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MATHEMATICAL PROBLEMS OF NONLINEARITY

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On the Centralizer and Conjugacy of Pseudo-Anosov Homeomorphisms

E. E. Chilina

The present paper is devoted to the study of the dynamics of mappings commuting with pseudo-Anosov surface homeomorphisms. It is proved that the centralizer of a pseudo-Anosov homeomorphism P consists of pairwise nonhomotopic mappings, each of which is a composition of a power of the pseudo-Anosov mapping and a periodic homeomorphism. For periodic mappings commuting with P, it is proved that their number is finite and does not exceed the number N_P, which is equal to the minimum among the number of all separatrices related to saddles of the same valency of P-invariant foliations. For a periodic homeomorphism J lying in the centralizer of P, it is also shown that, if the period of a point is equal to half the period of the homeomorphism J, then this point is located in the complement of the separatrices of saddle singularities. If the period of the point is less than half the period of J, then this point is contained in the set of saddle singularities. In addition, it is proved that there exists a monomorphism from the group of periodic maps commuting with a pseudo-Anosov homeomorphism to the symmetric group of degree N_P. Each permutation from the image of the monomorphism is represented as a product of disjoint cycles of the same length. Furthermore, it is deduced that a pseudo-Anosov homeomorphism with the trivial centralizer exists on each orientable closed surface of genus greater than 2. As an application of the results related to the structure of the centralizer of pseudo-Anosov homeomorphisms to their topological classification, it is proved that there are no pairwise distinct homotopic conjugating mappings for topologically conjugated pseudo-Anosov homeomorphisms.

Keywords: pseudo-Anosov homeomorphism, topological conjugacy, centralizer

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Ekaterina E. Chilina
k.chilina@yandex.ru
HSE University
ul. Bolshaya Pecherckaya 25/12, Nizhny Novgorod, 603155 Russia

Introduction

Let $f\colon M^n \to M^n$ be a homeomorphism on a topological manifold M^n of dimension n. Denote by $Z(f) = \{g\colon M^n \to M^n \mid gf = fg\}$ the *centralizer* of f (a set of all homeomorphisms on M^n commuting with f). The centralizer $Z(f)$ of $f\colon M^n \to M^n$ is called *trivial* if $Z(f) = \{f^n \mid n \in \mathbb{Z}\}$.

It is important to know the structure of the centralizer for a homeomorphism $f\colon M^n \to M^n$, as an example, for projecting f onto a topological space M^n/G by the natural projection $p_G\colon M^n \to M^n/G$, where G is an infinite cyclic group of homeomorphisms with a generator $g\colon M^n \to M^n$ acting freely and dicontinuously on M^n. If $g \in Z(f)$, then the projection $\widetilde{f}\colon M^n/G \to M^n/G$ of $f\colon M^n \to M^n$ onto M^n/G by the formula $\widetilde{f}(w) = p_G\left(f\left(p_G^{-1}(w)\right)\right)$ is a homeomorphism. Such constructions are used for realization of maps in some topological conjugacy classes of maps (see, for example, [1, 2]). Another application of mapping centralizers is the study of foliations (see, for example, [3, 4]). In particular, maps with the trivial centralizer are used for construction of structurally stable foliations (see, for example, [5]). Furthermore, the topological classification of mappings for some classes of dynamical systems requires knowledge of the structure of their centralizer (see, for example, [6]).

In 1967, S. Smale in his review [7] formulated the problem of describing the centralizer of an arbitrary diffeomorphism on a compact manifold M^n. At present, a significant number of results on this problem have been obtained. For example, N. Kopell [8] proved that there is an open and dense subset of circle diffeomorphisms in the space C^2 with the trivial centralizer. D. Palis and D. Yoccoz [9] proved that on a compact smooth manifold M^n the set of diffeomorphisms with the trivial centralizer is everywhere dense in the set of structurally stable (satisfying axiom A and the strong transversality condition) C^∞-diffeomorphisms with a periodic sink (source). At the same time, for C^1-diffeomorphisms on every compact manifold M^n, C. Bonatti, S. Crovisier, J. Vago, and G. M. Wilkinson in [10] constructed a nonempty open set of mappings containing a dense subset of diffeomorphisms with a nontrivial centralizer. D. Damjanovic, A. Wilkinson and D. Xu in [11] proved that the C^∞-centralizer of a partially hyperbolic diffeomorphism of a nilmanifold $M^3 \neq \mathbb{T}^3$ contains a normal subgroup which is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}\times(\mathbb{R}/\mathbb{Z})$.

In 1998 R. V. Plykin [12] proved that the centralizer of an Anosov diffeomorphism of an n-dimensional torus is isomorphic to the direct product $\mathbb{Z}^l \oplus F$, $l \in \mathbb{N}$, where F is a finite commutative group. In particular, in the case of a two-dimensional torus, the centralizer of an Anosov diffeomorphism is isomorphic to the group $\mathbb{Z}\oplus\mathbb{Z}_2$ and consists of powers of some Anosov diffeomorphism and their superposition with a periodic map of period 2.

Throughout this paper, M^2 is understood to be a closed orientable surface of genus greater than one.

A generalization of Anosov diffeomorphisms of a two-dimensional torus to surfaces of genus greater than one are pseudo-Anosov homeomorphisms (see the precise definitions in the section 1). A homeomorphism $P\colon M^2 \to M^2$ is called a *pseudo-Anosov map* with *dilatation* $\lambda > 1$ if on the surface M^2 there exists a pair of P-invariant measured transverse foliations $(\mathcal{F}_P^s, \mu^s)$, $(\mathcal{F}_P^u, \mu^u)$ with a set of saddle singularities $\mathcal{S}_P$ such that each saddle singularity from $\mathcal{S}_P$ has at least three separatrices and $\mu^s(P(\alpha)) = \lambda\mu^s(\alpha)$ $\left(\mu^u(P(\alpha)) = \lambda^{-1}\mu^u(\alpha)\right)$ for any arc α transverse to the foliation $\mathcal{F}_P^S$ $(\mathcal{F}_P^u)$.

In 1982, J. D. McCarthy published a preprint [13], in which he proved a theorem stating that the centralizer of a pseudo-Anosov homeomorphism of a surface is a finite extension of an infinite cyclic group.

The present paper is devoted to the study of dynamical properties of mappings commuting with pseudo-Anosov homeomorphisms and the structure of the group that they form.

For $q \in \mathbb{N}$, we denote by b_q the number of singular points of valency q of invariant foliations of a pseudo-Anosov homeomorphism P. The sequence $\mathcal{B}_P = \{b_q \colon q \in \mathbb{N}\}$ is called the *singular type* of P. Since the sequence $\mathcal{B}_P$ contains only a finite number of nonzero elements, for brevity we will further indicate only the nonzero elements of this sequence. We denote the smallest number of separatrices related to saddles of the same valency by

$$N_P = \min_{b_q \in \mathcal{B}_P} \{qb_q\}.$$

A homeomorphism $f \colon M^2 \to M^2$ is called *periodic* if there exists $n \in \mathbb{N}$ such that $f^n = \mathrm{id}$. The least such n is called the period of f. Let us denote by $Z_{Per}(P)$ the subset of all periodic homeomorphisms of the set $Z(P)$ (since the set $Z(P)$ contains the identity homeomorphism id_{M^2}, the set $Z(P)$ is nonempty) and by $|Z_{Per}(P)|$ the cardinality of the set $Z_{Per}(P)$.

By S_n we denote the symmetric group of degree $n \in \mathbb{N}$.

Theorem 1. *Let $P \colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism. Then the following statements are true*:

1) *the homeomorphism $J \in Z(P)$ is either a pseudo-Anosov or periodic homeomorphism*;
2) *$Z_{Per}(P)$ is a normal subgroup of $Z(P)$*;
3) *$|Z_{Per}(P)| \leqslant N_P$*;
4) *there is a monomorphism $\zeta_P \colon Z_{Per}(P) \to S_{N_P}$ such that*
$$\zeta_P(J) = (a_{11} \ldots a_{1n}) \cdots (a_{m1} \ldots a_{mn}) \in S_{N_P},$$
where n is a period of $J \in Z_{Per}(P)$ and $mn = N_P$;
5) *if $J \in Z_{Per}(P)$ is a homeomorphism of period n, then n divides qb_q for each $b_q \in \mathcal{B}_P$*;
6) *a homeomorphism $J \in Z(P)$ has the form $J = p^i \iota$, where p is the pseudo-Anosov homeomorphism with the singular type $\mathcal{B}_p = \mathcal{B}_P$, $i \in \mathbb{Z}$ and $\iota \in Z_{Per}(P)$.*

Let $f \colon M^2 \to M^2$ be a periodic homeomorphism of period n. A point $x \in M^2$ is called a point of period $k \in \mathbb{N}$ if k is the smallest natural number such that $f^k(x) = x$. It is easy to check that k divides n. Hence, $k \leqslant \frac{n}{2}$. We denote by Q_f^k the set of all points of period k and set $Q_f = \bigcup\limits_{1 \leqslant k \leqslant n/2} Q_f^k$.

Let $P \colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism. We recall that $\mathcal{S}_P$ is the set of saddle singularities of the foliations $\mathcal{F}_P^s$, $\mathcal{F}_P^u$, and denote by $\mathcal{L}_P$ the set of all separatrices of saddle singularities for these foliations (note that $\mathcal{L}_P \cap \mathcal{S}_P = \varnothing$).

Theorem 2. *Let $P \colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism and $J \in Z_{Per}(P)$ is a homeomorphism of period n. Then the following statements are true*:

1) *if J reverses the orientation of M^2 and $n = 2(2m-1)$, $m \in \mathbb{N}$, then $Q_J^{n/2} = \varnothing$ and $Q_J \subset \mathcal{S}_P$*;
2) *if J reverses the orientation of M^2 and $n = 4m$, $m \in \mathbb{N}$, then $Q_J^{n/2} \subset \left(M^2 \setminus \mathcal{L}_P\right)$ and $Q_J^k \subset \mathcal{S}_P$ for any $k < \frac{n}{2}$*;

3) *if J preserves the orientation of M^2 and $n = 2m - 1$, $m \in \mathbb{N}$, then $Q_J \subset \mathcal{S}_P$;*

4) *if J preserves the orientation of M^2 and $n = 2m$, $m \in \mathbb{N}$, then $Q_J^{n/2} \subset \big(M^2 \setminus \mathcal{L}_P\big)$ and $Q_J^k \subset \mathcal{S}_P$ for any $k < \frac{n}{2}$.*

Theorem 3. *Each closed orientable surface of genus $g \geqslant 3$ admits a pseudo-Anosov homeomorphism with the trivial centralizer.*

Theorem 4. *Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism. If the homeomorphism $J_1 \in Z(P)$ is homotopic to the homeomorphism $J_2 \in Z(P)$, then $J_1 = J_2$.*

Corollary. *Let P_1, $P_2\colon M^2 \to M^2$ be pseudo-Anosov homeomorphisms and there exist homeomorphisms H_1, $H_2\colon M^2 \to M^2$ such that $H_1P_1 = P_2H_1$, $H_2P_1 = P_2H_2$. If the homeomorphism H_1 is homotopic to the homeomorphism H_2, then $H_1 = H_2$.*

1. Main definitions and auxiliary statements

This section contains main definitions and auxiliary statements used in the paper.

A family $\mathcal{F} = \{L_i\colon i \in \mathcal{I}\}$ of path-connected subsets of a topological manifold M^n of dimension n is called a *k-dimensional foliation* if it satisfies the following three conditions:

- $L_i \cap L_j = \varnothing$ for any i, $j \in \mathcal{I}$ such that $i \neq j$;
- $\bigcup\limits_{i \in \mathcal{I}} L_i = M^n$;
- for any point $x \in M^n$ one can choose a local map (U_x, φ_x), $x \in U_x$, which we call the *local map of the foliation $\mathcal{F}$*, such that if $U_x \cap L_i \neq \varnothing$, $i \in \mathcal{I}$, then the path-connected components of the set $\varphi_x(U_x \cap L_i)$ have the form $\{(x_1, x_2, \ldots, x_n) \in \varphi_x(U_x);\ x_{k+1} = c_{k+1}, x_{k+2} = c_{k+2}, \ldots, x_n = c_n\}$, where the numbers c_{k+1}, c_{k+2}, $\ldots$, c_n are constant.

An element of a family $\mathcal{F}$ is called a *leaf*.

A foliation is called *transitive* on M^n if it contains a leaf that is everywhere dense on M^n.

A foliation $\mathcal{F}$ with a set of singularities $\mathcal{S}$ on a manifold M^n is a family of path-connected subsets of M^n such that the family $\mathcal{F} \setminus \mathcal{S}$ is a foliation of $M^n \setminus \mathcal{S}$.

A foliation $\mathcal{F}$ is called *invariant under the homeomorphism $P\colon M^n \to M^n$* (*P-invariant*) if P maps a leaf of $\mathcal{F}$ to a leaf of $\mathcal{F}$.

Throughout this paper, M^2 is a closed orientable surface.

A foliation W_q, $q \in \mathbb{N}$, on $\mathbb{R}^2$ with a standard saddle singularity at $O = (0, 0)$ and q separatrices is a foliation with a singularity set $\mathcal{S} = \{O\}$ on $\mathbb{R}^2$ and nonsingular leaves defined by the relation $(x_1 + ix_2)^q = (t + ic)^2$ if q is odd, and $(x_1 + ix_2)^{0.5q} = t + ic$ if q is even, where $t \in \mathbb{R}$, $c = \text{const}$. Rays $l_1, \ldots, l_q \in W_q$ for which $c = 0$ are called *separatrices* of O (see Fig. 1).

A one-dimensional foliation $\mathcal{F}$ on M^2 is called a *foliation with saddle singularities* if the set $\mathcal{S}$ of singularities of $\mathcal{F}$ is a finite number of points $s_1, \ldots, s_c$ and for any point s_i, $i \in \{1, \ldots, c\}$, there are a neighborhood $U_i \subset M^2$, a homeomorphism $\psi_i\colon U_i \to \mathbb{R}^2$ and a number $q_i \in \mathbb{N}$ such that $\psi_i(s_i) = O$ and $\psi_i(\mathcal{F} \cap U_i) = W_{q_i} \setminus \{O\}$. The leaf containing the curve $\psi_i^{-1}(l_j)$, $j \in \{1, \ldots, q_i\}$, is called a *separatrix* of the point s_i. The point s_i is called a *saddle singularity with q_i separatrices*, or a *singular point of valency q_i*.

Let $\mathcal{F}$ be a one-dimensional foliation with a set of singularities $\mathcal{S}$ on a manifold M^2. By an *arc α* on M^2 we mean a continuous image of the segment $[0, 1]$ (the interval $(0, 1)$ or the

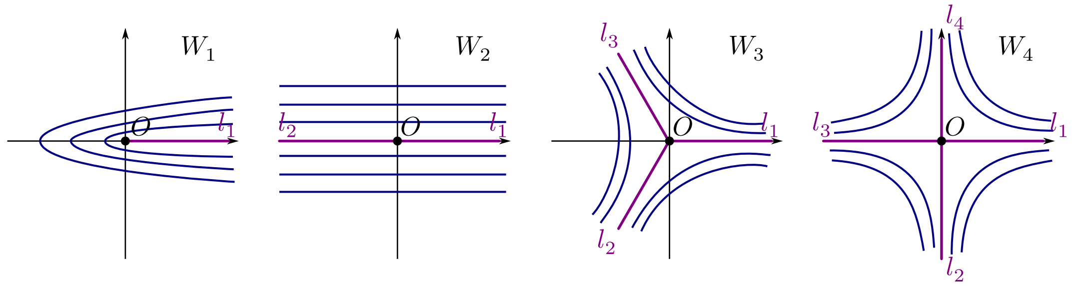

Fig. 1. Foliation W_q on $\mathbb{R}^2$ with a saddle singularity at O and q separatrices for $q = 1, 2, 3, 4$

half-interval $[0, 1)$, $(0, 1]$) on M^2. An arc without self-intersections $\alpha \subset M^2$ *is transversal to a foliation* $\mathcal{F}$ if for every nonsingular point $x \in \alpha$ of $\mathcal{F}$ there exists a local chart (U_x, φ_x) of $\mathcal{F}$ such that $\varphi_x(\alpha) = Ox_2$. Two one-dimensional foliations are called *transverse* to each other if they have a common set of singularities and any arc of any leaf of one of them is transversal to the other.

A transversal measure μ for a foliation $\mathcal{F}$ with saddle singularities on M^2 is a real-valued function that is defined on all arcs on M^2 and has the following properties:

1) $\mu(\alpha) \geqslant 0$ for any arc α on M^2;

2) $\mu(\alpha) = 0$ if and only if the arc α is completely contained in a leaf of the foliation $\mathcal{F}$;

3) if $\alpha_1 \cap \alpha_2 = \varnothing$, then $\mu(\alpha_1 \cup \alpha_2) = \mu(\alpha_1) + \mu(\alpha_2)$;

4) if α_0 and α_1 are two arcs transversal to $\mathcal{F}$ and connected by a homotopy $F\colon [0, 1] \times [0, 1] \to M^2$ such that $F([0, 1] \times \{0\}) = \alpha_0$, $F([0, 1] \times \{1\}) = \alpha_1$ and the arc $F(\{t\} \times [0, 1])$, $t \in [0, 1]$, is contained in some leaf of the foliation $\mathcal{F}$, then $\mu(\alpha_0) = \mu(\alpha_1)$ (see Fig. 2).

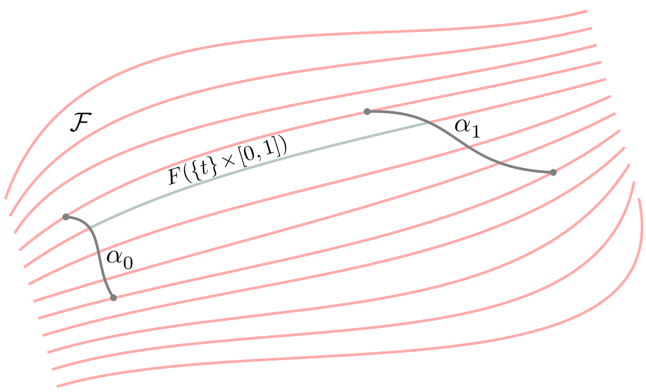

Fig. 2. Curves α_0 and α_1 are connected by homotopy F

A pair $(\mathcal{F}, \mu)$ consisting of a foliation $\mathcal{F}$ with saddle singularities and its transversal measure μ is called a measured foliation.

We recall that a homeomorphism $P\colon M^2 \to M^2$ is called a *pseudo-Anosov map* with *dilatation* $\lambda > 1$ if on the surface M^2 there exists a pair of P-invariant measured transverse foliations $(\mathcal{F}^s_P, \mu^s)$, $(\mathcal{F}^u_P, \mu^u)$ with a set of saddle singularities $\mathcal{S}_P$ such that each saddle singularity from $\mathcal{S}_P$ has at least three separatrices and $\mu^s(P(\alpha)) = \lambda\mu^s(\alpha)$ $\big(\mu^u(P(\alpha)) = \lambda^{-1}\mu^u(\alpha)\big)$ for any arc α transverse to the foliation $\mathcal{F}^S_P$ $(\mathcal{F}^u_P)$.

Statement 1 (see [14, Lemma 3.1]). *A homeomorphism P' that is topologically conjugate to a pseudo-Anosov homeomorphism P is also pseudo-Anosov. Moreover, the conjugating homeomorphism maps singular points of the P-invariant foliations to singular points of the same valency of the P'-invariant foliations.*

Statement 2 (see [14, Theorem 3.2]). *The set of periodic points of a pseudo-Anosov homeomorphism is everywhere dense on the surface.*

Statement 3 (see [14, p. 42]). *The dilatation of a pseudo-Anosov homeomorphism is a topological invariant.*

Statement 4 (see [14, Theorem 19.6]). *The set of topological conjugacy classes of pseudo-Anosov homeomorphisms of a fixed singular type with a dilatation bounded above is finite.*

Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism. For the point $x \in M^2$ we define the stable $W^s(x) = \left\{y \in M^2\colon d\left(P^n(x),\, P^n(y)\right) \to 0,\ n \to +\infty\right\}$ and unstable $W^u(x) = \left\{y \in M^2\colon d\left(P^n(x),\, P^n(y)\right) \to 0,\ n \to -\infty\right\}$ manifolds of point x. The following fact follows from the definition of pseudo-Anosov homeomorphism.

Statement 5. *Let P be a pseudo-Anosov homeomorphism. Then the stable (unstable) manifold of the point $x \notin \mathcal{S}_P$ is the leaf of the foliation $\mathcal{F}^s_P$ $(\mathcal{F}^u_P)$ containing x. The stable (unstable) manifold of the point $x \in \mathcal{S}_P$ is the union of x and its separatrices of the foliation $\mathcal{F}^s_P$ $(\mathcal{F}^u_P)$.*

A rectangle is a subset of $\Pi \subset M^2$ that is the image under the continuous mapping υ of the square $[0,\, 1] \times [0,\, 1]$ into M^2 with the following properties: υ is one-to-one on the interior of the square and maps horizontal segments of the square into arcs of the leaves of $\mathcal{F}^s_P$, and vertical segments into arcs of the leaves of $\mathcal{F}^u_P$. We denote by $\dot\Pi$ the image of the interior of the square. The images of the horizontal and vertical sides will be called the *contracting* and *stretching* sides of the rectangle Π.

A Markov partition for a pseudo-Anosov homeomorphism P is a finite family of rectangles $\widetilde\Pi = \{\Pi_1,\, \ldots,\, \Pi_n\}$ for which the following conditions are satisfied:

- $\bigcup\limits_i \Pi_i = M^2$; $\dot\Pi_i \cap \dot\Pi_j = \varnothing$ when $i \neq j$;
- $P(\partial^s\widetilde\Pi) \subset \partial^s\widetilde\Pi$ $(P(\partial_u\widetilde\Pi) \supset \partial_u\widetilde\Pi)$ for the union $\partial^s\widetilde\Pi$ $(\partial_u\widetilde\Pi)$ of all contracting (stretching) sides of parallelograms $\Pi_1,\, \ldots,\, \Pi_n$.

Statement 6 (see [15, Proposition 10.17]). *A pseudo-Anosov homeomorphism has a Markov-partition.*

Statement 7 (see [14, Lemma 3.5]). *Let P be a pseudo-Anosov homeomorphism and l^u (l^s) be the separatrix of a saddle singularity of the foliation $\mathcal{F}^u_P$ $(\mathcal{F}^s_P)$. Then for any arc α of any leaf of the foliation $\mathcal{F}^s_P$ $(\mathcal{F}^u_P)$ the intersection $l^u \cap \alpha$ $(l^s \cap \alpha)$ consists of a countable number of points, each of which is isolated in the inner topology l^u (l^s), i. e., the topology induced by the immersion $\mathbb{R} \to l^u$ $(\mathbb{R} \to l^s)$.*

Statement 8 (see [14, Exercise 3.3]). *Each leaf of the foliations $\mathcal{F}^s_P$ and $\mathcal{F}^u_P$ of a pseudo-Anosov homeomorphism P is everywhere dense on M^2.*

A foliation $\mathcal{F}$ is called *uniquely ergodic* if it admits a unique (up to multiplication by a constant) transversal measure.

Statement 9 (see [15, Theorem 12.1]). *The foliations $\mathcal{F}^s_P$ and $\mathcal{F}^u_P$ of a pseudo-Anosov homeomorphism P are uniquely ergodic.*

Statement 10 (The Euler – Poincaré formula, [14, Theorem 5.4]). *Let $\mathcal{B}_P = \{b_q\colon q \in \mathbb{N}\}$ be the singular type of a pseudo-Anosov homeomorphism $P\colon M^2 \to M^2$ on the surface M^2 of genus g. Then*

$$\sum_{b_q \in \mathcal{B}_P} (2-q)b_q = 2(2-2g).$$

Statement 11 (see [16]). *There exists a pseudo-Anosov homeomorphism of any singular type allowed by the Euler – Poincaré formula except for the singular type $\{b_3 = 1,\ b_5 = 1\}$.*

2. The structure of the centralizer of a pseudo-Anosov homeomorphism

This section contains a proof of Theorem 1 divided into lemmas.

Lemma 1. *Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism with measured foliations $(\mathcal{F}^s_P,\ \mu^s)$, $(\mathcal{F}^u_P,\ \mu^u)$. Then for any homeomorphism $J \in Z(P)$ the foliations $\mathcal{F}^u_P$ and $\mathcal{F}^s_P$ are J-invariant and the homomorphism $\eta_P\colon Z(P) \to \mathbb{R}_+$ is well defined by the formula $\eta_P(J) = \nu$, where $\mu^s(J(\alpha)) = \nu\mu^s(\alpha)$, $\mu^u(J(\alpha)) = \frac{1}{\nu}\mu^u(\alpha)$, α is an arc on M^2.*

Proof. Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism with measured foliations $(\mathcal{F}^s_P,\ \mu^s)$, $(\mathcal{F}^u_P,\ \mu^u)$ and $J \in Z(P)$. Since $P = JPJ^{-1}$, the homeomorphism J maps stable manifolds of the homeomorphism P to stable ones and unstable manifolds to unstable ones. Therefore, $J(\mathcal{F}^s_P) = \mathcal{F}^s_P$ and $J(\mathcal{F}^u_P) = \mathcal{F}^u_P$. For the foliation $\mathcal{F}^s_P$ ($\mathcal{F}^u_P$), we define a transversal measure $\widetilde{\mu}^s(\alpha) = \mu^s(J(\alpha))$ ($\widetilde{\mu}^u(\alpha) = \mu^u(J(\alpha))$), where α is an arc transverse to the foliation $\mathcal{F}^s_P$ ($\mathcal{F}^u_P$). Since the foliations $\mathcal{F}^s_P$, $\mathcal{F}^u_P$ are uniquely ergodic (Statement 9), there exist numbers ν^s, $\nu^u \in \mathbb{R}_+$ such that $\widetilde{\mu}^s = \nu^s\mu^s$ and $\widetilde{\mu}^u = \nu^u\mu^u$. Thus, $\mu^s(J(\alpha)) = \nu^s\mu^s(\alpha)$, $\mu^u(J(\alpha)) = \nu^u\mu^u(\alpha)$.

Since the pseudo-Anosov homeomorphism P has a Markov partition (see Proposition 6) consisting of $n \in \mathbb{N}$ rectangles $\Pi_1, \ldots, \Pi_n$, the measure $\mu^s \otimes \mu^u$ is well defined on each rectangle Π_i ($i \in \{1, \ldots, n\}$) by the formula

$$\mu^s \otimes \mu^u(\Pi_i) = \mu^s(\alpha_{s,i})\mu^u(\alpha_{u,i}) = \mu_i,$$

where $\alpha_{s,i}$ is the stretching side of the rectangle Π_i and $\alpha_{u,i}$ is the contracting side. Since the foliations $\mathcal{F}^s_P$, $\mathcal{F}^u_P$ are invariant under J, the set $J(\Pi_i)$ ($i \in \{1, \ldots, n\}$) is also a rectangle with measure

$$\mu^s \otimes \mu^u(J(\Pi_i)) = \mu^s(J(\alpha_{s,i}))\mu^u(J(\alpha_{u,i})) = \nu^s\nu^u\mu_i.$$

Thus,

$$\mu^s \otimes \mu^u\left(M^2\right) = \mu^s \otimes \mu^u\left(\bigcup_i \Pi_i\right) = \sum_i \mu_i$$

and

$$\mu^s \otimes \mu^u\left(J\left(M^2\right)\right) = \mu^s \otimes \mu^u\left(\bigcup_i (J(\Pi_i))\right) = \nu^s\nu^u\left(\sum_i \mu_i\right).$$

Since $J\left(M^2\right) = M^2$, we get

$$\nu^s \nu^u = 1.$$

Put $\nu^s = \nu$. Then $\mu^s(J(\alpha)) = \nu\mu^s(\alpha)$, $\mu^u(J(\alpha)) = \frac{1}{\nu}\mu^u(\alpha)$. Thus, the mapping

$$\eta_P\colon Z(P) \to \mathbb{R}_+$$

is well defined by the formula

$$\eta_P(J) = \nu,$$

where $\mu^s(J(\alpha)) = \nu\mu^s(\alpha)$, $\mu^u(J(\alpha)) = \frac{1}{\nu}\mu^u(\alpha)$, α is an arc on M^2.

Let J_1, $J_2 \in Z(P)$, $\eta_P(J_1) = \nu_1$ and $\eta_P(J_2) = \nu_2$, that is,

$$\mu^s(J_1(\alpha)) = \nu_1\mu^s(\alpha), \quad \mu^u(J_1(\alpha)) = \frac{1}{\nu_1}\mu^u(\alpha),$$
$$\mu^s(J_2(\alpha)) = \nu_2\mu^s(\alpha), \quad \mu^u(J_2(\alpha)) = \frac{1}{\nu_2}\mu^u(\alpha).$$

Then

$$\mu^s(J_1(J_2(\alpha))) = \nu_1\mu^s(J_2(\alpha)) = \nu_1\nu_2\mu^s(\alpha),$$
$$\mu^u(J_1(J_2(\alpha))) = \frac{1}{\nu_1}\mu^u(J_2(\alpha)) = \frac{1}{\nu_1\nu_2}\mu^u(\alpha).$$

Therefore, $\eta_P(J_1J_2) = \eta_P(J_1)\eta_P(J_2)$, that is, $\eta_P\colon Z(P) \to \mathbb{R}_+$ is a homomorphism. □

Lemma 2. *Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism and $J \in Z(P)$. Then the following statements hold*:

1) $\eta_P(J) \neq 1$ *if and only if J is a pseudo-Anosov homeomorphism*;

2) $\eta_P(J) = 1$ *if and only if J is a periodic homeomorphism.*

Proof. Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism, $J \in Z(P)$ and $\eta_P(J) = \nu$. We consider two cases separately: 1) $\nu \neq 1$ and 2) $\nu = 1$.

1) Let $\nu \neq 1$. By Lemma 1, the homeomorphism J has a pair of invariant transverse foliations $\mathcal{F}_P^s$, $\mathcal{F}_P^u$ with a common set of saddle singularities that have at least three separatrices and transversal measures μ^s, μ^u such that $\mu^s(J(\alpha)) = \nu\mu^s(\alpha)$, $\mu^u(J(\alpha)) = \nu^{-1}\mu^u(\alpha))$ for any arc α on M^2. Therefore, the homeomorphism J is a pseudo-Anosov map with dilatation $\nu > 1$ $\left(\frac{1}{\nu} > 1\right)$.

2) Let $\nu = 1$. Since the foliation $\mathcal{F}_P^u$ is invariant under $J \in Z(P)$, the separatrices of saddle singularities under the action of J are mapped into separatrices of saddle singularities. Finiteness of the set of separatrices implies that for a separatrix l^u of a saddle singularity z of the foliation $\mathcal{F}_P^u$ there exists $m \in \mathbb{N}$ such that $J^m\left(l^u\right) = l^u$ and $J^m(z) = z$.

Let us prove that $J^m(x) = x$ for any point $x \in l^u$. Let $[z, x]$ be the arc of the curve l^u bounded by the points z and x. Since $\mu^s\left(J^m[z, x]\right) = \mu^s([z, x])$, we have $J^m([z, x]) = [z, x]$. Therefore, $J^m(x) = x$.

Since the leaf l^u is everywhere dense on M^2 (see Statement 8) and $J^m|_{l^u} = \mathrm{id}$, we get that $J^m(x) = x$ for any $x \in M^2$ and J is a periodic homeomorphism. □

Thus, point 1 of Theorem 1 is proved. The proof of Lemma 2 implies Lemma 3.

Lemma 3. *Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism, l^u be a separatrix of a saddle singularity of the foliation $\mathcal{F}_P^u$ and $J \in Z_{Per}(P)$ be a homeomorphism of period n. Then the following statements hold:*

1) *the period of $x \in l^u$ is equal to n;*

2) *if $J(l^u) = l^u$, then $n = 1$.*

Lemma 4. *Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism. Then $Z_{Per}(P)$ is a normal subgroup of $Z(P)$.*

Proof. Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism and $\iota_1, \iota_2 \in Z_{Per}(P)$. Then $\eta_P\left(\iota_1\iota_2^{-1}\right) = \eta_P(\iota_1)\eta_P\left(\iota_2^{-1}\right) = 1$. By Lemma 2, $\iota_1\iota_2^{-1} \in Z_{Per}(P)$. This means that $Z_{Per}(P)$ is a subgroup of $Z(P)$.

Consider homeomorphisms $J \in Z(P)$ and $\iota \in Z_{Per}(P)$. Then $\eta_P\left(J\iota J^{-1}\right) = \eta_P(J)\eta_P(\iota)\eta_P^{-1}(J) = \eta_P(\iota) = 1$. By Lemma 2, $J\iota J^{-1} \in Z_{Per}(P)$. Thus, $Z_{Per}(P)$ is a normal subgroup of $Z(P)$. □

Thus, point 2 of Theorem 1 is proved.

Recall that $N_P = \min\limits_{b_q \in \mathcal{B}_P,\, b_q \neq 0} \{qb_q\}$, where $\mathcal{B}_P$ is the singular type of a pseudo-Anosov homeomorphism P (the smallest number of separatrices related to saddles of the same valency).

Lemma 5. *Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism. Then $|Z_{Per}(P)| \leqslant N_P$.*

Proof. Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism and $J_1, J_2 \in Z_{Per}(P)$. Denote by l^u a separatrix of a saddle singularity of the foliation $\mathcal{F}_P^u$. We put $l_1^u = J_1(l^u)$ and $l_2^u = J_2(l^u)$. Assume that $l_1^u = l_2^u$. Then $J_1^{-1}(J_2(l^u)) = l^u$. By Lemma 3, $J_1^{-1}J_2 = \mathrm{id}_{M^2}$. Then $J_1 \neq J_2$ if and only if $l_1^u \neq l_2^u$. Therefore, a homeomorphism $J \in Z_{Per}(P)$ is uniquely determined by the image of the separatrix l^u. Consider $\overline{q} \in \mathbb{N}$ such that $N_P = \overline{q}b_{\overline{q}}$. By Lemma 1, homeomorphisms J_1, J_2 map separatrices of saddle singularities of valency $\overline{q}$ to separatrices of saddle singularities of valency $\overline{q}$. Thus, $|Z_{Per}(P)| \leqslant N_P$. □

Thus, point 3 of Theorem 1 is proved.

Recall that S_n is the permutation group of degree $n \in \mathbb{N}$.

Lemma 6. *Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism. There is a monomorphism $\zeta_P\colon Z_{Per}(P) \to S_{N_P}$ such that*

$$\zeta_P(J) = (a_{11} \ldots a_{1n}) \cdots (a_{m1} \ldots a_{mn}) \in S_{N_P},$$

where n is a period of $J \in Z_{Per}(P)$ and $mn = N_P$.

Proof. Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism. Consider the separatrices of saddle singularities of the foliation $\mathcal{F}_P^s$ of valency $\overline{q}$ such that $N_P = \overline{q}b_{\overline{q}}$. Denote these separatrices by $l_1, \ldots, l_{N_P}$. Consider the homeomorphism $J \in Z_{Per}(P)$ of period n. Since J maps the separatrices of a saddle singularity of valency $\overline{q}$ to the separatrices of a saddle singularity of valency $\overline{q}$ (see Lemma 1), there is a permutation

$$\omega_J = \begin{pmatrix} l_1 & \cdots & l_{N_P} \\ J(l_1) & \cdots & J\left(l_{N_P}\right) \end{pmatrix}$$

which corresponds to $J \in Z_{Per}(P)$. It follows from Lemma 3 that the separatrix l_i for any $i \in \{1, \ldots, N_P\}$ has the same period n. Consequently, the permutation ω_J can be expressed as the product of $\frac{N_P}{n} = m$ disjoint cycles of length n. We define the mapping $\zeta_P\colon Z_{Per}(P) \to S_{N_P}$ by the rule $\zeta_P(J) = \omega_J$. Then

$$\zeta_P(J_1)\zeta_P(J_2) = \begin{pmatrix} l_1 & \cdots & l_{N_P} \\ J_1(l_1) & \cdots & J_1\left(l_{N_P}\right) \end{pmatrix} \begin{pmatrix} l_1 & \cdots & l_{N_P} \\ J_2(l_1) & \cdots & J_2\left(l_{N_P}\right) \end{pmatrix} =$$

$$= \begin{pmatrix} l_1 & \cdots & l_{N_P} \\ J_1(J_2(l_1)) & \cdots & J_1\left(J_2\left(l_{N_P}\right)\right) \end{pmatrix} = \zeta_P(J_1 J_2).$$

Therefore, the mapping ζ_P is a homomorphism. It follows from Lemma 3 that the kernel of the homomorphism ζ_P consists of the identity homeomorphism id_{M^2}. Therefore, the mapping ζ_P is a monomorphism. □

Thus, point 4 of Theorem 1 is proved.

Lemma 7. *Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism and $J \in Z_{Per}(P)$ be a homeomorphism of period n. Then n divides qb_q for each $b_q \in \mathcal{B}_P$.*

Proof. Let P be a pseudo-Anosov homeomorphism and $J \in Z_{Per}(P)$ be a homeomorphism of period n.

Let us prove that n divides qb_q for each $b_q \in \mathcal{B}_P$. Consider saddle singularities of valency q of the foliation $\mathcal{F}^u_P$. Since $JPJ^{-1} = P$, the homeomorphism J conjugates the pseudo-Anosov homeomorphism P to itself. Then, by Proposition 1, the homeomorphism J maps singular points of valency q of the foliation $\mathcal{F}^u_P$ to singular points of valency q of the foliation $\mathcal{F}^u_P$. Consequently, the homeomorphism J maps separatrices of saddle singularities of valency q to separatrices of saddle singularities of valency q. It follows from Lemma 3 that the orbit of any separatrix l^u of a saddle singularity of valency q consists of n curves. Thus, the set consisting of qb_q separatrices of saddle singularities of valency q is partitioned into $\frac{qb_q}{n}$ orbits, that is, n is a divisor of qb_q. □

Thus, point 5 of Theorem 1 is proved.

Lemma 8. *For any pseudo-Anosov homeomorphism $P\colon M^2 \to M^2$, there exists $\nu_P > 1$ such that $\eta_P(Z(P)) = \left\{\nu_P^i \mid i \in \mathbb{Z}\right\}$.*

Proof. Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism and $J \in Z(P)$.

We put $\eta_P^+(Z(P)) = \{\nu \in \eta_P(Z(P)) \mid \nu > 1\}$. It follows from Lemma 2 that any homeomorphism J from the preimage of $\eta_P^+(Z(P))$ is a pseudo-Anosov homeomorphism with dilatation $\eta_P(J)$. Consider an arbitrary $\lambda \in \eta_P^+(Z(P))$. Then, by Proposition 4, there exist finitely many topological conjugacy classes of pseudo-Anosov homeomorphisms on M^2 with dilatation less than λ. Since dilatation is a topological invariant of pseudo-Anosov homeomorphisms (Proposition 3), there are finitely many elements in the set $\eta_P^+(Z(P))$ that are less than λ. Therefore, there is a minimal element of the set $\eta_P^+(Z(P))$. We put $\nu_P = \min \eta_P^+(Z(P))$.

Let us prove that for any $J \in Z(P)$ there exists $n \in \mathbb{Z}$ such that $\eta_P(J) = \nu_P^n$.

If J is a periodic homeomorphism, $\eta_P(J) = \nu_P^0 = 1$, that is, $n = 0$.

Let J be a pseudo-Anosov homeomorphism such that $\eta_P(J) \in \eta_P^+(Z(P))$. Let us prove that there exists $i \in \mathbb{N}$ such that $\eta_P(J) = \nu_P^i$. If $\eta_P(J) = \nu_P$, then $i = 1$. If $\eta_P(J) \neq \nu_P$, we have $\eta_P(J) > \nu_P$. Note that $\frac{\eta_P(J)}{\nu_P} > 1$ and $\frac{\eta_P(J)}{\nu_P} \in \eta_P^+(Z(P))$. Let $m \in \mathbb{Z}$ be such that $\frac{\eta_P(J)}{\nu_P^m} > \nu_P$ and $\frac{\eta_P(J)}{\nu_P^{m+1}} \leqslant \nu_P$. Then $\frac{\eta_P(J)}{\nu_P^{m+1}} > 1$ and $\frac{\eta_P(J)}{\nu_P^{m+1}} \in \eta_P^+(Z(P))$. It follows from $\nu_P = \min \eta_P^+(Z(P))$ that $\frac{\eta_P(J)}{\nu_P^{m+1}} = \nu_P$. Therefore, $i = m+2$.

Since $\eta_P(J) \in \eta_P(Z(P))$ if and only if $\frac{1}{\eta_P(J)} \in \eta_P(Z(P))$, we have $\eta_P(Z(P)) = \big\{\nu_P^i \mid i \in \mathbb{Z}\big\}$. $\square$

Lemma 9. *Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism. Then a homeomorphism $J \in Z(P)$ has the form $J = p^i\iota$, where p is the pseudo-Anosov homeomorphism with the singular type $\mathcal{B}_p = \mathcal{B}_P$, $i \in \mathbb{Z}$ and $\iota \in Z_{Per}(P)$.*

Proof. Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism. Consider the homeomorphism $J \in Z(P)$. Put $\nu = \eta_P(J)$. By Lemma 8, there exists $\nu_P > 1$ such that $\eta_P(Z(P)) = \{\nu_P^i \mid i \in \mathbb{Z}\}$. Consequently, there exists $i \in \mathbb{Z}$ such that $\nu = \nu_P^i$. We choose a homeomorphism $p \in Z(P)$ such that $\eta_P(p) = \nu_P$ and put $\iota = p^{-i}J$. Then $\eta_P(\iota) = 1$ and, by Lemma 2, the homeomorphism ι is periodic. Thus, $J = p^i\iota$. It follows from Lemma 1 that $\mathcal{B}_p = \mathcal{B}_P$. $\square$

Thus, point 6 of Theorem 1 is proved.

3. Periodic mappings commuting with a pseudo-Anosov homeomorphism

This section contains a proof of Theorem 2 divided into lemmas.

Lemma 10. *Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism and $J \in Z_{Per}(P)$ be a mapping of period $n = 2m$, $m \in \mathbb{N}$. Then the period of $x \in M^2 \setminus (\mathcal{L}_P \cup \mathcal{S}_P)$ is equal to either n or m.*

Proof. Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism, $J \in Z_{Per}(P)$ be a mapping of period $n = 2m$, $m \in \mathbb{N}$, and $x \in M^2 \setminus (\mathcal{L}_P \cup \mathcal{S}_P)$. We denote by l^s the leaf of the foliation $\mathcal{F}_P^s$ containing the point x and by l^s_+, l^s_- connected components of the set $l^s \setminus \{x\}$. Let l^u be a separatrix of a saddle singularity of the foliation $\mathcal{F}_P^u$. By Proposition 7, the set $A = l^u \cap \alpha$ for any arc $\alpha \subset l^s$ consists of countably many points. We choose points $x_+ \in l^u \cap l^s_+$, $x_- \in l^u \cap l^s_-$ and denote by k the period of x. Since $J^k(x) = x$ and the foliation $\mathcal{F}_P^s$ is invariant under J, we have $J^k(l^s) = l^s$. There are two cases: 1) $J^k\left(l^s_+\right) = l^s_+$ and 2) $J^k\left(l^s_+\right) = l^s_-$.

Let us consider the case 1): $J^k\left(l^s_+\right) = l^s_+$. We denote by $[x, x_+]$ an arc on l_s with boundary points x and x_+. Since $\mu^s([x, x_+]) = \mu^s(J^k([x, x_+]))$ and $J^k(x) = x$, we have $J^k(x_+) = x_+$. Therefore, $J^k(L^u) = L^u$. It follows from Lemma 3 that the homeomorphism J has period k. Therefore, $k = n$.

Let us consider the case 2): $J^k\left(l^s_+\right) = l^s_-$. Then $J^{2k}\left(l^s_+\right) = l^s_+$. It follows from the previous case that $J^{2k}(x_+) = x_+$. By Lemma 3, the period of $x_+ \in l^u$ is equal to $2k$. Consequently, $k = m$. $\square$

Recall that, for a periodic homeomorphism $J\colon M^2 \to M^2$ of period n, Q_J^k is the set of all points of period k and Q_J is the set of all points of period less than n. For a pseudo-Anosov homeomorphism $P\colon M^2 \to M^2$, we remember that $\mathcal{S}_P$ is the set of saddle singularities of P-invariant foliations and $\mathcal{L}_P$ is the set of all separatrices of saddle singularities for P-invariant foliations (note that $\mathcal{L}_P \cap \mathcal{S}_P = \varnothing$).

Lemma 11. *Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism and $J \in Z_{Per}(P)$ be an orientation-reversing homeomorphism of period n. Then the following statements hold*:

1) *if* $n = 2(2k-1)$, $k \in \mathbb{N}$, *then* $Q_J^{n/2} = \varnothing$ *and* $Q_J \subset \mathcal{S}_P$;

2) *if* $n = 4k$, $k \in \mathbb{N}$, *then* $Q_J^{n/2} \subset \big(M^2 \setminus \mathcal{L}_P\big)$ *and* $Q_J^k \subset \mathcal{S}_P$ *for any* $k < \frac{n}{2}$.

Proof. Let $P\colon M^2 \to M^2$ be a pseudo-Anosov homeomorphism and $J \in Z_{Per}(P)$ be an orientation-reversing homeomorphism of period n. Note that $n = 2k$, where $k \in \mathbb{Z}$. We consider two cases separately: 1) $n = 2(2k-1)$, $k \in \mathbb{N}$; 2) $n = 4k$, $k \in \mathbb{N}$.

Consider the case 1) $n = 2(2k-1)$. Let us prove that $Q_J^{n/2} = \varnothing$. Assume the converse. Then $Q_J^{n/2}$ is an infinite set consisting of a disjoint union of simple closed curves (see [17, Corollary 2.4]). A separatrix l^u of a saddle singularity of the foliation $\mathcal{F}_P^u$ is everywhere dense on M^2 (see Proposition 8). Therefore, there exists a point $x \in l^u \cap Q_J^{n/2}$. By Lemma 3, the period of a point $x \in l^u$ is equal to n. This contradiction concludes that $Q_J^{n/2} = \varnothing$. Then, it follows from Lemma 3 and Lemma 10 that $Q_J \subset \mathcal{S}_P$.

Consider the case 2) $n = 4k$, $k \in \mathbb{N}$. Then it follows from Lemma 3 that $Q_J^{n/2} \subset \big(M^2 \setminus \mathcal{L}_P\big)$ and it follows from Lemma 10 that $Q_J^k \subset \mathcal{S}_P$ for any $k < \frac{n}{2}$. □

Thus, points 1 and 2 of Theorem 2 are proved.

Lemmas 3 and 10 imply the next lemma.

Lemma 12. *Let* $P\colon M^2 \to M^2$ *be a pseudo-Anosov homeomorphism and* $J \in Z_{Per}(P)$ *be an orientation-preserving homeomorphism of period* n. *Then the following statements hold*:

1) *if* $n = 2m-1$, $m \in \mathbb{N}$, *then* $Q_J \subset \mathcal{S}_P$;

2) *if* $n = 2m$, $m \in \mathbb{N}$, *then* $Q_J^{n/2} \subset \big(M^2 \setminus \mathcal{L}_P\big)$ *and* $Q_J^k \subset \mathcal{S}_P$ *for any* $k < \frac{n}{2}$.

Thus, points 3 and 4 of Theorem 2 are true.

4. Trivial centralizers

This section contains a proof of Theorem 3, which states that each closed orientable surface of genus $g \geqslant 3$ admits a pseudo-Anosov homeomorphism with the trivial centralizer.

Proof. Consider a closed orientable surface M^2 of genus $g \geqslant 3$. Statements 10 and 11 imply that there exists a pseudo-Anosov homeomorphism $P\colon M^2 \to M^2$ with the singular type $\mathcal{B}_P = \{b_3 = 1,\ b_4 = 1,\ b_{4g-5} = 1\}$ (note that $4g - 5 \geqslant 7$). By point 5 of Theorem 1, period n of a homeomorphism $J \in Z_{Per}(P)$ divides 3, 4 and $4g - 5$. Therefore, $n = 1$ and $Z_{Per}(P) = \mathrm{id}_{M^2}$.

Then it immediately follows from Theorem 1 that $Z(P) = \{p^n \mid n \in \mathbb{Z}\}$, where $p\colon M^2 \to M^2$ is a pseudo-Anosov homeomorphism with the singular type $\mathcal{B}_p = \mathcal{B}_P = \{b_3 = 1,\ b_4 = 1,\ b_{4g-5} = 1\}$ and with the trivial centralizer $Z(p) = Z(P) = \{p^n \mid n \in \mathbb{Z}\}$. □

5. Mapping class group of the pseudo-Anosov centralizer

This section contains a proof of Theorem 4 and a corollary.

A homeomorphism $f\colon M^2 \to M^2$ is called homotopically periodic if there exists $m \in \mathbb{N}$ such that f^m is homotopic to the identity map id_{M^2}. The least such m is called the homotopical order of f.

Statement 12 (see [18, Proposition 5.2.]). *Let $f\colon M^2 \to M^2$ be a homotopically periodic homeomorphism of homotopical order m on M^2 of genus $g \geqslant 2$. Then the period of any periodic homeomorphism which is homotopic to f is equal to m.*

Let P be a pseudo-Anosov homeomorphism. Let us prove Theorem 4, which states that if the homeomorphism $J_1 \in Z(P)$ is homotopic to the homeomorphism $J_2 \in Z(P)$, then $J_1 = J_2$.

Proof. Let P be a pseudo-Anosov homeomorphism and the homeomorphism $J_1 \in Z(P)$ is homotopic to the homeomorphism $J_2 \in Z(P)$. Then there exists a homeomorphism $h\colon M^2 \to M^2$ which is homotopic to the identity id_{M^2} such that $J_1 = hJ_2$. Note that $h = J_1 J_2^{-1} \in Z(P)$. Since h is homotopic to the identity id_{M^2}, it cannot be pseudo-Anosov (see, for example, [19, Theorem 4]). Therefore, h is a periodic homeomorphism (see point 1 of Theorem 1), which is homotopic to the identity id_{M^2}. Then, by Statement 12, $h = \mathrm{id}_{M^2}$. Thus, $J_1 = J_2$. □

Let P_1, $P_2\colon M^2 \to M^2$ be pseudo-Anosov homeomorphisms and there exist homeomorphisms H_1, $H_2\colon M^2 \to M^2$ such that $H_1P_1 = P_2H_1$, $H_2P_1 = P_2H_2$. Let us prove the corollary of Theorem 4, which states that, if the homeomorphism H_1 is homotopic to the homeomorphism H_2, then $H_1 = H_2$.

Proof. Let P_1, $P_2\colon M^2 \to M^2$ be pseudo-Anosov homeomorphisms and there exist homotopic homeomorphisms H_1, $H_2\colon M^2 \to M^2$ such that $H_1P_1 = P_2H_1$, $H_2P_1 = P_2H_2$. Then $P_1 = H_1^{-1}P_2H_1$ and $H_2H_1^{-1}P_2H_1 = P_2H_2$. Therefore, $H_2H_1^{-1}P_2 = P_2H_2H_1^{-1}$ and $H_2H_1^{-1} \in Z(P_2)$. Since the homeomorphism $H_2H_1^{-1}$ is homotopic to the identity id_{M^2} and $\mathrm{id}_{M^2} \in Z(P_2)$, Theorem 4 implies $H_2H_1^{-1} = \mathrm{id}_{M^2}$, that is, $H_1 = H_2$. □

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Conflict of interest

The author declares that she has no conflict of interest.

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