated by Khadzhiev's theorem (see [7]). Let $\{\lambda_1, \dots, \lambda_k\}$ be a basis in $\Gamma(X)$. We now choose highest vectors $F_{\lambda_1}, \dots, F_{\lambda_k}$ of the G-module $K[X]$ having weights $\lambda_1, \dots, \lambda_k$ respectively, and consider the set X of points at which none of the functions $F_{\lambda_1}, \dots, F_{\lambda_k}$ vanishes. This set is Zariski open. Hence the closure of a generic orbit is a multiplicity-free affine variety (since G/H is spherical), and its spectrum is $\Gamma(X)$. It remains to show that multiplication in all the algebras of regular functions on these closures has the same structure. This follows from the fact that the algebra of regular functions on the closure of an orbit of type G/H is a G-invariant subalgebra of $K[G/H]$. This last algebra is multiplicity-free, therefore the multiplicative structure on this subalgebra with given decomposition into irreducible G-modules is uniquely defined. This completes the proof of the proposition.

Remarks. (1) There exists a conjecture that spherical subgroups of a reductive group have no non-trivial deformations, so that, for an action with generic spherical orbit, a GS exists automatically. Hence our condition on the existence of a GS is probably not restrictive.

(2) The variety Y is not necessarily normal even if X is normal. For instance, let us consider the following action of a one-dimensional torus on the two-dimensional plane:

$$(x, y) \to (t^2 x, t^3 y).$$

Here Y is a semicubical parabola. In what follows we prove that if $\dim X/\!/G = 1$, then the variety Y is normal and coincides with the generic fibre of the quotient morphism $\pi\colon X \to X/\!/G$.

The following example suggested by Knop shows that if the generic orbit G/H is not spherical, then a generic closure does not necessarily exist.

We consider the action of SL_2 on the three-dimensional projective space $\mathbb{P}^3$ regarded as the projectivization of the space $M_{2\times 2}$ of matrices with SL_2 acting by left multiplication. In $\mathbb{P}^3$ there exist an open orbit of the type PSL_2 and a one-parameter family $\mathbb{P}^1 \times \mathbb{P}^1$ of orbits, where SL_2 acts on the first factor only.

Let Γ be a non-commutative finite subgroup of SL_2. We consider the quotient of $\mathbb{P}^3$ by the action of Γ induced by the action of Γ on $M_{2\times 2}$ by right multiplication. Again, there is a one-parameter family $\mathbb{P}^1 \times \mathbb{P}^1$ of SL_2-orbits in $\mathbb{P}^3/\Gamma$. We consider the product $(\mathbb{P}^3/\Gamma) \times \mathbb{P}^1$ with trivial action of SL_2 on the second factor and blow up the subvariety $\mathbb{P}^1 \times \Delta \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3/\Gamma \times \mathbb{P}^1$, where Δ is the diagonal in the product of the second and the third factors, and SL_2 acts only on the first copy of $\mathbb{P}^1$.

On the resulting variety, the embeddings of the homogeneous space SL_2/Γ correspond to blow-ups of various orbits of type SL_2/B, and it is impossible that almost all of them be isomorphic because the group of equivariant automorphisms of SL_2/Γ is finite.

To any embedding of the projective variety P obtained after the blow-up there corresponds a very ample bundle over P. By considering a sufficiently high tensor power of this bundle we can arrange that the corresponding affine cone over P be a normal affine variety. This cone is an example of a variety with generic orbit of type $(K^* \times SL_2)/\Gamma$ in which a generic closure does not exist.

The problem about the classes of quasi-affine homogeneous spaces in which a generic closure does exist seems to be of a certain interest.

3. Main construction

Let S be a normal affine embedding of the spherical homogeneous space $K^*\times G/H$ of the group $K^*\times G$ such that $S//G \cong A^1$. There arises a natural action of K^* on the quotient A^1, and we regard a unique fixed point of this action as the origin on the line A^1. The inverse image in S of any point of A^1 distinct from the origin is a spherical G-variety Y. We call Y the *generic fibre* for S.

We now describe several operations enabling one to construct new $qs1$-varieties from known ones.

1. *The operation of passage to a fibre product.*

Let U be a smooth affine curve and let $\varphi\colon U \to A^1$ be an étale morphism such that $\varphi^{-1}(0) = \{u\}$. We consider the fibre product $X = S \times_{A^1} U$, where

$$\begin{tikzcd} X \arrow[r] \arrow[d] & S \arrow[d] \\ U \arrow[r, "\varphi"] & A^1, \end{tikzcd}$$

defined by the condition $K[X] = K[U] \otimes_{K[A^1]} K[S]$. The variety X is a normal $qs1$-variety (since φ is étale) with generic fibre Y.

2. *The operation of gluing.*

Let C be a smooth affine curve covered by finitely many distinct affine Zariski open sets U_j such that if U is their intersection, then $U_j = U \cup \{c_j\}$ for each j, where $c_j \in C$.

Let $\{L_j\}$ be a collection of affine varieties and let $\{L_j \to U_j\}$ be a collection of morphisms such that all the inverse images of U in all the varieties L_j are isomorphic via isomorphisms compatible with the given morphisms. Then we identify the corresponding points in these isomorphic open subsets of the varieties L_j. We obtain a prevariety L and, in accordance with the affine criterion in [8], Chapter 1, § 2.3, the given morphisms can be 'glued together' into a morphism $L \to C$.

There is no canonical way to define isomorphisms of the open pieces, therefore the result of this gluing is not uniquely defined. We say that L *is obtained from the varieties L_j by the operation of gluing* if L can be obtained using the above construction for some choice of isomorphisms of isomorphic open subsets.

In our situation, let $\{S_j\}$ be a collection of spherical embeddings for $K^* \times G/H$ having the same generic fibre Y. For each U_j we consider the étale morphism used in the passage to the fibre product. Let X_j be the corresponding fibre product constructed over U_j. Then the isomorphic parts $U \times Y$ of these products can be identified ('glued'), so that we obtain a prevariety X.

Lemma 3.1. *The prevariety X is separated, that is, X is an algebraic variety.*

Proof. As is known (see [8], Chapter 1, § 2.5) a prevariety X is separated if and only if for any two morphisms $\varphi, \psi \colon Y \to X$ (here Y is an arbitrary prevariety) the set $\{y \in Y \mid \varphi(y) = \psi(y)\}$ is closed in Y. It is also shown in [8] that if any two points in X belong to the same open affine subset, then X is separated. Set $Z = \{y \in Y \mid \varphi(y) = \psi(y)\}$ for some morphisms $\varphi, \psi \colon Y \to X$ and let $z \in \overline{Z} \setminus Z$. Then the points $\varphi(z)$ and $\psi(z)$ cannot both belong to the same one of the affine varieties that are being glued together, because that contradicts the separation property of this affine variety. Therefore, we can assume that $\varphi(z) \in \pi^{-1}(c_{j_1})$ and $\psi(z) \in \pi^{-1}(c_{j_2})$, $j_1 \neq j_2$. We consider the composite of the morphisms $\pi \circ \varphi$ and $\pi \circ \psi \colon Y \to C$. Since C is separated, it follows that $Z' = \{y \in Y \mid (\pi \circ \varphi)(y) = (\pi \circ \psi)(y)\}$ is a closed subset of Y and $z \notin Z'$. On the other hand it is clear that $Z \subset Z'$. The contradiction thus obtained proves Lemma 3.1.

The following lemma proves that the variety X is affine.

The affine morphism lemma (see [9]). *Let $f \colon X \to Y$ be a morphism of algebraic varieties and let $\{V_i\}$ be an open covering of Y such that $f^{-1}(V_i)$ is affine for each i. Then the inverse image $f^{-1}(V)$ of each affine open set $V \hookrightarrow Y$ is affine.*

Proposition 2. *If $\mathrm{Aut}_G Y$ is finite, then the gluing is defined unambiguously.*

We need the following result for the proof of this proposition.

Lemma 3.2. *Each G-automorphism of Y can be uniquely extended to a G-automorphism of S that is trivial on the quotient $S//G$.*

Proof. Since $K[Y]$ is a multiplicity-free G-module, each G-automorphism acts on each of its irreducible submodules as multiplication by some constant. This automorphism can be extended in a natural way to $K[(A^1 \setminus \{0\}) \times Y]$ and then to $K[S]$, which is a G-invariant subalgebra. This completes the proof of Lemma 3.2.

Since $\mathrm{Aut}_G Y$ is finite, the identification of points in isomorphic open subsets of the varieties S_{j_1} and S_{j_2} can be defined by a single element of $\mathrm{Aut}_G Y$, and therefore it is unique up to isomorphisms of the glued varieties. This completes the proof of Proposition 2.

Remark. If $\mathrm{Aut}_G Y$ is infinite, then the gluing can be carried out in many ways. For instance, if G is a one-dimensional torus, then the question on the number of ways to carry out the identification is the problem of determining the number of linear bundles over a smooth affine curve that are locally trivial with respect to the Zariski topology.

As we show now, the suggested construction of a gluing is universal in a certain sense.

Definition. A $qs1$-variety X is said to be *simple* if the following conditions hold:

(1) there exists a GS $(G : X)$, which we denote by H;
(2) the variety X has trivial birational type, that is, it contains an open G-invariant subset X_0 isomorphic to $G/H \times U$ for some smooth curve U.

Condition (2) is equivalent to the triviality of some Galois cohomology class. If G is a torus, then this condition holds for every action. It can be shown that condition (2) holds if the group $N_G(H)/H$ is connected (see [7], § 2.7).

The following proposition shows that each $qs1$-variety can be obtained from a simple $qs1$-variety on taking the quotient by a finite group.

Proposition 3. *For each $qs1$-variety X with generic stabilizer H there exists a simple $qs1$-variety $\widetilde{X}$ with the same generic stabilizer and a finite group F acting on $\widetilde{X}$ by regular G-equivariant automorphisms such that*

$$X \cong \widetilde{X}//F.$$

Proof. Let X_1 be the open subset of X consisting of points with stabilizers conjugate to H and let $X_1^H \subset X_1$ be the set of H-fixed points. We consider an arbitrary curve in X_1^H that intersects the generic $N_G(H)$-orbits in X_1^H transversally. This curve defines a quasisection for the action $G : X$. There exists a natural dominant morphism $G/H \times U \to X$. We now consider the extension $K(X//G) \subset K(U)$ and continue it to a Galois extension $K(X//G) \subset K(U) \subset K(\widetilde{U})$ with Galois group F.

Let $\widetilde{K[X]}$ be the integral closure of $K[X]$ in $K(G/H \times \widetilde{U})$ and let $\widetilde{X} = \operatorname{Spec} \widetilde{K[X]}$. Then the morphism $\widetilde{X} \to X$ is finite and therefore surjective. Since X is normal, it follows that $X \cong \widetilde{X}//F$. Finally, the fact that $\widetilde{X}$ is birationally isomorphic to $G/H \times \widetilde{U}$ follows from the following standard lemma.

Lemma 3.3. *Let A be a finitely generated integral algebra over K, and let $F_1 = QA$ be its quotient field. If $F_1 \subset F_2$ is a finite extension of fields and B is the integral closure of A in F_2, then $QB = F_2$.*

We now return to simple $qs1$-varieties. We denote by Y the closure of the generic orbit G/H, which exists by Proposition 1. The algebra of regular functions on X_0 is $K[U] \bigotimes_K K[G/H]$, and $K[G/H]$ is multiplicity-free. By taking a smaller U if necessary we can assume that X contains an open G-invariant subset isomorphic to $U \times Y$. In what follows we denote this subset by X_0.

Let $C = X//G$ be the smooth affine curve $\operatorname{Spec} K[X]^G$ (it is smooth because X is normal). All components of the fibres of the quotient morphism $\pi\colon X \to C$ have codimension one in X. The following proposition shows that U can be regarded as a subset of C and that $\pi^{-1}(U) = X_0 \cong U \times Y$.

Proposition 4. *Let $f\colon X \to X_1$ be a dominant morphism of irreducible normal affine varieties and assume that the generic fibres of this morphism are connected (the last condition holds for the quotient morphism with respect to a connected reductive group). Then the generic fibres are irreducible.*

Proof. Let $\widetilde{K(X_1)}$ be the algebraic closure of $K(X_1)$ in $K(X)$. If $\widetilde{K(X_1)} \neq K(X_1)$, then also $\widetilde{K[X_1]} \neq K[X_1]$ by Lemma 3.3, where $\widetilde{K[X_1]}$ is the integral closure of $K[X_1]$ in $K(X)$. Since X is normal, it follows that $\widetilde{K[X_1]} \subset K[X]$, and therefore there exist regular morphisms $X \to \widetilde{X}_1 \to X_1$, where the latter morphism is finite and not birational. This contradicts the condition that the fibres are connected.

As regards the case where $\widetilde{K(X_1)} = K(X_1)$, see [10], vol. 1, Chapter 2, §6, Theorem 1. This completes the proof of the proposition.

Remark. The assertion of the proposition fails if we do not assume that X is normal. Indeed, we can consider the surface $x^2 = y^2 z$ and the projection onto the line with coordinate z.

We now consider the isotypical decomposition $K[Y] = \bigoplus_\lambda V_\lambda$. Then we have $K[X_0] = \bigoplus (K[U] \otimes_K V_\lambda)$, and therefore $K[X] = \bigoplus (T_\lambda \otimes_K V_\lambda)$ by the G-invariance, where $T_\lambda \subset K[U]$.

The algebra $K[X]$ is finitely generated, and we can assume that it is generated by elements of the form $c_{\lambda_i} v_{\lambda_i}$, $i = 1, \dots, s$, where $v_{\lambda_i} \in V_{\lambda_i}$ and $c_\lambda \in K[U]$. Considering a smaller set U if necessary, we can assume that $c_{\lambda_i} \in K[U]^*$.

Let $C \setminus U = \{y_1, \dots, y_t\}$. Passing to a smaller U if necessary, we can assume that

(1) $U_j = U \bigcup \{y_j\}$ is affine;
(2) there exists $\pi_j \in R_j$ such that $\pi_j = \varrho_j^*$, the map $\varrho_j \colon U_j \to A^1$ is étale, and $\varrho_j^{-1}(0) = \{y_j\}$ (here $R_j = K[U_j]$).

We denote the ring $K[U]$ by R. Then $\pi_j R_j$ is an ideal associated with y_j, and $R^* = R_j^* \times \{\pi_j^k\}_{k \in \mathbb{Z}}$. Let $X_j = \pi^{-1}(U_j)$. Then

$$K[X_j] = R_j[\dots, \pi_j^{r_l} V_{\lambda_l}, \dots]$$

for some finite set $P_j = \{(r_l, \lambda_l) \in \mathbb{Z} \times \Gamma(X)\}$ of pairs.

We now consider the variety

$$S_j = \operatorname{Spec} K[t, \dots, t^{r_l} V_{\lambda_l}, \dots], \quad \text{where} \quad K[t, \dots, t^{r_l} V_{\lambda_l}, \dots] \subset K[t] \otimes K[Y].$$

There is a natural action of the one-dimensional torus on S_j (in the variable t), and therefore S_j turns out to be a spherical variety for the group $K^* \times G$.

We have the following commutative diagram:

$$\begin{array}{ccccc} X_j & \xrightarrow{q} & U_j \times_{A_t^1} S_j & \xrightarrow{p_2} & S_j \\ \bigcup & & \bigcup & & \bigcup \\ U \times Y & \xrightarrow{\cong} & U_j \times_{A_t^1} (Y \times K^*) & \longrightarrow & Y \times K^* \end{array}$$

and the map q is an isomorphism. Since $\varrho_j \colon U_j \to A^1$ is étale, the projection p_2 is also étale. In particular, S_j is normal.

Thus, the variety X can be obtained in two steps: by passing from the fibre products of the spherical varieties S_j to the X_j and by gluing the X_j after that. In our discussion we have followed the line of reasoning of [5], where a similar argument had been carried out in the toric case. Thus, we have proved the following result.

Theorem 2.1. *Each simple $qs1$-variety with generic stabilizer H can be obtained by gluing normal spherical embeddings S_j of the homogeneous space $K^* \times G/H$ having the same typical fibre Y and such that $S_j /\!/ G = A^1$.*

Remark. In the spirit of [5], we can say that simple $qs1$-varieties are *spheroidal*.

Corollary. *The singularities of a $qs1$-variety are rational.*

Indeed, the singularities of spherical varieties are rational, as is shown in [11]. Hence the corollary follows from the fact that the morphisms constructed are étale and from Proposition 4, because the rationality of singularities is preserved by taking the quotient with respect to a finite (or, more generally, a reductive) group; see [7], Subsection 3.9.

In what follows we assume that the reader is acquainted with the concept of a coloured cone. As regards this concept, see [12].

Let C_j be a finite set of coloured cones that define embeddings S_j of the homogeneous space $K^* \times G/H$. A gluing of these embeddings over some curve into a single simple $qs1$-variety is possible if and only if the following conditions are satisfied: (a) $S_j//G = A^1$; (b) the typical fibres of all the embeddings S_j are isomorphic to the same G/H-embedding Y; (c) all the S_j are affine.

Definition. A qs-variety X is said to be *rigid* if the generic stabilizer H is such that $N_G(H) = H$.

If a spherical homogeneous space G/H is quasi-affine and $N_G(H)/H$ is finite, then it follows from the results of Brion and Knop that the subgroup H is reductive (see, for example, [13], Corollary 7.6). According to the Luna closedness criterion for orbits, the homogeneous space G/H admits in this case a unique affine embedding, which consists of a single orbit G/H. Therefore, generic G-orbits in X are closed, and each rigid qs-variety is a $qs1$-variety. Each rigid qs-variety is birationally trivial.

For rigid qs-varieties, we can use Proposition 2 to reformulate Theorem 2.1 as follows.

Theorem 2.2. *Each rigid qs-variety X with generic stabilizer H is uniquely determined by the following set of data*:

(1) *a smooth affine curve C (which is the quotient $X//G$)*;
(2) *a finite (possibly empty) set P of pairwise distinct points on the curve C*;
(3) *to each point in the set P there must correspond a coloured cone for the homogeneous space $K^* \times G/H$ that does not lie in the subspace corresponding to G/H and defines an affine embedding of $K^* \times G/H$.*

Conversely, every such set of data defines a rigid qs-variety of the group G.

Remark. The question on whether an embedding corresponding to a given coloured cone is affine can be readily solved using the criterion in [12].

Corollary. *If, under the assumptions of Theorem* 2.2, *it is additionally known that the homogeneous space G/H is of rank one, then it suffices to assign to each point in P a pair of coprime positive integers in place of a coloured cone. In this case all fibres of the quotient morphism $\pi\colon X \to C$ are irreducible.*

Proof. As is known [12], the dimension of the cone $\Upsilon_{G/H}$ of invariant valuations is the rank of the corresponding homogeneous space, and if $N_G(H)/H$ is finite, then $\Upsilon_{G/H}$ is strictly convex. We can identify $\Upsilon_{G/H}$ with the non-negative ray of the coordinate line; then the cone $\Upsilon_{K^*\times G/H}$ is the left half-plane (Fig. 1).

The set of colours of the space G/H is not empty because otherwise G/H would have to admit at least two affine embeddings, namely, $(0, \varnothing)$ and $(\Upsilon_{G/H}, \varnothing)$. Not

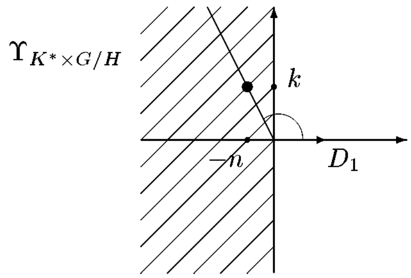

Figure 1

all the colours lie in the left half-plane, because otherwise there must exist an affine cone $(\Upsilon_{G/H}, \rho(D))$. The homogeneous space G/H is affine, therefore all its colours lie in the right half-plane. Considering the automorphism $t \to t^{-1}$ of the torus K^* we can assume that our coloured cone lies in the upper half-plane. It must contain all colours of G/H, therefore to define this cone it is necessary and sufficient to select a ray in the upper left quadrant. Since the ambient space is defined over the field of rationals, such a ray is uniquely determined by a pair of coprime positive integers. This ray corresponds to an irreducible divisor which is the fibre of the quotient morphism over a singular point, and therefore all fibres are irreducible. This completes the proof of the corollary.

It is clear that a gluing of the affine embeddings S_j can be carried out not only over an affine curve but also over an arbitrary smooth algebraic curve. Let us determine the class of G-varieties of complexity one obtained in this way.

In Mumford's geometric invariant theory one is acquainted with a concept of a 'good quotient'.

Definition. Let X be an algebraic G-variety. A morphism $\pi\colon X \to C$, where C is an algebraic space, is called a *good quotient* if

(a) π is G-equivariant with respect to the given action of G on X and the trivial action of G on C;
(b) π is an affine morphism;
(c) $\pi_*(O_X^G) \cong O_C$.

Condition (b) means that there exists a finite affine covering of C such that its inverse image is an affine covering of X. A good quotient is a categorical quotient, and therefore it is defined uniquely up to canonical isomorphism. For arbitrary actions on algebraic varieties this quotient seldom exists. It is necessary, in particular, that each point in X have an open affine invariant neighbourhood. If X is affine, then there always exists a good quotient $\pi\colon X \to \operatorname{Spec}(K[X]^G)$.

As is known, a good quotient can be obtained from the Mumford quotient for the set of points semistable under some embedding of X into a projective space if and only if C is a quasiprojective variety. In a more general situation, good quotients were studied by Bialynicki-Birula and Swiecicka (see [14] and the bibliography therein). In the same papers the problem of the description of all open invariant subsets admitting a good quotient was considered for an algebraic G-variety.

We can now describe a certain class of G-varieties with a good quotient irrespective of any embedding, in a rather constructive way.

Theorem 2.3. *A normal algebraic birationally trivial G-variety X of complexity one with generic stabilizer H admits a good quotient onto a smooth algebraic curve C if and only if it can be obtained by gluing over C some normal affine spherical embeddings S_j of the homogeneous space $K^* \times G/H$ that all have the same typical fibre Y and are such that $S_j//G = A^1$.*

To prove Theorem 2.3, it suffices to choose a covering of C in the same way as in the main construction and to apply the affine morphism lemma.

Since each algebraic curve is quasiprojective, the quotient in question can be obtained using the Mumford construction for an appropriate embedding in a projective space.

Corollary. *Each variety in Theorem* 2.3 *is quasiprojective.*

4. Properties of the fibres of the quotient morphism

Theorem 2.1 reduces the study of many geometric problems concerning varieties of complexity one with one-parameter families of spherical orbits to questions about spherical varieties, which are well understood. We now consider the problem of the normality and irreducibility of the fibres of the quotient morphism for simple $qs1$-varieties.

Let ω be the operation of passage from a G-algebra to the subalgebra consisting of its U-invariants (that is, $\omega\colon K[X] \to K[X]^U$), where U is a maximal unipotent subgroup of G. We denote by the same letter a similar passage $\omega\colon X \to \operatorname{Spec} K[X]^U$ proceeding on the level of G-varieties. If X is a $qs1$-variety for G, then $\operatorname{Spec} K[X]^U$ is a $qs1$-variety for a maximal torus T of G, and ω commutes with the passage to the algebra of regular functions on some fibre of the quotient morphism. Since the properties of an algebra to be integrally closed or integral are stable (using a term from [11]), the fibres of X are irreducible and normal if and only if the fibres of the T-variety $\operatorname{Spec} K[X]^U$ are stable. Therefore, we can restrict ourselves to actions of tori, and by Theorem 2.1 it suffices to study the case when X is a toric variety for an $(n+1)$-dimensional torus, where $n = \operatorname{rank} G$.

A toric variety is uniquely determined by a convex cone $C \subset \mathbb{Q}^{n+1}$ with linear span equal to the entire space $\mathbb{Q}^{n+1}$ that is generated by the vectors corresponding to all weights of the torus $(K^*)^{n+1}$ occurring in the weight decomposition of $K[X]$. This is the cone dual to the one usually considered in the theory of toric varieties. The condition $X//T = A^1$ means that C contains the positive ray of the last coordinate axis in $\mathbb{Q}^{n+1}$ but does not contain the negative ray (we assume here that T is embedded into $(K^*)^{n+1}$ with respect to the first n variables).

Definition. We call an action $T : X$ an *fp-action* (a fixed-point action) if the projection C_0 of C onto $\mathbb{Q}^n$ (the first n variables) lies in some closed subspace. Otherwise the action is called an *nfp-action.*

If G is semisimple, then the induced action $T : \operatorname{Spec}(K[X]^U)$ is an fp-action.

Lemma 4.1. *All fibres of the quotient morphism $\pi\colon X \to X//T = A^1$, except for the fibre over the origin, are toric varieties with respect to the torus T corresponding to the cone C_0.*

Proof. We have $K[X] = K[t, \dots, t^{r_l}\chi^{\lambda_l}, \dots]$. Hence

$$K[\pi^{-1}(A^1 \setminus \{0\})] = K[X][t^{-1}] = K[t, t^{-1}] \otimes_K K[\dots, \chi^{\lambda_l}, \dots] = K[t, t^{-1}] \bigotimes_K K[Y].$$

It follows from Lemma 4.1 that nfp-actions can be characterized by the condition that the generic fibre of the morphism $\pi\colon X \to X//T$ be the torus T itself.

Remark. If we have a finite set of cones in $\mathbb{Q}^{n+1}$ containing only the positive ray of the last coordinate axis such that their projections onto $\mathbb{Q}^n$ are the same, then the toric varieties corresponding to these cones can be glued together (in more than one way) into a single variety X, and each normal affine $(n+1)$-dimensional variety endowed with an effective action of the n-dimensional torus with one-dimensional quotient can be obtained in this way.

It remains to consider the fibre Y_0 of the morphism π over the origin. The algebra $K[Y]$ has a well-defined $\mathbb{Z}$-filtration: to each weight $(q_1, \dots, q_n)$, $q_i \in \mathbb{Z}$, belonging to C_0 we assign the least number $q_{n+1} \in \mathbb{Z}$ such that $(q_1, \dots, q_n, q_{n+1}) \in C$.

The passage from $K[X]$ to $K[Y_0]$ consists in considering the quotient by the ideal (t), which can be interpreted as the passage from $K[Y]$ to the associated algebra $\operatorname{gr} K[Y]$ with respect to the above filtration and the subsequent passage to the quotient by the maximal nilpotent ideal. After this, the only weights that remain in $K[Y]$ are those that are the projections of integral points lying in the proper faces of C. A product of eigenfunctions with respect to the torus that correspond to points in distinct faces of C corresponds to a point inside C, and therefore this product becomes a nilpotent function after the restriction to the fibre over the origin.

Hence it is easy to see that the number of irreducible components of the fibre Y_0 is the number of those projections of the faces of C onto $\mathbb{Q}^n$ having dimension n. These projections define a partition of C_0 into cones of the same dimension.

Proposition 5. (1) *For nfp-actions, the number of irreducible components of the fibre over the origin is the number of faces of the cone C.*

(2) *For fp-actions, the number of irreducible components of the fibre over the origin is at least the number of faces of the cone C minus n.*

Corollary 1. *All fibres of the quotient morphism corresponding to an action of a rank one semisimple group or an fp-action of the one-dimensional torus are irreducible.*

Thus, we have obtained another proof of Proposition 5 of [2].

Corollary 2. *For an nfp-action of an n-dimensional torus on an $(n+1)$-dimensional normal affine variety, the following conditions are equivalent*:

(1) *all fibres of the quotient morphism are irreducible*;
(2) *all fibres of the quotient morphism are isomorphic to the n-dimensional torus itself.*

In the case of fp-actions the irreducibility of the fibres means that the corresponding cone is simplicial. We also note that, as a rule, there is more than one way to extend an action of the n-dimensional torus on an $(n+1)$-dimensional variety to a locally transitive action of the $(n+1)$-dimensional torus. However, the number of faces of the resulting cone is always the same.

Now let C_i be some face of C having projection $C_{0,i}$ of maximal dimension n, $C_{0,i} \subset C_0$. Let $a_1x_1 + \dots + a_nx_n + a_{n+1}x_{n+1} = 0$ be an equation distinguishing C_i in C. Then the projections of integer points lying in C_i are distinguished among all points in $C_{0,i}$ with integer coordinates by the condition $\dfrac{a_1x_1 + \dots + a_nx_n}{a_{n+1}} \in \mathbb{Z}$, that is, they belong to a sublattice of finite index. Hence the irreducible component of the fibre over the origin corresponding to the face $C_{0,i}$ is a toric variety, although not for the torus T itself, but for its quotient with respect to the finite subgroup defined by the above sublattice. Hence we obtain the following result.

Proposition 6. *Assume that we have a simple $qs1$-action of a reductive group G. Then all irreducible components of each of the fibres of the quotient morphism are normal.*

Now, we proceed to the case $G = SL_2$.

5. Classification of (S, N)-varieties

We consider the classification of affine normal embeddings of the homogeneous space $K^* \times SL_2/N$ in the framework of the Luna–Vust theory.

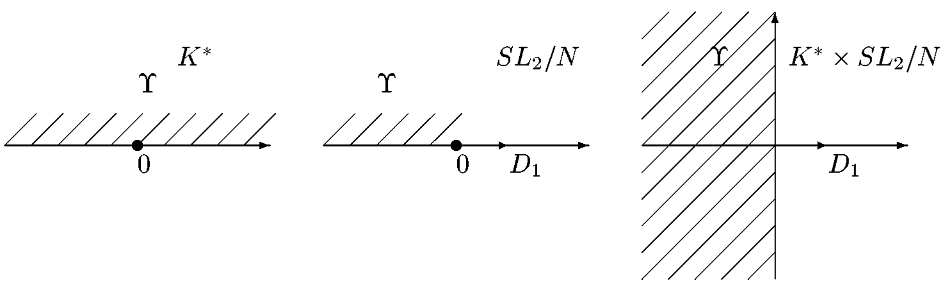

Figure 2

In Fig. 2 we depict the respective cones of invariant valuations Υ and the 'colours' for the homogeneous spaces K^*, SL_2/N, and $K^* \times SL_2/N$. For the required definitions, see [12]. The affine criterion in [12] shows that all normal affine embeddings of the homogeneous space $K^* \times SL_2/N$ are defined by coloured cones of one of the types depicted in Fig. 3.

The first two embeddings are $SL_2/N \times K^*$ and $SL_2/N \times K$ respectively. To define a cone of type (3) it suffices to specify a line in the upper left quadrant that is defined over the field of rationals. This is equivalent to the specification of a pair of coprime positive integers (n, k).

The embeddings corresponding to the cones of the types $(2')$ and $(3')$ can be obtained from the embeddings corresponding to (2) and (3) by means of the automorphism $t \to t^{-1}$ of K^*.

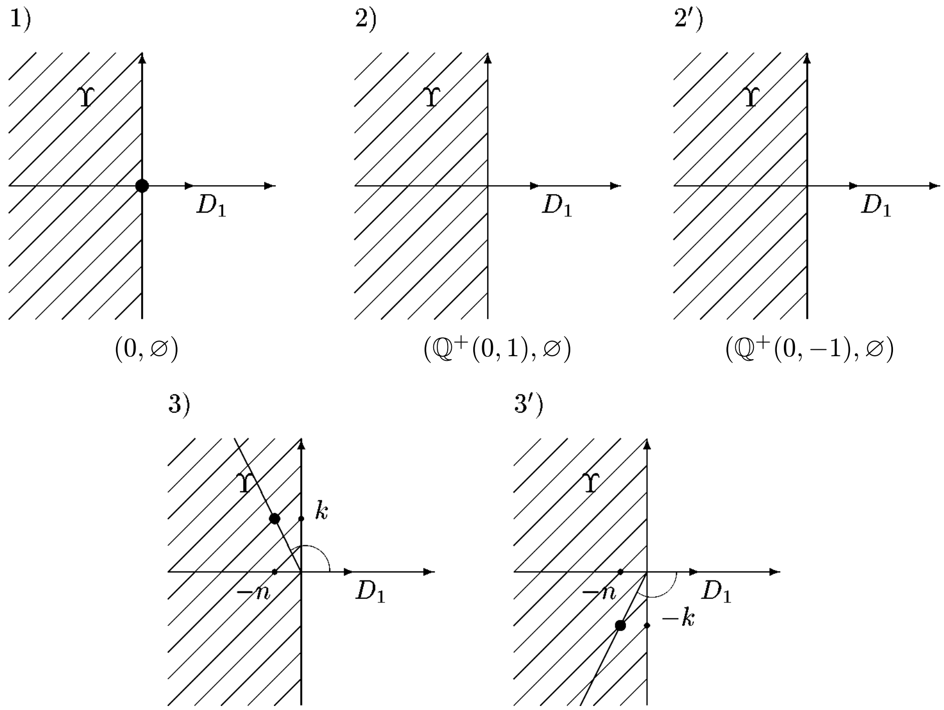

Figure 3

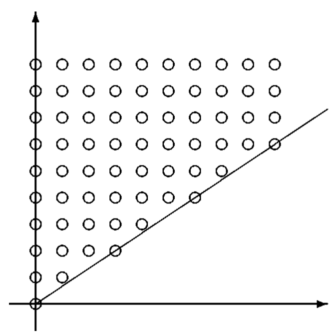

Figure 4

The algebra of regular functions on the variety corresponding to the embedding defined by a pair (n, k) can be described explicitly, as shown in Fig. 4.

Namely, we consider the cone C in $\mathbb{Q}^2$ contained between the positive ray of the axis Oy and the ray from the origin passing through the point (k, n) (recall that $k > 0$ and $n > 0$ by assumption). Then

$$K[X] = \{t^{r_2} V_{4r_1} \mid (r_1, r_2) \in C,\ r_1, r_2 \in \mathbb{Z}\} \subset K[A^1] \otimes K[SL_2/N], \tag{1}$$

where by V_{4r_1} we mean the $(4r_1 + 1)$-dimensional irreducible SL_2-module. The resulting variety X is normal because $\operatorname{Spec} k[X]^U$ is a toric variety by construction.

Using gluing and Proposition 2 we obtain the following result.

Theorem 3. *The (S,N)-varieties are in one-to-one correspondence with the following set of data*:

(1) *a smooth irreducible affine curve C*;
(2) *a finite collection W of points on this curve* (*possibly empty*);
(3) *a collection of pairs of coprime positive integers (k_1, n_1), ... , (k_m, n_m), where $M = \mid W \mid$, each pair attached to some point in W and called the mark of this point.*

The assertion of Theorem 3 is a special case of the corollary to Theorem 2.2, but we can present in this case a geometric construction of the variety corresponding to a pair of numbers (n,k). The corresponding variety is constructed in [2] and is denoted in that paper by $\operatorname{Norm} X_n^k$. We now recall the necessary definitions from [2]. Let V_2 be the module of the adjoint representation for SL_2 and let X^k be the quotient of V_2 by the group $\mathbb{Z}_{2k}$ acting as scalar multiplications by the $2k$th roots of unity. If we denote the standard coordinate functions in V_2 by a, b, and c, then $K[X^k] = K[a^i b^j c^l \mid i+j+l = 2k]$. We now consider the variety X_n^k that is the spectrum of the algebra $K[a^i b^j c^l (i+j+l=2k),\ z \mid z^n = (b^2-4ac)^k]$. The action of SL_2 on all these varieties is naturally induced by the action on V_2, and the element z is invariant. The valuation morphism $\operatorname{Norm} X_n^k \to X_n^k$ is bijective. We now find the cone corresponding to the embedding $\operatorname{Norm} X_n^k$.

Let ϵ be the fundamental weight of K^* and let λ be the fundamental weight of a maximal torus in SL_2. Since the closure of a B-invariant divisor on the homogeneous space $K^* \times SL_2/N$ contains the unique SL_2-fixed point of $\operatorname{Norm} X_n^k$, it follows that the 'colour' D_1 enters the cone under consideration. There exists precisely one divisor F on $\operatorname{Norm} X_n^k$ that is invariant with respect to $K^* \times SL_2$; namely, the fibre of the quotient morphism for SL_2 containing the fixed point. The function z is SL_2-invariant, and it is an eigenfunction of weight ϵ for the torus K^*.

Let q be the order of the zero of z on F. The function $\dfrac{a^{2k}}{(b^2-4ac)^k}$ is K^*-invariant, and it is semi-invariant of weight $4k\lambda$ with respect to SL_2. This function has a pole of order qn on F. Thus, we can associate with F a valuation that has the coordinates $(-qn/k, q)$ in our notation. The corresponding cone is defined by the pair (n,k).

The variety X_n^k can be obtained from X^k by passing to the fibre product with respect to the morphism $z \to z^n$ of quotients. Hence k corresponds to the order of the stabilizer of a point in a non-closed orbit of the singular fibre of X_n^k, while n is a kind of a 'rotation index' of the variety with respect to this singular fibre. A similar situation occurs in the theory of actions of compact groups on three-dimensional varieties. Namely, it is for the same reasons that oriented Seifert invariants were introduced in Raymond's classification of effective and smooth actions of the circle on smooth closed connected 3-varieties (see, for instance, [15]).

6. Classification of (S,T)-varieties

In a similar way to (S,N)-varieties we can classify (S,T)-varieties of birationally trivial type. The only modification required is the replacement of V_{4r_1} by V_{2r_1} in (1). Theorem 3 holds also in this case.

Consequently, there can be no orbits of type SL_2/N in a birationally trivial (S,T)-variety. This is also implied in étale slice theory because if X^T were reducible, then the slice in a neighbourhood of a closed orbit of type SL_2/N would be reducible and connected, and therefore not normal. On the other hand a point in the slice is normal if and only if it is normal in the variety.

The homogeneous space SL_2/T admits exactly one non-trivial SL_2-equivariant automorphism, and by Lemma 3.2 this automorphism can be extended to each (S,T)-variety. Passing to the quotient by this action of $\mathbb{Z}_2$ we obtain an (S,N)-variety.

Proposition 7. *Each (S,N)-variety admits a two-fold covering in the above way by a unique (S,T)-variety of birationally trivial type. Under this covering, the mark (k_i,n_i) on the (S,N)-variety corresponds to the mark $(2k_i,n_i)$ if n_i is odd and to $(k_i,n_i/2)$ if n_i is even. The fixed point is an isolated branching point in the first case, and there exists a branching divisor in the second.*

The proof follows from the fact that the action of $\mathbb{Z}_2$ commutes with the passage to a fibre product and gluing.

We now consider a birationally non-trivial (S,T)-variety X. The condition that X be non-trivial is equivalent to the condition that the curve X^T be irreducible. By Proposition 3, X has a twofold covering by a birationally trivial (S,T)-variety $\widetilde{X}$, and the covering morphism is the quotient morphism by $\mathbb{Z}_2$ (a two-fold covering is always a Galois covering). Such a covering is unique because the algebra of regular functions on $\widetilde{X}$ coincides with the integral closure of $K[X]$ in the canonically defined extension of degree two of the field $k(X)$.

Example. The three-dimensional space of the adjoint representation of SL_2 is a birationally non-trivial (S,T)-variety. It has a twofold covering that is a birationally trivial (S,T)-variety, the hypersurface $z^2=b^2-4ac$ in four-dimensional space. In our classification of birationally trivial (S,T)-varieties this hypersurface corresponds to a line with a single mark $(1,1)$ at the origin.

The SL_2-equivariant action of $\mathbb{Z}_2$ on $\widetilde{X}$ induces a non-trivial action of $\mathbb{Z}_2$ on the quotient $\widetilde{X}//SL_2=C$. As follows from the classification of birationally trivial (S,T)-varieties, to define a birationally trivial (S,T)-variety with required action of $\mathbb{Z}_2$ it is necessary and sufficient to define an irreducible smooth affine curve with non-trivial action of $\mathbb{Z}_2$ and to put marks on this curve with the only constraint: points belonging to the same $\mathbb{Z}_2$-orbit must be either both unmarked or marked by the same pair of numbers (in particular, there are no restrictions on the marks at $\mathbb{Z}_2$-fixed points). Such a system of marks is said to be $\mathbb{Z}_2$-*invariant*.

Theorem 4. *Birationally non-trivial (S,T)-varieties are in one-to-one correspondence with the following set of data*:

(1) *a smooth irreducible affine curve C endowed with a non-trivial action of $\mathbb{Z}_2$*;
(2) *a (maybe empty) $\mathbb{Z}_2$-invariant system of marks on C.*

Remarks. (1) The unmarked $\mathbb{Z}_2$-fixed points in C correspond to the fibres of type SL_2/N in X.

(2) If X is a birationally non-trivial (S,T)-variety, then the irreducible curve X^T is not necessarily smooth. In particular, it is certainly singular if not only $\mathbb{Z}_2$-fixed points are marked on C.

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