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BRIEF COMMUNICATIONS

On the Normality of the Closures of Spherical Orbits*

I. V. Arzhantsev UDC 512.74

Let X be a normal algebraic variety, let G be a reductive group acting on X by regular automorphisms, and let B be a Borel subgroup of G. The ground field k is assumed to be algebraically closed and $\operatorname{char}(k) = 0$. By the complexity of the action $G{:}X$ we mean the codimension of a generic B-orbit in X or, equivalently, the transcendence degree of the field of rational B-invariants $k(X)^B$. The actions of complexity zero are said to be spherical. An action of algebraic torus with an open orbit is an example of a spherical action. Such actions are called toric. As is known [1], every orbit on a toric variety has normal closure. The similar result holds for an arbitrary spherical variety. Here we prove that the closures of orbits are normal for a class of actions of complexity one.

Theorem. *Let a connected reductive group G act on a normal affine variety X with the generic stabilizer H, let the complexity of this action be one, and let the categorial quotient $X//G$ be one-dimensional. Then the closure of any G-orbit in X is normal.*

Remarks. 1) The theorem holds for nonaffine algebraic varieties if we replace the condition $\dim X//G = 1$ by the existence of the one-dimensional "good quotient" in the sense of Mumford.

2) Let us show that the hypothesis of the theorem cannot be weakened. Consider the action of the group SL_2 on the cubic binary forms as an example of a one-parameter family of orbits of complexity one. In this case, the generic orbit is isomorphic to SL_2/Z_3, the quotient is a line, the null-cone is the closure of the orbit of the form x^2y, and this closure is not normal. The above example fits for the embedding of a homogeneous space of complexity one if we involve the action of the group GL_2. In the case of one-parameter family of spherical orbits, under the condition $\dim X//G = 0$, we can consider the action of the one-dimensional torus on the plane given by the formula $(x,y) \to (t^2x, t^3y)$. For a two-parameter family of spherical orbits with two-dimensional quotient, one can take the action of the one-dimensional torus on the three-dimensional space, $(x,y,z) \to (t^{-1}x, t^2y, t^3z)$.

3) The algebra of invariants of a linear representation of a connected semisimple group with one-dimensional quotient is a ring of polynomials in one variable, and a polynomial generating it is irreducible. The last fact means that the null-cone of this representation is irreducible. Hence, it follows from the etale slice theory that, if, in the assumptions of the theorem, the group G is semisimple and the variety X is nonsingular, then any fiber of the morphism of passage to the quotient containing a fixed point is normal.

Proof of the Theorem. 1. Suppose that in X there is an open G-invariant subset isomorphic to $U \times G/H$, where U is a smooth curve with trivial G-action. Such actions $G{:}X$ are called birationally trivial. In [3] it is shown that, in this case, there exists an open affine covering $\{U_i\}$ of the curve $C = X//G$ such that, for every i, the following diagram is commutative:

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\ \phi\ } & S_i \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} \\ U_i & \xrightarrow{\ \phi_1\ } & A^1 \end{array}$$

where morphisms ϕ and ϕ_1 are etale, and $\pi^{-1}(U_i)$ is the preimage of S_i. Here S_i is an embedding of the

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spherical homogeneous space $k^* \times G/H$ of the group $k^* \times G$, and π and π_1 are the morphisms of passage to the quotient by G. Thus, the closure of any G-orbit in X is isomorphic to the closure of the corresponding G-orbit on some S_i. Every irreducible component of the fiber of the morphism of passage to the quotient for X (and for S_i) contains a dense spherical orbit [2]. Let us show that every such component is normal. The fiber $\pi_1^{-1}(a)$, $a \neq 0$, is normal and irreducible because it is generic [3]. If K is a component of $\pi_1^{-1}(0)$, then K is the closure of some $k^* \times G$-orbit on a spherical variety, and hence K is normal. Thus, every component is a spherical G-variety, which proves the theorem. In particular, the theorem is proved for any action of torus because, as is known, every action of the torus is birationally trivial.

2. For an arbitrary action $G{:}X$, consider the variety $Y = \mathrm{Spec}(k[X]^U)$, where U is a maximal unipotent subgroup of G. There exists a natural action of a maximal torus T of G on Y, and $X//G \cong C \cong Y//T$. According to the Luna–Vust normality criterion, the varieties X and Y are simultaneously normal or not. The action $T{:}Y$ satisfies the conditions of the theorem, and hence it suffices to prove that the algebra of U-invariants on an irreducible component Z_i of the fiber Z of the morphism $\pi\colon X \to C$ is T-isomorphic to the algebra of regular functions on some irreducible component $\widetilde{Z}_i$ of a fiber $\widetilde{Z}$ of the morphism $\tilde{\pi}\colon Y \to C$. Let $k[Z]$ be the reduced algebra of functions on the fiber. It is clear that $k[Z]^U$ is the reduced algebra of functions on the corresponding fiber $\widetilde{Z}$. The decomposition into the irreducible components corresponds to the choice of minimal prime ideals $I_1, \dots, I_k$ of the algebra $k[Z]$ satisfying the condition $I_1 \cap \dots \cap I_k = \{0\}$. On passing to the algebra of U-invariants we obtain $I_1^U \cap \dots \cap I_k^U = \{0\}$. We can readily show that $I_1^U, \dots, I_k^U$ are minimal prime ideals of the algebra $k[Z]^U$ and $(k[Z]/I_j)^U = k[Z]^U/I_j^U$. This proves the theorem.

Let a reductive group G act on an affine variety X. The set X^{sp} of points with spherical orbits is closed in X [2]. Assume that the closure of an orbit belonging to X^{sp} is not normal. If this orbit belongs to a one-parameter family of orbits of the same dimension in X^{sp} and the closure of this family contains more than one closed orbit, then we can conclude from the theorem that this closure is also not normal.

Example 1. Let V_d be the space of binary forms of degree d with the natural action of the group SL_2. Consider the SL_2-module $V = A_1 \oplus \dots \oplus A_l \oplus V_{2d_1} \oplus \dots \oplus V_{2d_k}$, where A_i are one-dimensional trivial SL_2-modules. Note that the generic stabilizer exists for any action of SL_2. Consider the set

$$L = \{(\alpha_1, \dots, \alpha_l, \alpha_{l+1} x^{d_1} y^{d_1}, \dots, \alpha_{l+k} x^{d_k} y^{d_k})\},$$

where $(\alpha_1, \dots, \alpha_{l+k})$ lies on a curve C in the $(l+k)$-dimensional space. Suppose that $X = \overline{SL_2 L}$. Each of the three-dimensional affine varieties with stable nontransitive SL_2-action can be represented in this way. Assume that there is a fixed point $x = (\alpha_1^0, \dots, \alpha_l^0, 0, \dots, 0)$ in X. If the curve C is given explicitly, then we can find an orbit whose closure contains x. It has the form

$$SL_2(\alpha_1^0, \dots, \alpha_l^0, \alpha_{l+1}^0 x^{2d_1}, \dots, \alpha_{l+p}^0 x^{2d_p}, 0, \dots, 0)$$

after a suitable reindexing of the coordinates. If $d_i > \gcd(d_1, \dots, d_p)$ for all $i = 1, \dots, p$, then the closure of this orbit is not normal, and hence the same holds for the variety X.

Example 2. In [6, Sec. 10], the normality of the closure of the G-orbit of a line from a maximal torus in the space of the adjoint representation of a simple group G is studied. Proposition 10.2 in [6] claims that this closure is normal in the case of $G = SO_n$, $n = 2k+1$, and a generic line. Let us give an example of a line L in the algebra $g = so_n$, $n = 2k+1$, for which $\overline{GL}$ is not normal. Assume that $L = \langle h_{\epsilon_k} \rangle$ is the line spanned by a semisimple element h_{ϵ_k}, see [7, p. 158 of the Russian edition], and $k = 2l+1$. Then the closure of the orbit of this line contains the orbit of a nilpotent element corresponding to the Jordan blocks of the sizes $(3, 2, \dots, 2)$. The orbit of such an element is spherical [8, Th. 4.4], and hence the orbits in its sheet are spherical as well. The closure of this orbit is not normal [9, Sec. 3.4, 16.2]. Therefore, the closure of the orbit of L is not normal. Note that, according to [6, Prop. 10.3], the cone $\pi^{-1}(\pi(L))$ is normal, where π is the morphism of passage to the quotient for the adjoint representation, and this cone is a Cohen–Macaulay variety. In particular, the Richardson necessary normality condition [6] holds for $\overline{GL}$, and thus the fact that $\overline{GL}$ is not normal cannot be obtained by the approach in [6].

Unfortunately, we could not prove the normality of one-parameter families of spherical orbits by our method.

Example 3. Let the one-dimensional torus act on the surface $x^2 = yz^3$ according to the formula $(x, y, z) \to (tx, t^2y, z)$. Then the fibers of the morphism of passage to the quotient are affine lines that are normal, and the quotient is normal as well, but the surface itself is not normal.

Finally, let us list the linear groups from two classes indicated below that satisfy the conditions of the theorem.

The irreducible representations of semisimple groups: SO_n, S^2SL_n, Λ^2SL_{2k}, $Spin_7$, $Spin_9$, G_2, E_6, $SL_n \otimes SL_n$, $SL_2 \otimes Sp_n$, $SL_4 \otimes Sp_4$.

Reducible representations of simple groups: $SL_n \oplus SL_n'$, $SL_n \oplus \Lambda^2SL_n$, $SL_{2k}' \oplus \Lambda^2SL_{2k}$, where $'$ stands for the dual representation.

One can also show that a reducible representation of a simple group cannot be spherical. The list of irreducible actions can be obtained from Theorem 3 of [4], and one can find the desired reducible representations of simple groups by the Elashvili tables [5].

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On the Convergence of the Trace of a Power of the Laplace–Beltrami Operator with a Potential on the Sphere S^n *

A. N. Bobrov and V. E. Podol'skii UDC 517.956.227

Let Δ_0 be the Laplace–Beltrami operator on the sphere $S^n \subset R^{n+1}$. We denote by $\Delta = \Delta_0 + (n-1)^2/4$ the normalized Laplace–Beltrami operator and by q the operator of multiplication by a smooth function on S^n. Let $\{\lambda_k\}_{k=0}^{\infty}$ and $\{\mu_k\}_{k=0}^{\infty}$ be the eigenvalues of the operators $-\Delta$ and $-\Delta + q$, respectively, enumerated in the ascending order of real parts, with multiplicity taken into account. The spectrum of the operator $-\Delta$ on S^n is well known [1]: it is formed by the points $\lambda_{ki} = (k + (n-1)/2)^2$, $k = 0, 1, \dots$, $i = 1, \dots, N_k$, where $N_k \sim ck^{n-1}$. For the eigenvalues of the operator $-\Delta + q$ we also use the double indexing μ_{ki} that agrees with the indexing of λ_{ki}.

In [2–5], V. Guillemin studied in detail the spectrum of the operator $-\Delta + q$ on M, where M is a symmetric space of rank one (see, e.g., [1]) and $q \in C^{\infty}(M)$; in particular, he showed that, for all M

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